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DISTINCT ZEROS AND SIMPLE ZEROS FOR THE FAMILY OF DIRICHLET L-FUNCTIONS

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Abstract

In this paper, we study the number of additional zeros of Dirichlet L-function caused by multiplicity by using Asymptotic Large Sieve. Then in asymptotic terms we prove that there are >80.13% of zeros of the family of Dirichlet L-functions which are distinct and >60.261% of zeros of the family of Dirichlet L-functions which are simple. In addition, assuming the Generalized Riemann Hypothesis, we improve these proportions to 83.216% and 66.433%.

1. Introduction

Let $L(s, \chi)$ be a Dirichlet L-function, where $s = \sigma + it$, $\chi \pmod q$ is a character. It is defined for $\sigma > 1$ by

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}. \tag{1.1}$$

Let $\chi \pmod q$ be a primitive character. The total number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ with $0 < \beta < 1$ and $|\gamma| \le T$, say $N(T, \chi)$, is known asymptotically very precisely

$$N(T, \chi) = \frac{T}{\pi} \log \frac{qT}{2\pi e} + O(\log qT), \quad T \ge 3. \tag{1.2}$$

We define the number of distinct zeros and the number of simple zeros of a single $L(s, \chi)$ as follows:

$$\begin{aligned} N_d(T, \chi) &= |\{\rho = \beta + i\gamma \colon -T < \gamma \le T,\ L(\rho, \chi) = 0\}|, \\ N_s(T, \chi) &= |\{\rho = \beta + i\gamma \colon -T < \gamma \le T,\ L(\rho, \chi) = 0, L'(\rho, \chi) \ne 0\}|. \end{aligned} \tag{1.3}$$

It is believed that $N_d(T, \chi) = N_s(T, \chi) = N(T, \chi)$, which means that all zeros of Dirichlet L-function are simple. This is known as the Simple Zero Conjecture.

As a special case, for the Riemann zeta-function, there is a long history of the study on the Simple Zero Conjecture, and the topic has been studied by a lot of papers (to see [**1**, **6**, **7**, **12**, **13**] for example).

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In 1995, Farmer [12] proved that at least 63.952% of zeros of the Riemann zeta-function are distinct. Farmer proved this by using a combination method which was based on proportions of simple zeros of the derivatives of the Riemann ξ-function.

It is known that >40.58% of zeros of the Riemann zeta-function are simple, which was proved in 2011 by Bui, Conrey and Young's work [2]. The work is based on Levinson's method (to see [5–7, 14]).

If assuming the Riemann Hypothesis, we know that >67.275% of zeros of the Riemann zeta-function are simple, which was proved by Cheer and Goldston [7] in 1993 (see also [17, 18]).

In 1998, under the assumption of the Riemann Hypothesis and the Generalized Lindelöf Hypothesis, Conrey *et al*. proved in [8] that >84.56% of zeros of the Riemann zeta-function are distinct and >70.37% of zeros of the Riemann zeta-function are simple. Their starting point is the observation that

$$\left|\sum_{-T\le\gamma\le T} M\left(\frac{1}{2}+i\gamma\right)\zeta'\left(\frac{1}{2}+i\gamma\right)\right|^2 \le N_s(T,1)\sum_{-T\le\gamma\le T}\left|M\left(\frac{1}{2}+i\gamma\right)\zeta'\left(\frac{1}{2}+i\gamma\right)\right|^2 \tag{1.4}$$

by Cauchy's inequality. The function $M(s)$ is taken to be a mollifier

$$M(s)=\sum_{n\le y}\mu(n)P\left(\frac{\log y/n}{\log y}\right)n^{-s}, \tag{1.5}$$

where $y=T^{1/2-\varepsilon}$ and $P(x)$ is a suitable smooth function. In 2013, Bui and Heath-Brown successfully improved the approach of Conrey *et al*. to remove the dependence on the Generalized Lindelöf Hypothesis. In their work [3], they obtained the same proportion only under the assumption of the Riemann Hypothesis.

For Dirichlet L-function, Conrey *et al*. invented the Asymptotic Large Sieve (to see [9]) and successfully used it in Levinson's method. Let $\Psi(x)$ be a non-negative function, which is smooth, compactly supported on $\mathbb{R}^+$. Put

$$N(T,Q)=\sum_{q\le Q}\frac{\Psi(q/Q)}{\phi(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}N(T,\chi), \tag{1.6}$$

where $Q\ge 3$ and $T\ge 3$. Here $\phi(n)$ is the Euler function and the superscript $*$ restricts the summation to primitive characters. Denoted by $N_s'(T,\chi)$ the number of simple zeros of $L(s,\chi)$, $\rho=\frac{1}{2}+i\gamma$ with $|\gamma|\le T$. Let $N_s'(T,Q)$ denote the same sum as (1.6), but with $N(T,\chi)$ replaced by $N_s'(T,\chi)$. Conrey *et al*. obtained the following result in [10].

Theorem 1.1 *For Q and T with* $(\log Q)^6\le T\le(\log Q)^A$ *we have*

$$N_s'(T,Q)\ge 0.5865N(T,Q), \tag{1.7}$$

where $A\ge 6$ is any constant, provided Q is sufficiently large in terms of A.

Remark In the main theorem of [10], the proportion is 0.56. This is because Conrey *et al*. took the polynomial $P(x)$ in their mollifier in the simplest form for convenience. They also mentioned that the proportion should be 0.5865 with the optimal choice of $P(x)$. So, we present Conrey *et al.*'s result with the proportion 0.5865 in Theorem 1.1.

In the proof of our results in this paper, we also need to use the mollifier. For convenience, we will choose the mollifier the same as the one used in Conrey *et al.* [**10**] in this paper. The mollifier may not be the best form for this problem, one may find some other possible choices of it in [**1**, **2**, **11**, **15**].

Conrey *et al.*'s result says that more than a half of zeros of the family of Dirichlet L-functions are on the critical line and simple, which may mean that the Riemann Hypothesis and the Simple Zero Conjecture for this family are more likely to be true than not.

If counting with suitable weights for the low-lying simple zeros, Chandee *et al.* [**4**] recently obtained a pretty good proportion for the low-lying simple zeros of primitive Dirichlet L-functions under the assumption of the Generalized Riemann Hypothesis (GRH). Let Φ be a smooth function that is real and compactly supported in (a, b) with $0 < a < b$, and

$$\hat{\Phi}(s) = \int_0^\infty \Phi(x)x^{s-1}\,\mathrm{d}x \tag{1.8}$$

be the Mellin transform of Φ. Let

$$N_\Phi(Q) = \sum_{q\le Q}\frac{W(q/Q)}{\phi(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\ q)}\sum_{\gamma_\chi}|\hat{\Phi}(i\gamma_\chi)|^2 \tag{1.9}$$

with W a smooth function, compactly supported in $(1, 2)$, and the last sum being over all non-trivial zeros $\frac{1}{2} + i\gamma_\chi$ of Dirichlet L-function $L(s, \chi)$. Chandee *et al.* proved the following result.

THEOREM 1.2 *Assume GRH and choose suitable Φ so that $\hat{\Phi}(ix) = (\sin x/x)^2$, then we can have*

$$\frac{1}{N_\Phi(Q)}\sum_{q\le Q}\frac{W(q/Q)}{\phi(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\ q)}\sum_{\substack{\gamma_\chi\\ \text{simple}}}\left(\frac{\sin\gamma_x}{\gamma_x}\right)^4 \ge \frac{11}{12} + o(1). \tag{1.10}$$

In this work, we focus on the number of distinct zeros and simple zeros of the family of Dirichlet L-functions. The work is motivated by Conrey *et al.*'s work. Different from Chandee *et al.*'s special weights for low-lying zeros, we count with the traditional weight as Conrey *et al.* did in [**10**]. As (1.6) we define

$$\begin{aligned} N_d(T, Q) &= \sum_{q\le Q}\frac{\Psi(q/Q)}{\phi(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\ q)} N_d(T, \chi),\\ N_s(T, Q) &= \sum_{q\le Q}\frac{\Psi(q/Q)}{\phi(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\ q)} N_s(T, \chi) \end{aligned} \tag{1.11}$$

for $Q \ge 3$ and $T \ge 3$. Our results are presented below in great details.

THEOREM 1.3 *For Q and T with $(\log Q)^6 \leq T \leq (\log Q)^A$ we have*

$$N_d(T, Q) \geq 0.8013 N(T, Q), \quad N_s(T, Q) \geq 0.60261 N(T, Q), \tag{1.12}$$

where $A \geq 6$ is any constant, provided Q is sufficiently large in terms of A.

If assuming that the Riemann Hypothesis for the family of these Dirichlet L-functions holds, we can obtain following better results both on distinct zeros and simple zeros.

THEOREM 1.4 *Assume the GRH. For Q and T with $(\log Q)^6 \leq T \leq (\log Q)^A$ we have*

$$N_d(T, Q) \geq 0.83216 N(T, Q), \quad N_s(T, Q) \geq 0.66433 N(T, Q), \tag{1.13}$$

where $A \geq 6$ is any constant, provided Q is sufficiently large in terms of A.

We note that the number of distinct zeros of the Dirichlet L-function $L(s, \chi)$ is closely related to the distribution of zeros of the function which is a combination of $L(s, \chi)$ and $L'(s, \chi)$ as follows:

$$G(s, \chi) = L(s, \chi) w_1(s, \chi) + L'(s, \chi) w_2(s, \chi), \tag{1.14}$$

where $w_1(s, \chi)$ and $w_2(s, \chi)$ can be any analytic functions. Observing this, we will prove Theorems 1.3 and 1.4 from two other results. One is an upper bound for the proportion of zeros in the right-hand side of the critical area for the family of some functions $G(s, \chi)$. To obtain this upper bound in Theorem 1.5, we will need to choose some appropriate $w_1(s, \chi)$ and $w_2(s, \chi)$ in (1.14) to make $G(s, \chi)$ have fewer zeros in the right-hand side. Another is a lower bound for the proportion of zeros of the family of $\xi'(s, \chi)$ on the critical line. These two results will be specified by Theorems 1.5 and 1.6 in the following.

Let $\mathcal{D}$ be the closed rectangle with vertices $\frac{1}{2} - iT, 3 - iT, \frac{1}{2} + iT, 3 + iT$. We are interested in such $G(s, \chi)$ with the form given by (1.14), which has less zeros in $\mathcal{D}$. Let $N_G(\mathcal{D}, \chi)$ denote the number of zeros of $G(s, \chi)$ in $\mathcal{D}$, including zeros on the left boundary and define

$$N_G(\mathcal{D}, Q) = \sum_{q \leq Q} \frac{\Psi(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} N_G(\mathcal{D}, \chi) \tag{1.15}$$

for $Q \geq 3$ and $T \geq 3$, the following theorem can be obtained.

THEOREM 1.5 *For sufficiently large T with $(\log Q)^6 \leq T \leq (\log Q)^A$, there are some $G(s, \chi)$ with the form given by (1.14), which make*

$$N_G(\mathcal{D}, Q) < 0.167835 N(T, Q). \tag{1.16}$$

Let

$$\xi(s, \chi) = H(s, \chi) L(s, \chi) \tag{1.17}$$

with

$$H(s,\chi)=\frac{1}{2}s(s-1)\left(\frac{q}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\frac{s+\frac{1}{2}(1-\chi(-1))}{2}\right). \tag{1.18}$$

In the critical area $0<\sigma<1$, the function $\xi(s,\chi)$ has the same zeros as the Dirichlet L-function $L(s,\chi)$, and for any positive integer m the number of zeros of $\xi^{(m)}(s,\chi)$ with $|t|<T$ in the critical area is $N(T,\chi)+O_j(\log qT)$ (to see Lemma 1.7 in the following). Let $N_{\xi^{(m)},c}(T,\chi)$ denote the number of zeros of $\xi^{(m)}(s,\chi)$ on the critical line and define

$$N_{\xi^{(m)},c}(T,Q)=\sum_{q\le Q}\frac{\Psi(q/Q)}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}^{*}N_{\xi^{(m)},c}(T,\chi), \tag{1.19}$$

as the definition of $N(T,Q)$ in (1.6) for $Q\ge 3$ and $T\ge 3$.

Let

$$\alpha_m=\liminf_{T\to\infty}\frac{N_{\xi^{(m)},c}(T,Q)}{N(T,Q)}. \tag{1.20}$$

We obtain the following lower bounds for α_m.

THEOREM 1.6 *For sufficiently large T with $(\log Q)^6\le T\le(\log Q)^A$, we have $\alpha_1>0.93828$, $\alpha_2>0.97314$, $\alpha_3>0.99203$.*

We only give numerical results of α_m for $m\le 3$ here since α_3 is very close to 1 already. For large m, it has been proved by Conrey in [5] that $\alpha_m=1+O(m^{-2})$ as $m\to\infty$. In the proof of Theorems 1.3 and 1.4, we will only use the numerical result of α_1.

1.1. *Proof of Theorems* 1.3 *and* 1.4 *from Theorems* 1.5 *and* 1.6

To study the number of distinct zeros and simple zeros of Dirichlet L-function, we firstly focus on the number of the additional zeros caused by multiplicity. Here the number of additional zeros caused by multiplicity means the number of zeros counted according to multiplicity minus one. We note that if ρ, a zero of $L(s,\chi)$, is a non-simple zero, it must be a zero of

$$G(s,\chi)=L(s,\chi)w_1(s,\chi)+L'(s,\chi)w_2(s,\chi) \tag{1.21}$$

with multiplicity reduced by at most one. Here $w_1(s,\chi)$, $w_2(s,\chi)$ can be any analytic functions. Hence the number of additional zeros of $L(s,\chi)$ caused by multiplicity in any region is not more than the number of zeros of $G(s,\chi)$ in the same region.

It is therefore important to find a such $G(s,\chi)$ that has less zeros in the critical area $(0<\mathrm{Re}(s)<1)$, but this is very hard. A more feasible way is to partition the critical area into some sub areas, then for each sub area we may find a $G(s,\chi)$ that has less zeros in the area. In this paper, we actually partition the critical area into the left part $(\mathrm{Re}(s)<\frac{1}{2})$ and the right part $(\mathrm{Re}(s)\ge\frac{1}{2})$.

Firstly, let us consider the left-hand side ($\mathrm{Re}(s) < \frac{1}{2}$). The estimation of the number of additional zeros in this side is only required in the proof of Theorem 1.3. We choose $G(s, \chi) = \xi'(s, \chi)$ for this side, where $\xi(s, \chi)$ is given by (1.17).

About the number of zeros of $\xi^{(j)}(s, \chi)$ for primitive character χ, we also need the following lemma, which can be proved similarly as Lemma 2 in [5].

Lemma 1.7 *Let χ be a primitive character. For any integer $j \geq 0$, all zeros of $\xi^{(j)}(s, \chi)$ satisfy $0 < \sigma < 1$. Let $N_{\xi^{(j)}}(T, \chi)$ denote the number of zeros of $\xi^{(j)}(s, \chi)$ with $-T \leq t \leq T$. Then*

$$N_{\xi^{(j)}}(T, \chi) = \frac{T}{\pi} \log \frac{qT}{2\pi e} + O_j(\log qT).$$

The functional equation for $L(s, \chi)$ says

$$h(\chi)\xi(s, \chi) = \bar{h}(\chi)\xi(1 - s, \bar{\chi}), \tag{1.22}$$

where

$$\frac{\bar{h}(\chi)}{h(\chi)} = \frac{\tau(\chi)}{i^{(1-\chi(-1))} q^{1/2}} \tag{1.23}$$

with $|h(\chi)| = 1$, $\tau(\chi) = \sum_{n=1}^{q} \chi(n) \exp(2\pi i n/q)$.

From (1.22) we may find that if ρ is a zero of $\xi'(s, \chi)$, then $1 - \rho$ is a zero of $\xi'(s, \overline{\chi})$. From this and Lemma 1.7 with $j = 1$, we can see that $N_{\xi', l}(T, \chi)$, the number of zeros of $\xi'(s, \chi)$ in the left-hand side, and $N_{\xi', l}(T, \overline{\chi})$, the number of zeros of $\xi'(s, \overline{\chi})$ in the left-hand side, satisfy

$$N_{\xi', l}(T, \chi) + N_{\xi', l}(T, \overline{\chi}) = N(T, \chi) - N_{\xi', c}(T, \chi) + O(\log qT). \tag{1.24}$$

Here $N_{\xi', c}(T, \chi)$ denotes the number of zeros of $\xi'(s, \chi)$ on the critical line, which satisfies $N_{\xi', c}(T, \chi) = N_{\xi', c}(T, \overline{\chi})$. In particular, if χ is a real character, it follows:

$$N_{\xi', l}(T, \chi) = \frac{1}{2}\big(N(T, \chi) - N_{\xi', c}(T, \chi)\big) + O(\log qT). \tag{1.25}$$

We now come to the right-hand side. Let $\mathbf{q} = q/\pi$ and $R > 0$, r be constants to be specified later. Denote

$$\begin{aligned} \psi_1(s, \chi) &= \sum_{n \leq X} \frac{\mu(n)\chi(n)}{n^{s+R/\log \mathbf{q}}} P_1\left(\frac{\log X/n}{\log X}\right), \\ \psi_2(s, \chi) &= \sum_{n \leq X} \frac{\mu(n)\chi(n)}{n^{s+R/\log \mathbf{q}}} P_2\left(\frac{\log X/n}{\log X}\right), \end{aligned} \tag{1.26}$$

where μ is the Möbius function, $X = \mathbf{q}^{\theta}$ with $0 < \theta < 1$. Here P_1, P_2 are polynomials with $P_1(0) = P_2(0) = 0$, $P_1(1) = P_2(1) = 1$, which will be specified later. Let $G(s, \chi)$ be defined by (1.14) with

$$w_1(s, \chi) = \psi_1(s, \chi) \qquad \text{and} \qquad w_2(s, \chi) = \frac{1}{r \log \mathbf{q}} \psi_2(s, \chi). \tag{1.27}$$

Recall that $\mathcal{D}$ is the closed rectangle with vertices $\frac{1}{2} - iT$, $3 - iT$, $\frac{1}{2} + iT$, $3 + iT$ and $N_G(\mathcal{D}, \chi)$ denote the number of zeros of $G(s, \chi)$ in $\mathcal{D}$, including zeros on the left boundary. Then, we have that the number of additional zeros of $L(s, \chi)$ caused by multiplicity in the right-hand side is not more than $N_G(\mathcal{D}, \chi)$.

From the above discussion, it follows that the number of additional zeros of $L(s, \chi)$ caused by multiplicity is not more than $N_{\xi', l}(s, \chi)$ in the left-hand side and $N_G(\mathcal{D}, \chi)$ in the right-hand side. So, by (1.24) we may have the following formula for the number of distinct zeros of $L(s, \chi)$,

$$N_d(T, \chi) + N_d(T, \overline{\chi}) \geq N(T, \chi) + N_{\xi', c}(T, \chi) - N_G(\mathcal{D}, \chi) - N_G(\mathcal{D}, \overline{\chi}) + O(\log qT). \tag{1.28}$$

In addition, the number of all non-simple zeros counted according to multiplicity is not more than $2N_{\xi', l}(T, \chi) + 2N_G(\mathcal{D}, \chi)$ since each non-simple zero has multiplicity at least 2. Thus for simple zeros we have

$$\begin{aligned} N_s(T, \chi) + N_s(T, \overline{\chi}) &\geq 2N(T, \chi) - 2N_{\xi', l}(T, \chi) - 2N_G(\mathcal{D}, \chi) - 2N_{\xi', l}(T, \overline{\chi}) - 2N_G(\mathcal{D}, \overline{\chi}) \\ &= 2N_{\xi', c}(T, \chi) - 2N_G(\mathcal{D}, \chi) - 2N_G(\mathcal{D}, \overline{\chi}) + O(\log qT). \end{aligned} \tag{1.29}$$

If assuming the Riemann Hypothesis for this $L(s, \chi)$, we do not need to care about the left-hand side since there is no zero in this side. Then, we have

$$N_d(T, \chi) \geq N(T, \chi) - N_G(\mathcal{D}, \chi). \tag{1.30}$$

Similarly, for simple zero we have

$$N_s(T, \chi) \geq N(T, \chi) - 2N_G(\mathcal{D}, \chi). \tag{1.31}$$

By averaging over the conductors q and primitive the characters χ, we have from (1.28) and (1.29) that

$$N_d(T, Q) \geq \left(\frac{1}{2} + o(1)\right) N(T, Q) + \frac{1}{2} N_{\xi', c}(T, Q) - N_G(\mathcal{D}, Q), \tag{1.32}$$

$$N_s(T, Q) \geq N_{\xi', c}(T, Q) - 2N_G(\mathcal{D}, Q) + o(1) N(T, Q). \tag{1.33}$$

Then by Theorems 1.5 and 1.6 we have

$$N_d(T,Q) \geq \left(\frac{1}{2} + o(1)\right)N(T,Q) + \frac{1}{2}N_{\xi',c}(T,Q) - N_G(\mathcal{D},Q)$$
$$> \left(\frac{1}{2} + 0.46914 - 0.167835\right)N(T,Q) > 0.8013N(T,Q), \tag{1.34}$$

and

$$N_s(T,Q) \geq N_{\xi',c}(T,Q) - 2N_G(\mathcal{D},Q) + o(1)N(T,Q)$$
$$> (0.93828 - 0.33567)N(T,Q) = 0.60261N(T,Q). \tag{1.35}$$

Hence we obtain Theorem 1.3.

If assuming the GRH, we have

$$N_d(T,Q) \geq N(T,Q) - N_G(\mathcal{D},Q), \tag{1.36}$$

$$N_s(T,Q) \geq N(T,Q) - 2N_G(\mathcal{D},Q). \tag{1.37}$$

Employing Theorems 1.5 and 1.6 we have

$$N_d(T,Q) \geq N(T,Q) - N_G(\mathcal{D},Q) > 0.83216N(T,Q), \tag{1.38}$$

and

$$N_s(T,Q) \geq N(T,Q) - 2N_G(\mathcal{D},Q) > 0.66433N(T,Q). \tag{1.39}$$

Hence we obtain Theorem 1.4.

In the following, we will first introduce an asymptotic large sieve result in Section 2. Then we will prove Theorem 1.5 in Section 3 and Theorem 1.6 in Section 4.

2. Asymptotic large sieve result

To obtain Theorems 1.5 and 1.6, we need to evaluate the following sum:

$$\sum_{q\leq Q}\frac{\Psi(q/Q)\log \mathbf{q}}{\phi(q)}$$
$$\times \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\int_{-T}^{T} L(\sigma_0+\alpha+it,\chi)L(\sigma_0+\beta-it,\bar{\chi})\psi_{i_1}(\sigma_0+it,\chi)\psi_{i_2}(\sigma_0-it,\bar{\chi})\,\mathrm{d}t \tag{2.1}$$

for $\alpha,\ \beta \ll 1/\log Q$ and $\sigma_0 = \frac{1}{2} - R/\log \mathbf{q}$. We evaluate the sum in this section from the following lemma, which is implied in the work of Conrey *et al*. [**10**].

LEMMA 2.1 *Suppose that $\psi_1(s,\chi)$ and $\psi_2(s,\chi)$ are defined as in (1.26). For sufficiently large Q, T with $\log^6 Q \le T \le \log^A Q$ and constant $A \ge 6$, let $X = \mathbf{q}^\theta$ with $0 < \theta < 1$, $\sigma_0 = \frac{1}{2} - R/\log \mathbf{q}$, $\alpha, \beta \ll 1/\log Q$ and $h_1 = h/(h,k)$, $k_1 = k/(h,k)$. Then, for $i_1, i_2 \in \{1,2\}$,*

$$\begin{aligned}
&\sum_{q\le Q} \frac{\Psi(q/Q)\log \mathbf{q}}{\phi(q)} \\
&\quad \times \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-T}^{T} L(\sigma_0+\alpha+it,\chi)L(\sigma_0+\beta-it,\bar{\chi}) \\
&\qquad\qquad \psi_{i_1}(\sigma_0+it,\chi)\psi_{i_2}(\sigma_0-it,\bar{\chi})\,\mathrm{d}t \\
&\quad \sim 2T\left(\sum_{q\le Q}\Psi(q/Q)\log \mathbf{q}\frac{\phi^*(q)}{\phi(q)}\right)\sum_{h\le X}\sum_{k\le X}\frac{\mu(h)\mu(k)}{(hh_1kk_1)^{1/2}} \\
&\quad F(\alpha,\beta,h_1,k_1)P_{i_1}\left(\frac{\log X/h}{\log X}\right)P_{i_2}\left(\frac{\log X/k}{\log X}\right)
\end{aligned} \tag{2.2}$$

holds uniformly in complex numbers α, $\beta \ll 1/\log Q$ and admits differentiations in α, β, where

$$\begin{aligned}
F(\alpha,\beta,h_1,k_1) &= h_1^{1/2-\sigma_0-\beta}k_1^{1/2-\sigma_0-\alpha}\zeta(2\sigma_0+\alpha+\beta) \\
&\quad + \frac{h_1^{\alpha+\sigma_0-\frac{1}{2}}k_1^{\beta+\sigma_0-\frac{1}{2}}}{\mathbf{q}^{2\sigma_0+\alpha+\beta-1}}\zeta(2-2\sigma_0-\alpha-\beta),
\end{aligned} \tag{2.3}$$

and $\phi^*(q)$ denotes the number of primitive characters (mod q).

We now evaluate the sum (2.1) from (2.2). Making the following variable changes

$$a = \left(\sigma_0+\alpha-\frac{1}{2}\right)\log \mathbf{q} \ll 1, \tag{2.4}$$

$$b = \left(\sigma_0+\beta-\frac{1}{2}\right)\log \mathbf{q} \ll 1, \tag{2.5}$$

we find that the formula (2.3) becomes to

$$\begin{aligned}
F(\alpha,\beta,h_1,k_1) &= h_1^{-b/\log \mathbf{q}}k_1^{-a/\log \mathbf{q}}\zeta\left(1+\frac{a+b}{\log \mathbf{q}}\right) \\
&\quad + \frac{h_1^{a/\log \mathbf{q}}k_1^{b/\log \mathbf{q}}}{e^{a+b}}\zeta\left(1-\frac{a+b}{\log \mathbf{q}}\right).
\end{aligned} \tag{2.6}$$

We also approximate $\zeta(s)$ near 1 by $(s-1)^{-1}$ getting the following asymptotic values

$$\zeta\left(1+\frac{a+b}{\log \mathbf{q}}\right) \sim \frac{\log \mathbf{q}}{a+b}, \quad \zeta\left(1-\frac{a+b}{\log \mathbf{q}}\right) \sim -\frac{\log \mathbf{q}}{a+b}. \tag{2.7}$$

Then, we get

$$F(\alpha,\beta,h_1,k_1)\sim\frac{\log\mathbf{q}}{a+b}\Big(h_1^{-b/\log\mathbf{q}}k_1^{-a/\log\mathbf{q}}-\mathrm{e}^{-a-b}h_1^{a/\log\mathbf{q}}k_1^{b/\log\mathbf{q}}\Big). \tag{2.8}$$

Substituting (2.8) into the right side of (2.2) we have

$$\begin{aligned}\sum_{h\le X}\sum_{k\le X}&\frac{\mu(h)\mu(k)}{(hh_1kk_1)^{1/2}}F(\alpha,\beta,h_1,k_1)P_{i_1}\left(\frac{\log X/h}{\log X}\right)P_{i_2}\left(\frac{\log X/k}{\log X}\right)\\&\sim\frac{\log\mathbf{q}}{a+b}\left\{\sum_{h\le X}\sum_{k\le X}\frac{\mu(h)\mu(k)}{h^{1/2}k^{1/2}h_1^{1/2+b/\log\mathbf{q}}k_1^{1/2+a/\log\mathbf{q}}}P_{i_1}\left(\frac{\log X/h}{\log X}\right)P_{i_2}\left(\frac{\log X/k}{\log X}\right)\right.\\&\left.\quad-e^{-a-b}\sum_{h\le X}\sum_{k\le X}\frac{\mu(h)\mu(k)}{h^{1/2}k^{1/2}h_1^{1/2-a/\log\mathbf{q}}k_1^{1/2-b/\log\mathbf{q}}}P_{i_1}\left(\frac{\log X/h}{\log X}\right)P_{i_2}\left(\frac{\log X/k}{\log X}\right)\right\}.\end{aligned}\tag{2.9}$$

To evaluate the sum over h and k within the brackets we appeal to Lemma 1 of [6], which says that

$$\begin{aligned}\sum_{h\le X}\sum_{k\le X}&\frac{\mu(h)\mu(k)}{h^{1/2}k^{1/2}h_1^{1/2+\omega_1}k_1^{1/2+\omega_2}}P_{i_1}\left(\frac{\log X/h}{\log X}\right)P_{i_2}\left(\frac{\log X/k}{\log X}\right)\\&\sim\frac{1}{\log X}\int_0^1\Big(P_{i_1}'(t)+\omega_1(\log X)P_{i_1}(t)\Big)\Big(P_{i_2}'(t)+\omega_2(\log X)P_{i_2}(t)\Big)\,\mathrm{d}t\end{aligned}\tag{2.10}$$

holds uniformly in complex numbers $\omega_1,\omega_2\ll(\log X)^{-1}$ and admits differentiations in ω_1,ω_2. Recall that $X=\mathbf{q}^\theta$. Substituting (2.10) into (2.9) we get

$$\begin{aligned}\sum_{h\le X}\sum_{k\le X}&\frac{\mu(h)\mu(k)}{(hh_1kk_1)^{1/2}}F(\alpha,\beta,h_1,k_1)P_{i_1}\left(\frac{\log X/h}{\log X}\right)P_{i_2}\left(\frac{\log X/k}{\log X}\right)\\&\sim\frac{1}{(a+b)\theta}\left(g_{i_1,i_2}(b,a)-\mathrm{e}^{-a-b}g_{i_1,i_2}(-a,-b)\right)\end{aligned}\tag{2.11}$$

with

$$g_{i_1,i_2}(a,b)=\int_0^1\Big(P_{i_1}'(t)+a\theta P_{i_1}(t)\Big)\Big(P_{i_2}'(t)+b\theta P_{i_2}(t)\Big)\,\mathrm{d}t. \tag{2.12}$$

Substituting (2.11) into (2.2) we have

$$
\begin{aligned}
&\sum_{q\leq Q}\frac{\Psi(q/Q)\log \mathbf{q}}{\phi(q)}\\
&\quad\times\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\, q)}\int_{-T}^{T} L(\sigma_0+\alpha+it,\chi)L(\sigma_0+\beta-it,\bar\chi)\psi_{i_1}(\sigma_0+it,\chi)\psi_{i_2}(\sigma_0-it,\bar\chi)\,\mathrm{d}t\\
&\quad\sim\frac{2T}{(a+b)\theta}\left(\sum_{q\leq Q}\Psi(q/Q)\log\mathbf{q}\frac{\phi^*(q)}{\phi(q)}\right)\left(g_{i_1,i_2}(b,a)-\mathrm{e}^{-a-b}g_{i_1,i_2}(-a,-b)\right) \qquad (2.13)
\end{aligned}
$$

holds uniformly in complex numbers $\alpha,\beta\ll(\log Q)^{-1}$, where $a=\left(\sigma_0+\alpha-\frac{1}{2}\right)\log\mathbf{q}$, $b=\left(\sigma_0+\beta-\frac{1}{2}\right)\log\mathbf{q}$. Moreover, (2.13) admits differentiations in α, β.

3. Proof of Theorem 1.5

Let $\mathcal{D}_1$ be the closed rectangle with vertices σ_0-iT, $3-iT$, σ_0+iT, $3+iT$, where

$$\sigma_0=\frac{1}{2}-R/\log\mathbf{q}. \qquad (3.1)$$

By a direct calculation we can see

$$L'(3+it,\chi)\psi_2(3+it,\chi)\ll 1 \qquad (3.2)$$

and

$$|L(3+it,\chi)\psi_1(3+it,\chi)|\geq 1-2\sum_{2}^{\infty}n^{-3}-\left(\sum_{2}^{\infty}n^{-3}\right)^2>1/3. \qquad (3.3)$$

Then, for any t, we have from (1.14) and (1.27) that $G(3+it,\chi)\neq 0$ when q is large enough. Determine $\arg G(\sigma+iT,\chi)$ by continuation left from $3+iT$ and $\arg G(\sigma-iT,\chi)$ by continuation left from $3-iT$. If a zero is reached on the upper edge, use $\lim G(\sigma+iT+i\varepsilon,\chi)$ as $\varepsilon\to+0$ and $\lim G(\sigma-iT-i\varepsilon,\chi)$ on the lower edge. Make horizontal cuts in $\mathcal{D}_1$ from the left side to the zeros of G in $\mathcal{D}_1$. Applying Littlewood's formula (to see [16]), we have

$$
\begin{aligned}
&\int_{-T}^{T}\log|G(\sigma_0+it,\chi)|\,\mathrm{d}t-\int_{-T}^{T}\log|G(3+it,\chi)|\,\mathrm{d}t\\
&\quad+\int_{\sigma_0}^{3}\arg G(\sigma+iT,\chi)\,\mathrm{d}\sigma-\int_{\sigma_0}^{3}\arg G(\sigma-iT,\chi)\,\mathrm{d}\sigma\\
&\quad=2\pi\sum_{\rho\in\mathcal{D}_1}\mathrm{dist}(\rho), \qquad (3.4)
\end{aligned}
$$

where $\mathrm{dist}(\rho)$ is the distance of ρ from the left side of $\mathcal{D}_1$.

Recall the definition of ψ_i for $i=1,2$. A direct calculation shows that $\psi_i(s,\chi)\ll\mathbf{q}$ for $\mathrm{Re}(s)>0$. Hence $G(s,\chi)\ll(T\mathbf{q})^2$ for $\mathrm{Re}(s)>0$. Then we have

$$\int_{\sigma_0}^{3} \arg G(\sigma + iT, \chi)\mathrm{d}\sigma = O(\log(T\mathbf{q})) \tag{3.5}$$

by using Jensen's theorem in a familiar way as in §9.4 of [19]. Similarly, we may have

$$\int_{\sigma_0}^{3} \arg G(\sigma - iT, \chi)\mathrm{d}\sigma = O(\log(T\mathbf{q})). \tag{3.6}$$

Using the estimation (3.2) and (3.3) in the definition of $G(s, \chi)$ we have

$$\int_{-T}^{T} \log|G(3 + it, \chi)|\mathrm{d}t = \int_{-T}^{T} \log|L(3 + it, \chi)\psi_1(3 + it, \chi)|\mathrm{d}t + O(T/\log \mathbf{q}). \tag{3.7}$$

Since for $\sigma > 1$

$$\log L(s, \chi) = - \sum \frac{\Lambda(n)\chi(n)}{n^s \log n}, \tag{3.8}$$

it follows taking the real part that

$$\int_{-T}^{T} \log|L(3 + it, \chi)|\mathrm{d}t \ll 1. \tag{3.9}$$

For the entire function $\psi_1(s)$, it is easy to see, for $\sigma \geq 3$,

$$|\psi_1(s) - 1| \leq \frac{1}{2^\sigma} + \frac{1}{3^\sigma} + \int_3^\infty \frac{\mathrm{d}\nu}{\nu^\sigma} \leq \frac{1}{2^\sigma} + \frac{5}{2}\frac{1}{3^\sigma} < 2^{1-\sigma}. \tag{3.10}$$

Therefore, $\log \psi_1(s)$ is analytic for $\sigma \geq 3$. Integrating on the contour $\sigma + iT,\ 3 \leq \sigma < \infty$; $3 + it,\ -T \leq t \leq T$; $\sigma - iT,\ 3 \leq \sigma < \infty$ gives

$$\begin{aligned}\int_{-T}^{T} \log|\psi_1(3 + it, \chi)|\mathrm{d}t &\leq \left|\int_{-T}^{T} \log \psi_1(3 + it, \chi)\mathrm{d}t\right| \\ &= \left|\int_3^\infty \log \psi_1(\sigma - iT, \chi)\mathrm{d}\sigma - \int_3^\infty \log \psi_1(\sigma + iT, \chi)\mathrm{d}\sigma\right| \\ &\leq \int_3^\infty \left|\log \psi_1(\sigma - iT, \chi)\right|\mathrm{d}\sigma + \int_3^\infty \left|\log \psi_1(\sigma + iT, \chi)\right|\mathrm{d}\sigma. \end{aligned} \tag{3.11}$$

Then employing (3.10) in the above we have

$$\begin{aligned}\int_{-T}^{T} \log|\psi_1(3 + it, \chi)|\mathrm{d}t &\leq 2\int_3^\infty \left|\psi_1(\sigma - iT, \chi) - 1\right|\mathrm{d}\sigma + 2\int_3^\infty \left|\psi_1(\sigma + iT, \chi) - 1\right|\mathrm{d}\sigma \\ &\leq 8\int_3^\infty \frac{\mathrm{d}\sigma}{2^\sigma} = O(1). \end{aligned} \tag{3.12}$$

Substituting (3.9), (3.12) into (3.7), we have

$$\int_{-T}^{T} \log|G(3+it,\chi)|\mathrm{d}t \ll T/\log \mathbf{q}. \tag{3.13}$$

Then using (3.5), (3.6), (3.13) in (3.4) and by the condition that $(\log Q)^6 \leq T \leq (\log Q)^A$ for some constant $A \geq 6$, we have

$$\int_{-T}^{T} \log|G(\sigma_0+it,\chi)|\mathrm{d}t + O(T/\log \mathbf{q}) = 2\pi \sum_{\rho\in\mathcal{D}_1} \mathrm{dist}(\rho). \tag{3.14}$$

Since $\mathcal{D} \subset \mathcal{D}_1$ and all zeros of G in closed rectangle $\mathcal{D}$ are at least distance $\frac{1}{2} - \sigma_0 = R/\log \mathbf{q}$ from the line $\sigma = \sigma_0$, it follows from (3.14) that

$$N_G(\mathcal{D},\chi) \leq \frac{\log \mathbf{q}}{2\pi R} \int_{-T}^{T} \log|G(\sigma_0+it,\chi)|\mathrm{d}t + O(T). \tag{3.15}$$

Summing both sides of (3.15) over $q \leq Q$ and primitive characters χ, we have

$$N_G(\mathcal{D},Q) \leq \frac{1}{4\pi R} \sum_{q\leq Q} \frac{\Psi(q/Q)\log \mathbf{q}}{\phi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-T}^{T} \log|G(\sigma_0+it,\chi)|^2\,\mathrm{d}t + o(N(T,Q)). \tag{3.16}$$

By the concavity of the Logarithm function we get

$$N_G(\mathcal{D},Q) \leq \left(\frac{1}{2R}\log c(\theta,r,R) + o(1)\right) N(T,Q), \tag{3.17}$$

where $c(\theta,r,R)$ is defined by

$$\begin{aligned} c(\theta,r,R) &\sum_{q\leq Q} \Psi(q/Q)\log \mathbf{q} \frac{\phi^*(q)}{\phi(q)} \\ &= \frac{1}{2T} \sum_{q\leq Q} \frac{\Psi(q/Q)\log \mathbf{q}}{\phi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-T}^{T} |G(\sigma_0+it,\chi)|^2\,\mathrm{d}t, \end{aligned} \tag{3.18}$$

and $\phi^*(q)$ denotes the number of primitive characters (mod q). In the process to obtain (3.17), we have used $\log \mathbf{q}T$ to replace $\log \mathbf{q}$. We can make this replacement because we have assumed that $(\log Q)^6 \leq T \leq (\log Q)^A$ for some constant $A \geq 6$.

It is obvious that $\Psi(q/Q)\log \mathbf{q}$ in (3.18) still satisfies the condition about $\Psi(q/Q)$ given in the definition of $N(T,Q)$. Hence we can evaluate the right side of (3.18) by using the Asymptotic Large Sieve result. Recalling the definition of $G(s,\chi)$ from (1.14), (1.26) and (1.27), we have

$$
\begin{aligned}
|G(\sigma_0+it,\chi)|^2 = {} & L(\sigma_0+it,\chi)L(\sigma_0-it,\bar\chi)\psi_1(\sigma_0+it,\chi)\psi_1(\sigma_0-it,\bar\chi)\\
& + \frac{1}{r\log \mathbf{q}}L'(\sigma_0+it,\chi)L(\sigma_0-it,\bar\chi)\psi_2(\sigma_0+it,\chi)\psi_1(\sigma_0-it,\bar\chi)\\
& + \frac{1}{r\log \mathbf{q}}L(\sigma_0+it,\chi)L'(\sigma_0-it,\bar\chi)\psi_1(\sigma_0+it,\chi)\psi_2(\sigma_0-it,\bar\chi)\\
& + \frac{1}{r^2\log^2 \mathbf{q}}L'(\sigma_0+it,\chi)L'(\sigma_0-it,\bar\chi)\psi_2(\sigma_0+it,\chi)\psi_2(\sigma_0-it,\bar\chi). \qquad (3.19)
\end{aligned}
$$

Substituting this into (3.18), we have

$$
c(\theta,r,R)\sum_{q\le Q}\Psi(q/Q)\log \mathbf{q}\frac{\phi^*(q)}{\phi(q)} = \frac{1}{2T}\left(D_{11}+\frac{1}{r\log\mathbf{q}}D_{21}+\frac{1}{r\log\mathbf{q}}D_{12}+\frac{1}{r^2\log^2\mathbf{q}}D_{22}\right), \qquad (3.20)
$$

where

$$
\begin{aligned}
D_{i_1i_2} = {} & \sum_{q\le Q}\frac{\Psi(q/Q)\log\mathbf{q}}{\phi(q)}\\
& \times \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\int_{-T}^{T} L^{(i_1-1)}(\sigma_0+it,\chi)L^{(i_2-1)}(\sigma_0-it,\bar\chi)\psi_{i_1}(\sigma_0+it,\chi)\psi_{i_2}(\sigma_0-it,\bar\chi)\,\mathrm{d}t. \qquad (3.21)
\end{aligned}
$$

Choosing $\alpha=\beta=0$ in (2.13), we get

$$
D_{11}\sim 2T\left(\sum_{q\le Q}\Psi(q/Q)\log\mathbf{q}\frac{\phi^*(q)}{\phi(q)}\right)\left(\frac{g_{1,1}(b,a)-\mathrm{e}^{-a-b}g_{1,1}(-a,-b)}{(a+b)\theta}\right)\Bigg|_{a=b=-R} \qquad (3.22)
$$

with $g_{i_1,i_2}(a,b)$ defined by (2.12). Differentiating (2.13) in α and choosing $\alpha=\beta=0$, we get

$$
D_{21}\sim 2T\log\mathbf{q}\left(\sum_{q\le Q}\Psi(q/Q)\log\mathbf{q}\frac{\phi^*(q)}{\phi(q)}\right)\partial_a\left(\frac{g_{2,1}(b,a)-\mathrm{e}^{-a-b}g_{2,1}(-a,-b)}{(a+b)\theta}\right)\Bigg|_{a=b=-R}. \qquad (3.23)
$$

Differentiating (2.13) in β and choosing $\alpha=\beta=0$, we get

$$
D_{12}\sim 2T\log\mathbf{q}\left(\sum_{q\le Q}\Psi(q/Q)\log\mathbf{q}\frac{\phi^*(q)}{\phi(q)}\right)\partial_b\left(\frac{g_{1,2}(b,a)-\mathrm{e}^{-a-b}g_{1,2}(-a,-b)}{(a+b)\theta}\right)\Bigg|_{a=b=-R}. \qquad (3.24)
$$

Differentiating (2.13) both in α, β and choosing $\alpha=\beta=0$, we get

$$D_{22} \sim 2T \log^2 \mathbf{q} \left(\sum_{q \le Q} \Psi(q/Q) \log \mathbf{q} \frac{\phi^*(q)}{\phi(q)} \right) \partial_{a,b} \left(\frac{g_{2,2}(b,a) - \mathrm{e}^{-a-b} g_{2,2}(-a,-b)}{(a+b)\theta} \right) \Bigg|_{a=b=-R}. \tag{3.25}$$

Substituting (3.22)–(3.25) into (3.20), we have

$$\begin{aligned} c(\theta, r, R) \sim {} & h_{1,1}(a,b)\Big|_{a=b=-R} + \frac{1}{r}\partial_a h_{2,1}(a,b)\Big|_{a=b=-R} \\ & + \frac{1}{r}\partial_b h_{1,2}(a,b)\Big|_{a=b=-R} + \frac{1}{r^2}\partial_{a,b} h_{2,2}(a,b)\Big|_{a=b=-R}, \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} h_{i_1,i_2}(a,b) = \frac{1}{\theta(a+b)} \bigg\{ & \int_0^1 \big(P'_{i_1}(t) + b\theta P_{i_1}(t)\big)\big(P'_{i_2}(t) + a\theta P_{i_2}(t)\big)\,\mathrm{d}t \\ & - \mathrm{e}^{-a-b} \int_0^1 \big(P'_{i_1}(t) - a\theta P_{i_1}(t)\big)\big(P'_{i_2}(t) - b\theta P_{i_2}(t)\big)\,\mathrm{d}t \bigg\}. \end{aligned} \tag{3.27}$$

Taking $\theta = 1 - \varepsilon$, $r = 1.154$, $R = 0.617$,

$$\begin{aligned} P_1(x) &= x - 0.158x(1-x) + 0.25x^2(1-x), \\ P_2(x) &= x - 0.492x(1-x) + 0.075x^2(1-x), \end{aligned} \tag{3.28}$$

and making $\varepsilon \to 0$, we get $c(\theta, r, R) = 1.230108\ldots$, then from (3.17) we have

$$N_G(\mathcal{D}, Q) \le 0.167835 N(T, Q). \tag{3.29}$$

We have restricted that P_1, P_2 are polynomials with degrees not bigger than 3 in the calculation above. Then optimizing the function $c(\theta, r, R)$ by *Mathematica* may get our coefficients. A more general choice of polynomials P_1, P_2 with bigger degrees and more items may obtain a better result, but the improvement seems very slight.

4. Proof of Theorem 1.6

In this section, we will study the number of zeros of the family of $\xi^{(m)}(s, \chi)$ on the critical line by using Asymptotic Large Sieve and Levinson's method. In 1983, Conrey [5] had made a careful study on zeros of derivative of the Riemann ξ-function on the critical line. It is natural that the method in [5] can be used here. However, in this section we will use the Levinson's method generalized by Conrey in [6] instead. We will see that this choice can produce a better result on the problem.

From the functional equation (1.22) it is easy to see that $h(\chi)\xi^{(n)}(s, \chi)$ is real for $s = \frac{1}{2} + it$ when n is even and is purely imaginary when n is odd. To obtain lower bound for α_m, let $g \neq 0$

be real, g_n, $n \geq 1$, be complex numbers with g_n real if $n-m$ is odd and g_n purely imaginary if $n-m$ is even. Let T be a large parameter and

$$\mathcal{L} = \log(T\mathbf{q}). \tag{4.1}$$

Now define

$$\eta(s,\chi) = g\xi^{(m)}(s,\chi)\mathcal{L}^{-m} + \sum_{n=0}^{N} g_n \xi^{(n)}(s,\chi)\mathcal{L}^{-n} \tag{4.2}$$

for some fixed N. One may find that the way in [5] is actually the case

$$\eta(s) = g\xi^{(m)}(s,\chi)\mathcal{L}^{-m} + \sum_{n=m}^{N} g_n \xi^{(n)}(s,\chi)\mathcal{L}^{-n}, \tag{4.3}$$

so it is natural for us to obtain a better result when take $\eta(s)$ as in (4.2). Then, for $s = \frac{1}{2} + it$, we have

$$\delta h(\chi)\xi^{(m)}(s,\chi) = \mathrm{Re}(h(\chi)\eta(s,\chi)) \tag{4.4}$$

for m is even and

$$\delta h(\chi)\xi^{(m)}(s,\chi) = \mathrm{Im}(h(\chi)\eta(s,\chi)) \tag{4.5}$$

for m is odd. So that on the critical line $\sigma = \frac{1}{2}$, $\xi^{(m)}(s,\chi) = 0$ if and only if $\mathrm{Re}(h(\chi)\eta(s,\chi)) = 0$ when m is even, and $\mathrm{Im}(h(\chi)\eta(s,\chi)) = 0$ when m is odd. Observe that for every change of π in the argument of $h(\chi)\eta(s,\chi)$ it must be the case that both $\mathrm{Re}(h(\chi)\eta(s,\chi))$ and $\mathrm{Im}(h(\chi)\eta(s,\chi))$ have at least one zero. We have that

$$N_{\xi^{(m)},c}(T,\chi) \geq \frac{1}{\pi}\Delta_C \arg(h(\chi)\eta(s,\chi)) = \frac{1}{\pi}\Delta_C \arg\eta(s,\chi), \tag{4.6}$$

where $\Delta_C \arg$ stands for the continuous variation of the argument as s runs over the critical line from $\frac{1}{2} - iT$ to $\frac{1}{2} + iT$, provided that the path does not cross a zeros of $\eta(s,\chi)$; if it crosses a zeros $\frac{1}{2} + i\gamma$, we alter the path to the arc $\{s\colon |s - \frac{1}{2} - i\gamma| = \varepsilon, \mathrm{Re}(s) \geq \frac{1}{2}\}$ in the neighborhood of $\frac{1}{2} + i\gamma$. Here $\varepsilon > 0$ should be taken very small to ensure that $\eta(s,\chi)$ has no zero in $\{s\colon |s - \frac{1}{2} - i\gamma| \leq \varepsilon, \mathrm{Re}(s) \geq \frac{1}{2}\}$ except for $\frac{1}{2} + i\gamma$.

To estimate the change in argument of $\eta(s,\chi)$ on the critical line, we let

$$\eta(s,\chi) = H(s,\chi)V(s,\chi), \tag{4.7}$$

where $H(s,\chi)$ is defined in (1.18) and

$$V(s,\chi) = \frac{g}{\mathcal{L}^m}\sum_{k=0}^{m}\binom{m}{k}\frac{H^{(m-k)}(s,\chi)}{H(s,\chi)}L^{(k)}(s,\chi) + \sum_{n=0}^{N}\frac{g_n}{\mathcal{L}^n}\sum_{k=0}^{n}\binom{n}{k}\frac{H^{(n-k)}(s,\chi)}{H(s,\chi)}L^{(k)}(s,\chi). \quad (4.8)$$

From the Stirling formula, for $|t| \geq 2$, we have

$$\arg H\left(\frac{1}{2}+it,\chi\right) = \frac{t}{2}\log\frac{|qt|}{2\pi e} + O(1), \quad (4.9)$$

and

$$\frac{H^{(m)}}{H}(s,\chi) = \left(\frac{1}{2}\log\frac{qs}{2\pi}\right)^m\left(1+O\left(\frac{1}{|t|}\right)\right) \quad (4.10)$$

for $t \geq 10$, $0 < \sigma < A_1$, here A_1 can be any positive constant (to see Lemma 1 of [5] for a proof of these formulas). Hence we may have

$$\Delta\arg\eta\left(\frac{1}{2}+it,\chi\right)\Big|_{-T}^{T} = T\log\frac{qT}{2\pi e} + \Delta\arg V\left(\frac{1}{2}+it,\chi\right)\Big|_{-T}^{T} + O(T) \quad (4.11)$$

and denote $V(s,\chi)$ by

$$V(s,\chi) = \left(1+O\left(\frac{1}{|t|}\right)\right) \times\left\{\left(Q_{01}\left(\frac{\log\frac{qs}{2\pi}}{2\mathcal{L}}+\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}s}\right)+\left(\frac{\log\frac{qs}{2\pi}}{2\mathcal{L}}+\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}s}\right)^m Q_{02}\left(\frac{\log\frac{qs}{2\pi}}{2\mathcal{L}}+\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}s}\right)\right)L(s,\chi)\right\} \quad (4.12)$$

with

$$Q_{01}(x) = \sum_{n=0}^{m-1} g_n x^n \quad (4.13)$$

and

$$Q_{02}(x) = g + \sum_{n=m}^{N} g_n x^{n-m}. \quad (4.14)$$

From (4.6), (4.11) and Lemma 1.7 with T replaced by $T(\log T)^{-10}$, we have that

$$\left| \Delta \arg V\left(\frac{1}{2}+it,\ \chi\right)\Big|_{-T(\log T)^{-10}}^{T(\log T)^{-10}} \right| \ll T(\log T)^{-9}+\mathcal{L}. \tag{4.15}$$

Then we employ the above formula into (4.11) and have

$$\begin{aligned}\Delta \arg \eta\left(\frac{1}{2}+it,\ \chi\right)\Big|_{-T}^{T} \geq T\log\frac{qT}{2\pi e} &- \left|\Delta \arg V\left(\frac{1}{2}+it,\ \chi\right)\Big|_{-T}^{-T(\log T)^{-10}}\right| \\ &- \left|\Delta \arg V\left(\frac{1}{2}+it,\ \chi\right)\Big|_{T(\log T)^{-10}}^{T}\right| + O(T+\mathcal{L}).\end{aligned} \tag{4.16}$$

For $T(\log T)^{-10} \leq |t| \leq T$, we may use the following approximation to $V(s,\ \chi)$,

$$V_1(s,\ \chi) = \left(Q_{01}\left(\frac{1}{2}+\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}s}\right) + \left(\frac{1}{2}+\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}s}\right)^m Q_{02}\left(\frac{1}{2}+\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}s}\right)\right)L(s,\ \chi). \tag{4.17}$$

We restrict that Q_{01} and Q_{02} satisfy the condition

$$Q_{01}\left(\frac{1}{2}\right) + \frac{1}{2^m}Q_{02}\left(\frac{1}{2}\right) = 1. \tag{4.18}$$

When we apply Littlewood's formula in the following as in Section 3, this restriction will ensure the integration on line $\sigma = 3$ to be a error item. Also we may present $V_1(s,\ \chi)$ in the following form:

$$U(s,\ \chi) = \left((1-\delta)Q_1\left(-\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}s}\right) + \delta\left(1+\frac{2}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}s}\right)^m Q_2\left(-\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}s}\right)\right)L(s,\ \chi), \tag{4.19}$$

where

$$\delta = \frac{1}{2^m}Q_{02}\left(\frac{1}{2}\right). \tag{4.20}$$

Then, the condition (4.18) is equivalent to

$$Q_1(0) = Q_2(0) = 1. \tag{4.21}$$

If we restrict that δ is real and $Q_1(x)$, $Q_2(x)$ are real polynomials, then the restriction of g_n is equivalent to

$$Q_2'(x) = Q_2'(1-x) \tag{4.22}$$

and $Q_1(x)$ has the form

$$Q_1(x) = \sum_{0 \le k \le m/2} a_k (1 - 2x)^{2k} \tag{4.23}$$

for m is odd and

$$Q_1(x) = \sum_{1 \le k \le m/2} a_k (1 - 2x)^{2k-1} \tag{4.24}$$

for m is even.

From the condition (4.21), $U(s, \chi)$ can be presented as a linear combination of $L^{(n)}(s, \chi)$ as follows:

$$U(s, \chi) = L(s, \chi) + \sum_{n=1}^{N} b_n \mathcal{L}^{-n} L^{(n)}(s, \chi), \tag{4.25}$$

where the coefficients b_n are constants decided by δ and the polynomials Q_1, Q_2, so these coefficients can be specified later.

Let $\mathcal{D}_2$ be the closed rectangle with vertices $\frac{1}{2} + iT(\log T)^{-10}$, $3 + iT(\log T)^{-10}$, $\frac{1}{2} + iT$, $3 + iT$ and $\mathcal{D}_3$ be the closed rectangle with vertices $\frac{1}{2} - iT(\log T)^{-10}$, $3 - iT(\log T)^{-10}$, $\frac{1}{2} - iT$, $3 - iT$. We employ $N_U(\mathcal{D}_2, \chi)$, $N_U(\mathcal{D}_3, \chi)$ and $N_U(\mathcal{D}, \chi)$ to be the number of zeros of $U(s, \chi)$ in $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}$, respectively, including zeros on the left boundary. Then by the Cauchy's argument principle we have

$$\begin{aligned} &\Delta \arg U(3 + it, \chi)\Big|_{T(\log T)^{-10}}^{T} + \Delta \arg U(\sigma + iT, \chi)\Big|_{3}^{1/2} \\ &\quad + \Delta \arg U\left(\frac{1}{2} + it, \chi\right)\Bigg|_{T}^{T(\log T)^{-10}} + \Delta \arg U\left(\sigma + iT(\log T)^{-10}, \chi\right)\Big|_{1/2}^{3} \\ &\quad \le 2\pi N_U(\mathcal{D}_2, \chi). \end{aligned} \tag{4.26}$$

It is not difficult to see from (4.25) that Re $U(3 + it, \chi) > 0$. Thus we have the first item in the above formula satisfies

$$\Delta \arg U(3 + it, \chi)\Big|_{T(\log T)^{-10}}^{T} = O(1). \tag{4.27}$$

Then using Jensen's theorem in a familiar way as in Section 3, we have

$$\Delta \arg U(\sigma + iT, \chi)\Big|_{3}^{1/2} = O(\log T), \tag{4.28}$$

and

$$\Delta \arg U\left(\sigma + iT(\log T)^{-10}, \chi\right)\Big|_{1/2}^{3} = O(\log T). \tag{4.29}$$

Employing this in (4.26) we have

$$\left|\Delta \arg U\left(\frac{1}{2}+it,\,\chi\right)\Big|_{T}^{T(\log T)^{-10}}\right| \le 2\pi N_U(\mathcal{D}_2,\,\chi)+O(\log T). \tag{4.30}$$

Similarly, we have

$$\left|\Delta \arg U\left(\frac{1}{2}+it,\,\chi\right)\Big|_{-T(\log T)^{-10}}^{-T}\right| \le 2\pi N_U(\mathcal{D}_3,\,\chi)+O(\log T). \tag{4.31}$$

Thus we have

$$\begin{aligned}\left|\Delta \arg U\left(\frac{1}{2}+it,\,\chi\right)\Big|_{T}^{T(\log T)^{-10}}\right| &+ \left|\Delta \arg U\left(\frac{1}{2}+it,\,\chi\right)\Big|_{-T(\log T)^{-10}}^{-T}\right| \\ &\le 2\pi\left(N_U(\mathcal{D}_2,\,\chi)+N_U(\mathcal{D}_3,\,\chi)\right)+O(\log T) \\ &\le 2\pi N_U(\mathcal{D},\,\chi)+O(\log T).\end{aligned} \tag{4.32}$$

Let

$$\psi(s,\,\chi)=\sum_{n\le X}\frac{\mu(n)\chi(n)}{n^{s+R/\mathcal{L}}}P\left(\frac{\log X/n}{\log X}\right) \tag{4.33}$$

be a mollifier as in (1.26). Here μ is the Möbius function, $X=\mathbf{q}^{\theta}$ with $0<\theta<1$ and P is a polynomial with $P(0)=0$, $P(1)=1$, which will be specified later. It is obvious that

$$N_U(\mathcal{D},\,\chi)\le N_{U\psi}(\mathcal{D},\,\chi), \tag{4.34}$$

where $N_{U\psi}(\mathcal{D},\,\chi)$ denotes the number of zeros of $U\psi(s,\,\chi)$ in $\mathcal{D}$, including zeros on the left boundary.

Employing (4.32), (4.34) and (4.16) into (4.6) we have

$$N_{\xi^{(m)},c}(T,\,\chi)\ge\frac{T}{\pi}\log\frac{qT}{2\pi e}-2N_{U\psi}(\mathcal{D},\,\chi)+O(T+\mathcal{L}). \tag{4.35}$$

The same to (3.15), we may have from the Littlewood's formula and a discussion as in Section 3 that

$$N_{U\psi}(\mathcal{D},\,\chi)\le\frac{\log\mathbf{q}}{2\pi R}\int_{-T}^{T}\log|U\psi(\sigma_0+it,\,\chi)|\,\mathrm{d}t+O(T). \tag{4.36}$$

Substituting this into (4.35) and summing both sides of the formula, we have

$$N_{\xi^{(m)},c}(T,Q) \geq N(T,Q)(1+o(1)) \\ -\frac{1}{2\pi R}\sum_{q\leq Q}\frac{\Psi(q/Q)\log \mathbf{q}}{\phi(q)}\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)}\int_{-T}^{T}\log|U\psi(\sigma_0+it,\chi)|^2\,\mathrm{d}t. \tag{4.37}$$

Recall that $(\log Q)^6 \leq T \leq (\log Q)^A$ for some constant $A \geq 6$. We refine the definition of U which was previously defined in (4.19) to

$$U(s,\chi)=\left\{(1-\delta)Q_1\left(-\frac{1}{\log \mathbf{q}}\frac{\mathrm{d}}{\mathrm{d}s}\right)+\delta\left(1+\frac{2}{\log \mathbf{q}}\frac{\mathrm{d}}{\mathrm{d}s}\right)^m Q_2\left(-\frac{1}{\log \mathbf{q}}\frac{\mathrm{d}}{\mathrm{d}s}\right)\right\}L(s,\chi). \tag{4.38}$$

Substitute this refined U into (4.37), and it is easy to see that the error caused by this refinement in (4.37) can be absorbed into the error term. Then using the concavity of the Logarithm function, we get

$$N_{\xi',c}(T,Q) \geq \left(1-\frac{1}{R}\log c_1(\theta,R)+o(1)\right)N(T,Q), \tag{4.39}$$

where $c_1(\theta,R)$ satisfies

$$c_1(\theta,R)\sum_{q\leq Q}\Psi(q/Q)\log \mathbf{q}\frac{\phi^*(q)}{\phi(q)}=\frac{1}{2T}\sum_{q\leq Q}\frac{\Psi(q/Q)\log \mathbf{q}}{\phi(q)}\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)}\int_{-T}^{T}|U\psi(\sigma_0+it,\chi)|^2\,\mathrm{d}t. \tag{4.40}$$

Then, using formula (2.13) in (4.40) as in Section 3, we get

$$c_1(\theta,R)\sim\big((1-\delta)Q_1(-\partial_a)+\delta(1+2\partial_a)Q_2(-\partial_a)\big) \\ \times\big((1-\delta)Q_1(-\partial_b)+\delta(1+2\partial_b)Q_2(-\partial_b)\big)\left(\frac{g(b,a)-\mathrm{e}^{-a-b}g(-a,-b)}{\theta(a+b)}\right)\Bigg|_{a=b=-R}, \tag{4.41}$$

where

$$g(a,b)=\int_0^1 (P'(t)+a\theta P(t))(P'(t)+b\theta P(t))\,\mathrm{d}t. \tag{4.42}$$

Taking $m=1$, $\theta=1-\varepsilon$, $R=0.746$, $\delta=0.771$,

$$\begin{aligned} Q_1(x)&=1,\\ Q_2(x)&=1-0.673x+0.369(x^2/2-x^3/3)-4.635(x^3/3-x^4/2+x^5/5),\\ P(x)&=x-0.482x(1-x)-0.392x^2(1-x)-0.262x^3(1-x) \end{aligned} \tag{4.43}$$

into (4.41) and making $\varepsilon\to 0$, we have from (4.39) that

$$N_{\xi',c}(T,Q) \geq 0.93828N(T,Q). \tag{4.44}$$

Taking $m=2$, $\theta=1-\varepsilon$, $R=1.259$, $\delta=0.862$,

$$\begin{aligned} Q_1(x) &= 1-2x, \\ Q_2(x) &= 1-0.481x-1.143(x^2/2-x^3/3)-5.04(x^3/3-x^4/2+x^5/5), \\ P(x) &= x-0.993x(1-x)-0.035x^2(1-x)-2.112x^3(1-x) \end{aligned} \tag{4.45}$$

into (4.41) and making $\varepsilon \to 0$, we have from (4.39) that

$$N_{\xi^{(2)},c}(T,Q) \geq 0.97314N(T,Q). \tag{4.46}$$

Taking $m=3$, $\theta=1-\varepsilon$, $R=1.69$, $\delta=0.886$,

$$\begin{aligned} Q_1(x) &= 0.183+0.817(1-2x)^2, \\ Q_2(x) &= 1-0.368x-1.98(x^2/2-x^3/3)-5.228(x^3/3-x^4/2+x^5/5), \\ P(x) &= x-1.181x(1-x)-0.715x^2(1-x)-3.67x^3(1-x) \end{aligned} \tag{4.47}$$

into (4.41) and making $\varepsilon \to 0$, we have from (4.39) that

$$N_{\xi^{(3)},c}(T,Q) \geq 0.99203N(T,Q). \tag{4.48}$$

Hence we have Theorem 1.6.

In (4.43), (4.45) and (4.47) polynomials Q_1, Q_2 should be taken to satisfy their restrictions (4.21)–(4.24) and polynomial P should satisfy $P(0)=0$, $P(1)=1$. In all cases above, we restrict that polynomial Q_2 has degree not bigger than 5 and polynomial P has degree not bigger than 4, then optimizing the function $c_1(\theta, R)$ by *Mathematica* may get our coefficients. A more general choice of polynomials Q_2, P with bigger degrees and more items may obtain a better result, but the improvement seems very slight.

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