Linear Algebra Done Right, 4th Edition Solutions

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Chapter 1: Vector Spaces

Exercises 1A: R^n and C^n

1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in C$

Proof.

$\alpha, \beta \in C \rightarrow \alpha = a + bi$ and $\beta = c + di$ where $a, b, c, d \in R$

LHS:

$\alpha + \beta = (a + bi) + (c + di) = (a + c) + (b + d)i$

RHS:

$\beta + \alpha = (c + di) + (a + bi) = (c + a) + (d + b)i$

Because $a, b, c, d \in R$, they obey commutativity

$\rightarrow \beta + \alpha = (c + a) + (d + b)i = (a + c) + (b + d)i = \alpha + \beta$

$\square$

2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in C$

Proof.

LHS:

$\alpha = a + bi, \beta = c + di, = e + fi$

$(\alpha + \beta) + \lambda = ((a + bi) + (c + di)) + (e + fi) =$

$(a + c) + (b + d)i + (e + fi)$

$(a + c + e) + (b + d + f)i$

RHS:

$\alpha + (\beta + \lambda) = (a + bi) + ((c + di) + (e + fi)) =$

$(a + bi) + ((c + e) + (d + f)i) = (c + e + a) + (d + f + b)$

By commutativity of real numbers $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$

$\square$

3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in C$

Proof.

LHS:

$\alpha\beta = (a+bi)(c+di) = (ac-bd) + (ad+bc)i$

$((ac-bd) + (ad+bc)i)\lambda = (ac-bd) + (ad+bc)i)(e+fi)$

$(ace-bde) + (ade+bce)i + (acf-bdf)i + (adf+bcf)(-1)$

$(ace-bde-adf-bcf) + (ade+bce+acf-bdf)i$

RHS:

$\alpha(\beta\lambda) = (a+bi)((c+di)(e+fi))$

$= (a+bi)((ce-df) + (cf+de)i)$

$= (cea-dfa-cfb-deb) + (cfa+dea+ceb-dfb)i$

By commutativity: $LHS = RHS$

$\square$

4. Show that $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in C$

Proof.

LHS:

$\lambda(\alpha+\beta) = (e+fi)((a+bi) + (c+di))$

$= (e+fi)((a+c) + (b+d)i)$

$(e+fi)(a+c) + (e+fi)(bi+di)$

$(ea+ce) + (fa+fc)i + (eb+ed)i + (-fb-fd)$

$(ea+ce-fb-fd) + (fa+fc+eb+ed)i$

RHS:

$\lambda\alpha + \lambda\beta = (e+fi)(a+bi) + (e+fi)(c+di)$

$(ea-fb) + (fa+eb)i + (ec-fd) + (fc+ed)i$

$= (ea-fb+ec-fd) + (fa+eb+fc+ed)i$

By Commutativity: $LHS = RHS$

$\square$

5. Show that for every $\alpha \in C$ there exists a unique $\beta \in C$ such that $\alpha+\beta = 0$

Proof.

Proving existence:

$\alpha = (a+bi), \beta = (-a-bi) \to \alpha + \beta = 0$

Proving uniqueness: assume that $\lambda \neq \beta$ is also an additive inverse

$\alpha + \beta = 0 = \alpha + \lambda \to \lambda = \beta$

Contradiction, only assumption made was $\lambda \neq \beta$, so $\lambda = \beta$

$\square$

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6. Show that for every $\alpha \in C$ with $\alpha \neq 0$ there exists a unique $\beta \in C$ such that $\alpha\beta = 1$

Proof.

Proving existence:

$$\alpha = (a + bi), \beta = \frac{1}{(a + bi)} \rightarrow \alpha\beta = 1$$

Proving uniqueness: assume that $\lambda \neq \beta$ also satisies this property

$\alpha\beta = 1 = 1(\alpha\lambda) \rightarrow \beta = \lambda$

Contradiction, only assumption made was $\lambda \neq \beta$, so $\lambda = \beta$

□

7. Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (TODO)

8. Find two distinct square roots of i (TODO)

9. Find $X \in R^4$ such that (4,-3,1,7) +2x = (5,9-6,8)

Proof.

$y = 2x \rightarrow (4, -3, 1, 7) + y = (4 + y_1, -3 + y_2, 1 + y_3, 7 + y_4)$

$y_1 = 1, y_2 = 12, y_3 = -7, y_4 = 1 \rightarrow (4, -3, 1, 7) + y = (5, 9, -6, 8)$

$x = \dfrac{y}{2} = (1, 12, -7, 1)/2 = (\dfrac{1}{2}, 6, \dfrac{-7}{2}, \dfrac{1}{2})$

□

10. Explain why there does not exist $\lambda \in C$ such that $\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, 32 - 9i)$

Proof. This is because λ is a scalar, not a list/vector. It is one number that will be multiplied across every element of the vector. Because the vector on the right hand side is not a multiple of the vector on the left hand side, no such λ exists to satisfy this. □

11. Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in F^n$

Proof.

LHS:

$(x + y) + z = ((x_1, ..., x_n) + (y_1, ..., y_n)) + (z_1, ...z_n)$

$(x_1 + y_1, ..., x_n + y_n) + (z_1, ...z_n)$

$(x_1 + y_1 + z_1, ...x_n + y_n + z_n)$

RHS:

$x + (y + z) = (x_1, ..., x_n) + ((y_1, ..., y_n) + (z_1, ...z_n))$

$(x_1, ..., x_n) + (y_1 + z_1, ..., y_n + z_n)$

$(y_1 + z_1 + x_1, ...y_n + z_n + x_n)$

By commutativity: $RHS = LHS$

□

12. Show that $(ab)x = a(bx)$ for all $x \in F^n$ and $a, b \in F$

Proof.

LHS:

Let $ab = c \rightarrow ab(x) = cx$

$= (cx_1, ..., cx_n)$

RHS:

$a(bx) = a(bx_1, ..., b_x n)$

$= (abx_1, ..., abx_n) = (cx_1, ...cx_n)$

□

13. Show that $1x = x$ for all $x \in F^n$

Proof.

$$1x = 1(x_1, ..., x_n) = (1x_1, ...1x_n) = (x_1, ...x_n) = x$$

□

14. Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in F$ and $x, y \in F^n$

Proof.

$LHS = \lambda(x + y) = \lambda(x_1 + y_1, ...x_n + y_n)$

$= (\lambda(x_1 + y_1), ..., \lambda(x_n + y_n) = (\lambda x_1 + \lambda y_1, ..., \lambda x_n + \lambda y_n)$

$= \lambda(x_1, ..., x_n) + \lambda(y_1, ..., y_n) = \lambda(x) + \lambda(y) = RHS$

□

15. Show that $(a + b)x = ax + bx$ for all $a, b \in F$ and $x \in F^n$

Proof.

Let $a + b = c \rightarrow LHS = cx$
$$= c(x_1, ... x_n) = (cx_1, ..., cx_n) = ((a+b)x_1, ..., (a+b)(x_n))$$
$$(ax_1 + bx_1, ..., ax_n + bx_n) = ax + bx = RHS$$

$\square$

Exercises 1B: Definition of Vector Space

References

1. Axler, S. (2024). Linear Algebra Done Right (4th ed.). Springer.