SPRINGER FIBERS AND THE DELTA CONJECTURE AT $t = 0$

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Abstract. We introduce a family of varieties $Y_{n,\lambda,s}$, SG: which we call the Δ-*Springer varieties*, that generalize the type A Springer fibers. We give an explicit presentation of the cohomology ring $H^*(Y_{n,\lambda,s})$ and show that there is a symmetric group action on this ring that generalizes the Springer action on the cohomology of a Springer fiber. The $\lambda = (1^k)$ case of this construction gives a new geometric realization for the expression in the Delta Conjecture when $t = 0$. We also prove that the top cohomology groups of these varieties give a generalization of the type A Springer correspondence to the setting of induced Specht modules. Finally, we generalize results of de Concini and Procesi. Precisely, we find a topological space $Y_{n,\lambda}$ whose cohomology ring is isomorphic to the coordinate ring of the scheme of "rank deficient" diagonal matrices.

To Do:

- ☐ Subsec 2.1, Background: Schubert cells
- ☐ Subsec 2.6, Background: $R_{n,\lambda,s}$
- ☐ Sec 3, Add a full example of affine paving
- ☐ Sec 5, Rewrite using new notations
- ☐ Sec 7, Define $Y_{n,\lambda}$, show its cohomology is $R_{n,\lambda}$, state that this cohomology is iso to the coord ring of the diagonal rank scheme
- ☐ Sec 8, Perhaps add an example of the partial orders on the set of admissible permutations, and the bijection with ordered set partitions in the case of $\lambda = (1^k)$ and $s = k$.

1. Introduction

In this article, we introduce a family of varieties generalizing the Springer fibers. We prove an explicit presentation of their cohomology rings generalizing the one given by Tanisaki for the cohomology ring of a Springer fiber, which coincides with the rings $R_{n,\lambda,s}$ introduced by the first author [11]. As a special case, our construction gives a new *compact* geometric realization of the expression in the Delta Conjecture in the case $t = 0$. We also prove a version of the Springer correspondence for this family of varieties, showing that their top cohomology groups have the S_n-module structure of an induced Specht module. Finally, we generalize work of de Concini and Procesi [7] by introducing a topological space whose cohomology ring coincides with the coordinate ring of the scheme-theoretic intersection of an Eisenbud–Saltman rank variety with diagonal matrices.

In the seminal work [21, 22], T.A. Springer introduced a family of varieties associated to any complete flag variety G/B, called Springer fibers, that have remarkable connections to

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the representation theory of Weyl groups. Springer proved that although the Weyl group does not act on a Springer fiber, it does act nontrivially on the cohomology ring of a Springer fiber. Furthermore, Springer proved that the highest degree nonzero cohomology group of a Springer fiber is (in type A) an irreducible representation of the Weyl group, and every irreducible representation appears this way. This is known as the *Springer correspondence*. We note that the S_n-action discussed in this paper differs from Springer's original construction by tensoring with the sign representation.

The graded S_n-module type of the cohomology ring of a Springer fiber was discovered by Hotta and Springer [17]. Under the Frobenius characteristic map Frob that associates a symmetric function to each S_n-module, the cohomology ring of a Springer fiber is sent to the *modified Hall-Littlewood symmetric function*

$$\mathrm{Frob}(H^*(\mathcal{B}^\lambda;\mathbb{Q});q) = \widetilde{H}_\lambda(x;q^2), \tag{1.1}$$

where the q on the left-hand side keeps track of the grading of the cohomology ring. A detailed analysis of this connection has been given by Garsia and Procesi [9], who were inspired by the explicit quotient ring presentations for $H^*(\mathcal{B}^\lambda;\mathbb{Q})$ discovered by De Concini-Procesi [7] and Tanisaki [23].

The Delta Conjecture of Haglund–Remmel–Wilson [12] predicts two combinatorial formulas for a particular symmetric function with q and t parameters $\Delta'_{e_{k-1}}e_n(q,t)$ coming from the theory of Macdonald polynomials. The conjecture is known to be true in several special cases, and the *rise* version of the conjecture has recently been proven in full generality [6]. Since $\Delta'_{e_{k-1}}e_n$ is conjectured to be Schur-positive, there is much interest in a natural algebraic or geometric construction of a (bigraded) S_n-module whose Frobenius characteristic is $\Delta'_{e_{k-1}}e_n$. Haglund–Rhoades–Shimozono [13] did this in the case $t=0$ by constructing a graded ring $R_{n,k}$ with a suitable S_n-action whose graded Frobenius characteristic is $\Delta'_{e_{k-1}}e_n(q,0)$ (after a minor twist).

Pawlowski and Rhoades [18] gave a parallel geometric interpretation by exhibiting a complex algebraic variety whose cohomology ring is $R_{n,k}$. Since the Hilbert series of $R_{n,k}$ is not symmetric, such a variety must be either non-compact or singular by Poincaré Duality. Pawlowski defined the non-compact smooth space of *spanning line arrangements*, n-tuples of lines in $\mathbb{C}^k$ that span $\mathbb{C}^k$,

$$X_{n,k} \coloneqq \{(L_1,\ldots,L_n)\in(\mathbb{P}^{k-1})^n \mid L_1+\cdots+L_n=\mathbb{C}^k\}. \tag{1.2}$$

They proved that

$$H^*(X_{n,k}) \cong R_{n,k}, \tag{1.3}$$

thus giving a connection between the expression in the Delta Conjecture at $t=0$ and geometry. Since the Poincaré series recording the graded dimensions of the ring $R_{n,k}$ is not symmetric, then by Poincaré duality, any complex variety whose cohomology ring is isomorphic to $R_{n,k}$ must either be noncompact or singular.

In this article, we introduce a compact and singular variety $Y_{n,(1^k),k}$, similar to a Springer fiber, whose cohomology ring is the Haglund–Rhoades–Shimozono ring $R_{n,k}$. Thus, the variety $Y_{n,(1^k),k}$ gives a new geometric realization of the expression in the Delta Conjecture when $t=0$. Furthermore, the family $Y_{n,(1^k),k}$ extends to a family of varieties $Y_{n,\lambda,s}$ generalizing the Springer fibers. This allows us to use techniques from the study of Springer fibers to analyze our varieties. Furthermore, it situates the study of $R_{n,k}$ and $\Delta'_{e_{k-1}}e_n$ in the context of the theory of Springer fibers and geometric representation theory.

As our main result, we prove an explicit presentation of the ring $H^*(Y_{n,\lambda,s})$ as a quotient of a polynomial ring, generalization Tanisaki's presentation for the cohomology ring of a Springer fiber [23]. This presentation coincides with the graded ring $R_{n,\lambda,s}$ recently introduced by the first author [11]. As a consequence, we see that the cohomology ring of $Y_{n,\lambda,s}$ has a graded S_n-module structure generalizing the classical one in the Springer fiber case.

We use results of the first author to prove a generalization of the Springer correspondence to the setting of induced Specht modules. We show that for $s > \ell(\lambda)$,

$$H^{2d}(Y_{n,\lambda,s};\mathbb{Q}) \cong \mathrm{Ind}\uparrow_{S_k}^{S_n}(S^\lambda), \tag{1.4}$$

where d is the dimension of the variety $Y_{n,\lambda,s}$ and S^λ is the irreducible Specht module indexed by λ. In the special case when $s = \ell(\lambda)$, we show that the top cohomology group is a skew Specht module $S^{\Lambda/(n-k)^{s-1}}$. We also prove that $Y_{n,\lambda,s}$ is equidimensional of complex dimension

$$d = \sum_i \binom{\lambda'_i}{2} + (s-1)(n-k), \tag{1.5}$$

and we give a characterization of the irreducible components of $Y_{n,\lambda,s}$.

Finally, we generalize results of de Concini and Procesi [7]. Let $\mathfrak{sl}_n$ be the Lie algebra of trace zero $n \times n$ matrices over $\mathbb{Q}$. Given $\lambda \vdash n$, define $O_\lambda \subseteq \mathfrak{sl}_n$ to be the set of $n \times n$ nilpotent matrices over $\mathbb{Q}$ with Jordan type λ, and let $\overline{O}_\lambda$ be its closure in $\mathfrak{sl}_n$. Let $\mathfrak{t} \subset \mathfrak{sl}_n$ be the Cartan subalgebra of diagonal matrices. Then de Concini and Procesi proved that

$$H^*(\mathcal{B}^\lambda;\mathbb{Q}) \cong \mathbb{Q}[\overline{O}_{\lambda'} \cap \mathfrak{t}], \tag{1.6}$$

where the right-hand side is the coordinate ring of the scheme-theoretic intersection of $\overline{O}_{\lambda'}$ and $\mathfrak{t}$. Given a partition λ of size at most n, let $\overline{O}_{n,\lambda}$ be the Eisenbud–Saltman rank variety (defined in Section 7). We define the direct limit space $Y_{n,\lambda} := \varinjlim_s Y_{n,\lambda,s}$ and prove that there is an isomorphism of graded rings and graded S_n-modules

$$H^*(Y_{n,\lambda};\mathbb{Q}) \cong \mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]. \tag{1.7}$$

In Section 2, we outline preliminary definitions and previous results. In Section 3, we define the variety $Y_{n,\lambda,s}$ and prove that it has an affine paving given by intersecting $Y_{n,\lambda,s}$ with Schubert cells. We then use this to compute the rank generating function of the cohomology ring. In Section 4, we analyze the case of $\lambda = \emptyset$ and prove that the variety $Y_{n,\emptyset,s}$ has the same cohomology ring as a product of projective spaces. In Section 5, we show that $Y_{n,\lambda,s}$ is the image of a projection down from a Spaltenstein variety, and we us this to prove a presentation of the cohomology ring of $Y_{n,\lambda,s}$ as a quotient ring. In Section 6, we prove our generalization of the Springer correspondence, and we characterize the irreducible components of $Y_{n,\lambda,s}$. In Section 7, we introduce $Y_{n,\lambda}$ and prove the isomorphism (1.7). Finally, in Section 8 we list some open problems.

2. Background

2.1. **Flag varieties and Schubert cells.** SG: To do: Finish this section

Given a vector space V, a *partial flag* is a nested sequence of vector subspaces of V,

$$V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_m). \tag{2.1}$$

Given a strictly increasing sequence of positive integers $n_1 \leq n_2 \leq \cdots \leq n_m \leq \dim(V)$, define the *partial flag variety* to be the set of partial flags of V such that the dimensions of the subspaces are given by the sequence $(n_i)_i$,

$$\mathrm{Fl}_{n_1,\ldots,n_m}(V) := \{V_\bullet = (V_1 \subset \cdots \subset V_m) \mid \dim(V_i) = n_i \text{ for } i \leq m\}. \tag{2.2}$$

In the case when $V = \mathbb{C}^n$ and $n_i = i$ for $i \leq n$, we recover the *complete flag variety*, denoted by $\mathrm{Fl}(n) = \mathrm{Fl}_{1,\ldots,n}(\mathbb{C}^n)$.

SG: To do: Define Schubert cells and Schubert varieties. State the fact that the PD classes of the Schubert varieties are a basis of cohomology here?

SG: Should we switch to indexing partial flag varieties by compositions and include the ambient space in our flag? If so, we should be careful about the Schubert conditions ($NV_i \subseteq V_i$ is different from $NV_i \subseteq V_{i-1}$ in that case)

2.2. **Chern classes.** Given a complex vector bundle E on a topological space X, the ith Chern class of E is a distinguished cohomology class $c_i(E) \in H^{2i}(X)$, where $c_0(E) = 1$. The Chern classes are invariants of the vector bundle, in the sense that if two vector bundles on X are isomorphic, then their Chern classes agree.

The sum of the Chern classes of a vector bundle $c(E) := 1 + c_1(E) + c_2(E) + \cdots$ is called the **total Chern class** of E. It has the following useful properties.

- Naturality: For any continuous map $f : X \to Y$ and any complex vector bundle E on Y, then $f^*(c(E)) = c(f^*(E))$, where the first f^* is the map on cohomology and $f^*(E)$ is the pullback of E.
- Additivity: Given a short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ on X, we have

$$c(E) = c(E')c(E''), \tag{2.3}$$

  where multiplication is via the cup product on cohomology.
- Vanishing: If r is the rank of E as a complex vector bundle, then $c_i(E) = 0$ for all $i > r$.
- Triviality: If $E \cong \mathbb{C}^r \times X$, a trivial vector bundle, then $c(E) = 1$.

In the case of $X = \mathrm{Fl}(\mathbb{C}^n)$, for each j there is the tautological vector bundle $\widetilde{V}_j$ whose fiber over $V_\bullet = (V_1, \ldots, V_n)$ is the vector space V_j. Borel [3] proved that the classes $c_1(\widetilde{V}_j/\widetilde{V}_{j-1})$ generate the cohomology ring $H^*(\mathrm{Fl}(\mathbb{C}^n))$ as a graded algebra. Moreover, there is an isomorphism of graded algebras,

$$H^*(\mathrm{Fl}(\mathbb{C}^n)) \cong \frac{\mathbb{Z}[x_1, \ldots, x_n]}{\langle e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)\rangle} \tag{2.4}$$

identifying $-c_1(\widetilde{V}_j/\widetilde{V}_{j-1})$ with x_j, where each variable x_j is considered to be degree 2. The quotient ring on the right-hand side of (2.4) is also known as the *coinvariant ring*.

2.3. **Affine paving.** An affine paving is another tool that we will use for working with cohomology. Given a complex algebraic variety X, an **affine paving** of X is a sequence of closed subvarieties

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_m = X \tag{2.5}$$

of X such that $X_i \setminus X_{i-1} \cong \bigsqcup_j A_{i,j}$ for locally closed subspaces $A_{i,j}$ such that for all i, j, $A_{i,j} \cong \mathbb{C}^k$ for some k. The affine spaces $A_{i,j}$ are called the **cells** of the affine paving.

An affine paving allows us to compute the ranks of the cohomology groups of X with compact support. When X is compact in the analytic topology, cohomology with compact support is the same as cohomology, so an affine paving gives us a way of computing the ranks of the cohomology groups.

Lemma 2.1. *Suppose X is a smooth or compact complex algebraic variety that has an affine paving. If $X_i \setminus X_{i-1} = \bigsqcup_{i,j} A_{i,j}$ is the decomposition of X into affine spaces, then*

$$H_c^{2k}(X) \cong \mathbb{Z}^{\#\{(i,j)\,|\,\dim_{\mathbb{C}}(A_{i,j})=k\}} \tag{2.6}$$

$$H_c^{2k+1}(X) = 0, \tag{2.7}$$

for all $k \geq 0$.

Under certain conditions, affine pavings can also be used to prove that the map on cohomology corresponding to a continuous map is injective or surjective.

Lemma 2.2. *Suppose X is a smooth compact complex algebraic variety and $Y \subseteq X$ is a closed subvariety of X. If Y and $X \setminus Y$ have affine pavings, then the map on cohomology*

$$H^*(X) \to H^*(Y) \tag{2.8}$$

induced by the inclusion $Y \subseteq X$ is surjective.

Proof. By Lemma 2.1, all odd cohomology groups of X and Y and all odd cohomology groups with compact support of $X \setminus Y$ are zero. By the long exact sequence for compactly supported cohomology associated to the diagram $Y \hookrightarrow X \hookleftarrow X \setminus Y$, we have short exact sequences

$$0 \to H_c^{2i}(X \setminus Y) \to H^{2i}(X) \to H^{2i}(Y) \to 0 \tag{2.9}$$

for all i. The surjectivity of the map on cohomology then follows from (2.9). □

Lemma 2.3. *Suppose $f : X \to Y$ is a surjective continuous map between compact complex algebraic varieties. Suppose that Y has an affine paving such that for each cell $A_{i,j}$,*

$$f^{-1}(A_{i,j}) \cong Z_{i,j} \times A_{i,j} \tag{2.10}$$

for some nonempty compact complex algebraic variety $Z_{i,j}$ with an affine paving. Then the map on cohomology

$$H^*(Y) \to H^*(X) \tag{2.11}$$

is injective.

Proof. Since Y has an affine paving, and $f^{-1}(A_{i,j}) \cong Z_{i,j} \times A_{i,j}$, it can be seen that X has an affine paving with cells $C \times A_{i,j}$, where C runs over all cells of $Z_{i,j}$. Therefore, $H_*(X)$ is freely generated by the fundamental classes $[C \times A_{i,j}]$.

Since $Z_{i,j}$ is compact, there is a cell of $Z_{i,j}$ consisting of a single point, $C = \{\mathrm{pt}\}$. Letting $f_* : H_*(X) \to H_*(Y)$ be the map on homology induced by f, we have

$$f_*([\{\mathrm{pt}\} \times A_{i,j}]) = [A_{i,j}], \tag{2.12}$$

hence f_* is surjective. By the Universal Coefficient Theorem, the map f^* is the dual of f_*, which is thus injective. □

2.4. **Springer fibers.** Given a partition λ of n, let N_λ be a $n \times n$ nilpotent matrix whose Jordan block sizes are recorded by λ. The **Springer fiber** associated to λ is

$$\mathcal{B}^\lambda := \{V_\bullet \in \mathrm{Fl}(n) \mid N_\lambda V_i \subseteq V_i \text{ for all } i \leq n\}. \tag{2.13}$$

Springer proved that although S_n does not act on $\mathcal{B}^\lambda$, it does act on the cohomology ring of $\mathcal{B}^\lambda$. We note that in this article, the action on the cohomology ring we consider differs from the one originally constructed by Springer by tensoring with the sign representation.

A remarkable property of this action is that it gives a geometric construction of the irreducible Specht modules. Indeed, the dimension of $\mathcal{B}^\lambda$ as a complex variety is

$$n(\lambda) := \sum_i \binom{\lambda'_i}{2}, \tag{2.14}$$

and the top nonzero cohomology group of $\mathcal{B}^\lambda$ as an S_n-module is

$$H^{2n(\lambda)}(\mathcal{B}^\lambda; \mathbb{Q}) \cong S^\lambda. \tag{2.15}$$

Therefore, in Lie type A there is a bijection, known as the *Springer correspondence*, between Springer fibers and the irreducible S_n-modules, up to isomorphism.

Hotta and Springer [17] proved that the map on cohomology induced by the inclusion $\mathcal{B}^\lambda \subseteq \mathrm{Fl}(n)$,

$$H^*(\mathrm{Fl}(n)) \to H^*(\mathcal{B}^\lambda), \tag{2.16}$$

is surjective and S_n-equivariant. Hence, by surjectivity the cohomology ring $H^*(\mathcal{B}^\lambda)$ is generated by the cohomology classes $c_1(\widetilde{V}_i/\widetilde{V}_{i-1})$. Here, we are abusing notation and writing $\widetilde{V}_i$ for the restriction of this vector bundle to $\mathcal{B}^\lambda$.

There is an explicit presentation of $H^*(\mathcal{B}^\lambda)$ as a quotient ring extending Borel's theorem [7, 23]. For all $i \leq n$, let $p_i(\lambda) = \lambda'_n + \lambda'_{n-1} + \cdots + \lambda'_{n-i+1}$, where $\lambda'_i = 0$ for all $i > \lambda_1$. Given $S \subseteq \{x_1, \ldots, x_n\}$, define $e_d(S)$ to be the sum of all square-free products of variables in S of degree d. Define the following ideal and quotient ring,

$$I_\lambda := \langle e_d(S) \mid d > |S| - p_{|S|}(\lambda) \rangle, \tag{2.17}$$

$$R_\lambda := \mathbb{Q}[x_1, \ldots, x_n]/I_\lambda. \tag{2.18}$$

Here, and throughout the paper, we consider R_λ to be a graded ring where each variable x_j is in degree 2. Tanisaki proved that there is an isomorphism of graded rings

$$H^*(\mathcal{B}^\lambda; \mathbb{Q}) \cong R_\lambda \tag{2.19}$$

given by identifying the cohomology class $-c_1(\widetilde{V}_j/\widetilde{V}_{j-1})$ with the variable x_j.

For example, when $\lambda = (2,1)$, then $p_1(\lambda) = 0$, $p_2(\lambda) = 1$, and $p_3(\lambda) = 3$. Therefore, I_λ is generated by $e_d(S)$ where $3 \geq d > 0$ and $|S| = 3$, or $2 \geq d > 1$ and $|S| = 2$, so

$$I_{(2,1)} = \langle e_1(x_1,x_2,x_3), e_2(x_1,x_2,x_3), e_3(x_1,x_2,x_3), e_2(x_1,x_2), e_2(x_1,x_3), e_2(x_2,x_3) \rangle \tag{2.20}$$

$$= \langle x_1 + x_2 + x_3,\ x_1x_2 + x_1x_3 + x_2x_3,\ x_1x_2x_3,\ x_1x_2,\ x_1x_3,\ x_2x_3 \rangle \tag{2.21}$$

$$= \langle x_1 + x_2 + x_3,\ x_1x_2,\ x_1x_3,\ x_2x_3 \rangle, \tag{2.22}$$

and $H^*(\mathcal{B}^{(2,1)}) \cong R_{(2,1)} = \mathbb{Z}[x_1, x_2, x_3]/I_{(2,1)}$.

2.5. **Symmetric functions and the Delta Conjecture.** The representation theory of the group S_n is closely related to the theory of symmetric functions. A **symmetric function** is a formal power series in the infinite variable set $\{x_1, x_2, \dots\}$ that is invariant under any permutation of the variables. For $\lambda \vdash n$, let $e_\lambda(x)$ and $s_\lambda(x)$ denote the *elementary symmetric functions* and *Schur symmetric functions*, which form bases of the ring of symmetric functions.

The Frobenius characteristic map gives a connection between symmetric functions and representations of S_n, which we define next. Given $\lambda \vdash n$, let S^λ be the irreducible S_n-module indexed by λ, also known as a *Specht module*. Given a finite-dimensional vector space V over $\mathbb{Q}$ which has the structure of a S_n-module, it decomposes as a direct sum of Specht modules

$$V \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{c_\lambda} \tag{2.23}$$

for some nonnegative integers c_λ. The *Frobenius characteristic* of V is defined to be the symmetric function

$$\mathrm{Frob}(V) = \sum_{\lambda \vdash n} c_\lambda s_\lambda(x). \tag{2.24}$$

Given a graded S_n-module $V = \bigoplus_{i=0}^m V_i$ with finite-dimensional direct summands V_i, the *graded Frobenius characteristic* of V is

$$\mathrm{Frob}(V;q) = \sum_{i=0}^{m} \mathrm{Frob}(V_i) q^i. \tag{2.25}$$

One well-known family of symmetric functions is the *Macdonald symmetric functions* $\widetilde{H}_\lambda(x;q,t)$. In Mark Haiman's groundbreaking work [14, 15], he proved that $\widetilde{H}_\lambda(x;q,t)$ is the Frobenius character of the bigraded *Garsia-Haiman module* [8]. One piece of Haiman's analysis uses linear operators Δ'_f on the space of symmetric functions [1] whose eigenbasis is the set of Macdonald functions $\widetilde{H}_\lambda(x;q,t)$.

Some major open problems following Haiman's work involve finding combinatorial and geometric interpretations for evaluations of this operator when f is a complete elementary symmetric function. One such problem, called the Delta Conjecture [12], predicts two combinatorial formulas for the q,t symmetric function $\Delta'_{e_k} e_n(q,t)$ in terms of combinatorial statistics on *parking functions*. The Delta Conjecture has recently been proven by D'Adderio–Mellit [6] and Blasiak–Haiman–Morse–Pun–Seelinger [2].

There has been an ongoing search for algebraic and geometric interpretations of the expression $\Delta_{e_k} e_n(q,t)$ in the Delta Conjecture. Haglund, Rhoades, and Shimozono [13] found an algebraic interpretation when $t = 0$. Precisely, they defined the following ring $R_{n,k}$ depending on two positive integers $k \leq n$,

$$R_{n,k} = \frac{\mathbb{Z}[x_1, \dots, x_n]}{\langle x_1^k, \dots, x_n^k, e_n, e_{n-1}, \dots, e_{n-k+1} \rangle}. \tag{2.26}$$

When $k = n$, it can be checked that $R_{n,n}$ is equal to the usual coinvariant ring (2.4). They proved that the graded Frobenius characteristic of $R_{n,k}$ is $\Delta'_{e_{k-1}} e_n(q,0)$ at $t = 0$, up to a minor twist:

$$\operatorname{Frob}(R_{n,k};q) = \omega \circ \operatorname{rev}_q(\Delta'_{e_{k-1}} e_n(q,0)), \tag{2.27}$$

where ω and rev_q are simple idempotent operators on symmetric functions. Pawlowski and Rhoades then found a parallel geometric construction for $R_{n,k}$ as the cohomology ring of a space of spanning line arrangements [18]. An algebraic interpretation for the q,t symmetric function $\Delta_{e_{k-1}} e_n$ has been conjectured by Zabrocki [25] in terms of the bigraded super-diagonal coinvariant ring.

2.6. **The rings $R_{n,k}$ and $R_{n,\lambda,s}$.** We recall the definition and properties of the ring $R_{n,\lambda,s}$ introduced by the first author in [11], which simultaneously generalizes the cohomology ring of a Springer fiber $H^*(\mathcal{B}^\lambda)$ and the Haglund–Rhoades–Shimozono ring $R_{n,k}$.

Fix $k \leq n$, a partition $\lambda \vdash k$, and $s \geq \ell(\lambda)$. Let $p^n_m(\lambda) = \lambda'_n + \cdots + \lambda'_{n-m+1}$, where $\lambda'_i = 0$ for all $i > \lambda_1$. The ideal $I_{n,\lambda,s}$ and ring $R_{n,\lambda,s}$ are defined as follows,

$$I_{n,\lambda,s} = \langle x_1^s, \ldots, x_n^s \rangle + \langle e_d(S) \mid S \subseteq \{x_1, \ldots, x_n\},\, d > |S| - p^n_{|S|}(\lambda) \rangle, \tag{2.28}$$

$$R_{n,\lambda,s} = \mathbb{Z}[x_1, \ldots, x_n]/I_{n,\lambda,s}. \tag{2.29}$$

It can be checked that

- When $n = k$, then $I_{n,\lambda,s} = I_\lambda$ for any s, thus $R_{n,\lambda,s} = R_\lambda$ in this case.
- When $\lambda = (1^k)$ and $s = k$, then $I_{n,(1^k),k} = I_{n,k}$, thus $R_{n,(1^k),k} = R_{n,k}$.

For a further example, let $n = 4$, $\lambda = (2,1)$, and $s = 2$. Then $I_{4,(2,1),2}$ is generated by x_i^2 for $i = 1,2,3,4$ and the polynomials $e_d(S)$ for $S \subseteq \{x_1, \ldots, x_4\}$ such that

$$\begin{aligned} &d = 2 \text{ and } |S| = 4, \qquad && d = 3 \text{ and } |S| = 4, \\ &d = 4 \text{ and } |S| = 4, \qquad && d = 3 \text{ and } |S| = 3. \end{aligned}$$

We have

$$\begin{aligned} I_{4,(2,1),2} &= \langle x_1^2, x_2^2, x_3^2, x_4^2, e_2, e_3, e_4, e_3(x_1,x_2,x_3), e_3(x_1,x_2,x_4), e_3(x_1,x_3,x_4), e_3(x_2,x_3,x_4) \rangle, \\ &= \langle x_1^2,\, x_2^2,\, x_3^2,\, x_4^2,\, x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4, \\ &\qquad x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4, \\ &\qquad x_1x_2x_3x_4,\, x_1x_2x_3,\, x_1x_2x_4,\, x_1x_3x_4,\, x_2x_3x_4 \rangle, \end{aligned}$$

and $R_{4,(2,1),2} = \mathbb{Z}[x_1,x_2,x_3,x_4]/I_{4,(2,1),2}$.

Let $I^{\mathbb{Q}}_{n,\lambda,s}$ be the ideal in $\mathbb{Q}[x_1, \ldots, x_n]$ given by the same generators as $I_{n,\lambda,s}$, and let $R^{\mathbb{Q}}_{n,\lambda,s} = \mathbb{Q}[x_1, \ldots, x_n]/I^{\mathbb{Q}}_{n,\lambda,s}$. The first author has proven a basis of $R^{\mathbb{Q}}_{n,\lambda,s}$ generalizing the *Artin basis* of the coinvariant ring. Define $\mathcal{A}_{1,\emptyset,s} = \{1, x_1, \ldots, x_1^{s-1}\}$ and $\mathcal{A}_{1,(1),s} = \{1\}$. Let the set $\mathcal{A}_{n,\lambda,s}$ be defined recursively as follows,

$$\mathcal{A}_{n,\lambda,s} := \bigsqcup_{i=1}^{\ell(\lambda)} x_n^{i-1} \mathcal{A}_{n-1,\lambda^{(i)},s} \sqcup \bigsqcup_{i=\ell(\lambda)+1}^{s} x_n^{i-1} \mathcal{A}_{n-1,\lambda,s}, \tag{2.30}$$

where for $1 \leq i \leq \ell(\lambda)$, $\lambda^{(i)}$ is the partition obtained by sorting the parts of

$$(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_{\ell(\lambda)})$$

and deleting a trailing zero if necessary. Then $\mathcal{A}_{n,\lambda,s}$ is a $\mathbb{Q}$-basis of $R^{\mathbb{Q}}_{n,\lambda,s}$ [11, Theorem 3.18].

SG: Should we move the following lemmata into the body of the paper? They are straightforward from the results in my thesis.

Lemma 2.4. *The set $\mathcal{A}_{n,\lambda,s}$ represents a $\mathbb{Z}$-basis of $R_{n,\lambda,s}$.*

Proof. The proof of Lemma 3.14 in [11] also proves that $\mathcal{A}_{n,\lambda,s}$ is a $\mathbb{Z}$-spanning set of $R_{n,\lambda,s}$. Since $\mathcal{A}_{n,\lambda,s}$ represents a $\mathbb{Q}$-linearly independent subset of $R^{\mathbb{Q}}_{n,\lambda,s}$, then it also represents a $\mathbb{Z}$-linearly independent subset of $R_{n,\lambda,s}$. $\square$

Given $V = \bigoplus_{i\geq 0} V_i$ a graded free $\mathbb{Z}$-module with graded pieces V_i of finite rank $\mathrm{rk}(V_i)$, let the *Hilbert–Poincaré series* of the module V be

$$\mathrm{Hilb}(V;q) := \sum_{i\geq 0} \mathrm{rk}(V_i) q^i. \tag{2.31}$$

By Lemma 2.4, $R_{n,\lambda,s}$ is a free $\mathbb{Z}$-module. Under our convention that x_i has degree 2 for all i, we have the following recursive formula for the Hilbert series, which follows immediately by Lemma 2.4.

Lemma 2.5. *We have*

$$\mathrm{Hilb}(R_{n,\lambda,s};q) = \sum_{i=1}^{\ell(\lambda)} q^{2(i-1)}\mathrm{Hilb}(R_{n-1,\lambda^{(i)},s};q) + \sum_{i=\ell(\lambda)+1}^{s} q^{2(i-1)}\mathrm{Hilb}(R_{n-1,\lambda,s};q). \tag{2.32}$$

Since the set of generators of the homogeneous ideal $I_{n,\lambda,s}$ is closed under the action of S_n permuting variables, $R_{n,\lambda,s}$ inherits the structure of a graded S_n-module. In order to prove our generalization of the Springer correspondence, we make use of a formula for the graded Frobenius characteristic of $R_{n,\lambda,s}$ proven in [11]. We state the formula and define the associated combinatorial objects in Section 6 where we need it.

3. Definition of $Y_{n,\lambda,s}$ and an affine paving

AW: We should say this is an analogue of the Tymoczko and Precup results for Hessenberg varieties and that we're using a similar approach.

In this section, we define a family of varieties $Y_{n,\lambda,s}$ that generalize the Springer fibers. We construct an affine paving of $Y_{n,\lambda,s}$ by intersecting it with Schubert cells, analogous to the affine pavings for Hessenberg varieties constructed by Precup and Tymoczko [19, 20, 24]. We then use this affine paving to show that $H^*(Y_{n,\lambda,s})$ and $R_{n,\lambda,s}$ have the same Hilbert-Poincaré series.

Let $k \leq n$, where k is a nonnegative integer and n is a positive integer, let $\lambda \vdash k$, and let $s \geq \ell(\lambda)$. Define $\Lambda := \Lambda(n,\lambda,s) := (n-k+\lambda_1,\ldots,n-k+\lambda_s)$ and $K := |\Lambda| = s(n-k)+k$. We define a variety $Y_{n,\lambda,s}$, which is our main object of study.

Definition 3.1. Let N_Λ be a nilpotent matrix of Jordan type Λ. Define

$$Y_{n,\lambda,s} := \{V_\bullet \in \mathrm{Fl}_{1,\ldots,n}(\mathbb{C}^K) \mid N_\Lambda V_i \subseteq V_i \text{ for } i \leq n, \text{ and } \mathrm{im}(N_\Lambda^{n-k}) \subseteq V_n\}, \tag{3.1}$$

where $\mathrm{im}(N_\Lambda^{n-k})$ is the image of the linear map $N_\Lambda^{n-k} : \mathbb{C}^K \to \mathbb{C}^K$.

SG: How about the Δ-*Springer variety*??

$[\Lambda] = [\Lambda(n,\lambda,s)] =$

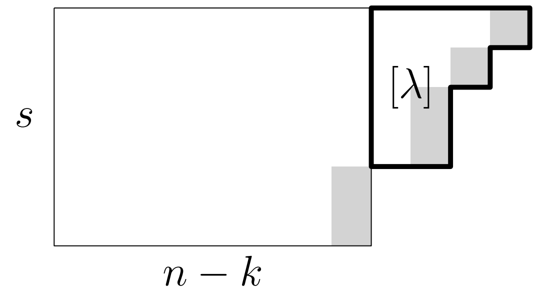

Figure 1. The Young diagram $[\Lambda]$, which has a copy of $[\lambda]$ in the upper right corner, highlighted in bold. The cells in the right edge of $[\Lambda]$ are shaded.

Remark 3.2. Since N_Λ is nilpotent, it can be checked that the set of conditions $N_\Lambda V_i \subseteq V_i$ for $i \leq n$ on a partial flag $V_\bullet \in \mathrm{Fl}_{1,\dots,n}(\mathbb{C}^K)$ is equivalent to the set of conditions $N_\Lambda V_i \subseteq V_{i-1}$ for $i \leq n$, where $V_0 := 0$. Therefore, the variety $Y_{n,\lambda,s}$ can alternatively be defined as

$$Y_{n,\lambda,s} = \{V_\bullet \in \mathrm{Fl}_{1,\dots,n}(\mathbb{C}^K) \,|\, N_\Lambda V_i \subseteq V_{i-1} \text{ for } i \leq n, \text{ and } \mathrm{im}(N_\Lambda^{n-k}) \subseteq V_n\}. \tag{3.2}$$

We use these two definitions interchangeably throughout the paper.

It can be checked that the isomorphism type of $Y_{n,\lambda,s}$, both as a variety and as a topological space, depends only on Λ and not on the choice of N_Λ. It will be convenient to specify particular choices for N_Λ, which we do next.

We denote by $[\Lambda]$ the Young diagram of Λ, following the English convention, considered as the set

$$[\Lambda] = \{(i,j) \mid 1 \leq i \leq \ell(\Lambda), 1 \leq j \leq \Lambda_i\}.$$

The cells in column $n-k+1$ and to the right form a copy of the Young diagram of λ, which we denote by $[\lambda]$. We think of a filling of the Young diagram as a function $T : [\Lambda] \to \mathbb{Z}_{>0}$. We say that a cell of $[\Lambda]$ or a label of T is on the **right edge** if it is right most in its row. See Figure 1 for an illustration of $[\Lambda]$ and $[\lambda]$, where the cells in the right edge of $[\Lambda]$ are shaded.

For any filling T of $[\Lambda]$ satisfying the following conditions,

(S1) T is a bijection between $[\Lambda]$ and $\{1, 2, \dots, K\}$,
(S2) $T(i,j) \leq k$ for all $(i,j) \in [\lambda]$,

we define a variety Y_T, as follows. Fix an ordered basis $f_1, \dots, f_K \in \mathbb{C}^K$, let $F_i = \mathrm{span}\{f_1, \dots, f_i\}$ for all i with $1 \leq i \leq K$, and define N_T to be the nilpotent endomorphism where $N_T(f_{T(i,\Lambda_i)}) = 0$ for $i \leq s$, and $N_T(f_{T(i,j)}) = f_{T(i,j+1)}$ for $i \leq s$ and $j < \Lambda_i$. Note that N_T has Jordan type Λ by construction. Define

$$Y_T := Y_{n,\lambda,s,T} := \{V_\bullet \in \mathrm{Fl}_{1,\dots,n}(\mathbb{C}^K) \mid N_T V_i \subseteq V_i \text{ for all } i, \text{ and } F_k \subseteq V_n\}, \tag{3.3}$$

which is a specific instance of the variety $Y_{n,\lambda,s}$.

In order to show that the intersection of Y_T with the Schubert decomposition of $\mathrm{Fl}_{1,\dots,n}(\mathbb{C}^K)$ is a paving by affines, we must first specify T further. We say that T is (n,λ,s)**-Schubert compatible** if (S1), (S2), and the following conditions hold:

(S3) T is decreasing along each row from left to right.
(S4) For $(i,j) \in [\lambda]$, the label $T(i,j)$ is greater than all labels in column $j+1$.

$$T = \begin{array}{|c|c|c|c|} \hline 8 & 5 & 3 & 1 \\ \hline 7 & 4 & 2 \\ \cline{1-3} 9 & 6 \\ \cline{1-2} \end{array} \qquad \begin{array}{l} f_8 \xrightarrow{N_T} f_5 \xrightarrow{N_T} f_3 \xrightarrow{N_T} f_1 \xrightarrow{N_T} 0 \\ f_7 \xrightarrow{N_T} f_4 \xrightarrow{N_T} f_2 \xrightarrow{N_T} 0 \\ f_9 \xrightarrow{N_T} f_6 \xrightarrow{N_T} 0 \end{array}$$

FIGURE 2. A Schubert-compatible filling T of $\Lambda(5,(2,1),3)$, and the action of N_T on the basis vectors.

(S5) The labels in the right edge of T form an increasing sequence when read from top to bottom.

(S6) Whenever $T(a,b) > T(c,d)$ for $b,d > 1$, then $T(a,b-1) > T(c,d-1)$.

If n, λ, and s are obvious from context, we will simply say T is **Schubert compatible**.

Example 3.3. Let $n=5$, $\lambda=(2,1)$, and $s=3$. Let T be the Schubert-compatible filling of $\Lambda(5,(2,1),3)$ in Figure 2. Then $Y_{5,(2,1),3}$ is the variety of partial flags $V_\bullet = (V_1,V_2,V_3,V_4,V_5) \in \mathrm{Fl}_{1,2,3,4,5}(\mathbb{C}^9)$ such that the following conditions hold:

$$N_T V_i \subseteq V_i \text{ for } i \leq 5, \tag{3.4}$$

$$V_5 \supseteq F_3 = \mathrm{span}\{f_1, f_2, f_3\}. \tag{3.5}$$

For example, the partial flag

$$\mathrm{span}\{f_1\} \subset \mathrm{span}\{f_1,f_2\} \subset \mathrm{span}\{f_1,f_2,f_4\} \subset \mathrm{span}\{f_1,f_2,f_3,f_4\} \subset \mathrm{span}\{f_1,f_2,f_3,f_4,f_7\}.$$

is in $Y_{5,(2,1),3}$.

Example 3.4. We construct a Schubert-compatible filling T as follows. Let the *reading order* of $[\Lambda]$ be the ordering of the cells given by scanning down the columns of $[\Lambda]$ from right to left. For $(i,j) \in [\Lambda]$, if (i,j) is the pth cell in the reading order, then let $T(i,j) = p$. It can be checked that T is a Schubert-compatible filling. See the left-most filling in Figure 3 for an example of such a filling with $n=7$, $\lambda=(2,2)$, and $s=4$.

Lemma 3.5. *Suppose T is a Schubert-compatible filling. If $j < \Lambda_i$, then*

$$N_T(F_{T(i,j)} \setminus F_{T(i,j)-1}) \subseteq F_{T(i,j+1)} \setminus F_{T(i,j+1)-1}.$$

Proof. We have $N_T\, f_{T(i,j)} = f_{T(i,j+1)}$ by definition. Let $f_{T(a,b)} \in F_{T(i,j)}$ with $T(a,b) < T(i,j)$. If $b < \Lambda_a$, then since $T(a,b) < T(i,j)$, by (S6) we have $T(a,b+1) < T(i,j+1)$, and hence $N_T\, f_{T(a,b)} = f_{T(a,b+1)} \in F_{T(i,j+1)}$. Otherwise, if $b = \Lambda_a$, then $N_T\, f_{T(a,b)} = 0$. In either case, we have $N_T F_{T(i,j)} \subseteq F_{T(i,j+1)}$.

If $v \in F_{T(i,j)} \setminus F_{T(i,j)-1}$, then the expansion of v in the f basis has a nonzero $f_{T(i,j)}$ coefficient. Therefore, the expansion of $N_T\, v$ in the f basis has a nonzero $f_{T(i,j+1)}$ coefficient, so $N_T\, v \notin F_{T(i,j+1)-1}$. The lemma then follows. $\square$

For $1 \leq i \leq s$, define a **flattening function** $\mathrm{fl}_T^{(i)}$ and a filling $T^{(i)}$ as follows. If $i \leq \ell(\lambda)$, then $\mathrm{fl}_T^{(i)}$ is the unique order-preserving function with the following domain and codomain,

$$\mathrm{fl}_T^{(i)} : [K] \setminus \{T(i,\Lambda_i)\} \to [K-1], \tag{3.6}$$

$$T = \begin{array}{|c|c|c|c|c|}\hline 13 & 9 & 5 & 3 & 1 \\ \hline 14 & 10 & 6 & 4 & 2 \\ \hline 15 & 11 & 7 \\ \cline{1-3} 16 & 12 & 8 \\ \cline{1-3}\end{array} \qquad T^{(1)} = \begin{array}{|c|c|c|c|c|}\hline 13 & 9 & 5 & 3 & 1 \\ \hline 12 & 8 & 4 & 2 \\ \cline{1-4} 14 & 10 & 6 \\ \cline{1-3} 15 & 11 & 7 \\ \cline{1-3}\end{array} \qquad T^{(3)} = \begin{array}{|c|c|c|c|}\hline 8 & 5 & 3 & 1 \\ \hline 9 & 6 & 4 & 2 \\ \hline 11 & 7 \\ \cline{1-2} 12 & 10 \\ \cline{1-2}\end{array}$$

FIGURE 3. The Schubert-compatible filling T of $[\Lambda] = [\Lambda(7,(2,2),4)]$ determined by reading order and the fillings $T^{(1)}$ and $T^{(3)}$, which are also Schubert compatible.

and if $i > \ell(\lambda)$, then $\mathrm{fl}_T^{(i)}$ is the unique order preserving function

$$\mathrm{fl}_T^{(i)} : [K] \setminus (\{T(i,\Lambda_i)\} \cup \{T(i',1) \mid i' \neq i\}) \to [K-s]. \tag{3.7}$$

For $i \leq \ell(\lambda)$, let $T^{(i)}$ be the filling obtained by deleting the last box in row i, applying $\mathrm{fl}_T^{(i)}$ to the label in each cell, and reordering the rows so that the labels of the cells in the new right edge are increasing from top to bottom. For $i > \ell(\lambda)$, we also delete each cell $(i',1)$ for $i' \neq i$ and shift row i' to the left by one unit before applying $\mathrm{fl}_T^{(i)}$ to form $T^{(i)}$.

See Figure 3 for an example of a Schubert compatible filling T and the fillings $T^{(1)}$ and $T^{(3)}$. When constructing $T^{(3)}$, the cells labeled by $7, 13, 14$, and 16 are deleted, and rows $1, 2$ and 4 are shifted left by one unit. The cells are relabeled as follows: $\mathrm{fl}_T^{(3)}(8) = 7$, $\mathrm{fl}_T^{(3)}(9) = 8$, $\mathrm{fl}_T^{(3)}(10) = 9$, $\mathrm{fl}_T^{(3)}(11) = 10$, $\mathrm{fl}_T^{(3)}(12) = 11$, and $\mathrm{fl}_T^{(3)}(15) = 12$. Then rows 3 and 4 are swapped to obtain $T^{(3)}$. It can be checked that both $T^{(1)}$ and $T^{(3)}$ are Schubert compatible.

Lemma 3.6. *If $i \leq \ell(\lambda)$, then $T^{(i)}$ is $(n-1,\lambda^{(i)},s)$-Schubert compatible. If $i > \ell(\lambda)$, then $T^{(i)}$ is $(n-1,\lambda,s)$-Schubert compatible.*

Proof. By (S4), the labeling $T^{(i)}$ is of partition shape after sorting the rows by the labels in the right edge. It is immediate from the definitions that if $i \leq \ell(\lambda)$ then $T^{(i)}$ is of shape $\Lambda(n-1,\lambda^{(i)},s)$ and if $i > \ell(\lambda)$, then $T^{(i)}$ is of shape $\Lambda(n-1,\lambda,s)$. It also follows by construction that (S1) and (S2) hold for $T^{(i)}$.

Since the operations of deleting a cell, applying the flattening function to the labels, and possibly shifting a row to the left all preserve (S3), then $T^{(i)}$ has property (S3). Since (S4) only concerns labels of $[\lambda]$, and all cells of $[\lambda]$ are shifted left during the process of constructing $T^{(i)}$, then $T^{(i)}$ also satisfies (S4). The property (S5) is automatically satisfied by construction. Finally, $T^{(i)}$ satisfies (S6) since deleting a cell, relabeling, swapping rows, and shifting a row to the left all preserve the property (S6). Therefore, $T^{(i)}$ is Schubert compatible. $\square$

The set of injective maps $w : [n] \to [K]$ indexes the Schubert cells of $\mathrm{Fl}_{1,\ldots,n}(\mathbb{C}^K)$. Given such a map, we say that w is **admissible** with respect to T if both of the following hold.

(A1) The image of the map w contains $[k]$.
(A2) For $i \leq n$, if $w(i) = T(a,b)$ for $b < \Lambda_a$, then $T(a,b+1) \in \{w(1),\ldots,w(i-1)\}$.

Lemma 3.7. *Assume T is a Schubert-compatible filling. Then $C_w \cap Y_T \neq \emptyset$ if and only if w is admissible.*

Proof. If w is admissible, then the partial flag $V_\bullet$ defined by $V_i = \langle f_{w(1)}, \dots, f_{w(i)} \rangle$ is in $C_w \cap Y_T$, so $C_w \cap Y_T \neq \emptyset$. Therefore, it suffices to prove that if $C_w \cap Y_T \neq \emptyset$, then w is admissible.

Given an injective map $w : [n] \to [K]$, recall that

$$C_w = \{V_\bullet \in \mathrm{Fl}_{1,\dots,n}(\mathbb{C}^K) \mid \dim(V_i \cap F_j) = \#\{p \leq i \mid w(p) \leq j\}\}. \tag{3.8}$$

Given $V_\bullet \in \mathrm{Fl}_{1,\dots,n}(\mathbb{C}^K)$, then $F_k \subseteq V_n$ if and only if $\dim(V_n \cap F_k) = k$. Therefore, $F_k \subseteq V_n$ for some $V_\bullet \in C_w$ if and only if (A1) holds.

Suppose $C_w \cap Y_T \neq \emptyset$, and let $V_\bullet \in C_w \cap Y_T$. Suppose there exists a $i \leq n$ such that $w(i) = T(a,b)$ with $b < \Lambda_a$. Then $\dim(V_i \cap F_{T(a,b)}) > \dim(V_i \cap F_{T(a,b)-1})$, so $V_i \cap (F_{T(a,b)} \setminus F_{T(a,b)-1}) \neq \emptyset$. By Lemma 3.5, we have $N_T(F_{T(a,b)} \setminus F_{T(a,b)-1}) \subseteq F_{T(a,b+1)} \setminus F_{T(a,b+1)-1}$. Hence,

$$N_T V_i \cap (F_{T(a,b)} \setminus F_{T(a,b+1)-1}) \neq \emptyset \tag{3.9}$$

and since $N_T V_i \subseteq V_{i-1}$, then

$$V_{i-1} \cap (F_{T(a,b+1)} \setminus F_{T(a,b+1)-1}) \neq \emptyset, \tag{3.10}$$

so $T(a, b+1) = w(i')$ for some $i' \leq i-1$. Hence, (A2) holds and w is admissible. ☐

We define a linear transformation related to N_T that we use throughout the paper.

Definition 3.8. Define the nilpotent endomorphism N_T^t of $\mathbb{C}^K$ on the basis $\{f_1, \dots, f_K\}$ as follows,

$$N_T^t \, f_{T(i,j)} := \begin{cases} f_{T(i,j-1)} & \text{if } j > 1, \\ 0 & \text{if } j = 1. \end{cases} \tag{3.11}$$

Our notation is motivated by the fact that the matrix for N_T^t with respect to the ordered basis $\{f_i\}$ is the transpose of the matrix for N_T. The transformation N_T^t has the crucial property that

$$N_T N_T^t \, f_{T(i,j)} = \begin{cases} f_{T(i,j)} & \text{if } j > 1 \\ 0 & \text{if } j = 1. \end{cases} \tag{3.12}$$

Lemma 3.9. *Let T be Schubert compatible, w be admissible, and $w(1) = T(i, \Lambda_i)$. Given $v \in \mathrm{span}\{f_{T(h,\Lambda_h)} \mid h < i\}$, then the linear transformation $U_v : \mathbb{C}^K \to \mathbb{C}^K$ defined by*

$$U_v(f_{T(p,q)}) = \begin{cases} f_{T(p,q)} + (N_T^t)^{\Lambda_i - q} \, v & \textit{if } p = i \\ f_{T(p,q)} & \textit{otherwise.} \end{cases} \tag{3.13}$$

is upper triangular with 1s along the diagonal such that $N_T U_v = U_v N_T$.

Proof. In the case $p = i$, the nonzero components of the vector $(N_T^t)^{\Lambda_i - q} v$ are $f_{T(j, \Lambda_j - (\Lambda_i - q))}$ for $j < i$. By (S5), we have $T(j, \Lambda_j) < T(i, \Lambda_i)$ for all $j < i$. Therefore, by applying (S6) $\Lambda_i - q$ many times, we have $T(j, \Lambda_j - (\Lambda_i - q)) < T(i,q) = T(p,q)$. Hence, T_v is upper triangular with 1s along the diagonal.

Given $f_{T(p,q)}$ such that $p \neq i$, then $N_T U_v f_{T(p,q)} = N_T f_{T(p,q)} = U_v N_T f_{T(p,q)}$, where the second equality follows from the fact that either $N_T f_{T(p,q)} = f_{T(p,q+1)}$ or 0. On the other hand, if $p = i$, then

$$U_v N_T f_{T(i,q)} = \begin{cases} f_{T(i,q+1)} + (N_T^t)^{\Lambda_i - q - 1} v & \text{if } q < \Lambda_i \\ 0 & \text{if } q = \Lambda_i, \end{cases} \tag{3.14}$$

Likewise, we have

$$N_T U_v f_{T(i,q)} = N_T f_{T(i,q)} + N_T (N_T^t)^{\Lambda_i - q} v \tag{3.15}$$

If $q = \Lambda_i$, then this is equal to $N_T v = 0$. Otherwise, $q < \Lambda_i$ and we have

$$N_T U_v f_{T(i,q)} = N_T f_{T(i,q)} + (N_T N_T^t)(N_T^t)^{\Lambda_i - q - 1} v = f_{T(i,q+1)} + (N_T^t)^{\Lambda_i - q - 1} v, \tag{3.16}$$

where the second equality follows from (3.12). Hence, $N_T U_v = U_v N_T$ and the proof is complete. ☐

Lemma 3.10. *Let T be Schubert compatible, w be admissible, and $w(1) = T(i, \Lambda_i)$. We have*

$$C_w \cap Y_T \cong \mathbb{C}^{i-1} \times (C_{\mathrm{fl}_T^{(i)}(w)} \cap Y_{T^{(i)}}).$$

Proof. Since

$$\mathrm{span}\{f_{T(j,\Lambda_j)} \,|\, j < i\} \cong \mathbb{C}^{i-1}, \tag{3.17}$$

we may identify the two spaces as affine varieties. Define linear maps

$$\psi^{(i)} : \mathbb{C}^{K-1} \to \mathbb{C}^K \quad \text{for } i \le \ell(\lambda), \tag{3.18}$$

$$\psi^{(i)} : \mathbb{C}^{K-s} \to \mathbb{C}^K \quad \text{for } i > \ell(\lambda), \tag{3.19}$$

by $\psi^{(i)}(f_j) := f_{(\mathrm{fl}_T^{(i)})^{-1}(j)}$, and extend linearly. Given $v \in \mathrm{span}\{f_{T(j,\Lambda_j)} \,|\, j < i\}$ and $V_\bullet \in C_{\mathrm{fl}_T^{(i)}(w)} \cap Y_{T^{(i)}}$, define $\Phi(v, V_\bullet)$ to be

$$(\mathrm{span}\{f_{w(1)} + v\}, \mathrm{span}\{f_{w(1)} + v\} + U_v \psi^{(i)}(V_1), \ldots, \mathrm{span}\{f_{w(1)} + v\} + U_v \psi^{(i)}(V_{n-1})). \tag{3.20}$$

Claim: The partial flag $\Phi(v, V_\bullet)$ is in $C_w \cap Y_T$, so Φ is a well-defined map

$$\Phi : \mathrm{span}\{f_{T(j,\Lambda_j)} \,|\, j < i\} \times (C_{\mathrm{fl}_T^{(i)}(w)} \cap Y_{T^{(i)}}) \to C_w \cap Y_T. \tag{3.21}$$

It can be checked that since $V_\bullet \in C_{\mathrm{fl}_T^{(i)}(w)}$, then $\Phi(0, V_\bullet) \in C_w$. Observe that $\Phi(v, V_\bullet) = U_v \Phi(0, V_\bullet)$, where U_v acts on each subspace in the partial flag $\Phi(0, V_\bullet)$. Since U_v is upper triangular with 1s along the diagonal by Lemma 3.9, it preserves the Schubert cell C_w, so $\Phi(v, V_\bullet) \in C_w$. In particular, since w is admissible then the nth part of the partial flag $\Phi(v, V_\bullet)$ contains F_k. Furthermore, it can be checked that $N_T \psi^{(i)}(w) - \psi^{(i)} N_T^{(i)}(w) \in \ker(N_T)$ for all w in the domain of $\psi^{(i)}$. Combining this with Lemma 3.9, we have

$$N_T(\mathrm{span}\{v\} + U_v \psi^{(i)}(V_j)) = N_T U_v \psi^{(i)}(V_j) = U_v \psi^{(i)} N_{T^{(i)}}(V_j) \subseteq U_v \psi^{(i)}(V_{j-1}). \tag{3.22}$$

Hence, $\Phi(v, V_\bullet) \in C_w \cap Y_T$, which proves the claim.

To show that Φ is an isomorphism, it suffices to show that Φ has an inverse. Define linear maps

$$\phi^{(i)} : \mathbb{C}^K \to \mathbb{C}^{K-1} \quad \text{for } i \le \ell(\lambda), \tag{3.23}$$

$$\phi^{(i)} : \mathbb{C}^K \to \mathbb{C}^{K-s} \quad \text{for } i > \ell(\lambda), \tag{3.24}$$

by $\Phi^{(i)}(f_j) \coloneqq f_{\mathrm{fl}_T^{(i)}(j)}$ if j is in the domain of $\mathrm{fl}_T^{(i)}$ and 0 otherwise. Given $V_\bullet \in C_w \cap Y_T$, there exist unique vectors $v_1, \ldots, v_n$ with $v_i \in V_i$ such that the coefficient of $f_{w(i)}$ in v_i is 1 and the coefficient of $f_{w(i)}$ in v_j is 0 for all $i < j$. Define $V_i' = \mathrm{span}\{v_2, \ldots, v_{i+1}\}$ for all i. Then it can be checked that the inverse of Φ is

$$\Phi^{-1}(V_\bullet) = (v_1 - f_{w(1)}, (\phi^{(i)} U^{-1}_{v_1 - f_{w(1)}}(V_1'), \ldots, \phi^{(i)} U^{-1}_{v_1 - f_{w(1)}}(V_{n-1}'))). \tag{3.25}$$

Moreover, since U_v can be represented by a unipotent upper triangular matrix whose coordinates are regular functions on Y_T, then the same is true of U_v^{-1}, and hence both Φ and Φ^{-1} are algebraic maps, so Φ is an isomorphism of algebraic varieties. $\square$

Theorem 3.11. *If T is Schubert compatible, then the intersections $C_w \cap Y_{n,\lambda,s,T}$ for w admissible are the cells of an affine paving of $Y_{n,\lambda,s,T}$.*

Proof. Since the Schubert cells C_w are the cells of an affine paving of $\mathrm{Fl}_{1,\ldots,n}(\mathbb{C}^K)$, it suffices to show that each nonempty intersection $C_w \cap Y_{n,\lambda,s,T}$ is isomorphic to an affine space $\mathbb{C}^d$ for some d. By Lemma 3.10, $C_w \cap Y_{n,\lambda,s,T}$ is nonempty if and only if w is admissible. We proceed by induction on n to show that each of these intersection is an affine space. In the base case when $n = 1$, either $\lambda = \emptyset$ or $\lambda = (1)$. In the first case, $Y_{1,\emptyset,s,T} = \mathbb{P}^{s-1}$ for any Schubert-compatible T, the admissible w are in bijection with $[s]$, and the nonempty intersections $C_w \cap Y_{1,\emptyset,s,T}$ can be identified with usual cells of $\mathbb{P}^{s-1}$. In the second case, $Y_{1,(1),s,T}$ is a point, and the only nonempty intersection is a point. The inductive proof then follows by applying Lemma 3.10. $\square$

Corollary 3.12. *We have*

$$\mathrm{Hilb}(H^*(Y_{n,\lambda,s}); q) = \sum_{i=1}^{\ell(\lambda)} q^{2(i-1)} \mathrm{Hilb}(H^*(Y_{n-1,\lambda^{(i)},s}); q) + \sum_{i=\ell(\lambda)+1}^{s} q^{2(i-1)} \mathrm{Hilb}(H^*(Y_{n-1,\lambda,s}); q).$$

Proof. Let T be Schubert compatible. By Lemma 2.1, the q^{2i} coefficient of $\mathrm{Hilb}(H^*(Y_{n,\lambda,s}); q) = \mathrm{Hilb}(H^*(Y_{n,\lambda,s,T}); q)$ is the number of cells $C_w \cap Y_{n,\lambda,s,T}$ for w admissible that are complex dimension i. It can be checked that for $1 \le i \le s$, then $\{\mathrm{fl}^{(i)}(w) \,|\, w \text{ admissible}\}$ is the set of admissible injective maps for $T^{(i)}$. Thus, the subspaces $C_{\mathrm{fl}^{(i)}(w)} \cap Y_{T^{(i)}}$ are the cells of an affine paving for $Y_{T^{(i)}}$, by Theorem 3.11. The corollary then follows from Lemma 3.10. $\square$

Corollary 3.13. *The cohomology ring $H^*(Y_{n,\lambda,s,T})$ is a graded free $\mathbb{Z}$-module concentrated in even degrees, whose rank generating function is equal to* $\mathrm{Hilb}(R_{n,\lambda,s}; q^2)$.

Proof. By Theorem 3.11, $Y_{n,\lambda,s,T}$ has a paving by affines where each cell is a copy of complex affine space. By Lemma 2.1, all of the odd cohomology group vanish, and $H^{2i}(Y_{n,\lambda,s,T})$ is a free $\mathbb{Z}$-module of rank equal to the number of cells of complex dimension i in the paving.

We prove that the cohomology ring of $Y_{n,\lambda,s,T}$ and $R_{n,\lambda,s}$ have the same rank generating function by induction on n. In the case when $n = 1$, then either $\lambda = \emptyset$ and $Y_{1,\emptyset,s,T} = \mathbb{P}^{s-1}$, or $\lambda = (1)$ and $Y_{1,(1),s,T} = \mathbb{P}^0$. In the first case, the rank generating function of $H^*(Y_{1,\emptyset,s,T})$ is $1 + q^2 + \cdots + q^{2(s-1)}$. On the other hand, $R_{1,\emptyset,s} = \mathbb{Z}[x]/(x^s)$, so the lemma holds in this case. In the second case, the rank generating function of $H^*(Y_{1,(1),s,T})$ is 1, and $R_{1,(1),1}$ is the trivial 1-dimensional ring, so the lemma holds in the base case.

Suppose $n > 1$. By Lemma 2.5 and Corollary 3.12, the Hilbert series of $R_{n,\lambda,s}$ and the Hilbert series of $H^*(Y_{n,\lambda,s})$ satisfy the same recursion. Hence, the two q-series must be equal by induction on n. $\square$

Lemma 3.14. *Let T be a (n, λ, s)-Schubert compatible filling of $\Lambda(n, \lambda, s)$, and let T' be a $(n, \emptyset, s)$-Schubert compatible filling of $\Lambda(n, \emptyset, s) = (n^s)$ such that every entry of the ith row of T is in the ith row of T'. Then the linear map $j : \mathbb{C}^K \hookrightarrow \mathbb{C}^{ns}$, which is the inclusion of the first K coordinates, induces a closed embedding*

$$\iota : Y_{n,\lambda,s,T} \hookrightarrow Y_{n,\emptyset,s,T'}, \tag{3.26}$$

defined by sending the flag $V_\bullet \in Y_{n,\lambda,s,T}$ to the flag $(j(V_1), \ldots, j(V_n))$.

Proof. The proof follows from the fact that the entries of T in row i are right justified in row i of T'. SG: What details should we add? □

Lemma 3.15. *If T is a Schubert-compatible filling, the space $Y_{n,\emptyset,s,T'} \setminus \iota(Y_{n,\lambda,s,T})$ has an affine paving.*

Proof. By Theorem 3.11, the intersections $C_w \cap Y_{n,\lambda,s,T}$ for w admissible with respect to T are the cells of an affine paving of $Y_{n,\lambda,s,T}$, and the intersections $C_v \cap Y_{n,\emptyset,s}$ for v admissible with respect to T' are the cells of an affine paving of $Y_{n,\emptyset,s,T'}$.

Given such a cell $C_w \cap Y_{n,\lambda,s,T}$ with w admissible with respect to T, define $w' : [n] \to [ns]$ by extending the codomain of w to $[ns]$. Then w' is admissible with respect to T', and it can be checked that $\iota(C_w \cap Y_{n,\lambda,s,T}) = C_{w'} \cap Y_{n,\emptyset,s,T'}$. Therefore, $Y_{n,\emptyset,s,T'} \setminus \iota(Y_{n,\lambda,s,T})$ has an affine paving given by removing the cells of the form $C_{w'} \cap Y_{n,\emptyset,s,T'}$ from the affine paving of $Y_{n,\emptyset,s,T'}$. □

Corollary 3.16. *The closed embedding ι induces a surjection on cohomology,*

$$H^*(Y_{n,\emptyset,s,T}) \twoheadrightarrow H^*(Y_{n,\lambda,s,T'}). \tag{3.27}$$

Proof. This follows immediately by Lemma 2.2, Theorem 3.11, and Lemma 3.15. □

4. The case of $\lambda = \emptyset$

In this section, we analyze the variety $Y_{n,\lambda,s}$ in the case when λ is the empty partition $\emptyset$. We prove that this space is an iterated projective bundle in Lemma 4.1. We then prove that $Y_{n,\emptyset,s}$ has the same cohomology ring as $(\mathbb{P}^{s-1})^n$ in Lemma 4.2.

For all $i \leq n$, let $\widetilde{V}_i$ be the tautological rank i vector bundle on $\mathrm{Fl}_{1,\ldots,n}(\mathbb{C}^{ns})$ for $i \leq n$. We abuse notation and also denote by $\widetilde{V}_i$ the restriction of $\widetilde{V}_i$ to the subvariety $Y_{n,\emptyset,s}$.

Lemma 4.1. *Let T be a $(n, \emptyset, s)$-Schubert compatible filling such that the labels in the first column are $ns - s + 1, \ldots, ns$ in some order, and let T' be the result of deleting the first column of T. Then the map*

$$Y_{n,\emptyset,s,T} \to Y_{n-1,\emptyset,s,T'}, \tag{4.1}$$

given by forgetting the last subspace in the partial flag, is a $\mathbb{P}^{s-1}$-bundle map.

Proof. By our assumptions, the nilpotent transformation $N_{T'}$ is the restriction of N_T to the subspace $\mathrm{span}\{f_1, \ldots, f_{n(s-1)}\} \subseteq \mathbb{C}^{ns}$. Given any $V_\bullet \in Y_{n,\emptyset,T}$, then we have

$$V_{n-1} \subseteq \mathrm{span}\{f_1, \ldots, f_{n(s-1)}\}. \tag{4.2}$$

Therefore, $(V_1, \ldots, V_{n-1}) \in Y_{n-1,\emptyset,s,T'}$, so the map (4.1) is well-defined.

Given a subspace $V \subseteq \mathbb{C}^{ns}$, let $N_T^{-1}(V)$ be the preimage of V under the map $N_T : \mathbb{C}^{ns} \to \mathbb{C}^{ns}$. Observe that given $(V_1, \ldots, V_{n-1}) \in Y_{n-1,\emptyset,s,T'}$, an extension of this partial flag to

$(V_1,\dots,V_{n-1},W)\in \mathrm{Fl}_{1,\dots,n}(\mathbb{C}^K)$ is in $Y_{n,\emptyset,s}$ if and only if $W\subseteq N_T^{-1}(V_{n-1})$. We claim that for any subspace $V\subseteq \operatorname{span}\{e_1,\dots,e_{n(s-1)}\}$ of dimension $n-1$, then

$$\dim_{\mathbb{C}}(N_T^{-1}(V)) = s+n-1. \tag{4.3}$$

Indeed, define a linear map

$$\varphi = N_T|_{N_T^{-1}(V)} : N_T^{-1}(V)\to V, \tag{4.4}$$

which is the restriction of N_T. It is clear that this map is surjective map, so (4.3) follows by rank-nullity and the fact that $\dim(\ker(N_T)) = s$.

Let $\widetilde{N_T^{-1}V}_{n-1}$ be the rank $s+n-1$ vector bundle on $Y_{n-1,\emptyset,s,T'}$ whose fiber over $V_\bullet$ is $N_T^{-1}(V_{n-1})$, and let $\widetilde{V}_{n-1}$ be the rank $n-1$ tautological vector bundle on $Y_{n-1,\emptyset,s,T'}$. We have an isomorphism

$$Y_{n,\emptyset,s,T}\cong \mathbb{P}(\widetilde{N_T^{-1}V}_{n-1}/\widetilde{V}_{n-1}), \tag{4.5}$$

defined by sending $V_\bullet$ to the line V_n/V_{n-1} over the point $(V_1,\dots,V_{n-1})$ of $Y_{n-1,\emptyset,s,T'}$. Hence, $Y_{n,\emptyset,s,T}$ is a $\mathbb{P}^{s-1}$-bundle over $Y_{n-1,\emptyset,s,T'}$ via the forgetting map (4.1). $\square$

We note that the variety $Y_{n,\emptyset,s}$ is a special case of a Steinberg variety, as defined in [4, 20]. Its cohomology ring is known to be isomorphic to the ring of $(S_1\times\cdots S_1\times S_{n(s-1)})$-invariants of the cohomology ring of the Springer fiber $H^*(\mathcal{B}^\Lambda)$ [4]. It is not hard to prove the next lemma using this fact, but we instead give a self-contained proof for the sake of completeness.

Lemma 4.2. *There is an isomorphism*

$$H^*(Y_{n,\emptyset,s})\cong \frac{\mathbb{Z}[x_1,\dots,x_n]}{\langle x_1^s,\dots,x_n^s\rangle}, \tag{4.6}$$

that identifies x_i with $c_1(\widetilde{V}_i/\widetilde{V}_{i-1})$

Proof. We may assume, without loss of generality, that the hypotheses in Lemma 4.1 continue to hold. We proceed by induction on n. In the case $n=1$, the lemma follows from the fact that $Y_{1,\emptyset,s,T}=\mathbb{P}^{s-1}$. Suppose by way of induction that

$$H^*(Y_{n-1,\emptyset,s,T'})\cong \frac{\mathbb{Z}[x_1,\dots,x_{n-1}]}{\langle x_1^s,\dots,x_{n-1}^s\rangle}. \tag{4.7}$$

Let us denote $E := \widetilde{N^{-1}V}_{n-1}/\widetilde{V}_{n-1}$. By (4.5), we have an isomorphism

$$Y_{n,\emptyset,s,T}\cong \mathbb{P}(E), \tag{4.8}$$

so that $\widetilde{V}_n/\widetilde{V}_{n-1}\cong \mathcal{O}_E(1)$. Hence, by Grothendieck's construction of Chern classes, we have

$$H^*(Y_{n,\emptyset,s,T})\cong \frac{H^*(Y_{n-1,\emptyset,s,T'})[x_n]}{\langle x_n^s + c_1(E)x_n^{s-1}+\cdots+c_s(E)\rangle}. \tag{4.9}$$

It suffices to prove $c(E)=1$. Indeed, observe that if $V_\bullet\in Y_{n-1,\emptyset,s,T'}$, then $V_{n-1}\subseteq \operatorname{span}\{e_1,\dots,e_{n(s-1)}\} = \operatorname{im}(N_T)$. Let $\mathbb{C}^{ns}$ and $\operatorname{im}(N_T)$ be the corresponding trivial vector bundles on $Y_{n-1,\emptyset,s,T'}$. Consider the following short exact sequence of vector bundles,

$$0\to E\to \mathbb{C}^{ns}/\widetilde{V}_{n-1}\to \operatorname{im}(N_T)/\widetilde{V}_{n-1}\to 0, \tag{4.10}$$

where the second map is the composition $E \hookrightarrow \mathbb{C}^{ns} \twoheadrightarrow \mathbb{C}^{ns}/\widetilde{V}_{n-1}$, and the third map is induced by N_T. Then we have the following identity of Chern classes,

$$c(E) = \frac{c(\mathbb{C}^{ns}/\widetilde{V}_{n-1})}{c(\mathrm{im}(N_T)/\widetilde{V}_{n-1})} = c(\mathbb{C}^{ns}/\mathrm{im}(N_T)) = 1, \tag{4.11}$$

which completes the proof. □

5. Spaltenstein varieties and the cohomology of $Y_{n,\lambda,s}$

SG: To do: Rewrite this section using Alex's notation

In this section, we prove that there is a cellular surjective map from a Spaltenstein variety to $Y_{n,\lambda,s}$. We use this fact together with work of Brundan and Ostrik on the cohomology ring of a Spaltenstein variety [5] to prove that the cohomology ring of $Y_{n,\lambda,s}$ is isomorphic to $R_{n,\lambda,s}$, stated as Theorem 5.6.

Let us outline our strategy for proving that the cohomology ring of $Y_{n,\lambda,s}$ is isomorphic to $R_{n,\lambda,s}$. First, by Corollary 3.16 we know that $H^*(Y_{n,\lambda,s})$ is a quotient of the ring

$$\frac{\mathbb{Z}[x_1,\dots,x_n]}{\langle x_1^s,\dots,x_n^s\rangle}. \tag{5.1}$$

By Corollary 3.13, the rings $R_{n,\lambda,s}$ and $H^*(Y_{n,\lambda,s})$ are free $\mathbb{Z}$-modules with the same rank generating function. Therefore, it suffices to prove that for each generator $e_d(S)$ of $I_{n,\lambda,s}$ with $S \subseteq \{x_1,\dots,x_n\}$, the same polynomial in the first Chern classes $x_i = c_1(\widetilde{V}_i/\widetilde{V}_{i-1})$ vanishes in $H^*(Y_{n,\lambda,s})$. To do this, we exhibit an injection from $H^*(Y_{n,\lambda,s})$ into the cohomology of a Spaltenstein variety, and we prove that the $e_d(S)$ polynomials in the first Chern classes vanish in the cohomology ring of the Spaltenstein variety.

Let us recall the definition of a Spaltenstein variety. Given an $m\times m$ nilpotent matrix N_ν of Jordan type $\nu \vdash m$ and a composition $\mu \vDash m$ of length ℓ, the *Spaltenstein variety* is

$$\mathcal{B}^\nu_\mu := \{V_\bullet \in \mathrm{Fl}_{\mu_1,\,\mu_1+\mu_2,\dots,\,m}(\mathbb{C}^m) \,|\, N_\nu V_i \subseteq V_{i-1} \text{ for } i \le \ell\}. \tag{5.2}$$

Let $X_j = \{x_{\mu_1+\cdots+\mu_{j-1}+1},\dots,x_{\mu_1+\cdots+\mu_j}\}$. Given $1\le i_1<\cdots<i_p\le \ell$, let

$$e_d(X;i_1,\dots,i_p) := e_d(X_{i_1}\cup X_{i_2}\cup\cdots\cup X_{i_p}). \tag{5.3}$$

Furthermore, let I^ν_μ be the following ideal of $\mathbb{Z}[x_1,\dots,x_m]$,

$$I^\nu_\mu := \langle e_d(X;i_1,\dots,i_p) \,|\, 1\le i_1<\cdots<i_p\le\ell \text{ and } d>\mu_{i_1}+\cdots+\mu_{i_p}-\nu'_{\ell-p+1}-\cdots-\nu'_m\rangle. \tag{5.4}$$

Brundan and Ostrik [5] proved the following isomorphism of graded rings,

$$H^*(\mathcal{B}^\nu_\mu) \cong \frac{\mathbb{Z}[x_1,\dots,x_m]^{S_\mu}}{I^\nu_\mu}. \tag{5.5}$$

Let us take $m=K$, $\nu=\Lambda$, and $\mu=(1^n,s-1,s-1,\dots,s-1)$, where $s-1$ is repeated $n-k$ many times, so that $\ell=2n-k$. Observe that $\Lambda'_{n-k+i}=\lambda'_i$ for $i\ge 0$. Further observe that for each $j\le n$, then $X_j=\{x_j\}$. Taking $1<i_1<\cdots<i_p\le n$, then $e_d(x_{i_1},\dots,x_{i_p})\in I^\Lambda_\mu$ for

$$d > p - \Lambda'_{(2n-k)-p+1} - \cdots - \Lambda'_K, \tag{5.6}$$

or equivalently

$$d > p - \lambda'_{n-p+1} - \cdots - \lambda'_n. \tag{5.7}$$

The next lemma follows immediately from these observations.

Lemma 5.1. *With μ as above, we have $I_{n,\lambda,s} \subseteq I^\Lambda_\mu$.*

Observe that there is a map

$$\pi : \mathcal{B}^\Lambda_\mu \to Y_{n,\lambda,s} \tag{5.8}$$

given by projecting onto the first n parts of the partial flag. Indeed, if $V_\bullet \in \mathcal{B}^\Lambda_\mu$, then $V_{2n-k} = \mathbb{C}^K$ by definition. Since $N_\Lambda V_i \subseteq V_{i-1}$ for all i, then $\operatorname{im}(N_\Lambda^{n-k}) = N_\Lambda^{n-k} V_{2n-k} \subseteq V_n$, so $\pi(V_\bullet) \in Y_{n,\lambda,s}$. In order to show that the map π is a surjective cellular map, we need the following two lemmata, the second of which is a strengthening of Lemma 3.9 that only holds for a subclass of Schubert-compatible fillings.

Lemma 5.2. *Let T be a Schubert-compatible filling. If $j > 1$, then*

$$N_T^t(F_{T(i,j)} \setminus F_{T(i,j)-1}) \subseteq F_{T(i,j-1)} \setminus F_{T(i,j-1)-1}. \tag{5.9}$$

Sketch. The proof is an application of (S6), similar to the proof of Lemma 3.5. □

Lemma 5.3. *Let T be a Schubert compatible filling with the property that $T(i',j') > T(i,j)$ if $j' < j$. Let w be admissible with respect to T, and let $V_\bullet \in Y_{n,\lambda,s,T} \cap C_w$. For all $p \leq n$, let v_p be a vector in $V_p \setminus V_{p-1}$ whose $f_{w(p)}$ coefficient is 1. Then there exists a unipotent upper triangular matrix U such that*

$$U f_{w(p)} = v_p \tag{5.10}$$

for all $p \leq n$, and

$$U N_T f_{T(i,j)} = N_T U f_{T(i,j)} \tag{5.11}$$

for all $T(i,j) \notin \{w(1), \ldots, w(n)\}$.

Proof. Define the action of U on the basis $\{f_i\}$ as follows. Let (i_p, j_p) be the coordinates of the label $w(p)$ in T, so $w(p) = T(i_p, j_p)$. For $p \leq n$, we take (5.10) as a definition,

$$U f_{w(p)} \coloneqq v_p. \tag{5.12}$$

For each $T(i,j) \notin \{w(1), \ldots, w(n)\}$, let $p \leq n$ be maximal such that $i = i_p$. If such a p exists, define

$$U f_{T(i,j)} \coloneqq (N^t)^{j_p - j} v_p. \tag{5.13}$$

If such a p does not exist, define

$$U f_{T(i,j)} \coloneqq f_{T(i,j)}. \tag{5.14}$$

The matrix U is unipotent upper triangular by Lemma 5.2.

It remains to show that (5.11) holds. Let $T(i,j) \notin \{w(1), \ldots, w(n)\}$, and let $p \leq n$ be maximal such that $i = i_p$. If such a p exists, then $j_p > j$ by admissibility condition (A2), and we have

$$U N_T f_{T(i,j)} = U f_{T(i,j+1)} = (N_T^t)^{j_p - j - 1} v_p, \tag{5.15}$$

$$N_T U f_{T(i,j)} = N_T (N_T^t)^{j_p - j} v_p. \tag{5.16}$$

Therefore, in order for (5.11) to hold, we need that $N_T(N_T^t)^{j_p-j}v_p = (N_T^t)^{j_p-j-1}v_p$. By (3.12), this holds if and only if the coefficient of $f_{T(i',1)}$ in $(N_T^t)^{j_p-j-1}v_p$ is zero for all $i' \leq s$. Hence, it suffices to show the coefficient of $f_{T(i',j_p-j)}$ in v_p is zero for all $i' \leq s$. Indeed, if the coefficient of $F_{T(i',j_p-j)}$ in v_p were nonzero, then since $v_p \in F_{T(i_p,j_p)}$, we have $T(i', j_p - j) \leq T(i_p, j_p)$. However, since $j_p - j < j_p$, this is impossible by our restriction on T in the hypotheses of the lemma. Hence, (5.11) holds in the case that $i = i_p$ for some p.

Suppose $i \neq i_p$ for all p. Recall that we have defined $Uf_{T(i,j)} = f_{T(i,j)}$. If $j < \Lambda_i$ then $Uf_{T(i,j+1)} = f_{T(i,j+1)}$, and so $UN_Tf_{T(i,j)} = Uf_{T(i,j+1)} = f_{T(i,j+1)} = N_TUf_{T(i,j)}$. If $j = \Lambda_i$, then $Nf_{T(i,j)} = 0$, and we again have $UNf_{T(i,j)} = NUf_{T(i,j)}$. This verifies (5.11), and the proof is complete. □

Lemma 5.4. *The map π is surjective. Let T be Schubert compatible and w be admissible. Then we have*

$$\pi^{-1}(C_w \cap Y_{n,\lambda,s,T}) \cong (C_w \cap Y_{n,\lambda,s,T}) \times \mathcal{B}^{\Lambda'}_{(s-1)^{n-k}}, \tag{5.17}$$

where Λ' is the partition obtained by deleting cells labeled $w(1), \dots, w(n)$ from T, and then recording the row sizes of the remaining cells in weakly decreasing order.

Proof. Let $E_i = \operatorname{span}\{f_{w(1)}, \dots, f_{w(i)}\}$ for $i \leq n$, which form the unique partial flag in C_w such that each subspace is spanned by a subset of the f basis. Let $N_T|_{\mathbb{C}^K/E_n}$ be the nilpotent endomorphism induced by N_T on the quotient space $\mathbb{C}^K/E_n$, which has Jordan type Λ'. For each $p \leq n$, let (i_p, j_p) be the coordinates of $w(p)$ in T, so $w(p) = T(i_p, j_p)$. Let E' be the span of all of the f_i basis vectors which are not in E_n.

Let $V_\bullet \in C_w \cap Y_{n,\lambda,s,T}$. By Lemma 5.3, there is a unipotent upper triangular matrix U such that for all $p \leq n$,

$$UE_p = V_p, \tag{5.18}$$

and for all $e' \in E'$, we have

$$N_TUe' = UN_Te'. \tag{5.19}$$

Consider $\mathcal{B}^{\Lambda'}_{(s-1)^{n-k}}$ as the set of partial flags $(W_1, \dots, W_{n-k})$ fixed by the nilpotent transformation $N_T|_{\mathbb{C}^K/E_n}$. We claim that for any $(V_1, \dots, V_n, V_{n+1}, \dots, V_{2n-k}) \in \pi^{-1}(C_w \cap Y_{n,\lambda,s,T})$, then

$$(U^{-1}V_{n+1}/E_n, \dots, U^{-1}V_{2n-k}/E_n) \in \mathcal{B}^{\Lambda'}_{(s-1)^{n-k}}. \tag{5.20}$$

Indeed, it is evident that it is in the partial flag variety $\mathcal{B}_{(s-1)^{n-k}}$. To show it is in $\mathcal{B}^{\Lambda'}_{(s-1)^{n-k}}$, it suffices to prove that $N_TU^{-1}V_{n+i} \subseteq U^{-1}V_{n+i-1}$ for $i \geq 1$. Indeed, since $U^{-1}V_{n+i} \supseteq E_n$, we have the vector space decomposition

$$U^{-1}V_{n+i} = E_n \oplus (U^{-1}V_{n+i} \cap E'). \tag{5.21}$$

By (5.19), it follows that $N_T(U^{-1}V_{n+i} \cap E') = U^{-1}N_T U(U^{-1}V_{n+i} \cap E') = U^{-1}N_T(V_{n+i} \cap UE')$. Hence,

$$N_T U^{-1} V_{n+i} = N_T E_n + N_T(U^{-1}V_{n+i} \cap E') \tag{5.22}$$
$$= N_T E_n + U^{-1} N_T(V_{n+i} \cap UE') \tag{5.23}$$
$$\subseteq E_n + U^{-1} N_T V_{n+i} \tag{5.24}$$
$$\subseteq U^{-1} V_{n+i-1}. \tag{5.25}$$

Thus, (5.20) holds.

By (5.20), we have a map

$$\pi^{-1}(C_w \cap Y_{n,\lambda,s,T}) \to (C_w \cap Y_{n,\lambda,s,T}) \times \mathcal{B}^{\Lambda'}_{(s-1)^{n-k}}, \tag{5.26}$$

defined by sending $(V_1, \ldots, V_n, V_{n+1}, \ldots, V_{2n-k})$ to

$$((V_1, \ldots, V_n), (U^{-1}V_{n+1}/E_n, \ldots, U^{-1}V_{2n-k}/E_n)), \tag{5.27}$$

where U depends on $(V_1, \ldots, V_n)$, as defined above. Furthermore, it can be checked that the coordinates of the matrix representing U are regular algebraic functions on the partial flag variety. Since U is unipotent, then the coordinates of U^{-1} are also regular algebraic functions, hence (5.26) is a map of algebraic varieties.

A similar calculation following from (5.19) shows that there is a map of algebraic varieties

$$(C_w \cap Y_{n,\lambda,s,T}) \times \mathcal{B}^{\Lambda'}_{(s-1)^{n-k}} \to \pi^{-1}(C_w \cap Y_{n,\lambda,s,T}), \tag{5.28}$$

defined by sending $((V_1, \ldots, V_n), (W_1, \ldots, W_{n-k}))$ to

$$(V_1, \ldots, V_n, UW_1 + V_n, \ldots, UW_{n-k} + V_n), \tag{5.29}$$

where U depends on $(V_1, \ldots, V_n)$, as defined above. It can be checked that (5.28) is a map of varieties and that (5.26) and (5.28) are mutual inverses of each other. Thus, the isomorphism (5.17) follows.

To show that π is surjective, it suffices to show that $\mathcal{B}^{\Lambda'}_{(s-1)^{n-k}} \neq \emptyset$. This follows from the fact that $\nu^{\Lambda'}$ can be partitioned into $n-k$ vertical strips of size $s-1$ SG: (should we add this detail or give a reference?). The proof is thus complete. □

Lemma 5.5. *The map on cohomology induced by π,*

$$\pi^* : H^*(Y_{n,\lambda,s}) \to H^*(\mathcal{B}^{\Lambda}_{\mu}), \tag{5.30}$$

is injective.

Proof. By Theorem 3.11, for any Schubert compatible T, $Y_{n,\lambda,s,T}$ is paved by the affines spaces $C_w \cap Y_{n,\lambda,s,T}$ for w admissible, so $H_*(Y_{n,\lambda,s}) = H_*(Y_{n,\lambda,s,T})$ is freely generated by the classes $[C_w \cap Y_{n,\lambda,s,T}]$. By Lemma 5.4, we have an isomorphism $\pi^{-1}(C_w \cap Y_{n,\lambda,s,T}) \cong (C_w \cap Y_{n,\lambda,s,T}) \times \mathcal{B}^{\Lambda'}_{(s-1)^{n-k}}$. By [5], each Spaltenstein variety $\mathcal{B}^{\Lambda'}_{(s-1)^{n-k}}$ has a paving by affine spaces. The proof is then completed by applying Lemma 2.3. □

Theorem 5.6. *We have a degree-doubling isomorphism of graded rings*

$$R_{n,\lambda,s} \cong H^*(Y_{n,\lambda,s}) \tag{5.31}$$

given by sending x_i to $c_1(\widetilde{V}_i/\widetilde{V}_{i-1})$.

Proof. By Corollary 3.16, we have a surjection,

$$\frac{\mathbb{Z}[x_1,\dots,x_n]}{\langle x_1^s,\dots,x_n^s\rangle} \twoheadrightarrow H^*(Y_{n,\lambda,s}), \tag{5.32}$$

given by sending x_i to $c_1(\widetilde{V}_i/\widetilde{V}_{i-1})$. By Lemma 5.1, we have that the cohomology class represented by $e_d(S)$ in $H^*(\mathcal{B}_\mu^\Lambda)$ for $S \subseteq \{x_1,\dots,x_n\}$ is zero if $d > |S| - p_{|S|}^n(\lambda)$. By Lemma 5.5 and naturality of Chern classes, then the cohomology class represented by $e_d(S)$ in $H^*(Y_{n,\lambda,s})$ is zero as well. Hence, the map (5.32) descends to a map

$$R_{n,\lambda,s} \twoheadrightarrow H^*(Y_{n,\lambda,s}). \tag{5.33}$$

Since both of these rings are free $\mathbb{Z}$-modules and have the same rank generating function by Corollary 3.13, this map is an isomorphism, and the proof is complete. ☐

6. Irreducible components

In this section, we characterize the irreducible components of $Y_{n,\lambda,s}$. We show that the number of irreducible components is equal to $\binom{n}{k} \cdot \#\mathrm{SYT}(\lambda)$ when $s > \ell(\lambda)$.

Given a sequence of partitions $\mu^\bullet = (\emptyset \subseteq \mu^1 \subseteq \mu^2 \subseteq \cdots \subseteq \mu^n)$ such that μ^i/μ^{i-1} consists of at most one box for all i, define the following subset of $Y_{n,\lambda,s}$,

$$Y_{n,\lambda,s}^{\mu^\bullet} = \{V_\bullet \in Y_{n,\lambda,s} \mid N|_{V_i \cap \mathrm{im}(N^{n-k})} \text{ has Jordan type } \mu^i\}. \tag{6.1}$$

Lemma 6.1. *The subspaces $Y_{n,\lambda,s}^{\mu^\bullet}$ form a partition of $Y_{n,\lambda,s}$ into irreducible subspaces. If $s > \ell(\lambda)$, then $Y_{n,\lambda,s}^{\mu^\bullet}$ is nonempty if and only if $\mu^n = \lambda$. If $s = \ell(\lambda)$, then $Y_{n,\lambda,s}^{\mu^\bullet}$ is nonempty if and only if we have that $\mu^n = \lambda$ and $\mu^i = \mu^{i+1}$ implies $\mu_s^i = \lambda_s$. Furthermore, if $Y_{n,\lambda,s}^{\mu^\bullet}$ is nonempty, then*

$$\dim_{\mathbb{C}}(Y_{n,\lambda,s}^{\mu^\bullet}) = n(\lambda) + (n-k)(s-1). \tag{6.2}$$

Proof. If $V_\bullet \in C_{(i_1,\dots,i_n)}$, then the Jordan type of $N|_{V_j \cap \mathrm{im}(N^{n-k})}$ is $\lambda^{(i_{j+1},\dots,i_n)}$. Therefore,

$$Y_{n,\lambda,s}^{\mu^\bullet} = \bigsqcup_{(i_1,\dots,i_n)} C_{(i_1,\dots,i_n)} \tag{6.3}$$

where the union is over all sequences $(i_1,\dots,i_n)$ such that $\mu^j = \lambda^{(i_{j+1},\dots,i_n)}$ for $0 \le j \le n$. It can then be seen from this that $Y_{n,\lambda,s}^{\mu^\bullet}$ is a product of projective spaces and affine spaces, hence it is irreducible. Furthermore, $Y_{n,\lambda,s}^{\mu^\bullet}$ is nonempty if and only if there exists at least one such sequence $(i_1,\dots,i_n)$. If $s > \ell(\lambda)$, the existence of such a sequence is equivalent to $\mu^n = \lambda$. If $s = \ell(\lambda)$, then the existence of such a sequence is equivalent to $\mu^n = \lambda$ and $\mu^i = \mu^{i+1}$ implies $\mu_s^i = \lambda_s$.

Suppose that $Y_{n,\lambda,s}^{\mu^\bullet}$ is nonempty, and recall that $\dim_{\mathbb{C}}(C_{(i_1,\dots,i_n)}) = \sum_{j\le n}(i_j - 1)$. Then the dimension of $Y_{n,\lambda,s}^{\mu^\bullet}$ is the maximum of $\sum_{j\le n}(i_j-1)$ over all $(i_1,\dots,i_n)$ such that $\mu^j = \lambda^{(i_{j+1},\dots,i_n)}$ for all $0 \le j \le n$. To verify the dimension formula (6.2), first observe that $\sum_{j\le n}(i_j-1)$ is maximal if and only if the following two conditions hold.

- Whenever μ^j/μ^{j-1} is a single box, then i_j is the row index of the lowest box in that column of μ^j,
- Whenever μ^j/μ^{j-1} is empty, then $i_j = s$.

□

Theorem 6.2. *The space $Y_{n,\lambda,s}$ is equidimensional of dimension $n(\lambda)+(n-k)(s-1)$. The subspaces $\overline{Y^{\mu^\bullet}_{n,\lambda,s}}$ for which $Y^{\mu^\bullet}_{n,\lambda,s}$ is nonempty form a complete set of irreducible components. In the case $s>\ell(\lambda)$, there are $\binom{n}{k}\cdot\#\mathrm{SYT}(\lambda)$ many irreducible components.*

The next theorem is a characterization of the top cohomology group of $Y_{n,\lambda,s}$. This result generalizes Springer's theorem that the top cohomology group is an irreducible.

Theorem 6.3. *Let $d=\dim(Y_{n,\lambda,s})=n(\lambda)+(n-k)(s-1)$, and consider S_k as the subgroup of S_n permuting the elements of $[k]$. For $s>\ell(\lambda)$, we have an isomorphism of S_n-modules*

$$H^{2d}(Y_{n,\lambda,s};\mathbb{Q})\cong \mathrm{Ind}\uparrow_{S_k}^{S_n}(S^\lambda). \tag{6.4}$$

For $s=\ell(\lambda)$, we have

$$H^{2d}(Y_{n,\lambda,s};\mathbb{Q})\cong S^{\Lambda/(n-k)^{s-1}}. \tag{6.5}$$

Proof. The case when $s>\ell(\lambda)$ follows immediately by combining Theorem 5.6 with [10, Corollary 3.3.15], which says that the top degree component of $R_{n,\lambda,s}$ is isomorphic to $\mathrm{Ind}\uparrow_{S_k}^{S_n}(S^\lambda)$.

Let us now assume $s=\ell(\lambda)$. Combining Theorem 5.6 and Theorem **??**, we have that

$$\mathrm{Frob}(H^{2d}(Y_{n,\lambda,s};\mathbb{Q});q)=\sum_{\substack{\varphi\in\mathrm{ECI}_{n,\lambda,s},\\ \mathrm{inv}(\varphi)=d}}\mathbf{x}^\varphi. \tag{6.6}$$

It can be checked that $\mathrm{inv}(\varphi)=d$ if and only if $\varphi_{i,j}>\varphi_{i+1,j}$ for $i<\lambda'_j$ and all basement cells are in column $\ell(\lambda)-1$.

It suffices to prove that the right-hand side of (6.6) is equal to $s_{\Lambda/(n-k)^{\ell(\lambda)-1}}(x)$. Let $\alpha=(\alpha_1,\dots,\alpha_n)$ be a weak composition of n. The x^α coefficient of the right-hand side of (6.6) is the number of $\varphi\in\mathrm{ECI}_{n,\lambda,s}$ of type α such that $\mathrm{inv}(\varphi)=d$.

Given $\varphi\in\mathrm{ECI}_{n,\lambda,s}$ of type α such that $\mathrm{inv}(\varphi)=d$, define a labeling T of the Young diagram of skew shape $\Lambda/(n-k)^{\ell(\lambda)-1}$ by labeling the ith row of T from left to right with the labels of the ith column of φ, read from top to bottom, and then replacing each label j with $n+1-j$. Since each column of φ is weakly increasing from top to bottom, each row of T is weakly increasing from left to right. Furthermore, since the rows of φ are strictly decreasing from left to right, the columns of T are strictly increasing from top to bottom. Therefore, T is a semi-standard Young tableau of type $(\alpha_n,\alpha_{n-1},\dots,\alpha_1)$. Moreover, φ can easily be reconstructed from T.

Therefore, the coefficient of x^α in the right-hand side of (6.6) is equal to the coefficient of $x^{(\alpha_n,\alpha_{n-1},\dots,\alpha_1)}$ in $s_{\Lambda/(n-k)^{\ell(\lambda)-1}}(x)$. Since both the right-hand side of (6.6) and the skew-Schur function are symmetric, we conclude that these two symmetric functions are equal. Therefore, the isomorphism (6.5) holds, and the proof is complete. □

7. The space $Y_{n,\lambda}$

In this section, we construct a topological space $Y_{n,\lambda}$ whose cohomology ring is isomorphic to $R_{n,\lambda}$. We then state a generalization of a theorem of de Concini and Procesi that relates Springer fibers to the scheme of diagonal "nilpotent" matrices.

For any n and $\lambda \vdash k$, define the topological space $Y_{n,\lambda}$ as follows. Let N be a nilpotent operator on the $\mathbb{C}$-vector space $\mathbb{C}^\infty$ with countably infinite dimension that has Jordan type

$$(n-k+\lambda_1, \dots, n-k+\lambda_{\ell(\lambda)}, n-k, n-k, \dots).$$

Then $\mathrm{im}(N^{n-k})$ has dimension k. It can be checked that

$$Y_{n,\lambda} \coloneqq \{V_\bullet \in \mathrm{Fl}_{1,\dots,n}(\mathbb{C}^\infty) \,|\, NV_i \subseteq V_i \text{ for } i \leq n \text{ and } \mathrm{im}(N^{n-k}) \subseteq V_n\}. \tag{7.1}$$

We have closed embeddings

$$Y_{n,\lambda,\ell(\lambda)} \subseteq Y_{n,\lambda,\ell(\lambda)+1} \subseteq \cdots \subseteq Y_{n,\lambda,s} \subseteq \cdots.$$

It can be checked that $Y_{n,\lambda}$ is the direct limit of these topological spaces,

$$Y_{n,\lambda} \cong \varinjlim_s Y_{n,\lambda,s}.$$

Let us recall the Universal Coefficient Theorem for Cohomology. It states that given any space X, there exists a split exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(X), \mathbb{Z}) \to H^i(X) \xrightarrow{h} \mathrm{Hom}(H_i(X), \mathbb{Z}) \to 0$$

where the map h is defined as follows: given $\varphi : C_i(X) \to \mathbb{Z}$ an i-cocycle, then $\delta\varphi = 0$, where δ is the differential map on singular cocycles. Therefore, we have $\varphi|_{B_i(X)} = 0$, so φ induces a map $\overline{\varphi} : Z_i(X)/B_i(X) \to \mathbb{Z}$. The map h is defined by $h([\varphi]) \coloneqq \overline{\varphi}$.

Theorem 7.1. *We have $H^*(Y_{n,\lambda}) \cong R_{n,\lambda}$ as graded rings.*

Proof. By Theorem 5.6, we have $H^*(Y_{n,\lambda,s}) \cong R_{n,\lambda,s}$ From the definitions of $R_{n,\lambda}$ and $R_{n,\lambda,s}$, it can be checked that

$$R_{n,\lambda} \cong \varprojlim_s H^*(Y_{n,\lambda,s}), \tag{7.2}$$

where the inverse limit is the limit in the category of graded rings. Therefore, it suffices to show that the natural map induced by the inclusions $Y_{n,\lambda,s} \subseteq Y_{n,\lambda}$,

$$H^i(Y_{n,\lambda}) \to \varprojlim_s H^i(Y_{n,\lambda,s}), \tag{7.3}$$

is an isomorphism for all i.

Since each $Y_{n,\lambda,s}$ is a T_1 space, then $Y_{n,\lambda,s}$ satisfies the hypotheses of [16, Proposition 3.33], so the following natural map is an isomorphism

$$\varinjlim_s H_i(Y_{n,\lambda,s}) \xrightarrow{\sim} H_i(Y_{n,\lambda}) \tag{7.4}$$

for all i. Since each of the spaces $Y_{n,\lambda,s}$ has an affine paving, then $H_i(Y_{n,\lambda,s})$ is a free $\mathbb{Z}$-module for all i, so $\mathrm{Ext}^1_{\mathbb{Z}}(H_i(Y_{n,\lambda,s}), \mathbb{Z}) = 0$ for all i. Therefore, $H^i(Y_{n,\lambda,s}) \cong \mathrm{Hom}(H_i(Y_{n,\lambda,s}), \mathbb{Z})$ by the Universal Coefficient Theorem for Cohomology. Hence, we have

$$Hom(H_i(Y_{n,\lambda}), \mathbb{Z}) \xrightarrow{\sim} Hom(\varinjlim_s H_i(Y_{n,\lambda,s}), \mathbb{Z}) \tag{7.5}$$

$$\xrightarrow{\sim} \varprojlim_s Hom(H_i(Y_{n,\lambda,s}), \mathbb{Z}) \tag{7.6}$$

$$\xleftarrow{\sim} \varprojlim_s H^i(Y_{n,\lambda,s}). \tag{7.7}$$

By (7.4) and the fact that $H_i(Y_{n,\lambda,s}) \cong H_i(Y_{n,\lambda,s+1})$ for $s > i/2$, then $H_i(Y_{n,\lambda})$ is a free $\mathbb{Z}$-module for all i, so $\mathrm{Ext}^1_{\mathbb{Z}}(H_i(Y_{n,\lambda}), \mathbb{Z}) = 0$. Hence, by the Universal Coefficient Theorem for Cohomology, we have that

$$H^i(Y_{n,\lambda}) \cong Hom(H_i(Y_{n,\lambda}), \mathbb{Z}) \tag{7.8}$$

for all i, so

$$H^i(Y_{n,\lambda}) \cong Hom(H_i(Y_{n,\lambda}), \mathbb{Z}) \cong \varprojlim_s H^i(Y_{n,\lambda,s}). \tag{7.9}$$

In order to finish the proof, it must be checked that the composition of the isomorphisms in (7.9) is the same as the natural map (7.3). This is a routine check, and we omit it. □

8. Future Work

Question 1. There is a s-dimensional torus action on $Y_{n,\lambda,s}$, given by scaling the vectors in each generalized eigenspace of N by the same constant. The space $Y_{2,\emptyset,2}$ is not a GKM variety because it does not have finitely many one-dimensional orbits with respect to this action. Is it still possible to compute its equivariant cohomology ring? Brundan and Ostrik reference Goresky-Macpherson's characterization of equiv cohomology of Springer fibers in terms of subspace arrangements, and extensions of those results.

Question 2. What are the cell closures for the paving of $Y_{n,(1^k),k}$? Is there a nice description of the corresponding "Bruhat poset"?

Question 3. Can we identify the cohomology classes of the cell closures in the case of $Y_{n,(1^k),k}$?

Question 4. The space $Y_{n,\lambda,s}$ is usually singular because it has many irreducible components. Under what condition are all of the irreducible components smooth? Is this true for $Y_{n,(1^k),k}$, in particular?

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References

[1] F. Bergeron, A. M. Garsia, M. Haiman, and G. Tesler, *Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions*, vol. 6, 1999, pp. 363–420. Dedicated to Richard A. Askey on the occasion of his 65th birthday, Part III.
[2] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. Seelinger, *A proof of the Extended Delta Conjecture*. arXiv:2102.08815, 2021, 2102.08815.
[3] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2), 57 (1953), pp. 115–207.
[4] W. Borho and R. MacPherson, *Partial resolutions of nilpotent varieties*, in Analysis and topology on singular spaces, II, III (Luminy, 1981), vol. 101 of Astérisque, Soc. Math. France, Paris, 1983, pp. 23–74.
[5] J. Brundan and V. Ostrik, *Cohomology of Spaltenstein varieties*, Transform. Groups, 16 (2011), pp. 619–648.
[6] M. D'Adderio and A. Mellit, *A proof of the compositional Delta conjecture*. arXiv:2011.11467, 2020, 2011.11467.

[7] C. De Concini and C. Procesi, *Symmetric functions, conjugacy classes and the flag variety*, Invent. Math., 64 (1981), pp. 203–219.

[8] A. M. Garsia and M. Haiman, *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A., 90 (1993), pp. 3607–3610.

[9] A. M. Garsia and C. Procesi, *On certain graded S_n-modules and the q-Kostka polynomials*, Adv. Math., 94 (1992), pp. 82–138.

[10] S. T. Griffin, *Combinatorics and Representation Theory of Rank Varieties, Springer Fibers, and Hyperplane Arrangements*, PhD thesis, University of Washington, August 2020.

[11] ——, *Ordered set partitions, Garsia-Procesi modules, and rank varieties*, Trans. Amer. Math. Soc., (Accepted, 2020).

[12] J. Haglund, J. B. Remmel, and A. T. Wilson, *The Delta Conjecture*, Trans. Amer. Math. Soc., 370 (2018), pp. 4029–4057.

[13] J. Haglund, B. Rhoades, and M. Shimozono, *Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture*, Adv. Math., 329 (2018), pp. 851–915.

[14] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc., 14 (2001), pp. 941–1006.

[15] ——, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math., 149 (2002), pp. 371–407.

[16] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[17] R. Hotta and T. A. Springer, *A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups*, Invent. Math., 41 (1977), pp. 113–127.

[18] B. Pawlowski and B. Rhoades, *A flag variety for the Delta Conjecture*, Trans. Amer. Math. Soc., 372 (2019), pp. 8195–8248.

[19] M. Precup, *Affine pavings of Hessenberg varieties for semisimple groups*, Selecta Math. (N.S.), 19 (2013), pp. 903–922.

[20] M. Precup and J. Tymoczko, *Hessenberg varieties of parabolic type*, arXiv e-prints, (2017), arXiv:1701.04140.

[21] T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math., (1976), pp. 173–207.

[22] ——, *A construction of representations of Weyl groups*, Invent. Math., 44 (1978), pp. 279–293.

[23] T. Tanisaki, *Defining ideals of the closures of the conjugacy classes and representations of the Weyl groups*, Tôhoku Math. J. (2), 34 (1982), pp. 575–585.

[24] J. S. Tymoczko, *Linear conditions imposed on flag varieties*, Amer. J. Math., 128 (2006), pp. 1587–1604.

[25] M. Zabrocki, *A module for the Delta conjecture*. arXiv:1902.08966, 2019.