STRONG TOPOLOGICAL COMPLETENESS OF GL BEYOND ω

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Abstract. Provability logic GL is known to be complete for the set of finite, converse well-founded, irreflexive trees. Strong completeness does not hold, though. To obtain it, one needs to consider a slight modification of the Kripke semantic (ω-bouquets), with the help of which one can also establish strong completeness of GL for the order topology on ordinals $\geq \omega^\omega + 1$. We show that GL over a language with uncountably many propositional variables cannot be strongly complete for ω-bouquets and neither for countable ordinals. However, for each infinite cardinal κ, GL with κ variables is strongly complete for bouquets of size κ and for the subspaces of the interval topology on κ. Additionally, we establish the strong completeness of the logic $\mathsf{GL.3}$ with respect to ultralinear bouquets, which we introduce in the paper.

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1. Introduction

The interest in provability logic stems from the investigations of Gödel's incompleteness theorems. Löb [Lö55] formulated three conditions on the provability predicate of Peano Arithmetic that form a useful modification of the conditions that Hilbert and Bernays [HB39] introduced for their proof of Gödel's second incompleteness theorem. Friedman [Fri75] posed the problem of axiomatizing the set of valid arithmetical formulæ built from expressions of the form "φ is provable" by means of Boolean connectives and provability assertions. Boolos [Boo75] (and independently Bernardi, Montagna, and van Benthem) proved that Löb's axiomatization was complete when restricting to closed (i.e., variable-free) formulæ, building on work of Segerberg [Seg71] on the Kripke semantics of Löb's logic GL. Solovay [Sol76] later extended Boolos' theorem to a completeness theorem of GL for its arithmetical interpretation.

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Although arithmetical completeness is a crucial fact to show that GL is 'the' logic of provability, we need completeness with respect to some simpler models to be able to work with GL. GL is known to be Kripke complete with respect to the class of finite irreflexive trees and topologically complete with respect to any ordinal $\geq\omega^\omega$ with its order topology. Things get trickier if we are to ask if there is a model in a given class that (locally) satisfies a given set of formulæ consistent with the logic, if the answer is positive, we say that the logic has *the strong completeness property* with respect to the class of models. It is known that the strong Kripke completeness fails for GL and holds for scattered topological spaces (in fact one can restrict to the countable ordinals with their order topology), GL is also known to be strongly complete with respect to ω-bouquet, who are topological structures reminiscent to Kripke frames.

We consider a modal language with uncountably many variables. This allows to consider sets of formulæ of uncountable sizes. If κ is a cardinal we say that GL^κ is the logic with the same set of axiom schemata as GL over the language with κ-many variables. We show that the strong completeness for this logic fails with respect to countable models fails (both ω-bouquets and ordinals), moreover it fails for the ordinal spaces in general.

We shall prove:

Theorem A.

(1) GL^κ *is not strongly complete with respect to the order topology on ordinals for any uncountable λ-bouquet for each $\lambda < \kappa$.*

(2) GL^κ *is not strongly complete w.r.t. order topology on ordinals for any uncountable κ.*

However, we show:

Theorem B. *For each $\kappa > \aleph_0$*

(1) GL^κ *is strongly complete with respect to the class of κ-bouquets.*

(2) GL^κ *is strongly complete with respect to the subsets of $\kappa+1$, i.e. with the class*

$$\{(K, \tau_K) : \kappa \in K \subset \kappa+1, \tau_K = \{U \cap K : U \in \tau_\iota\}\},$$

where τ_ι is the order topology on Ord.

Theorem A warrants the restrictions in Theorem B, showing that the statement is the strongest we can get.

Another result we present here is that GL is strongly complete with ω-bouquets and countable ordinals of a chosen size, namely

Theorem C. *Let α be a countable ordinal and let Γ be a non-degenerate (i.e. consistent with $\Diamond^n\top$ for each n) set of formulæ consistent with* GL*, then*

(1) *there is an ω-bouquet with rank α which satisfies Γ;*
(2) *there is a valuation $v : P(\omega^\alpha+1) \to \mathsf{var}$, such that $(\omega^\alpha+1, \omega^\alpha) \Vdash_v \Gamma$;*

$\mathsf{GL.3}$ is an extension of the logic GL obtained by adding the axiom schema $.3 = \Box(\Box\varphi \to \psi) \vee (\Box\psi \wedge \psi \to \varphi)$. This logic was studied by Solovay, who shown that it is complete with respect to the set theoretic interpretation, where $\Box\varphi$ is interpreted as "φ holds in all transitive models of ZFC". $\mathsf{GL}.3$ is Kripke complete with respect to the set of finite strict linear orders, however the strong completeness fails akin to the GL case. Aguilera and Pakhomov [AP25] obtained set-theoretic completeness for the polymodal generalization $\mathsf{GLP}.3$ and Aguilera with the author found new natural topological models that give completeness to $\mathsf{GL}.3$ [AS24a]. We employ the machinery used for results about GL^κ to obtain similar strong completeness results for $\mathsf{GL}.3^\kappa$. We work with trees where each node can only have infinitely many, one or zero immediate successors, and for evaluation of modal formulæ we fix an ultrafilter instead of the cobounded filter (see Definition 10). We call such models *ultralinear κ-bouquets.* We prove the following:

Theorem D.

(1) Let Γ be a consistent set of $\mathsf{GL}.3^\kappa$ formulæ, then there is an ultralinear κ-bouqet $(B,<)$ with a valuation $v:\mathsf{var}^\kappa\to P(B)$ and the root r such that $B,r\Vdash_v\Gamma$.

(2) Let Γ be a consistent set of $\mathsf{GL}.3$-formula, such that $\Diamond^n\top\in\Gamma$ for every n, then for each countable ordinal α there is an ultralinear ω-bouquet $(B,<)$ with the root r of rank α and a valuation $v:\mathsf{var}\to P(B)$ such that $B,r\Vdash_v\Gamma$.

In this article we do not focus on building ordinal spaces for $\mathsf{GL}.3$, since it requires a different machinery and in fact is a subject of our different work [AS24a]. However in order for this overview to be complete, we state the result as well:

Fact 1 (Aguilera, S.)**.**

(1) Assume $V=L[\mathcal{U}]$, where $\mathcal{U}$ is an infinite set of normal measures, such that for each n, there is a measure of Mitchell order n, then $\mathsf{GL}.3$ is complete with respect to the normal topology on Ord.

(2) Assume $\mathsf{ZF}+\mathsf{AD}$, then $\mathsf{GL}.3$ is complete with respect to the club topology on Ord.

2. Preliminaries

2.1. **Kripke semantics.** For a cardinal κ we define the following modal language:

$$\mathcal{L}^\kappa = p \mid \varphi\wedge\psi \mid \neg\varphi \mid \Box\varphi$$

where $p\in\mathsf{var}^\kappa$ and $\varphi,\psi\in\mathcal{L}^\kappa$ and var^κ is a set of propositional variables with $|\mathsf{var}^\kappa|=\kappa$. [1]

Definition 2. Logic GL^κ is the minimal set of $\mathcal{L}^\kappa$-formulæ closed under modus ponens and Nec and containing the following axioms:

(1) classical tautologies;
(2) $\Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi)$;
(3) $\Box(\Box\varphi\to\varphi)\to\Box\varphi$;

We write GL for GL^ω.

Definition 3. *Kripke frame* is a tuple $F=\langle W,<\rangle$, where W is a set and $<\subset W\times W$. Given a Kripke frame F and a function $v:\mathsf{var}^\kappa\to PW$, we say that $M=\langle F,v\rangle$ is a Kripke model, which yields the following interpretation $[\![\cdot]\!]$ of the modal formulæ:

[1] One can additionally assume $\mathsf{var}^\kappa\subset H_\kappa$, i.e. each variable is hereditarily smaller than κ.

- $[\![\bot]\!] = \emptyset$;
- $[\![p]\!] = v(p)$, where $p \in \mathsf{var}^\kappa$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$;
- $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$;
- $[\![\Diamond\varphi]\!] = \{x : \exists y(x < y \wedge y \in [\![\varphi]\!])\}$;

Conventionally, $\Box\varphi = \neg\Diamond\neg\varphi$. We say that a formula φ holds at a point x in a model M if $x \in [\![\varphi]\!]$, in which case we write $M, x \Vdash \varphi$. We write $M \Vdash \varphi$ to mean $M, x \Vdash \varphi$ for some $x \in M$ and $M \models \varphi$ to mean $M, x \Vdash \varphi$ for all $x \in F$ and $F \models \varphi$ to mean $M \models \varphi$ for all models of the form $M = \langle F, v\rangle$ we might occasionally write $F, x \Vdash_v \varphi$ instead if $(F, v), x \Vdash \varphi$ or even omit the index, if v is clear form the context.

We say that a Kripke frame is a GL-*frame* if $F \models \mathsf{GL}$, one can show that $(F, <)$ is a GL-frame if $<$-is converse well-founded, irreflexive and transitive relation on F.

Given a GL-frame F, then for each point $x \in W$ we assign *rank* $\rho(x)$, which is 0 for each element that has no successors, and otherwise it is

$$\rho(x) = \sup\{\rho(y) + 1 : y \text{ is an immediate successor of } x\}.$$

Given a rooted tree T with the root r we let $\rho(T) = \rho(r)$.

Fact 4 (Segerberg)**.** *$\mathsf{GL} \vdash \varphi$ is the logic of GL-frames, moreover GL is complete with respect to finite irreflexive trees.*

However, the following example shows that we cannot attain *strong completeness* for Kripke frames. Indeed, letting

$$(*) \qquad \Gamma = \{\Diamond p_0\} \cup \{\Box(p_i \to \Diamond p_{i+1}) : i < \omega\},$$

one can see that Γ necessitates the model to have an infinite $<$-chain, thus such model cannot be a model of GL.

2.2. **Topological semantics.** Despite the failure of the strong completeness for the Kripke semantics, in [AFD17] the strong topological completeness (with respect to countable ordinal spaces) was established via the strong topological completeness with respect to Kripke-like structures, namely *ω-bouqets*. For our convenience we introduce this notion in the topological fashion.

Definition 5. Let (X, τ) be a topological space. We let

$$d_\tau(A) = \{x : \forall U \in \tau_i \exists y \neq x(y \in U \cap A)\}$$

for $A \subset X$, we call d_τ the *derivative operator*, we omit the subscript τ if there is no risk of confusion. We call (X, τ) a *scattered* topological space if any $A \subset X$ has an isolated point. If X is a scattered space, then *the rank function* $\mathrm{rk}(x) = \min\{\alpha : x \notin d^{\alpha+1}X\}$ is well-defined for all $x \in X$.

Definition 6. A topological model is a pair $\langle (X, \tau), v\rangle$, where (X, τ) is a topological space and $v : \mathsf{var}^\kappa \to PX$, which yields the interpretation:

- $[\![p]\!] = v(p)$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$;
- $[\![\neg\psi]\!] = X \setminus [\![\psi]\!]$;
- $[\![\Diamond\varphi]\!] = d([\![\varphi]\!])$;

Note that $[\![\Box\varphi]\!] = \tilde{d}([\![\varphi]\!]) = X \setminus d(X \setminus [\![\varphi]\!])$, in other words $X, x \models \Box\varphi$ if and only if there is a punctured neighborhood U of x, such that $X, y \models \varphi$ for any $y \in U$.

It is known that (X, τ) validates GL if and only if (X, τ) is a scattered space. In particular, one can construe Kripke frames as topological spaces generated by the set of all upsets, namely all $A \subset X$ such that $\forall x, y(x \in A \wedge xRy \to y \in A)$. Such topological spaces are scattered and retain validity from the Kripke interpretation, thus GL is complete with respect to the class of scattered spaces.

A natural type of scattered spaces is the class of ordinals with their order topology also known as the interval topology.

Definition 7. We call *order topology* τ_ι the topology on Ord generated by the intervals (α, β) for each $\alpha < \beta$, for $\gamma \in \mathsf{Ord}$ we let (γ, τ_ι) denote $(\gamma, \{U \cap \gamma : U \in \tau_\iota\})$. One can assume $\mathsf{Ord} = 2^{2^\kappa}$ whenever we work with the logic GL^κ.

It is known that GL is complete with respect tho the class of countable ordinals with the interval topology, moreover it is complete with respect to a singleton:

Theorem 8 (Abashidze, Blass)**.** *Let $\Omega \geq \omega^\omega$, then GL is complete with respect to the class $\{(\Omega, \tau_\iota)\}$.*

2.3. **Bouquets.** Now we define κ-bouqets, which are reminiscent to Kripke frames with a slightly different interpretation rule and ordering of the nodes. We loosen the requirement for a Box-formula φ to be satisfied. The original definition of ω-bouquet in [AFD17] says that if a node of the tree has ω-many successors, then it is enough to satisfy φ at all but finitely many of them. Dually $\Diamond\varphi$ hods if $\varphi \vee \Diamond\varphi$ holds at infinitely many immediate successors. We can similarly define the satisfaction relation in the case of λ-many immediate successors. In this case we say that $\Box\varphi$ holds if $\varphi \wedge \Box\varphi$ holds in a co-bounded set of the node's immediate successors, dually $\Diamond\varphi$ holds if $\varphi \vee \Diamond\varphi$ holds in a cofinal set of the node's immediate successors. Note that if we are to deal with a node with λ-many successors for a singular λ, being cofinal is not permutation invariant. To amend this we additionally fix an explicit enumeration of the immediate successors of each node. Moreover, one can note that the definition can be generalized by taking some other than co-bounded filter F_λ on λ.

Definition 9. Let X be a set and $R \subset X$ is a converse well-founded tree with the root r, for each cardinal λ, let $\mathcal{F} = \{F_\lambda\}$ be a class of filters on each cardinal λ. For $x \in X$ let Y_x be the set of immediate successors of x and $e_x : Y_x \to |Y_x|$ be some bijective enumeration. For $x \in X$ let $R_x = R \cap \{y \in X : y = x \vee xRy\}^2$. Then we set $R^\mathcal{F} = R^\mathcal{F}_r$ where for each $x \in X$, $R^\mathcal{F}_x$ is a set of relations on X, such that $A \in R^\mathcal{F}_x$, if and only if

(1) $|Y_x| = n < \aleph_0$, $Y_x = \{y_i\}_{i<n}$ and for each $i < n$ there is $A_i \in R^\mathcal{F}_{y_i}$ such that $\bigcup_{i<n}\{\{(x, y_i)\} \cup A_i\} \subset A$, or
(2) $|Y_x| = \lambda \geq \aleph_0$ and $Y_x = \{y_i\}_{i<\lambda}$ such that $e_x(y_i) = i$ and there is $D \in F_\lambda$ such that for each $i \in D$ there is $A_i \in R^\mathcal{F}_{y_i}$ such that $\bigcup_{i \in D}(\{(x, y_i)\} \cup A_i) \subset A$;

Definition 10 ($(\mathcal{F}, \kappa)$-bouquet)**.** Let (T, R) be a converse well-founded tree of size κ, and let $\rho : T \to \mathsf{Ord}$ be the rank function on T with respect to the upset topology.

We define a new topology, $\sigma^\mathcal{F}_R$, to be the least topology containing every $U \in R^\mathcal{F}_x$ for each $x \in T$. We say a topological space (T, σ) is a $(\mathcal{F}, \kappa)$-*bouquet* if there exists a binary relation R on T such that (T, R) is a converse well-founded tree and $\sigma = \sigma^\mathcal{F}_R$.

We call a space (T, σ) a *κ-bouquet* if it is a $(\mathcal{F}, \kappa)$-bouquet where $\mathcal{F}$ is such that $\forall \lambda, F_\lambda$ is a cobounded filter on λ.

The number of immediate successors is called *degree* of the node and *rank* $\rho(x)$ of a node $x \in T$ is defined inductively as $\rho(x) = 0$ if it has no immediate successors and $\rho(x) = \sup\{\rho(y_\alpha) + 1 : y_\alpha$ is an immediate successor of $x\}$.

Claim 11. *Let (T, σ) be an $(\mathcal{F}, \kappa)$-bouquet, then for each $x \in T$, $\mathrm{rk}_\sigma(x) = \rho(x)$ (see Definition 5).*

Proof. Note that a x is isolated in (T, σ) if and only if it has no immediate successors. Thus, a simple induction on $\rho(x)$ proves the statement of the claim. $\square$

Fact 12 ([AFD17])**.** GL *is strongly complete with respect to the class of ω-bouquets.*

3. Two Incompleteness Lemmata

We prove two lemmata comprise the incompleteness Theorem A, this will warrant the statement of the later coming theorems, by showing that the strong completeness cannot be attained for the so far considered models.

Lemma 13. *There exists a consistent set Γ of GL^κ-formulæ such that for any ordinal $\lambda \leq \kappa$ for any $v : \mathsf{var}^\kappa \to {}^\lambda 2$, $(\lambda, \tau_\iota) \not\Vdash_v \Gamma$, moreover if κ is regular, then for any $\lambda \in \mathsf{Ord}$ with $cof(\lambda) < \kappa$, $(\mathsf{Ord}, \tau_\iota), \lambda \not\Vdash \Gamma$.*

Proof. Let
$$\Gamma = \{\Diamond p_0\} \cup \{\Box(p_\alpha \to \Diamond p_\beta) : \alpha < \beta < \kappa\}.$$

For the sake of contradiction take δ and $v : \mathsf{var}^\kappa \to {}^\lambda 2$, such that $\delta \Vdash \Gamma$. That is $\delta \in dv(p_0)$ and $\delta \in \tilde{d}(v(p_\alpha)^c \cup dv(p_\beta))$ for each $\alpha < \beta < \kappa$. Since $\delta \in \tilde{d}(v(p_\alpha)^c \cup dv(p_\beta))$, there is $\delta_{\alpha\beta} < \delta$ such that $(\delta_{\alpha\beta}, \delta) \subset v(p_\alpha)^c \cup dv(p_\beta)$. We let $F : [\kappa]^2 \to \lambda : (\alpha, \beta) \mapsto \delta_{\alpha\beta}$. Then there is an infinite $A \subset \lambda$ with $F([A]^2) = \delta' < \delta$. Let $A' = \{\alpha_i\}_{i<\omega} \subset A$ be increasing and $\gamma \in (\delta', \delta)$. The assumption that $\delta \Vdash \Diamond p_0$ implies there is $\beta_0 \in (\delta', \delta)$ with rank $\rho_0 = \rho(\beta_0)$ with $\beta_0 \Vdash p_0$, hence $\beta_0 \Vdash \Diamond p_{\alpha_1}$. Inductively, for each $i < \omega$ there exists $\beta_i \in (\delta', \delta)$ with $\beta_i \Vdash \Diamond p_{\alpha_i}$, moreover $\rho(\beta_{i+1}) < \rho(\beta_i)$, which amounts to an infinite decreasing sequence of ordinals.

The 'moreover' part is proven similarly. $\square$

Lemma 14. *Let κ be an uncountable cardinal, then there is a consistent set of GL^κ-formulæ such that for any $\eta \in \mathsf{Ord}$ for any $v : {}^\eta 2 \to \mathsf{var}^\kappa$, $(\eta, \tau, v) \not\Vdash \Gamma$.*

Proof. Consider the following set of GL^κ-formulæ.

$$\Gamma' = \{\Diamond p_0\} \cup \{\Box(p_\alpha \to \Diamond p_\beta) : \alpha < \beta < \omega_1\}$$

Let $\Gamma = \Gamma' \cup \{\Diamond q, \Box\Box\neg q\}$. The set is consistent with GL^κ, moreover if $\mathsf{Ord}, \alpha \Vdash \Gamma'$ then $\mathrm{cof}\,(\alpha) \geq \omega_1$. Now $\mathsf{Ord}, \alpha \Vdash \Diamond q$ implies $A = [\![q]\!] \cap \alpha$ is an unbounded subset of α, let A' be the set of the limit points of A (i.e. $\beta \in A'$ if and only if $\sup A \cap \beta = \beta$), then A' is unbounded in α and $A' \subset [\![\Diamond q]\!]$, hence $\alpha \Vdash \Diamond\Diamond q$, a contradiction. $\square$

4. Strong completeness

This section comprises a proof of Theorem B. Namely, we for a GL-consistent set of $\mathcal{L}^\kappa$-formulæ, we show how to produce an κ-bouquet and an and a subset of $\kappa + 1$ with valuations of the propositional variables, and bouquets for the strong completeness the strong completeness of GL^κ. Although we gave a topological definition for κ-bouquets, one can construe them as a modification of Kripke frames. The following claim states the similarity of these structures in a precise way.

Claim 15. *Given a formula θ, let $(T,<)$ be a well-founded tree with the root r with the corresponding $(\mathcal{F},\kappa)$-bouquet (T,σ) and let $v : \mathsf{var}^\kappa \to P(T)$ be a valuation, then for any $x \in T$, $T, x \Vdash_v \theta$ if and only if:*

- $\theta = p \in \mathsf{var}$ *and* $x \in v(p)$, *or*
- $\theta = \varphi \wedge \psi$ *and* $T, x \Vdash \varphi$ *and* $T, x \Vdash \psi$, *or*
- $\theta = \neg\varphi$ *and* $T, x \nVdash \varphi$, *or*
- $\theta = \Diamond\varphi$ *and either*
  - $\deg(x)$ *is finite and* $T, y \Vdash \varphi \vee \Diamond\varphi$ *for some immediate successor* y *of* x, *or*
  - $\deg(x) = \lambda \geq \aleph_0$ *and the set* $\{i < \lambda : i = r(y) \wedge T, y \models \varphi \vee \Diamond\varphi\} \in (F_\lambda)^+$;

  or
- $\theta = \Box\varphi$ *implies*
  - $\deg(x)$ *is finite and* $T, y \models \varphi \wedge \Box\varphi$ *for all immediate successors* y *of* x, *or*
  - $\deg(x) = \lambda \geq \aleph_0$ *and* $\{i < \lambda : i = r(y) \wedge T, y \models \varphi \wedge \Box\varphi\} \in F_\lambda$;

Proof. The proof is by simultaneous induction on the formulæ complexity and the rank of x. The statement clearly holds for Booleans and the propositional variables, note that the statement for $\theta = \Box\psi$ follows from the statement for $\theta = \Diamond\varphi$.

Assume now $\theta = \Diamond\varphi$ and $x \Vdash \theta$ for some $x \in T$. Let $Y = \{y_i : i < \lambda\}$ enumerate all immediate successors of x according to e. For the sake of contradiction assume $\{i < \lambda : y_i \models \varphi \vee \Diamond\varphi\} \cap C = \emptyset$ for some $C \in F_\lambda$, thus $C \subset \{i < \lambda : y_i \Vdash \neg\varphi \wedge \Box\neg\varphi\}$. It follows that $y_i \Vdash \neg\varphi$ and there is a neighborhood U_i of y_i such that $U_i \setminus \{y_i\} \subset v(\neg\varphi)$. Thus, $\bigcup_{i\in C}\{y_i\} \cup U_i$ is a punctured neighborhood of y contained in $v(\neg\varphi)$, thus $x \Vdash \Box\neg\varphi$, a contradiction.

For the opposite implication, we fix an open neighborhood U of x and letting

$$Y_\varphi = \{i < \lambda : y_i \Vdash \varphi\}, \quad Y_{\Diamond\varphi} = \{i < \lambda : B, y_i \Vdash \Diamond\varphi\}$$

assume $Y_\varphi \cup Y_{\Diamond\varphi} \in (F_\lambda)^+$. If $Y_\varphi \in (F_\lambda)^+$, then $\{i < \lambda : y_i \in U \wedge y_i \Vdash \varphi\} \neq \emptyset$ and so $U \cap v(\varphi) \neq \emptyset$. If $Y_{\Diamond\varphi} \in F_\lambda$ then letting $U_i = U \cap R_{y_i}$, by the induction hypothesis we have $U_i \cap v(\varphi) \in (F_\lambda)^+$ whenever $i \in Y_{\Diamond\varphi} \cap \{i < \lambda : y_i \in U\}$. Thus, $U \cap v(\varphi) \neq \emptyset$. □

The next lemma is a key step for binding the strong κ-bouquet completeness with the strong topological completeness. κ-bouquet for Γ shall be obtained by bundling smaller bouquet who satisfy certain subsets of Γ. Thus, if we know how to build d-maps for these smaller bouquets, we shall be able to produce the ordinal space that embeds into the κ-bouquet.

Lemma 16. *Let κ be an uncountable cardinal and let $\{(B_j, <_j, e_j)\}_{j<\kappa}$ be a sequence of pairwise disjoint $<\kappa$-bouquets. For each $j < \kappa$ we let $\lambda_j < \kappa$ be a limit ordinal, and $\lambda_j \in \Lambda_j \subset \lambda_j + 1$ be such that there is a d-map $f_j : (\Lambda_j, \tau_\iota) \to (B_j, <_j)$. Let $(B, <, e)$ be a κ-bouquet such that $B = \{r\} \cup \bigcup_{j<\kappa} B_i$, for $b_1, b_2 \in B$, $b_1 < b_2$ if and only if*

- $b_1 = r$ *and* $b_2 \in B_j$ *for some* $j < \kappa$; *or*
- $b_1, b_2 \in B_j$ *for some* j *and* $b_1 <_j b_2$;

$e(b) = 0$ if $b = r$ and $e(b) = e_j(b)$ is $b \in B_i$ for some j. Then there is an ordinal λ of cardinality κ and $\lambda \in \Lambda \subset \lambda + 1$ such that there is a d-map $f : \Lambda \to B$.

Proof. Let $\lambda = \sum_{j<\kappa}(\lambda_j + 1)$, let

$$h_j : \lambda_j + 1 \to \lambda : \alpha \mapsto \sum_{i<j} \lambda_j + 1 + \alpha$$

be the isomorphic embedding. Note that $h_j : (\lambda_j+1, \tau_\iota) \to (\lambda, \tau_\iota)$ is rank preserving for each $j < \kappa$. Since the map is injective, it is obviously a d-map. Letting $\Lambda = \{\lambda\} \cup \bigcup_{j<\kappa} h_j[\Lambda_j]$, then we set

$$f : \alpha \mapsto \begin{cases} f_j(\beta), & \alpha = h_j(\beta); \\ r, & \alpha = \lambda; \end{cases}$$

Since h_j, f_j are d-maps for each $j < \kappa$, then clearly f is open continuous at every point of $\Lambda \setminus \{\lambda\}$ and for each $b \in B \setminus \{r\}$, $f^{-1}[\{b\}]$ is discrete. Note also that $f^{-1}[\{b\}] = \{\lambda\}$, and so f is pointwise discrete.

The last part is to check openness and continuity for the neighborhoods of λ. Take $U = (\gamma; \lambda] \cap \Lambda$ for some $\gamma < \lambda$, let $i = \min\{j : \lambda_j > \gamma\}$, then the image is a union $f[U] = f\big[(\gamma; \lambda_i] \cap \Lambda\big] \cup f\big[\Lambda \setminus (\lambda_i + 1)\big]$. Since $f\big[(\gamma; \lambda_i] \cap \Lambda\big]$ is open by the hypothesis that f_i is open, whereas $f\big[\Lambda \setminus (\lambda_i + 1)\big] = \{r\} \cup \bigcup_{j>i} T_j$ is open in B. Now given a basic open neighborhood V of the root of B, then $V = \{r\} \cup \bigcup_{j>i} T_i$ for some $i < \kappa$, thus $f^{-1}[V] = \Lambda \setminus (\lambda_i + 1)$ is open. □

Lemma 17. *Let κ be an uncountable cardinal and Γ be a maximal GL-consistent set of $\mathcal{L}^\kappa$-formulæ, there are*

(1) κ-bouquet $(B, <, e)$ with the root r and a valuation $v : \mathsf{var}^\kappa \to {}^B 2$ such that $B, r \Vdash_v \Gamma$; and
(2) $\kappa \in K \subset \kappa + 1$ and a d-map $f : (K, \tau_\iota) \to (B, \sigma_<)$ with $f(\kappa) = r$, thus $K, \kappa \Vdash_{v'} \Gamma$, where $v'(p) = f^{-1}[v(p)]$ for each $p \in \mathsf{var}^\kappa$;

Proof. The proof by induction on κ. If $\kappa = \omega$, then the claim follows from Fact 12. Fix $\kappa > \aleph_0$ and a maximal GL-consistent set Γ of $\mathcal{L}^\kappa$-formulæ. Let $\Phi = \{\varphi : \Box\varphi \in \Gamma\} = \{\varphi_i\}_{i<\kappa}$, $\Psi = \{\psi : \Diamond\psi \in \Gamma\} = \{\psi_i\}_{i<\kappa}$ and for each $\psi \in \Psi$, the set $\{i < \kappa : \psi = \psi_i\}$ is cofinal in κ. For every $i < \kappa$ we let $\Gamma(i) = \{\psi_i\} \wedge \{\varphi_j \wedge \Box\varphi_j\}_{j<i}$ whenever $i < \kappa$, then $|\Gamma(i)| = |i+1| < \kappa$. One can see that for every $i < \kappa$, $\Gamma \vdash \Diamond \bigwedge \Delta$ for any finite $\Delta \subset \Gamma(i)$, and so $\Gamma(i)$ is consistent. By the induction hypothesis, there is one $|i|$-bouquet $(B_i, <_i, v_i, e_i)$ with the root r_i such that $r_i \Vdash \Gamma(i)$. Now let $(B, <, v, e)$ be a κ-bouquet with the root r such that $B = r \cup \bigcup_{i<\kappa} B_i$, for $b_1, b_2 \in B$, $b_1 < b_2$ if and only if

- $b_1 = r$ and $b_2 \in B_i$ for some i; or
- $b_1, b_2 \in B_i$ for some i and $b_1 <_i b_2$;

and $e(b) = e_i(b)$ if $b \in B_i \setminus \{r_i\}$, $e(b) = i$ if $b = r_i$ and $e(b) = 0$ if $b = r$. It is straightforward to check that $B, r \Vdash \Gamma$. Note that by the second part of the induction hypothesis, there is $|i| \in I \subset |i| + 1$ with a d-map $f_i : (I, \tau_\iota) \to (B_i, <_i)$. Thus, Lemma 16, there is $\kappa \in K \subset \kappa + 1$ and a d-map $f : (K, \tau_\iota) \to (B, \sigma_<)$ with $f(\kappa) = r$. □

Corollary 18 (Strong completeness)**.** *Let κ be an uncountable cardinal, then GL^κ is strongly complete with respect to the class of κ-bouquets and to the class $\{(K, \tau_\iota) : \kappa \in K \subset \kappa + 1\}$.*

5. Countable bouquets

In this section we show a finer strong completeness result for GL^ω, namely for each GL-consistent set of formulæ Γ and countable ordinal β we find an ω-bouquet $(T,<)$ with the root r and a valuation $v : \mathsf{var} \to P(T)$, such that $(T,<), r \Vdash_v \Gamma$ and a valuation $\nu : \mathsf{var} \to P(\omega^\beta + 1)$ such that $(\omega^\beta + 1, \tau_\iota), \omega^\beta \Vdash_\nu \Gamma$

Definition 19. Let β be a countable ordinal. We call an ω-bouquet $(T,<)$ a (ω,β)-bouquet if the rank of its root r is β.

Lemma 20. *Let Γ be a consistent set of GL-formulæ such that $\{\Diamond^n \top : n < \omega\} \subset \Gamma$ and let α be a countable ordinal, then there are*

(1) an (ω,α)-bouquet $(T,<)$ with the root r and a valuation $v : \mathsf{var} \to {}^T 2$, such that $(T,<), r \Vdash_v \Gamma$;

(2) a sujective d-map $j : \omega^\alpha + 1 \to T$ such that $j(\omega^\alpha) = r$;

Proof. The proof is by induction on α. If $\alpha = \omega$ the statement follows from Fact 12. Let $\alpha = \beta + 1$. Take $\Gamma_\omega = \{\psi_i \vee \Diamond\psi_i\}_{i<\omega} \cup \{\varphi_i \wedge \Box\varphi_i\}_{i<\omega}$. If Γ_ω is consistent and $\{\Diamond^k \top : k < \omega\} \subset \Gamma_\omega$, then we can apply the induction hypothesis on β to Γ_ω and get an (ω,β)-bouquet $(T_\beta, <_\beta)$ with the root r_β and a valuation $v_\beta : \mathsf{var} \to {}^{T_\beta}2$, such that $(T_\beta, <_\beta), r_\beta \Vdash_{v_\beta} \Gamma_\omega$. Then letting $T = \{r\} \cup T_\beta$, $<$ be such that for all $x, y \in T$, $x < y$ if $x, y \in T_\beta$ and $x <_\beta y$ or $x = r$ and $y \in T_\beta$ and for all $p \in \mathsf{var}$, $v(p) = v_\beta(p) \cup \{r : p \in \Gamma\}$, we have an (ω,β)-bouquet $(T,<)$ with the root r and a valuation $v : \mathsf{var} \to {}^T 2$, such that $(T,<), r \Vdash_v \Gamma$. Hence, it is left to prove the following claim.

Claim 21. *Γ_ω is consistent and contains $\{\Diamond^k \top : k < \omega\}$.*

Proof. Since for any k, $\Diamond^{k+1}\top \in \Gamma$ it follows that $\Diamond^k \top \vee \Diamond^{k+1} \in \Gamma_\omega$, which proves the former and the latter statements. As for consistency, we show that any finite $\Gamma' \subset \Gamma_\omega$ is consistent from the fact that $\Gamma \vdash \Diamond \bigwedge \Gamma'$. Assume that $\Gamma \vdash \Box \bigvee \neg\Gamma'$. Note that $\Gamma \vdash \Box\varphi_i$, $\Gamma \vdash \Box\Box\varphi_i$ for each $i < \omega$. Then, $\Gamma \vdash \Box \bigwedge_{\varphi_i \wedge \Box\varphi_i \in \Gamma'} (\varphi_i \wedge \Box\varphi_i)$ let $\Delta_1 = \{\varphi_i \wedge \Box\varphi_i\}_{\varphi_i \wedge \Box\varphi_i \in \Gamma'}$ and $\Delta_2 = \{\varphi_i \vee \Diamond\psi_i\}_{\psi_i \vee \Diamond\psi_i \in \Gamma'}$, then $\Gamma \vdash \Box \bigwedge \Delta_1$ and $\Gamma \vdash \Box(\bigvee \neg\Delta_1 \vee \bigvee \neg\Delta_2)$, which results to $\Gamma \vdash \neg\Diamond(\bigwedge \Delta_2)$, which contradicts consistency of Γ. $\square$

This proves (1) for successor steps.

Now, by (2) of the induction hypothesis we have a d-map $j_\beta : \omega^\beta + 1 \to T_\beta$. Fix a cofinal in ω^β sequence $\langle \delta_i \rangle_{i<\omega}$ and define $j : \omega^\alpha + 1 \to T$ by

$$j(x) = \begin{cases} j_\beta(\delta_i + \alpha), & x = \omega^\beta \cdot i + 1 + \alpha,\ \alpha \leq \omega^\beta,\ i < \omega; \\ r, & x = \omega^\alpha; \end{cases}$$

Since j_β is a d-map, each open neighborhood of $x \in T_\beta$ is open in $(\omega^\alpha + 1, \tau_\iota)$, as well as each open neighborhood of $\gamma < \omega^\alpha$ is open in T, moreover j is pointwise disctrete which follow from j_β being pointwise discrete and $j^{-1}[\{r\}] = \{\omega^\alpha\}$. Now note that if $V \subset T$ is a neighborhood of r in B, then it is of the form $\{r\} \cup U$ where U is a neighborhood of r_β – the root of T_β. $j_\beta^{-1}[U]$ contains $(\delta; \omega^\beta]$ for some $\delta < \omega^\beta$. Since $\{\delta_i\}_{i<\omega}$ is cofinal in ω^β, there is n, such that $\delta_i > \delta$ for all $i > n$. Hence, $(\delta_{n+1}; \omega^\alpha) \subset j^{-1}[U]$ and $(\delta_{n+1}; \omega^\alpha] \subset j^{-1}[V]$. Now if one take an open basic neighborhood of ω^α given by $(\delta, \omega^\alpha]$ for some $\delta < \omega^\alpha$, then $\{r\} \cup U \subset j[(\delta, \omega^\alpha]]$ for some U open in T_β. Hence j is indeed a d-map

If α is limit, we fix a cofinal sequence $\{\alpha_n\}_{n<\omega}$. For each $n < \omega$ we set $\Gamma(n) = \{\psi_n\} \cup \{\varphi_i \wedge \Box\varphi_i\}_{i<n}$, for each $n < \omega$, $\Gamma(n)$ is consistent. Let $\Gamma'(n)$ be a maximal consistent extension with $\{\Diamond^k\top : k < \omega\} \subset \Gamma'(n)$ if $\{\Diamond^k\top : k < \omega\}$ is consistent with $\Gamma(n)$ and some maximal consistent extension otherwise. Let $N = \{n < \omega : \{\Diamond^i\top : i < \omega\}$ is consistent with $\Gamma(n)\}$.

Claim 22. *The set N is infinite.*

Proof. Fix $n, l \leq \omega$ and assume that

$$\mathsf{GL} \vdash \psi_n \wedge \bigwedge_{i \leq n} \varphi_i \wedge \Box\varphi_i \to \Box^l\bot$$

$$\mathsf{GL} \vdash \Box(\psi_n \wedge \bigwedge_{i \leq n} \varphi_i \wedge \Box\varphi_i \to \Box^l\bot)$$

$$\mathsf{GL} \vdash \Box(\psi_n \wedge \bigwedge_{i \leq n} \varphi_i \wedge \Box\varphi_i) \to \Box^{l+1}\bot$$

now since $\Diamond^{l+1}\top \in \Gamma$ it follows that $\Diamond(\neg\psi_n \vee \bigvee_{i \leq n} \neg\varphi_i) \in \Gamma$. By construction, $\Box\varphi_i \in \Gamma$ for each $i < \omega$, it follows $\Diamond\neg\psi_i \in \Gamma$, then $\neg\psi_i = \psi_m$ for some $m < \omega$. We claim that $\Gamma(m)$ is consistent with $\{\Diamond^k\top : k < \omega\}$, otherwise for some k

$$\mathsf{GL} \vdash \psi_m \wedge \bigwedge_{i \leq m} \varphi_i \wedge \Box\varphi_i \to \Box^k\bot$$

$$\mathsf{GL} \vdash \neg\psi_i \wedge \bigwedge_{i \leq m} (\varphi_i \wedge \Box\varphi_i) \to \Box^k\bot$$

it follows then that

$$\mathsf{GL} \vdash \bigwedge_{i \leq n'} (\varphi_i \wedge \Box\varphi_i) \to \Box^{l'}\bot$$

$$\mathsf{GL} \vdash \bigwedge_{i \leq n'} (\Box\varphi_i \wedge \Box\Box\varphi_i) \to \Box^{l+1}\bot$$

for $n' = \max(n, m)$ and $l' = \max(l, k)$, then $\Box^{l'+1} \in \Gamma$, a contradiction. Thus, there are infinitely many n such that $\Gamma(n)$ is consistent with $\{\Diamond^k\top : k < \omega\}$. $\Box$

Let $N = \{n_i : i < \omega\}$ be the order preserving enumeration of N. For each $n < \omega$ we let $(T_n, <_n, v_n)$ be an (ω, α_i)-bouquet with the root r_n such that $T_n, r_n \Vdash \Gamma(n)$ if there is i such that $n_i = n$ and $(T_n, <_n, v_n)$ be some bouquet with the root r_n such that $T_n, r_n \Vdash \Gamma(n)$ otherwise.

We let $(T, <, v)$ be the (ω, α)-bouquet with the root r, where $T = \{r\} \cup \bigcup_n T_n$ such that for each $x, y \in T$, $x < y$ if there is n such that $x, y \in T_n$ and $x <_n y$ or $x = r$ and for each $p \in \mathsf{var}$ we set $v(p) = \{r : p \in \Gamma\} \cup \bigcup_n v_n(p)$. One can see that $T, r \Vdash_v \Gamma$.

Similarly, from (2) of the induction hypothesis, there is a d-map $j_n : T_n \to \omega^{\beta_n}+1$ for each n, moreover if there is i such that $n = n_i$, then $\beta_n = \alpha_i$ and $\beta_n < \alpha$ otherwise. Hence $\sum_n \omega^{\beta_n} = \omega^\alpha$. Now we define the map $j : \omega^\alpha + 1 \to T$ as follows:

$$j(\gamma) = \begin{cases} j_n(\delta) & \gamma = \sum_{k<n} \omega^{\beta_k} + 1 + \delta,\ \delta \leq \omega^{\beta_n} \\ r, & \gamma = \omega^\alpha; \end{cases}$$

The proof that j is a d-map is the same as for the case of α being successor. $\Box$

Corollary 23. *For each countable β, $\mathsf{GL} + \{\Diamond^k\top : k < \omega\}$ is locally complete with respect to:*

(1) *the class (ω, β)-bouqets;*
(2) *the class $\{(\omega^\beta + 1, \tau_\iota)\}$;*

6. The logic GL.3

We introduced the notion of bouquet relativized to the filter we use to evaluate validity of modal formulæ in the nodes with infinite degree, however the main results employed only bouquets with cobounded filters. In this section we make use ultrafilters instead to obtain the strong completeness of GL.3. Note that without the Axiom of Choice there may be no non-principle ultrafilters on ω, so the results in this section are independent of ZF.

Definition 24. The logic GL.3 is the minimal set of formulæ closed under modus ponens and containing $\mathsf{GL} + \Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \wedge \psi \to \varphi)$.

Fact 25. GL.3 *is sound and complete with respect to the class of finite strict linear orders or* lines.

Claim 26. GL.3 *is not strongly complete with respect to the class of finite strict linear orders.*

Proof. See Example (∗). □

In the light of last claim we introduce semantics which is a modification of the notion of bouquet.

Definition 27. We call a $(\mathcal{F}, \kappa)$-bouquet B *linear* if each node of the underlying tree has zero, one or infinitely many immediate successors. If additionally each $F \in \mathcal{F}$ is a non-principal ultrafilter, we call such bouquet an *ultralinear κ-bouquet*.

Note that we do nor write 'ultralinear $(\mathcal{F}, \kappa)$-bouquet', since the following proofs do only rely on the fact that the filters are non-principal ultrafilters.

Lemma 28. *The logic* GL.3 *is sound with respect to the class of ultralinear κ-bouquets.*

Proof. GL is obviously sound, since a refinement of a scattered space is scattered. We have to show that $B \models \Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \wedge \psi \to \varphi)$ whenever B is an ultralinear ω-bouquet. From here on we reason by induction on the rank of the node $x \in B$ (see Definition 10).

Case I. x has one immediate successor y and by the induction hypothesis on y we have the following cases:

(1) $y \Vdash \Box(\Box\varphi \to \psi)$ and $y \Vdash \neg\Box\varphi$;
(2) $y \Vdash \Box(\Box\varphi \to \psi)$ and $y \Vdash \Box\varphi$ and $y \Vdash \psi$;
(3) $y \Vdash \Box(\Box\varphi \to \psi)$ and $y \Vdash \Box\varphi$ and $y \Vdash \neg\psi$;
(4) $y \Vdash \Box(\Box\psi \wedge \psi \to \varphi)$ and $y \Vdash \neg\Box\psi \vee \neg\psi$;
(5) $y \Vdash \Box(\Box\psi \wedge \psi \to \varphi)$ and $y \Vdash \Box\psi \wedge \psi$ and $y \Vdash \varphi$;
(6) $y \Vdash \Box(\Box\psi \wedge \psi \to \varphi)$ and $y \Vdash \Box\psi \wedge \psi$ and $y \Vdash \neg\varphi$;

One can see that in cases (1), (2), (6) $x \Vdash \Box(\Box\varphi \to \psi)$ and otherwise $x \Vdash \Box(\Box\psi \wedge \psi \to \varphi)$.

Case II. x has λ-many successors $\{y_i\}_{i<\lambda}$ where $\lambda \leq \kappa$. The same case distinction on y_i for U_λ-a.e. i completes the argument.

□

As was mentioned before, $\mathsf{GL.3}$ is complete with respect to the set of strict finite linear orders, thus each frame is of the form $(n,<)$, where $<$ is the strict ordering of the natural numbers.

Theorem 29. *The logic* $\mathsf{GL.3}$ *is strongly complete with respect to the class of ultralinear ω-bouquets.*

Proof. Let Γ be (without loss of generality maximal) $\mathsf{GL.3}$ consistent set of formulæ. We let $\langle\varphi_i\rangle_{i<\omega}$ enumerate all formulæ φ such that $\Box\varphi \in \Gamma$ and $\langle\psi_i\rangle_{i<\omega}$ enumerate all formulæ ψ such that $\Diamond\psi \in \Gamma$, let $\Gamma(n) = \{\varphi_i \wedge \Box\varphi_i\}_{i<n} \cup \{\psi_i \vee \Diamond\psi_i\}_{i<n}$, whenever $n < \omega$.

Claim 30. $\Gamma(n)$ *is consistent with* $\mathsf{GL.3}$.

Proof. Assume it is not, then

$$\mathsf{GL.3} \vdash \bigwedge_{i<n} \varphi_i \wedge \Box\varphi_i \to \bigvee_{i<n} \neg\psi_i \wedge \Box\neg\psi_i.$$

Applying normality and transitivity, we get

$$\mathsf{GL.3} \vdash \bigwedge_{i<n} \Box\varphi_i \to \Box \bigvee_{i<n} \neg\psi_i \wedge \Box\neg\psi_i.$$

Trivially,

$$\mathsf{GL.3} \vdash \bigwedge_{i<n} \Diamond\psi_i \to \bigwedge_{i<n} \Diamond\psi_i$$

Combining, we get

$$\mathsf{GL.3} \vdash \bigwedge_{i<n} \Diamond\psi_i \wedge \bigwedge_{i<n} \Box\varphi_i \to \bigwedge_{i<n} \Diamond\psi_i \wedge \Box \bigvee_{i<n} \neg\psi_i \wedge \Box\neg\psi_i.$$

The antecedent belongs to Γ, a contradiction. □

Now that $\Gamma(n)$ is consistent for each $n < \omega$, there is a valuation $v_n : \mathsf{var} \to P(n)$ such that $(n,<,v_n), 0 \Vdash \Gamma(n)$, we let $(L_n, <_n, \nu_n)$ to be a copy of $(n,<,v_n)$ for each n with r_n being a copy of 0. We define L to be an ultralinear bouquet with the root r by letting $L = \{r\} \cup \bigcup_n L_n$ and for each $x, y \in L$, we set $x < y$ if $x, y \in L_n$ for some n and $x <_n y$ or $x = r$ and $y \neq r$, and for each $p \in \mathsf{var}$, $v(p) = \{r : p \in \Gamma\} \cup \bigcup_n v_n(p)$. Now, if we apply Claim 15 to ultrafilters, we can see that $r \Vdash \Box\theta$ if and only if $r_n \Vdash \theta \wedge \Box\theta$ for almost all n and $r \Vdash \Diamond\theta$ if and only if $r_n \Vdash \theta \vee \Diamond\theta$ for almost all n. Thus, it is easy to see that $L, r \Vdash_v \Gamma$. □

Claim 31. *It is consistent with* ZF*, that* $\mathsf{GL.3}$ *is not strongly complete with respect to the class of linear $(\{U\},\omega)$-bouquets for any filter U on ω.*

Proof. It is consistent with ZF that there is no non-principal ultrafilters on ω, thence U is either principle or not an ultrafilter. The former case is clear by Claim.. G_δ. Whereas the latter case means that there is a $B \subset \omega$ such that $B \notin U$ and for any $A \in U$, $B \cap A \neq \emptyset$ (a set of positive measure but not measure one). We build a bouquet $(T,<,v)$ as follows. We define a model $(T,<,v)$ as follows. Let $T = \{r\} \cup \{a_i : i < \omega\}$ and for each $x, y \in T$, we put $x < y$ whenever $x = r$ and $y = a_i$ for some i and let v be such that $v(p) = \{a_i : i \in B\}$. Then $T, r \Vdash_v \Box\Box\bot \wedge \Diamond p \wedge \Diamond\neg p$, which is inconsistent with $\mathsf{GL.3}$. □

Now we state theorems similar to those in Section 4. Letting similarly $\mathsf{GL.3}^\kappa$ be the logic $\mathsf{GL.3}$ over κ many variables.

Theorem 32. $\mathsf{GL.3}^\kappa$ *is strongly complete with respect to the class of ultralinear* $(\mathcal{U},\kappa)$*-bouquets with* $\mathcal{U} = \{U_\lambda : \lambda \leq \kappa \wedge \lambda \text{ is a cardinal}\}$*, where* U_λ *is an ultrafilter on* λ*.*

Likewise, for a given countable ordinal β, we say that B is an ultralinear $\{\{U\},\omega,\beta\}$-bouquet, if it is a $\{\{U\},\omega\}$-bouquet and the rank of the root is β.

Theorem 33. *For each countable* β*,* $\mathsf{GL} + \{\Diamond^k\top : k < \omega\}$ *is locally complete with respect to:*

(1) the class (ω,β)*-bouqets;*
(2) the class $\{(\omega^\beta + 1, \tau_\iota)\}$*;*

The proofs for these two theorems are exactly as in Section 4.

Note also that in the current section we do not provide any results regarding ordinal completness, because if we were to pullback the ultralinear topology to the ordinals, it wouldn't correspond to any natural topology on ordinals. However in [AS24b] the author and Aguilera present some nice completness results regarding the measurable topology and the club topology.

7. Cardinal Characteristics

In this section we want to point out an interesting observation regarding the strong completeness and set theory.

Definition 34. By $[\omega]^\omega$ we denote the set of all infinite subsets of ω. We call $A \subset [\omega]^\omega$ an *almost disjoint family* if for any $a, b \in A$, the set $a \cap b$ is finite, we call an almost disjoint family A *maximal*, if for any $c \in [\omega]^\omega$, there is $a \in A$ such that $c \cap a$ is infinite. We let

$$\mathfrak{a} = \min\{|A| : A \text{ is an almost disjoint family}\}$$

to be the *almost disjointness number*.

It is well known, that the cardinality of $\omega_1 \leq \mathfrak{a} \leq \mathfrak{c}$, where $\mathfrak{c}$ is the cardinlaity of the continuum. Moreover, the precise size of $\mathfrak{a}$ is independent from ZFC, namely there are models of ZFC with $\mathfrak{a}$ having different size.

Let now $\mathcal{L}^\omega_\alpha$ be the closure of $\mathcal{L}^\omega$ over conjunctions and disjunctions of length α. Let $\mathcal{B}$ be the class of all ω-bouquets.

Proposition 35. *If* α *is an uncountable cardinal, then the* $\mathcal{L}^\omega_\alpha$*-logic of* ω*-bouquets is undecidable in* ZFC*, i.e. there is a* $\mathcal{L}^\omega_\alpha$*-formula* φ*, such that* ZFC *cannot prove or refute that* $\forall B \in \mathcal{B}, B \models \varphi$*.*

Proof. First, we note that the formula

$$\varphi_0 = \Box\Box\bot \wedge \bigwedge_{i<\alpha} \Diamond p_i \wedge \bigwedge_{i<j<\alpha} \neg\Diamond(p_i \wedge p_i)$$

is satisfied at the root of an ω-bouquet, the bouquet is of the form $\{r\} \cup \{x_k : k < \omega\}$ and $r < x_k$ for all k and the set $\{v(p_i) : i < \alpha\}$ is an almost disjoint family. Now we want to force it to be maximal. We want to say that no $v(q)$ with $r \Vdash \Diamond q$, can be almost disjoint from $v(p_i)$ whenever $i < \alpha$. Here we need an infinite disjunction

$$\varphi_1 = \bigvee_{i<\alpha} \neg\Diamond(p_i \wedge q)$$

Thus if there is a maximal disjoint family of size α, then there is $B \in \mathcal{B}$ and $v : \mathsf{Vars} \to P(B)$ such that $B, r \Vdash_v \varphi_0 \wedge \varphi_1$ and if there no such family, then $\varphi_0 \wedge \varphi_1$ cannot be satisfied at any ω-bouquet. □

Although the result is quite straightforward, we want to address the following question:

Question 36. *Is there a natural non-classical (e.g. intutionistic or many-valued) version of the logic* GL*, whose strong completeness with respect to some sensible generalization of the class of* ω*-bouquets is undecidable by* ZFC*?*

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