Linear Algebra Done Right, 4th Edition Solutions

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Chapter 1: Vector Spaces

Exercises 1A: R^n and C^n

1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in C$

Proof.

$\alpha, \beta \in C \rightarrow \alpha = a + bi$ and $\beta = c + di$ where $a, b, c, d \in R$
LHS:
$\alpha + \beta = (a + bi) + (c + di) = (a + c) + (b + d)i$
RHS:
$\beta + \alpha = (c + di) + (a + bi) = (c + a) + (d + b)i$
Because $a, b, c, d \in R$, they obey commutativity
$\rightarrow \beta + \alpha = (c + a) + (d + b)i = (a + c) + (b + d)i = \alpha + \beta$

□

2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in C$

Proof.

LHS:
$\alpha = a + bi, \beta = c + di, = e + fi$
$(\alpha + \beta) + \lambda = ((a + bi) + (c + di)) + (e + fi) =$
$(a + c) + (b + d)i + (e + fi)$
$(a + c + e) + (b + d + f)i$
RHS:
$\alpha + (\beta + \lambda) = (a + bi) + ((c + di) + (e + fi)) =$
$(a + bi) + ((c + e) + (d + f)i) = (c + e + a) + (d + f + b)$
By commutativity of real numbers $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$

□

3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in C$

Proof.

LHS:
$\alpha\beta = (a+bi)(c+di) = (ac-bd)+(ad+bc)i$
$((ac-bd)+(ad+bc)i)\lambda = (ac-bd)+(ad+bc)i)(e+fi)$
$(ace-bde)+(ade+bce)i+(acf-bdf)i+(adf+bcf)(-1)$
$(ace-bde-adf-bcf)+(ade+bce+acf-bdf)i$
RHS:
$\alpha(\beta\lambda) = (a+bi)((c+di)(e+fi))$
$= (a+bi)((ce-df)+(cf+de)i)$
$= (cea-dfa-cfb-deb)+(cfa+dea+ceb-dfb)i$
By commutativity: $LHS = RHS$

□

4. Show that $\lambda(\alpha+\beta) = \lambda\alpha+\lambda\beta$ for all $\lambda, \alpha, \beta \in C$

Proof.

LHS:
$\lambda(\alpha+\beta) = (e+fi)((a+bi)+(c+di))$
$= (e+fi)((a+c)+(b+d)i)$
$(e+fi)(a+c)+(e+fi)(bi+di)$
$(ea+ce)+(fa+fc)i+(eb+ed)i+(-fb-fd)$
$(ea+ce-fb-fd)+(fa+fc+eb+ed)i$
RHS:
$\lambda\alpha+\lambda\beta = (e+fi)(a+bi)+(e+fi)(c+di)$
$(ea-fb)+(fa+eb)i+(ec-fd)+(fc+ed)i$
$= (ea-fb+ec-fd)+(fa+eb+fc+ed)i$
By Commutativity: $LHS = RHS$

□

5. Show that for every $\alpha \in C$ there exists a unique $\beta \in C$ such that $\alpha+\beta = 0$

Proof.

Proving existence:
$\alpha = (a+bi), \beta = (-a-bi) \to \alpha+\beta = 0$
Proving uniqueness: assume that $\lambda \neq \beta$ is also an additive inverse
$\alpha+\beta = 0 = \alpha+\lambda \to \lambda = \beta$
Contradiction, only assumption made was $\lambda \neq \beta$, so $\lambda = \beta$

□

6. Show that for every $\alpha \in C$ with $\alpha \neq 0$ there exists a unique $\beta \in C$ such that $\alpha\beta = 1$

Proof.

Proving existence:

$\alpha = (a + bi), \beta = \dfrac{1}{(a + bi} \rightarrow \alpha\beta = 1$

Proving uniqueness: assume that $\lambda \neq \beta$ also satisies this property

$\alpha\beta = 1 = 1(\alpha\lambda) \rightarrow \beta = \lambda$

Contradiction, only assumption made was $\lambda \neq \beta$, so $\lambda = \beta$

□

7. Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (TODO)

8. Find two distinct square roots of i (TODO)

9. Find $X \in R^4$ such that (4,-3,1,7) +2x = (5,9-6,8)

Proof.

$$y = 2x \rightarrow (4, -3, 1, 7) + y = (4 + y_1, -3 + y_2, 1 + y_3, 7 + y_4)$$
$$y_1 = 1, y_2 = 12, y_3 = -7, y_4 = 1 \rightarrow (4, -3, 1, 7) + y = (5, 9, -6, 8)$$
$$x = \frac{y}{2} = (1, 12, -7, 1)/2 = (\frac{1}{2}, 6, \frac{-7}{2}, \frac{1}{2})$$

□

10. Explain why there does not exist $\lambda \in C$ such that $\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, 32 - 9i)$

Proof. This is because λ is a scalar, not a list/vector. It is one number that will be multiplied across every element of the vector. Because the vector on the right hand side is not a multiple of the vector on the left hand side, no such λ exists to satisfy this. □

11. Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in F^n$

Proof.

$$
\begin{aligned}
&\text{LHS:}\\
&(x+y)+z = ((x_1,...,x_n)+(y_1,...,y_n))+(z_1,...z_n)\\
&(x_1+y_1,...,x_n+y_n)+(z_1,...z_n)\\
&(x_1+y_1+z_1,...x_n+y_n+z_n)\\
&\text{RHS:}\\
&x+(y+z) = (x_1,...,x_n)+((y_1,...,y_n)+(z_1,...z_n))\\
&(x_1,...,x_n)+(y_1+z_1,...,y_n+z_n)\\
&(y_1+z_1+x_1,...y_n+z_n+x_n)\\
&\text{By commutativity: } RHS = LHS
\end{aligned}
$$

□

12. Show that (ab)x = a(bx) for all $x \in F^n$ and $a, b \in F$

Proof.

$$
\begin{aligned}
&\text{LHS:}\\
&\text{Let } ab = c \to ab(x) = cx\\
&= (cx_1,...,cx_n)\\
&\text{RHS:}\\
&a(bx) = a(bx_1,...,b_xn)\\
&= (abx_1,...,abx_n) = (cx_1,...cx_n)
\end{aligned}
$$

□

13. Show that $1x = x$ for all $x \in F^n$

Proof.

$$1x = 1(x_1,...,x_n) = (1x_1,...1x_n) = (x_1,...x_n) = x$$

□

14. Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in F$ and $x, y \in F^n$

Proof.

$$
\begin{aligned}
&LHS = \lambda(x+y) = \lambda(x_1+y_1,...x_n+y_n)\\
&= (\lambda(x_1+y_1),...,\lambda(x_n+y_n) = (\lambda x_1+\lambda y_1,...,\lambda x_n+\lambda y_n)\\
&= \lambda(x_1,...,x_n)+\lambda(y_1,...,y_n) = \lambda(x)+\lambda(y) = RHS
\end{aligned}
$$

□

15. Show that $(a+b)x = ax+bx$ for all $a, b \in F$ and $x \in F^n$

Proof.

$$\begin{aligned}&\text{Let } a+b=c \to LHS = cx\\ &= c(x_1, ...x_n) = (cx_1, ..., cx_n) = ((a+b)x_1, ..., (a+b)(x_n))\\ &(ax_1+bx_1, ..., ax_n+bx_n) = ax+bx = RHS\end{aligned}$$

□

Exercises 1B: Definition of Vector Space

1. Prove that $-(-\nu) = \nu$ for every $\nu \in V$

Proof.

$$\begin{aligned}\text{Given } \nu - \mu &= \nu + (-\mu)\\ \nu - (-(-\nu)) &= \nu + (-(-(-\nu)))\\ &= \nu + (-\nu) = 0\\ -\nu + (-1(-\nu)) &= 0\\ -(-\nu) &= \nu\end{aligned}$$

□

2. Suppose $a \in F, \nu \in V$ and $a\nu = 0$ Prove that $a = 0$ or $\nu = 0$

Proof.

$$\begin{aligned}&\text{Take the contrapositive, negate hypothesis and conclusion}\\ &\text{By DeMorgan's law: } \neg(a=0 \vee \nu=0) = a \neq 0 \wedge \nu \neq 0\\ &\text{where the conclusion becomes: } \neg(a\nu=0) \implies a\nu \neq 0\\ &\text{The product of two non-zero elements of any } F^n \text{ cannot be } 0\end{aligned}$$

□

3.

References

1. Axler, S. (2024). Linear Algebra Done Right (4th ed.). Springer.