# Seminar in Directed Graphical Models and Causality

## 1. Conditional Independence and Directed Acyclic Graphs

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August 11, 2021

# Outline

# Densities

- Let $X \in \mathbf{R}^m$ and $Y \in \mathbf{R}^n$ be random vectors, where $m, n \in \mathbf{N}$.
- Assume $f(x, y)$ is the joint density function of $(X, Y)$ with respect to product measure $\lambda = \lambda_X \otimes \lambda_Y$, where $\lambda_X$ and $\lambda_Y$ are measures in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively.
- Marginal distributions $P^X$ and $P^Y$ have densities

$$f_X(x) = \int f(x, y)\, d\lambda_Y(y) \quad \text{and } f_Y(y) = \int f(x, y)\, d\lambda_X(x)$$

# Conditional Densities/Distributions

## Definition

**Conditional density** of $X$ given $Y = y$ is

$$f_{X|Y}(x|y) = \begin{cases} \frac{f(x,y)}{f_Y(y)} & \text{if } f_Y(y) > 0 \\ \text{any density } f_0(x) & \text{otherwise} \end{cases}$$

The **conditional distribution** of $X$ given $Y = y$ is

$$P^{X|Y=y}(A) \equiv P(X \in A|Y = y) := \int_A f(x|y)\, d\lambda_X(x) \quad \forall A \in \mathbf{R}^m \text{ Borel set.}$$

# Independence

- $X$ and $Y$ are called **independent** if

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

  $\forall A, B$ Borel sets and write $X \perp\!\!\!\perp Y$.

- The following is a characterization of independence

$$X \perp\!\!\!\perp Y \iff f(x, y) = f_X(x) f_Y(y) \quad [\lambda - a.e.]$$

# Conditional independence

- Let $Z$ be random vector in $\mathbf{R}^k$, where $k \in \mathbf{N}$.

## Definition

$X$ and $Y$ are called **conditionally independent** given $Z$ if

$$P(X \in A, Y \in B|Z = z) = P(X \in A|Z = z)P(Y \in B|Z = z) \quad [P^Z - a.e]$$

$\forall A, B$ Borel sets and write $X \perp\!\!\!\perp Y|Z$.

# Question 1

Given a joint density of random vector (X, Y, Z) as

$$f(x, y, z) = \frac{1}{C}(x-z)^4 x^2 y^6 (y-z)^8,$$

where constant $C$ ensures that we have a valid density. Is the following relation true?

$$X \perp\!\!\!\perp Y | Z.$$

# Properties of Conditional Independence

- Assume $f(x, y, z)$ is the joint density of $(X, Y, Z)$ with respect to product measure $\lambda = \lambda_X \otimes \lambda_Y \otimes \lambda_Z$.

## Lemma

*The followings are equivalent and the equations hold $P^{X,Y,Z}$-a.e.:*

1. $X \perp\!\!\!\perp Y | Z$
2. $f(x, y|z) = f(x|z)f(y|z)$
3. $f(x|y, z) = f(x|z)$
4. $f(x, y, z) = \frac{f(x,z)f(y,z)}{f(z)}$
5. $f(x, y, z) = g(x, z)h(y, z)$ *for some measurable functions $g$ and $h$*
6. $f(x|y, z) = g(x, z)$ *for some measurable function $g$*

# Proof of Lemma

$1 \iff 2$: If $X \perp\!\!\!\perp Y|Z$, then

$$\begin{aligned}
P(X \in A, Y \in B|Z = z) &= P(X \in A|Z = z)P(Y \in B|Z = z) \\
&= \int_A f(x|z)\, d\lambda_X(x) \ \int_B f(y|z)\, d\lambda_Y(y) \\
&= \int_{A \times B} f(x|z)f(y|z)\, d(\lambda_X \otimes \lambda_Y)(x, y)
\end{aligned}$$

where $A$ and $B$ are arbitrary Borel sets. So, $f(x, y|z) = f(x|z)f(y|z)$ almost surely. If $f(x, y|z) = f(x|z)f(y|z)$ a.s., then

$$\begin{aligned}
P(X \in A, Y \in B|Z = z) &= \int_{A \times B} f(x, y|z)\, d(\lambda_X \otimes \lambda_Y)(x, y) \ = \int_{A \times B} f(x|z)f(y|z)\, d(\lambda_X \otimes \lambda_Y)(x, y) \\
&= \int_A f(x|z)\, d\lambda_X(x) \ \int_B f(y|z)\, d\lambda_Y(y) \\
&= P(X \in A|Z = z)P(Y \in B|Z = z).
\end{aligned}$$

# Proof of Lemma

$2 \iff 3$:

$$f(x,y|z) = f(x|z)f(y|z) \iff \frac{f(x,y,z)}{f(z)} = \frac{f(x,z)f(y,z)}{f(z)f(z)}$$
$$\iff \frac{f(x,y,z)}{f(y,z)} = \frac{f(x,z)}{f(z)} \iff f(x|y,z) = f(x|z),$$

where we are considering all the cases when the denominator is not zero and the equations hold almost surely. From the definition of conditional density the zero cases are trivial.

$3 \iff 4$:

$$f(x|y,z) = f(x|z) \iff \frac{f(x,y,z)}{f(y,z)} = \frac{f(x,z)}{f(z)} \iff f(x,y,z) = \frac{f(x,z)f(y,z)}{f(z)}.$$

# Proof of Lemma

$3 \implies 6$: Denote $g(x,z) := f(x|z)$.

$6 \implies 5$: Denoting $h(y,z) := f(y,z)$ we have $f(x,y,z) = f(x|y,z)f(y,z) = g(x,z)h(y,z)$.

$5 \implies 4$: We have

$$\begin{aligned}
\frac{f(x,z)f(y,z)}{f(z)} &= \frac{\int f(x,y,z)\,d\lambda_Y(y)\ \int f(x,y,z)\,d\lambda_X(x)}{\int f(x,y,z)\,d(\lambda_X \otimes \lambda_Y)(x,y)} \\
&= \frac{g(x,z)h(y,z)\int h(y,z)\,d\lambda_Y(y)\ \int g(x,z)\,d\lambda_X(x)}{\int g(x,z)\,d\lambda_X(x)\ \int h(y,z)\,d\lambda_Y(y)} \\
&= g(x,z)h(y,z) = f(x,y,z).
\end{aligned}$$

$\square$

# Question 1 (now should be easy)

Given a joint density of random vector (X, Y, Z) as

$$f(x, y, z) = \frac{1}{C}(x - z)^4 x^2 y^6 (y - z)^8,$$

where constant $C$ ensures that we have a valid density. Is the following relation true?

$$X \perp\!\!\!\perp Y | Z.$$

# General Properties of Conditional Independence

**(C1) "Symmetry":**

$$X \perp\!\!\!\perp Y|Z \iff Y \perp\!\!\!\perp X|Z.$$

**(C2) "Decomposition":**

$$X \perp\!\!\!\perp Y|Z \implies h(X) \perp\!\!\!\perp Y|Z \text{ for any measurable function } h.$$

In particular, $(X, W) \perp\!\!\!\perp Y|Z \implies X \perp\!\!\!\perp Y|Z$.

**(C3) "Weak union":**

$$X \perp\!\!\!\perp Y|Z \implies X \perp\!\!\!\perp Y|(Z, h(X)) \text{ for any measurable function h.}$$

In particular, using also (C2) we obtain $(X, W) \perp\!\!\!\perp Y|Z \implies X \perp\!\!\!\perp Y|(Z, W)$.

**(C4) "Contraction":**

$$X \perp\!\!\!\perp Y|Z \text{ and } X \perp\!\!\!\perp W|(Y, Z) \iff X \perp\!\!\!\perp (W, Y)|Z.$$

# Proof of (C1) and (C2)

(C1): For all Borel sets $A$, $B$ and for all values of $z$ we have

$$P(X \in A, Y \in B|Z = z) = P(X \in A|Z = z)P(Y \in B|Z = z)$$

(C2): For all Borel sets $A$, $B$ and for all values of $z$ we have

$$\begin{aligned} P(h(X) \in A, Y \in B|Z = z) &= P(X \in h^{-1}(A), Y \in B|Z = z) \\ &= P(X \in h^{-1}(A)|Z = z)P(Y \in B|Z = z) \\ &= P(h(X) \in A|Z = z)P(Y \in B|Z = z) \end{aligned}$$

So, $h(X) \perp\!\!\!\perp Y|Z$.

# Proof of (C3) and (C4)

(C3): The proof is only for last equation when we have densities

$$f(x|y,z,w) = \frac{f(x,w|y,z)}{f(w|y,z)} = \frac{f(x,w|z)}{f(w|y,z)} = \frac{f(x,w|z)}{f(w|z)} = f(x|w,z)$$

So, $X \perp\!\!\!\perp Y|(Z,W)$.

(C4): If $X \perp\!\!\!\perp Y|Z$ and $X \perp\!\!\!\perp W|(Y,Z)$, then

$$P^{X|(W,Y,Z)=(w,y,z)} = P^{X|(Y,Z)=(y,z)} = P^{X|Z=z} \quad [P^{(W,Y,Z)} - a.s.]$$

So, $X \perp\!\!\!\perp (W,Y)|Z$. Now if $X \perp\!\!\!\perp (W,Y)|Z$ from (C3) we have $X \perp\!\!\!\perp W|Y,Z$ and from (C2) we have $X \perp\!\!\!\perp Y|Z$.

# Intersection "Axiom"

From (C4) we have

$$X \perp\!\!\!\perp (W, Y)|Z \implies X \perp\!\!\!\perp W|(Y, Z) \text{ and } X \perp\!\!\!\perp (W, Y)|Z \implies X \perp\!\!\!\perp Y|(W, Z).$$

**(C5) "Intersection":**

Assume that we have a joint density $f(x, y, w, z)$ with respect to $\lambda = \lambda_X \otimes \lambda_Y \otimes \lambda_W \otimes \lambda_Z$ such that $f(y, w, z) > 0$ $[\lambda - a.e.]$. Then,

$$X \perp\!\!\!\perp (W, Y)|Z \iff X \perp\!\!\!\perp W|(Y, Z) \text{ and } X \perp\!\!\!\perp Y|(W, Z)$$

# Proof of (C5)

From previous slide we need only the reverse implication $\Longleftarrow$. From the Lemma we have

$$f(x,y,w,z) = \frac{f(x,w,z)f(y,w,z)}{f(w,z)} = \frac{f(x,y,z)f(y,w,z)}{f(y,z)}$$

Since $f(y,w,z) > 0$ almost surely we have

$$\frac{f(x,w,z)}{f(w,z)} = \frac{f(x,y,z)}{f(y,z)} \implies f(x,w,z)f(y,z) = f(x,y,z)f(w,z).$$

From the marginalization we have

$$f(x,w,z)f(z) = f(x,w,z)\int f(y,z)\,d\lambda_Y(y) = \int f(x,y,z)f(w,z)\,d\lambda_Y(y) = f(x,z)f(w,z).$$

So, from the Lemma we have $X \perp\!\!\!\perp W|Z$. Using (C4) with $X \perp\!\!\!\perp Y|(W,Z)$ we obtain $X \perp\!\!\!\perp (W,Y)|Z$. $\square$

# Terminology and Notation for DAGs

## Definition

A graph $\mathcal{G} = (\mathbf{V}, \mathcal{E})$ consists of a finite set of nodes $\mathbf{V}$ and edges $\mathcal{E} \subseteq \mathbf{V} \times \mathbf{V}$ of ordered pairs of distinct nodes.

- Given a set of random variables $\mathbf{X} = (X_1, \ldots, X_p)$, $\mathbf{V} := \{1, \ldots, p\}$ and a graph $\mathcal{G} = (\mathbf{V}, \mathcal{E})$ we associate every random variable $X_j$ with node $j \in \mathbf{V}$.
- The joint distribution of $\mathbf{X}$ is denoted by$P^X$ and marginal distribution of $\mathbf{X}_j$ by $P^{\mathbf{X}_j}$.
- A graph $\mathcal{G}_1 = (\mathbf{V}_1, \mathcal{E}_1)$ is called a **subgraph** of $\mathcal{G}$ if $\mathbf{V}_1 \subseteq \mathbf{V}$ and $\mathcal{E}_1 \subseteq \mathcal{E}$.
- If $\mathcal{G}_1$ is a subgraph of $\mathcal{G}$ we write $\mathcal{G}_1 \leq \mathcal{G}$ and if $\mathcal{E}_1 \neq \mathcal{E}$ we say $\mathcal{G}_1$ is **proper subgraph** of $\mathcal{G}$.

# Example 1

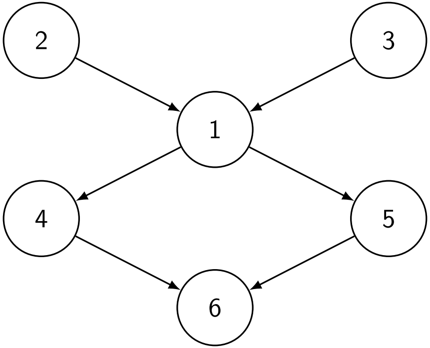


Figure 1: Graph $\mathcal{G} = (\{1, 2, 3, 4, 5, 6\}, \{(2, 1), (3, 1), (1, 4), (1, 5), (4, 6), (5, 6)\})$.

# Terminology and Notation for DAGs

- A node $i$ is called a child of $j$ if $(j, i) \in \mathcal{E}$ and is called a parent, if $(i, j) \in \mathcal{E}$.
- If $(i, j) \in \mathcal{E}$ we also write $i \to j$.
- Children of $j$ is denoted by $\mathbf{CH}_j^{\mathcal{G}} := \{i \in \mathbf{V} : (j, i) \in \mathcal{E}\}$ and parents of $j$ by $\mathbf{PA}_j^{\mathcal{G}} := \{i \in \mathbf{V} : (i, j) \in \mathcal{E}\}$.
- Two nodes $i$ and $j$ are called **adjacent** if $(j, i) \in \mathcal{E}$ or $(i, j) \in \mathcal{E}$ and if both holds we say the edge between $i$ and $j$ is **undirected**, otherwise **directed**.
- A graph is called **complete** if every two nodes are adjacent. **Cliques** of a graph $\mathcal{G}$ are the maximal complete subgraphs of $\mathcal{G}$ (here maximal in a sense of set inclusion).
- A **path** in $\mathcal{G}$ is a sequence of distinct nodes $j_1, \ldots, j_n$ such that $j_k$ and $j_{k+1}$ are adjacent $\forall k = 1, \ldots, n-1$ and $n \geq 2$. If $j_k \to j_{k+1}$ $\forall k = 1, \ldots, n-1$ path is called **directed** from $j_1$ to $j_n$.

# Example 2

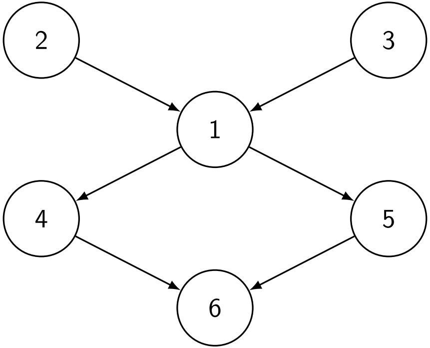


Figure 2: Graph $\mathcal{G} = (\{1, 2, 3, 4, 5, 6\}, \{(2, 1), (3, 1), (1, 4), (1, 5), (4, 6), (5, 6)\})$.

Some (directed) paths are

$$1 \to 4 \to 6 \leftarrow 5, \quad 3 \to 1 \to 4 \to 6, \quad 5 \to 6$$

# Terminology and Notation for DAGs

- We say $j$ is a **descendant** of $i$ if there is a directed path from $i$ to $j$ and denote all the descendants of $j$ by $\mathbf{DE}_j^{\mathcal{G}}$ and all non-descendants by $\mathbf{ND}_j^{\mathcal{G}}$. Note that descendants and non-descendants do not contain the node.
- $j_k$ is called a **collider** in the path if $j_{k-1} \to j_k$ and $j_{k+1} \to j_k$.
- $\mathcal{G}$ is called a **Partially Directed Acyclic Graph (PDAG)** if there is no directed cycle, i.e., if there is no pair (i, j) such that there are directed paths from i to j and from j to i.
- $\mathcal{G}$ is called **Directed Acyclic Graph (DAG)** if all edges are directed and there is no cycle in $\mathcal{G}$.

# Terminology and Notation for DAGs

- Three nodes $i, j, k$ are called **immorality** or **v-structure** if one of them, say $j$ is a child of the others and these parents are not adjacent: $i \to j$, $k \to j$ and $(k, i) \notin \mathcal{E}, (i, k) \notin \mathcal{E}$.
- The **skeleton** of graph $\mathcal{G}$ is the set of all edges without taking the direction into account, that is all $(i, j)$ such that $i \to j$ or $j \to i$.

# Example 3

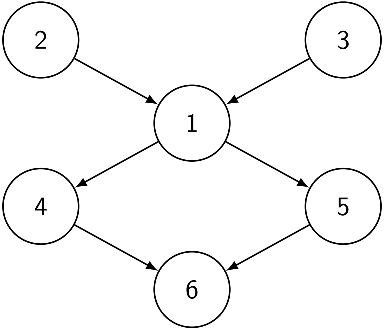


Figure 3: Graph $\mathcal{G} = (\{1, 2, 3, 4, 5, 6\}, \{(2, 1), (3, 1), (1, 4), (1, 5), (4, 6), (5, 6)\})$.

- Descendants of node 1 are $\{4,\ 5,\ 6\}$.
- 6 is a collider in the path $1 \rightarrow 4 \rightarrow 6 \leftarrow 4$
- 4, 5, 6 is a v-structure

# Local Markov Property

### Definition

The joint distribution $P^X$ of $\mathbf{X}$ is said to be **Local Markov with respect to the DAG** $\mathcal{G}$ if

$$\forall v \in \mathbf{V}: \quad v \perp\!\!\!\perp \mathbf{V} \setminus \{\{v\} \cup \mathbf{PA}_v^{\mathcal{G}} \cup \mathbf{DE}_v^{\mathcal{G}}\} | \mathbf{PA}_v^{\mathcal{G}}.$$

# Example 4

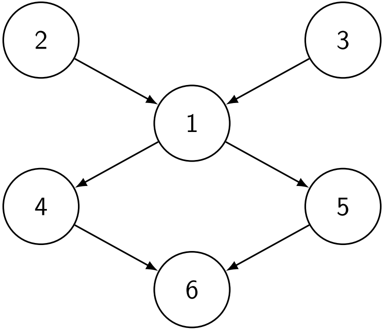


Figure 4: Graph $\mathcal{G} = (\{1, 2, 3, 4, 5, 6\}, \{(2, 1), (3, 1), (1, 4), (1, 5), (4, 6), (5, 6)\})$.

- From Local Markov Property we have $\{5\} \perp\!\!\!\perp \{2, 3, 4\} | \{1\}$.

# d-separation

## Definition

In a DAG $\mathcal{G} = (\mathbf{V}, \mathcal{E})$, a path between $i$ and $j$ is **blocked** by $\mathbf{S} \subsetneq \mathbf{V}$ $(i, j \notin \mathbf{S})$ whenever there is a node $k$ in the path and one of the following holds:

1. $k \in \mathbf{S}$ and $k$ is not a collider in the path, or
2. $k \notin \mathbf{S}$ and $k$ is a collider in the path and $\forall l \in \mathbf{DE}_k^{\mathcal{G}} \implies l \notin \mathbf{S}$.

## Definition

Given disjoint subsets $\mathbf{A}, \mathbf{B}, \mathbf{C}$, we say $\mathbf{A}$ and $\mathbf{B}$ are **d-separated** by $\mathbf{C}$ if every path between nodes in $\mathbf{A}$ and $\mathbf{B}$ is blocked by $\mathbf{C}$.

# Example 5

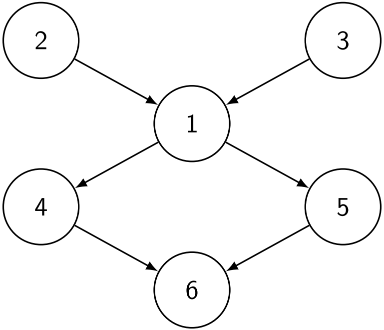


Figure 5: Graph $\mathcal{G} = (\{1, 2, 3, 4, 5, 6\}, \{(2, 1), (3, 1), (1, 4), (1, 5), (4, 6), (5, 6)\})$.

- Are $\{2\}$ and $\{4, 6\}$ d-separated by $\{1\}$?
- Are $\{2\}$ and $\{3\}$ d-separated by $\{1\}$?

# Markov Property and Faithfulness

## Definition

The joint distribution $\mathcal{L}(\mathbf{X})$ of $\mathbf{X}$ is said to be **(Global) Markov with respect to the DAG** $\mathcal{G}$ if

$$\mathbf{A}, \mathbf{B} \text{ d-sep. by } \mathbf{C} \implies \mathbf{A} \perp\!\!\!\perp \mathbf{B} | \mathbf{C}.$$

for all disjoint sets $\mathbf{A}, \mathbf{B}, \mathbf{C} \subseteq \mathbf{V}$.

## Definition

The joint distribution $\mathcal{L}(\mathbf{X})$ is said to be **faithful to the DAG** $\mathcal{G}$ if

$$\mathbf{A}, \mathbf{B} \text{ d-sep. by } \mathbf{C} \impliedby \mathbf{A} \perp\!\!\!\perp \mathbf{B} | \mathbf{C}.$$

for all disjoint sets $\mathbf{A}, \mathbf{B}, \mathbf{C} \subseteq \mathbf{V}$.

# Example 6

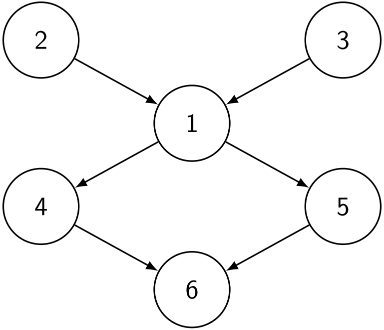


Figure 6: Graph $\mathcal{G} = (\{1,2,3,4,5,6\}, \{(2,1),(3,1),(1,4),(1,5),(4,6),(5,6)\})$.

- From Global Markov Property we have $\{2,3\} \perp\!\!\!\perp \{4,5,6\} | \{1\}$.

# Markov Equivalence class and Causal Minimality

- A distribution satisfies **causal minimality** with respect to graph $\mathcal{G}$ if it is Markov with respect to $\mathcal{G}$, but not to any proper subgraph of $\mathcal{G}$.
- Let's denote $\mathcal{M}(\mathcal{G}) := \{P^{\mathbf{X}} : P^{\mathbf{X}} \text{ is Markov w.r.t. } \mathcal{G}\}$ all the distributions which are Markov with respect to $\mathcal{G}$.
- Two DAGs $\mathcal{G}_1$ and $\mathcal{G}_2$ are called **Markov equivalent** if $\mathcal{M}(\mathcal{G}_1) = \mathcal{M}(\mathcal{G}_2)$.
- The above holds if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy same set of d-separations.
- The set of all DAGs that are Markov equivalent to some DAG is called Markov equivalence class.

# Equivalence of Markov Properties

**Theorem**

*Let $\mathcal{G}$ be a DAG. The joint distribution $P^X$ is Local Markov w.r.t. $\mathcal{G}$ if and only if $P^X$ is Global Markov w.r.t. $\mathcal{G}$.*

**Proof.**

In the next lecture. $\square$

# Exercises

1. Let $\mathcal{G}$ be a DAG and $A, B$ any non adjacent nodes. Prove that there is a set of nodes $\mathbf{S}$ such that $A$ and $B$ are d-separated given $\mathbf{S}$.
2. Given a DAG $\mathcal{G} = (\mathbf{V}, \mathcal{E})$ and any non adjacent nodes $L$ and $W$ in $\mathbf{V}$. Then, for any set of nodes $\mathbf{R}$ in $\mathbf{V}$ such that $\mathbf{R} \subset \mathbf{ND}_W^{\mathcal{G}}$
$$L, W \text{ d-sep. by } \mathbf{S} \cup \mathbf{R},$$
where $\mathbf{S} := \mathbf{PA}_L^{\mathcal{G}} \cup \mathbf{PA}_W^{\mathcal{G}}$.
3. If $P^{\mathbf{X}}$ is Markov and faithful with respect to graph $\mathcal{G}$, then $P^{\mathbf{X}}$ satisfies causal minimality with respect to $\mathcal{G}$. (Hint: use exercise 1)

# References

Some of the statements and proofs I have taken from Prof. Dr. Mathias Drton lecture in "Graphical Models in Statistics" at TUM.