Linear Algebra Done Right, 4th Edition Solutions

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Chapter 1: Vector Spaces

Exercises 1A: R^n and C^n

1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in C$

Proof.

$\alpha, \beta \in C \rightarrow \alpha = a + bi$ and $\beta = c + di$ where $a, b, c, d \in R$
LHS:
$\alpha + \beta = (a + bi) + (c + di) = (a + c) + (b + d)i$
RHS:
$\beta + \alpha = (c + di) + (a + bi) = (c + a) + (d + b)i$
Because $a, b, c, d \in R$, they obey commutativity
$\rightarrow \beta + \alpha = (c + a) + (d + b)i = (a + c) + (b + d)i = \alpha + \beta$

□

2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in C$

Proof.

LHS:
$\alpha = a + bi, \beta = c + di, = e + fi$
$(\alpha + \beta) + \lambda = ((a + bi) + (c + di)) + (e + fi) =$
$(a + c) + (b + d)i + (e + fi)$
$(a + c + e) + (b + d + f)i$
RHS:
$\alpha + (\beta + \lambda) = (a + bi) + ((c + di) + (e + fi)) =$
$(a + bi) + ((c + e) + (d + f)i) = (c + e + a) + (d + f + b)$
By commutativity of real numbers $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$

□

3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in C$

Proof.

LHS:
$\alpha\beta = (a+bi)(c+di) = (ac-bd)+(ad+bc)i$
$((ac-bd)+(ad+bc)i)\lambda = (ac-bd)+(ad+bc)i)(e+fi)$
$(ace-bde)+(ade+bce)i+(acf-bdf)i+(adf+bcf)(-1)$
$(ace-bde-adf-bcf)+(ade+bce+acf-bdf)i$
RHS:
$\alpha(\beta\lambda) = (a+bi)((c+di)(e+fi))$
$=(a+bi)((ce-df)+(cf+de)i)$
$=(cea-dfa-cfb-deb)+(cfa+dea+ceb-dfb)i$
By commutativity: $LHS = RHS$

□

4. Show that $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in C$

Proof.

LHS:
$\lambda(\alpha+\beta) = (e+fi)((a+bi)+(c+di))$
$=(e+fi)((a+c)+(b+d)i)$
$(e+fi)(a+c)+(e+fi)(bi+di)$
$(ea+ce)+(fa+fc)i+(eb+ed)i+(-fb-fd)$
$(ea+ce-fb-fd)+(fa+fc+eb+ed)i$
RHS:
$\lambda\alpha+\lambda\beta = (e+fi)(a+bi)+(e+fi)(c+di)$
$(ea-fb)+(fa+eb)i+(ec-fd)+(fc+ed)i$
$=(ea-fb+ec-fd)+(fa+eb+fc+ed)i$
By Commutativity: $LHS = RHS$

□

5. Show that for every $\alpha \in C$ there exists a unique $\beta \in C$ such that $\alpha+\beta = 0$

Proof.

Proving existence:
$\alpha = (a+bi), \beta = (-a-bi) \to \alpha+\beta = 0$
Proving uniqueness: assume that $\lambda \neq \beta$ is also an additive inverse
$\alpha+\beta = 0 = \alpha+\lambda \to \lambda = \beta$
Contradiction, only assumption made was $\lambda \neq \beta$, so $\lambda = \beta$

□

References

1. Axler, S. (2024). Linear Algebra Done Right (4th ed.). Springer.