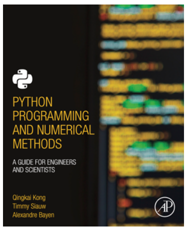

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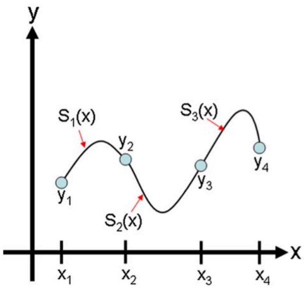
Cubic Spline Interpolation

In **cubic spline interpolation** (as shown in the following figure), the interpolating function is a set of piecewise cubic functions. Specifically, we assume that the points (x_i, y_i) and (x_{i+1}, y_{i+1}) are joined by a cubic polynomial $S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$ that is valid for $x_i \le x \le x_{i+1}$ for $i = 1, \dots, n-1$. To find the interpolating function, we must first determine the coefficients a_i, b_i, c_i, d_i for each of the cubic functions. For n points, there are $n-1$ cubic functions to find, and each cubic function requires four coefficients. Therefore we have a total of $4(n-1)$ unknowns, and so we need $4(n-1)$ independent equations to find all the coefficients.



First we know that the cubic functions must intersect the data the points on the left and the right:

$$\begin{aligned} S_i(x_i) &= y_i, \quad i = 1, \dots, n-1, \\ S_i(x_{i+1}) &= y_{i+1}, \quad i = 1, \dots, n-1, \end{aligned}$$

which gives us $2(n-1)$ equations. Next, we want each cubic function to join as smoothly with its neighbors as possible, so we constrain the splines to have continuous first and second derivatives at the data points $i = 2, \dots, n-1$.

$$\begin{aligned} S_i'(x_{i+1}) &= S_{i+1}'(x_{i+1}), \quad i = 1, \dots, n-2, \\ S_i''(x_{i+1}) &= S_{i+1}''(x_{i+1}), \quad i = 1, \dots, n-2, \end{aligned}$$

which gives us $2(n-2)$ equations.

Two more equations are required to compute the coefficients of $S_i(x)$. These last two constraints are arbitrary, and they can be chosen to fit the circumstances of the interpolation being performed. A common set of final constraints is to assume that the second derivatives are zero at the endpoints. This means that the curve is a "straight line" at the end points. Explicitly,

$$\begin{aligned} S_1''(x_1) &= 0 \\ S_{n-1}''(x_n) &= 0. \end{aligned}$$

In Python, we can use *scipy's* function *CubicSpline* to perform cubic spline interpolation. Note that the above constraints are not the same as the ones used by scipy's *CubicSpline* as default for performing cubic splines, there are different ways to add the final two constraints in scipy by setting the *bc_type* argument (see the help for *CubicSpline* to learn more about this).

TRY IT! Use *CubicSpline* to plot the cubic spline interpolation of the data set *x = [0, 1, 2]* and *y = [1, 3, 2]* for $0 \leq x \leq 2$.

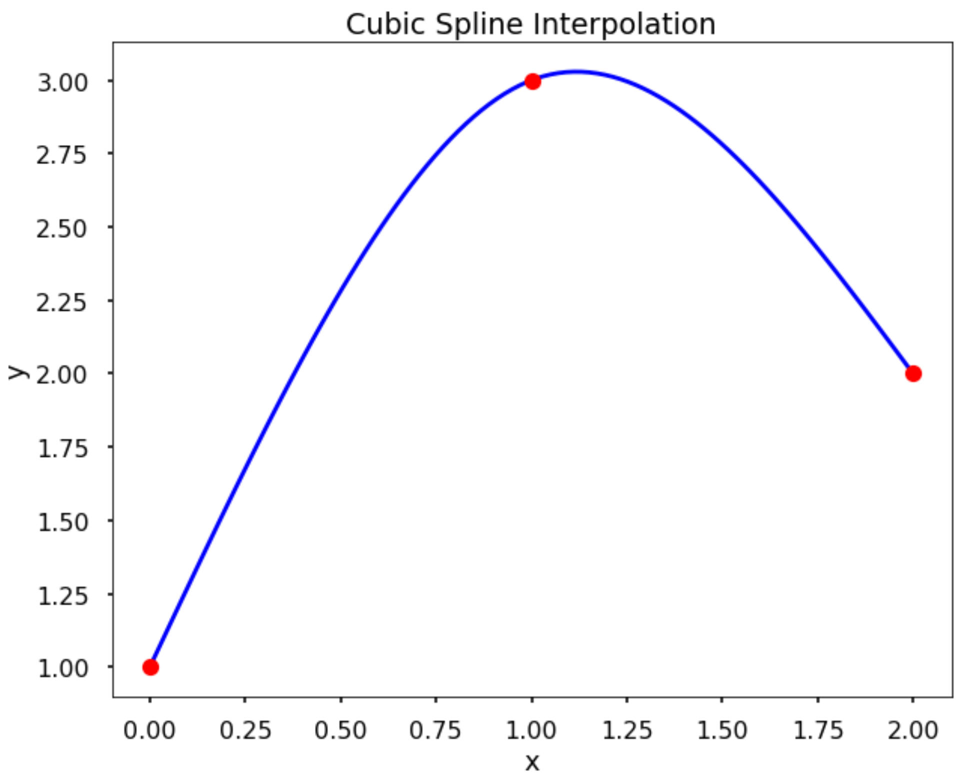
```
from scipy.interpolate import CubicSpline
import numpy as np
import matplotlib.pyplot as plt

plt.style.use('seaborn-poster')
```

```
x = [0, 1, 2]
y = [1, 3, 2]

# use bc_type = 'natural' adds the constraints as we described above
f = CubicSpline(x, y, bc_type='natural')
x_new = np.linspace(0, 2, 100)
y_new = f(x_new)
```

```
plt.figure(figsize = (10,8))
plt.plot(x_new, y_new, 'b')
plt.plot(x, y, 'ro')
plt.title('Cubic Spline Interpolation')
plt.xlabel('x')
plt.ylabel('y')
plt.show()
```



To determine the coefficients of each cubic function, we write out the constraints explicitly as a system of linear equations with $4(n-1)$ unknowns. For n data points, the unknowns are the coefficients a_i, b_i, c_i, d_i of the cubic spline, S_i joining the points x_i and x_{i+1}.

For the constraints $S_i(x_i) = y_i$ we have: $

$$\begin{array}{ccccc} a_1x_1^3+ & b_1x_1^2+ & c_1x_1+ & d_1 = & y_1, \\ a_2x_2^3+ & b_2x_2^2+ & c_2x_2+ & d_2 = & y_2, \\ \cdots \\ a_{n-1}x_{n-1}^3+ & b_{n-1}x_{n-1}^2+ & c_{n-1}x_{n-1}+ & d_{n-1} = & y_{n-1}. \end{array}$$

$

For the constraints $S_i(x_{i+1}) = y_{i+1}$ we have: $ $$\begin{array}{lllll} a_1 x_2^3 + & b_1 x_2^2 + & c_1 x_2 + & d_1 = & y_2, \\ a_2 x_3^3 + & b_2 x_3^2 + & c_2 x_3 + & d_2 = & y_3, \\ & & \cdots & & \\ a_{n-1} x_n^3 + & b_{n-1} x_n^2 + & c_{n-1} x_n + & d_{n-1} = & y_n. \end{array}$$ $

For the constraints $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$ we have:

$ $$\begin{array}{llllll} 3a_1 x_2^2 + & 2b_1 x_2 + & c_1 - & 3a_2 x_2^2 - & 2b_2 x_2 - & c_2 = 0, \\ 3a_2 x_3^2 + & 2b_2 x_3 + & c_2 - & 3a_3 x_3^2 - & 2b_3 x_3 - & c_3 = 0, \\ & & & \cdots & & , \\ 3a_{n-2} x_{n-1}^2 + & 2b_{n-2} x_{n-1} + & c_{n-2} - & 3a_{n-1} x_{n-1}^2 - & 2b_{n-1} x_{n-1} - & c_{n-1} = 0. \end{array}$$ $

For the constraints $S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$ we have:

$$\begin{array}{lllll} 6a_1 x_2 + & 2b_1 - & 6a_2 x_2 - & 2b_2 = & 0, \\ 6a_2 x_3 + & 2b_2 - & 6a_3 x_3 - & 2b_3 = & 0, \\ + & & \ldots - & & \\ 6a_{n-2} x_{n-1} + & 2b_{n-2} - & 6a_{n-1} x_{n-1} - & 2b_{n-1} = & 0. \end{array}$$

Finally for the endpoint constraints $S_1''(x_1) = 0$ and $S_{n-1}''(x_n) = 0$, we have: $ $$\begin{array}{ll} 6a_1 x_1 + & 2b_1 = 0, \\ 6a_{n-1} x_n + & 2b_{n-1} = 0. \end{array}$$ $

These equations are linear in the unknown coefficients a_i, b_i, c_i, and d_i. We can put them in matrix form and solve for the coefficients of each spline by left division. Remember that whenever we solve the matrix equation $Ax = b$ for x, we must make be sure that A is square and invertible. In the case of finding cubic spline equations, the A matrix is always square and invertible as long as the x_i values in the data set are unique.

TRY IT! Find the cubic spline interpolation at *x = 1.5* based on the data *x = [0, 1, 2], y = [1, 3, 2]*.

First we create the appropriate system of equations and find the coefficients of the cubic splines by solving the system in matrix form.}

The matrix form of the system of equations is: $ $$\left[\begin{array}{llllllll} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 4 & 2 & 1 \\ 3 & 2 & 1 & 0 & -3 & -2 & -1 & 0 \\ 6 & 2 & 0 & 0 & -6 & -2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 2 & 0 & 0 \end{array}\right]\left[\begin{array}{c} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{array}\right] = \left[\begin{array}{c} 1 \\ 3 \\ 3 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$ $

```
b = np.array([1, 3, 3, 2, 0, 0, 0, 0])
b = b[:, np.newaxis]
A = np.array([[0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 1, 1, 1], [1, 1, 1, 1, 0, 0, 0, 0], \
              [0, 0, 0, 0, 8, 4, 2, 1], [3, 2, 1, 0, -3, -2, -1, 0], [6, 2, 0, 0, -6, -2, 0, 0],\
              [0, 2, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 12, 2, 0, 0]])
```

```
np.dot(np.linalg.inv(A), b)
```

```
array([[-0.75],
       [ 0.  ],
       [ 2.75],
       [ 1.  ],
       [ 0.75],
       [-4.5 ],
       [ 7.25],
       [-0.5 ]])
```

Therefore, the two cubic polynomials are

$$S_1(x) = -.75x^3 + 2.75x + 1, \quad for \quad 0 \le x \le 1 \; and$$
$$S_2(x) = .75x^3 - 4.5x^2 + 7.25x - .5, \quad for \quad 1 \le x \le 2$$

So for $x = 1.5$ we evaluate $S_2(1.5)$ and get an estimated value of 2.7813.

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