NOTES ON THE PROJECTIVE ORDINALS

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Contents

1. Introduction

Definition 1 (Projective ordinal)**.** For each $n < \omega$ we set

$$\boldsymbol{\delta}^1_n = \{\alpha : \alpha \text{ is the length of a } \boldsymbol{\Delta}^1_n\text{-prewellordering of } {}^\omega\omega\}$$

The main goal of these notes is to establish:

Fact 2. *For any $n < \omega$.*

(1) $\boldsymbol{\delta}^1_n$ *is a regular, measurable cardinal;*
(2) $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$*;*
(3) $\boldsymbol{\delta}_{2n+1} = \kappa^+_{2n+1}$*, where κ_{2n+1} is a cardinal of cofinality ω;*

First we recall the Coding Lemma, which is one of the main instruments we employ here. We use the following useful statement as it was given in [Kec11], a more general statmement can be found in [Jac10].

Definition 3. Let $\leq$ be a prewellordering of (a subset of) the reals with $|\leq| = \xi$ and the corresponding norm $\varphi : \mathrm{Field}(\leq) \to \mathsf{Ord}$ and let $f : \xi \to P({}^\omega\omega)$ be a function. Then

$$\mathrm{Code}(f, \leq) = \{(a, b) \in {}^\omega\omega \times {}^\omega\omega : a \in \mathrm{Field}(\leq) \wedge b \in f(\varphi(a))\}$$

Similarly, if $A \in {}^n\xi$, then

$$\mathrm{Code}(A, \leq) = \{\vec{a} \in {}^n({}^\omega\omega) : (\varphi(a_0), \dots, \varphi(a_{n-1})) \in A)\}$$

Fact 4 (The Coding Lemma; Moschovakis)**.** *Let $\leq$ be a $\boldsymbol{\delta}^1_n$ prewellordering of a subset of ${}^\omega\omega$ of length ξ, and let $f : \xi \to P({}^\omega\omega)$ be some function. Then there is a function $g : \xi \to P({}^\omega\omega)$ such that* $\mathrm{Code}(g, \leq)$ *is $\boldsymbol{\Sigma}^1_n$ such that $\forall \eta < \xi(g(\eta) \subset f(\eta))$ and $\forall \eta < \xi(f(\eta) \neq \emptyset \to g(\eta) \neq \emptyset)$, we call such function a* choice subfunction.

Proof. See [Kec11]. □

Corollary 5. *If $\leq$ is a $\mathbf{\Delta}^1_n$ prewellordering of a subset of the reals with $|\leq| = \xi$ and the corresponding norm $\varphi : \mathrm{Field}(\leq) \to \mathsf{Ord}$. Then for any $A \subset {}^m\xi$, $\mathrm{Code}(A, \leq)$ is $\mathbf{\Delta}^1_n$.*

Proof. For simplicity assume $m = 1$. We pick two distinct reals a_0, a_1 and let

$$f : \xi \to P({}^\omega\omega) : \eta \mapsto \begin{cases} \{a_0\}, & \text{if } \eta \in A; \\ \{a_1\}, & \text{otherwise}; \end{cases}$$

By The Coding Lemma there is a choice subfunction $g : \xi \to P({}^\omega\omega)$ of f with $\mathrm{Code}(g, \leq) \in \mathbf{\Sigma}^1_n$, in fact the only choice subfunction of f is f and so $\mathrm{Code}(f, \leq) \in \mathbf{\Sigma}^1_n$. It follows that

$$\varphi(a) \in A \iff (a_0, a) \in \mathrm{Code}(f, \leq) \iff a \in \mathrm{Field}(\leq) \wedge (a_1, a) \notin \mathrm{Code}(f, \leq).$$

□

Fact 6. *For each $n < \omega$, $\boldsymbol{\delta}^1_n$ is a cardinal.*

Proof. Assume there is a bijection $f : \xi \to \boldsymbol{\delta}^1_n$. For $\alpha, \beta < \xi$ we let $\alpha <_* \beta$ if $f(\alpha) < f(\beta)$. Let $\prec$ be a $\mathbf{\Delta}^1_n$-prewellordering of ${}^\omega\omega$ of rank ξ. Let $F : {}^\omega\omega \to \xi$ be the corresponding norm. We let $P \subset {}^\omega\omega \times {}^\omega\omega$ be such that $P(x, y)$ if and only if $F(x) <_* F(y)$. Obviously P yields a prewellordering of the reals of length $\boldsymbol{\delta}^1_n$. By the Coding Lemma there a $\mathbf{\Delta}^1_n$ choice subfunction Q that gives $\mathbf{\Delta}^1_n$ prewellordering of length $\boldsymbol{\delta}^1_n$. A contradiction. □

1.1. **Successors.** The goal of this subsection is to prove the second clause of the theorem.

Definition 7. Let $A \subset {}^\omega\omega$, we call a map $\varphi : A \to \mathsf{Ord}$ a *norm*. The length of φ is the length of the prewellordering on A induced by φ. We say that φ is a Γ-norm if $\leq^*_\varphi$ and $<^*_\varphi$ are in Γ, i.e.

$$a \leq^*_\varphi b \iff a \in A \wedge [b \notin A \vee \varphi(a) \leq \varphi(b)]$$

$$a <^*_\varphi b \iff a \in A \wedge [b \notin A \vee \varphi(a) < \varphi(b)]$$

If any $A \in \Gamma$ has a Γ-norm, we say that Γ has the *prewellordering property.*

Fact 8 (Martin)**.** *For all $n \geq 0$, $\mathbf{\Pi}^1_{2n+1}, \mathbf{\Sigma}^1_{2n+2}$ have the scale and the prewellordering property, whereas $\mathbf{\Pi}^1_{2n+2}, \mathbf{\Sigma}^1_{2n+1}$ do not.*

Definition 9. Let $A, B \subset {}^\omega\omega$, we say that $A \leq_W b$ if there is a continuous function $f : {}^\omega\omega \to {}^\omega\omega$, such that for all $x \in {}^\omega\omega$, $x \in A \leftrightarrow f(x) \in B$.

Fact 10 (AD)**.** *For any $A, B \subset {}^\omega\omega$, either $A \leq_W B$ or $B \leq_B {}^\omega\omega \setminus A$.*

Proof. For $A, B \subset {}^\omega\omega$ we define a game $G_W(A, B)$, where Player I plays a real x, Player II plays a real y. II wins if $x \in A \leftrightarrow y \in B$. If II has a winning strategy τ, then we have $\tau(x) \in B \leftrightarrow x \in A$, and so $A \leq_W B$. Otherwise, let σ be a winning strategy for I, then $y \in B \leftrightarrow \sigma(x) \notin A$ and so $B \leq_W {}^\omega\omega \setminus A$. □

Corollary 11. *For any closed under continuous preimages Γ, any $A \in \Gamma \setminus \Delta$ is Γ-complete.*

Theorem 12 (Moschovakis)**.** *If φ is a $\mathbf{\Pi}^1_{2n+1}$-norm on a $\mathbf{\Pi}_{2n+1}$-set, then the length of φ is $\boldsymbol{\delta}^1_{2n+1}$.*

Definition 13. A *scale* is on a set $A \subset {}^\omega\omega$ is a sequence of norms $\{\varphi_i\}_{i<\omega}$ on A such that for every sequence $\{a_i\}_{i<\omega} \in {}^\omega A$, if

(1) $\lim_{i\to\infty} a_i = a$, and
(2) for each n there is an ordinal λ_n such that $\varphi_n(a_i)$ is eventually λ_n.

Then $a \in A$ and for all n, $\varphi_n(a) \leq \lambda_n$. The scale $\{\varphi_i\}_{i<\omega}$ is a λ-scale if $|\varphi_i| < \lambda$ and it is a $\mathbf{\Gamma}$-scale if $<^*_{\varphi_n}$ and $\leq^*_{\varphi_n}$ as predicates on $\omega \times {}^\omega\omega \times {}^\omega\omega$ are in $\mathbf{\Gamma}$.

Definition 14. Let X be a set we say that $T \subset {}^{<\omega}X$ is a *tree* if it is closed under initial segments, namely $s \in T$ implies $s\restriction n \in T$ for any $n < |s|$. We say that $b \in {}^\omega X$ is a branch through T, if for all $n < \omega$, $b\restriction n \in T$, we denote by $[T]$ the set of all branches through T. If $[T]$ is empty, we say that T is wellfounded, we say that T is illfounded otherwise. Let T be a tree on $\omega \times X$[1], *projection of the tree* T is the set $p[T] = \{a \in {}^\omega\omega : \exists b \in {}^\omega X, (a, b) \in [T]\}$.

If $\{\varphi_i\}_{i<\omega}$ is a λ-scale on A, then we define the *tree assosoated with the scale* is the tree on $\omega \times \lambda$ defined by

$$((k_0, \ldots, k_n), (\xi_0, \ldots, \xi_n)) \in T \iff \exists a \in A,\ \forall i \leq n(a(i) = k_i \wedge \phi_i(a) = \xi_i)$$

Fact 15. *For A, T, as above $p[T] = A$.*

Proof. Obviosly, $A \subset p[T]$. Now for $a \in p[T]$ we let $f : \omega \to \lambda$ such that $(a, f) \in [T]$. Thence $\forall n, (a\restriction n, f\restriction n) \in T$, which is witnessed by a sequence a_n that witnesses $a \in A$. $\square$

Definition 16. $A \subset {}^\omega\omega$ is λ-*Suslin*, if there is T on $\omega \times \lambda$ such that $A = p[T]$

Note that if a set has a λ-scale, then it is λ-Suslin.

Fact 17. *For each $n < \omega$,*

- *every $\mathbf{\Sigma}^1_{2n+2}$-set is $\boldsymbol{\delta}^1_{2n+1}$-Suslin;*
- *every $\mathbf{\Sigma}^1_{2n+1}$-set is κ_{2n+1}-Suslin, where κ_{2n+1} is a cardinal below $\boldsymbol{\delta}^1_{2n+1}$.*

Proof. Let $\langle , \rangle : \omega \times \kappa \to \kappa$ be a bijective coding of pairs with decoding functions $(\cdot)_0$ and $(\cdot)_1$. Then if $B \subset {}^\omega\omega \times {}^\omega\omega$ is κ-Suslin, we have a tree T on $\omega \times \omega \times \kappa$ such that $B = p[T]$ but, let $A = \{a : (a, b) \in B\}$ and let $(s, t) \in T'$ if and only if $(s, [t]_0, [t]_1) \in T$, then $A = p[T']$ and so A is κ-Suslin as well. Since $\mathbf{\Pi}^1_{2n+1}$ has the scale property, it is $\boldsymbol{\delta}_{2n+1}$-Suslin. Thus, by the previous observation, $\mathbf{\Sigma}^1_{2n+2}$ is $\boldsymbol{\delta}_{2n+1}$-Suslin.

For the second claim, we show that every $\mathbf{\Pi}^1_{2n}$-set is κ-Suslin for some fixed $\kappa < \boldsymbol{\delta}^1_{2n+1}$. Take A to be a $\mathbf{\Pi}^1_{2n}$-comlpete and let $\{\varphi_n\}_{i<\omega}$ be a $\mathbf{\Pi}_{2n+1}$ scale on A. $|\varphi_n| < \boldsymbol{\delta}^1_{2n+1}$. Note that $\boldsymbol{\delta}^1_{2n+1}$ is of uncountable cofinality and so $\{\varphi_n\}_{n<\omega}$ is a $< \boldsymbol{\delta}^1_{2n+1}$ scale, thus A is $< \boldsymbol{\delta}^1_{2n}$-Suslin. $\square$

Definition 18. We let $\mathbf{B}_\lambda$ denote the set of sets of reals containing all open sets, closed under complements and closed under $<\lambda$ well-ordered unions. We say that A is λ-*Borel*, if $A \in \mathbf{B}_\lambda$.

Fact 19 (Separation of Suslin sets)**.** *If $A, B \subset {}^\omega\omega$ are κ-Suslin and disjoint, then there is a κ^+-Borel set C that separates them.*

[1] We usually construe trees on the products as sets of pairs of tuples instead of tuples of pairs.

Proof. Let T, S be such that $A = p[T], B = p[S]$, we define the tree U by letting $(s, u, v) \in U$ if and only if $(s, u) \in T \wedge (s, v) \in S$. Since A and B were taken to be disjoint, U is well-founded. We define $C_{s,u,v}$ by induction as a set that separates $A_{s,u}$ and $B_{s,v}$, where

$$A_{s,u} = \{a \supset s : \exists f \supset u, (a, f) \in T\}$$
$$B_{s,v} = \{a \supset s : \exists f \supset v, (a, f) \in S\}$$

Note that $A_{s,u} = \bigcup_{n,\xi} A_{s^\frown n, u^\frown \xi}$ and $B_{s,v} = \bigcup_{n,\eta} A_{s^\frown n, u^\frown \eta}$ and obviously $A_{\emptyset,\emptyset} = A, B_{\emptyset,\emptyset} = B$. That is if we take $D_{n,m,\xi,\eta}$ to be a set that separates $A_{s^\frown n, u^\frown \xi}$ and $B_{s^\frown m, v^\frown \eta}$, Then, readily $C_{s,u,v} = \bigcup_{n,\xi} \bigcap_{m,\eta} D_{n,m,\xi,\eta}$ separating $A_{s,u}$ and $B_{s,v}$.

Assume now that we have defined $C_{s^\frown n, u^\frown \xi, v^\frown \eta}$, for each $(s^\frown n, u^\frown \xi, v^\frown \eta) \in U$. Note that U is well-founded, hence induction can be applied, where the base case is straightforward by the definition of U.

Case I: $n = m$ and $(s^\frown n, u^\frown \xi, v^\frown \eta) \in U$. Then, $D_{n,m,\xi,\eta} = C_{s^\frown n, u^\frown \xi, v^\frown \eta}$.

Case II: $n = m$ and $(s^\frown n, u^\frown \xi, v^\frown \eta) \notin U$, this means that $A_{s^\frown n, u^\frown \xi} = \emptyset$ or $B_{s^\frown n, v^\frown \eta} = \emptyset$, so they can be trivially separated.

Case III. $n \neq m$. In this case they can be separated by an open set $U_{s^\frown n}$.

That is, C is obtained by less then κ^+ intersections and unions of open sets. $\square$

Corollary 20. *If A and ${}^\omega\omega \setminus A$ are κ-Suslin, then $A \in \mathbf{B}_{\kappa^+}$.*

Fact 21. *For all $n < \omega$, $\mathbf{B}_{\boldsymbol{\delta}^1_{2n+1}} = \boldsymbol{\Delta}^1_{2n+1}$*

Proof. $\boldsymbol{\Delta}^1_{2n+1} \subset \mathbf{B}_{\boldsymbol{\delta}^1_{2n+1}}$ follows from the fact that any $\boldsymbol{\Delta}^1_{2n+1}$ set is $\kappa_{2n+1}(< \boldsymbol{\delta}^1_{2n+1})$-Suslin, thus it is enough to show that $\boldsymbol{\Delta}^1_{2n+1}$ is closed under $< \boldsymbol{\delta}^1_{2n+1}$ unions. For the sake of contradiction we assume there is some sequence $\{A_\xi\}_{\xi<\theta}$ such that $A = \bigcup_{\xi<\theta} A_\xi \notin \boldsymbol{\Delta}^1_{2n+1}$, where $\theta < \boldsymbol{\delta}^1_{2n+1}$, thus there is a $\boldsymbol{\Delta}_{2n+1}$ prewellordering $\leq$ of the reals with the corresponding norm φ. We may assume that the sequence is monotonous and continuous. Note that $a \in A$ if and only if there is $\xi < \theta$ such that $a \in A_\xi$. Let $f : \theta \to P({}^\omega\omega) : \xi \mapsto \{a \in {}^\omega\omega : a \text{ is a } \boldsymbol{\Delta}_{2n+1} \text{ code for } A_\xi\}$, which means $(a)_0$ and $(a)_1$ are $\boldsymbol{\Sigma}^1_{2n+1}$ and $\boldsymbol{\Delta}^1_{2n+1}$ codes for A_ξ. Generally, if y is a $\boldsymbol{\Delta}^1_{2n+1}$ code for a set B, we write $\boldsymbol{\Delta}_y$ for this B. Let g be a $\boldsymbol{\Sigma}^1_{2n+1}$ choice subfunction of f. Now we have $a \in A \iff \exists x, \exists y((x, y) \in \mathrm{Code}(g, \leq) \wedge a \in \boldsymbol{\Delta}_y)$, thus A is $\boldsymbol{\Sigma}^1_{2n+1}$, indeed it is $\boldsymbol{\Sigma}^1_{2n+1}$ complete, by the Wadge Lemma.

Let $\psi(a)$ be the unique ξ such that $a \in A_{\xi+1} \setminus A_\xi$. We claim that it is a $\boldsymbol{\Sigma}^1_{2n+1}$ norm. Indeed, $a \leq^*_\psi b$ if and only if $(a, b) \in \bigcup_{\xi,\theta}(A_{\xi+1} \times A_{\xi+1} \setminus A_\xi) \iff \exists x, y((x, y))$ as well as $a <^*_\psi b$ if and only if $(a, b) \in \bigcup_{\xi,\theta}(A_\xi \times A_{\xi+1} \setminus A_\xi)$. Both definitions are $\boldsymbol{\Sigma}^1_{2n+1}$. Thus, we obtained a $\boldsymbol{\Sigma}^1_{2n+1}$ norm on a $\boldsymbol{\Sigma}^1_{2n+1}$-complete set, which contradicts with the fact that $\boldsymbol{\Sigma}^1_{2n+1}$ does not have the prewellordering property. $\square$

Definition 22. If T is a tree on a set X and $u \in X^{<\omega}$, then we let $T_u = \{v \in X^{<\omega} : u^\frown v \in T\}$. If T is a tree on $\omega \times X$ and $a \in {}^\omega\omega$, we let $T(a) = \{u \in {}^{<\omega}X : (a\restriction|u|, u) \in T\}$.

For a tree J, we write $|J| < \eta$ if J is well-founded and has rank $<\eta$.

Fact 23 (Sierpinski)**.** *If $A \subset {}^\omega\omega$ is κ-Suslin then $A \in \mathbf{B}_{\kappa^{++}}$*

Proof. Let T be a tree on $\omega \times \kappa$ such that $A = p[T]$. For each $\xi < \kappa^+$ and $u \in \kappa^{<\omega}$ we let

$$A^\xi_u = \{a : |T(a)_u| < \xi\}$$

Now note that if $|u| = n$, then

$$A_u^0 = \{a : (a \restriction n, u) \notin T\}$$

$$A_u^{\xi+1} = A_u^\xi \cup \bigcup_{\eta<\kappa} A_{u^\frown\eta}^\xi$$

and

$$A_u^\lambda = \bigcup_{\xi<\lambda} A_u^\xi$$

Thus, each of these sets are in $\mathbf{B}_{\kappa^+}$ and so

$$a \notin A \iff a \notin p[T] \iff \mathrm{wf}(T(a)) \iff \exists \xi < \kappa^+ |T(a)| = \xi \iff \exists \xi < \kappa^+, a \in A_\emptyset^\xi$$

□

Fact 24. *If A is κ-Suslin and $cof(\kappa) > \omega$, then $A \in \mathbf{B}_{\kappa^+}$*

Proof. The proof is similar to the previous one. Note that $a \in A \iff \neg\,\mathrm{wf}(T(a)) \iff \exists \xi < \kappa$ such that $T(a)^\xi$ is not well-founded. Where T^ξ is T restricted to ordinals $< \xi$. The one can apply the argument from the previous proof to T^ξ. □

Fact 25. *For all $n < \omega$, $\boldsymbol{\delta}_{2n+1} = \kappa_{2n+1}^+$, where κ_{2n+1}^+ is a cardinal of countable cofinality.*

Proof. Let κ_{2n+1} be the least κ such that every $\mathbf{\Sigma}_{2n+1}^1$ is κ-Suslin. Assume $(\kappa_{2n+1})^{++} \leq \boldsymbol{\delta}_{2n+1}^1$, then every $\mathbf{\Sigma}_{2n+1}^1$ set is $\mathbf{B}_{\boldsymbol{\delta}_{2n+1}^1}$, then it is $\mathbf{\Delta}_{2n+1}^1$, a contradiction. Assume now that the cofinality of κ_{2n+1} is uncountable, then by 24 we have every $\mathbf{\Sigma}_{2n+1}^1$ set is $\mathbf{B}_{\kappa^+} = \mathbf{\Delta}_{2n+1}^1$, a contradiction. □

Fact 26 (Kunen-Martin)**.** *If $\prec \subset {}^\omega\omega \times {}^\omega\omega$ is wellfounded and admits a κ-scale, then $|\prec| < \kappa^+$.*

In fact this holds if we assume $\prec$ is κ-Suslin, but we stick to the weaker statement fact since it is enough for our goals and the proof is a bit nicer in this case, for the more general version, see [Kec11].

Proof. First we define a tree on ${}^\omega\omega$ given by the set of all $\prec$ descending sequences:

$$T_\prec = \{(a_0, \dots, a_n) : a_0 \succ \cdots \succ a_n\}$$

One can see by induction that

$$|a|_\prec = |(a_0, \dots, a_n, a)|_{T_\prec}$$

this implies, $|\prec| \leq |T|$. We construct an order preserving embedding $f : T \to S$, where S is a well-founded tree on $\omega \times \omega \times \lambda$. This would be enough to see that $|\prec| \leq \kappa^+$ since so is $|S|$.

Let $\{\varphi_i\}_{i<\omega}$ be a κ-scale on $\prec$, for $a \succ b$ then we let

$$\psi_n(a, b) = \ulcorner a_0, b_0, \varphi_0(a, b), \dots, a_n, b_n, \varphi_n(a, b) \urcorner$$

Then if the sequences $\{a^i\}_{i<\omega}, \{b^i\}_{i<\omega}$, $a^i \succ b^i$ for each i and for all n, $\psi_n(a^i, b^i)$ is eventually constant, then the limit (a, b) of $\{(a_i, b_i)\}_{i<\omega}$ exists and $a \succ b$.

We let now

$$
\begin{aligned}
&f(\langle\rangle) = \langle\rangle \\
&f(\langle a^0\rangle) = \langle\rangle \\
&f(\langle a^0, a^1\rangle) = \langle \psi_0(a^0, a^1)\rangle \\
&f\left(\langle a_0, a_1, a_2\rangle\right) = \left\langle \psi_0\left(a_0, a_1\right), \psi_1\left(a_0, a_1\right), \psi_1\left(a_1, a_2\right), \psi_0\left(a_1, a_2\right)\right\rangle, \\
&\dots \\
&f\left(\langle a_0, \dots, a_{n-1}, a_n\rangle\right) = f\left(\langle a_0, \dots, a_{n-1}\rangle\right)^\frown \\
&{}^\frown\langle \psi_{n-1}\left(a_0, a_1\right), \psi_{n-1}\left(a_1, a_2\right), \dots, \psi_{n-1}\left(a_{n-1}, a_n\right), \\
&\ \psi_{n-2}\left(a_{n-1}, a_n\right), \dots, \psi_0\left(a_{n-1}, a_n\right)\rangle.
\end{aligned}
$$

Now letting S be the downward closure of the image of f, we have $f : T_\prec \to S$ is order preserving. It is left to show that S is well-founded. Assume not, then there is a sequence:

$$f\left(\left\langle a_0^0\right\rangle\right) \sqsubset f\left(\left\langle a_0^1, a_1^1\right\rangle\right) \sqsubset f\left(\left\langle a_0^2, a_1^2, a_2^2\right\rangle\right) \sqsubset \dots$$

which means by the definition of f, that we have:

$$
\begin{array}{llllll}
\psi_0\left(a_0^1, a_1^1\right) & & & & & \\
\psi_0\left(a_0^2, a_1^2\right) & \psi_1\left(a_0^2, a_1^2\right) & \psi_1\left(a_1^2, a_2^2\right) & \psi_0\left(a_1^2, a_2^2\right) & & \\
\psi_0\left(a_0^3, a_1^3\right) & \psi_1\left(a_0^3, a_1^3\right) & \psi_1\left(a_1^3, a_2^3\right) & \psi_0\left(a_1^3, a_2^3\right) & \psi_2\left(a_0^3, a_1^3\right) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Here, each column converges to a pair $a_i \succ a_{j+1}$, yielding an infinite $\prec$-descending sequence of reals.

Fact 27. *For all $n < \omega$, $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$*

Proof. Take a $\boldsymbol{\Pi}_{2n+1}$-norm on a $\boldsymbol{\Pi}^1_{2n+1}$ set φ, we have $lh(\varphi) = \boldsymbol{\delta}^1_{2n+1}$ and since $\boldsymbol{\Pi}^1_{2n+1} \subset \boldsymbol{\Delta}^1_{2n+1}$, we have $\boldsymbol{\delta}^1_{2n+1} < \boldsymbol{\delta}^1_{2n+2}$ and $(\boldsymbol{\delta}^1_{2n+1})^+ \leq \boldsymbol{\delta}^1_{2n+2}$. Since every $\boldsymbol{\Sigma}^1_{2n+2}$ (in particular $\boldsymbol{\Delta}^1_{2n+2}$) relation is $\boldsymbol{\delta}^1_{2n+1}$-Suslin, we have $(\boldsymbol{\delta}^1_{2n+1})^+ \geq \boldsymbol{\delta}^1_{2n+2}$. □

Corollary 28. *For each $n < \omega$, $\boldsymbol{\delta}^1_n < \boldsymbol{\delta}^1_{n+1}$.*

Proof. If $n = 2n + 1$, then it follows from the previous fact. Otherwise we know that δ_{2n+1} is a successor of a cardinal of cofinality ω, who cannot be a projective one. □

Corollary 29. *For all $n < \omega$,*

$$\boldsymbol{\delta}^1_n = \{\xi : \xi \text{ is a } \boldsymbol{\Sigma}^1_n \text{ prewellordering}\}$$

Proof. For odd n, $\boldsymbol{\Sigma}^1_n$ is κ_n-Suslin and for even it is $\boldsymbol{\delta}^1_{n-1}$-Suslin, thus its length is below $\boldsymbol{\delta}^1_n$ by Kunen-Martin Thorem. □

Corollary 30. *$\boldsymbol{\delta}^1_1 = \omega_1$ and $\boldsymbol{\delta}^1_2 = \omega_2$.*

2. Regularity of the projective ordinals

Fact 31. *For each $n < \omega$, $\boldsymbol{\delta}^1_n$ is regular*

Proof. Assume there is a cardinal $\lambda < \boldsymbol{\delta}^1_n$ such that there is a cofinal in $\boldsymbol{\delta}^1_n$ sequence $\{\xi_i : i < \lambda\}$. Let $\prec_i$ be a $\boldsymbol{\Delta}^1_n$ pre prewellordering of the reals of length ξ_i and $\prec'$ of length λ and φ_i, φ' be the corresponding norms. We set:

$$f : \lambda \to P({}^\omega\omega) : i \mapsto \{a : a \text{ is a code for a well-founded relation of length } \xi_i\}$$

It follows from The Coding Lemma that there is a choice subfunction g of f with $\mathrm{Code}(g, \prec')$ being $\boldsymbol{\Sigma}^1_n$. We let $<$ be a relation on ${}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega$ defined by $(a, b, c) < (a', b', c')$ if $a = a'$, $b = b'$, $b = g(a)$ and $(c, c') \in W_b$. Thus, we obtained a $\boldsymbol{\Sigma}^1_n$ prewellordering of length $\boldsymbol{\delta}^1_n$ which contradicts Corollary 29. □

3. Measurability

In this section we give a full proof of measurability of ω_1 witnessed by the club filter, whereas for an arbitrary $\boldsymbol{\delta}^1_n$, we define a filter and show that it is actually a $< \boldsymbol{\delta}^1_n$-complete non-principle ultrafilter, which is actually the ω-club filter, i.e. a set A is in the filter if and only if there is an unbounded ω-closed subset $C \subset A$. This section lacks a proof of normality of the measure. In the next section we show measurability of all regular cardinals under $\aleph_{\epsilon_0}$ using the fact that $\boldsymbol{\delta}^1_{2n+1}$ has the strong partition property.

Note that the facts in this section we cannot simply refer to the completeness and the normality of club filters, because those proofs employ AC to choose club subsets of sets in the filter.

3.1. **The Case of ω_1.** First we outline a game on the countable ordinals to give intuition behind the proof. Fix $A \subset \omega_1$.We let I and II produce sequences of ordinals $\{\alpha_i : i < \omega\}$ and $\{\beta_i : i < \omega\}$ respectively with $\alpha_0 < \beta_0 < \ldots \alpha_i < \beta_i < \ldots$ and if a player violates this monotonicity requirement, they instantly loose. Otherwise, we let II win if $\sup\{\alpha_i, \beta_i : i < \omega\} \in A$. Note that if A contains a club C, then II can simply play ordinals from C. If II has a winning strategy τ, then for each η we define $\xi_\eta = \sup\{\tau(\alpha_0, \beta_0, \ldots, \alpha_i, \beta_i) : \alpha_0 < \beta_0 < \cdots < \alpha_i < \beta_i < \eta\}$. Since ξ_η has countable cofinality, the function $f : \omega_1 \to \omega_1 : \eta \to \xi_\eta$ is well-defined and unbounded, and so the set $C = \{\alpha : \beta < \alpha \to f(\beta) < \alpha\}$ is a club and for each $\gamma \in C$, I can legitimately play a cofinal in γ sequence when II play according to τ, thus witnessing $\gamma \in A$ and so $C \subset A$. If the game was determined, it would simply follow that the club filter on ω_1 is a measure (using AC_ω). Now we have to encode this game to be a game on the natural numbers.

Definition 32. For $a \in {}^\omega\omega$, we let $R_a = \{(m, n) \in \omega \times \omega : a(\ulcorner m, n \urcorner) = 0\}$. Then $\mathrm{WO} = \{a \in {}^\omega\omega : R_a \text{ is a wellordering}\}$, also for an $\alpha < \omega_1$, we let $\mathrm{WO}_{\leq\alpha} = \{a \in {}^\omega\omega : R_a \text{ is a wellordering of length } \leq \alpha\}$.

Fact 33. WO *is* $\boldsymbol{\Pi}^1_1$*-complete and* $x \mapsto |R_x|$ *is a* $\boldsymbol{\Pi}^1_1$*-norm (of length* ω_1*). And for each countable ordinal* α*,* $\mathrm{WO}_{\leq\alpha}$ *is Borel.*

Fact 34 (Boundedness lemma)**.** *If* $A \subset \mathrm{WO}$ *is analytic, then* $A \subset \mathrm{WO}_{\leq\alpha}$ *for some countable* α*.*

Proof. A particular case of Corollary 29. □

Fact 35 (Basic Coding Lemma; Solovay)**.** *Assume* $Z \subset \mathrm{WO} \times {}^\omega\omega$*, then there is a* $\boldsymbol{\Sigma}^1_2$ *choice subset* Z' *of* Z*. Moreover, there is* $X \subset {}^\omega\omega \times {}^\omega\omega$ *which is* $\boldsymbol{\Sigma}^1_1$ *and* $Z' = X \cap (\mathrm{WO} \times {}^\omega\omega)$.

In the rest of the subsection we prove:

Fact 36 (Solovay)**.** AD *implies that* ω_1 *is measurable as witnessed by the club filter.*

Definition 37. Fix $A \subset \omega_1$, we let $G(A)$ be the following game: I produces a real a which is construed as a sequence of reals $\{a_i\}_{i<\omega}$ and II plays a real which is construed as a sequence of reals $\{b_i\}_{i<\omega}$, we let II win if

(1) $\exists i < \omega, a_i \notin \mathrm{WO} \vee b_i \notin \mathrm{WO}$ implies the smallest such i is such that $a_i \notin \mathrm{WO}$, or
(2) there is $i < \omega, |x_j| < |y_j| < |x_{j+1}|$ for all $j < i$ and $|x_{j+1}| \geq |y_{j+1}|$, or
(3) $\forall i < \omega, a_i, b_i \in \mathrm{WO}$ and $\sup\{|a_i|, |b_i|\}_{i<\omega} \in A$.

The first two conditions are needed to simply force the players to play an increasing sequence of ordinals.

Claim 38. *II has a winning strategy if and only if A contains a club as well as I has a winning strategy if and only if A is disjoint from a club. Consequently, the club filter on* ω_1 *is an ultrafilter.*

Proof. The 'as well as' part follows from the symmetry of the game. Assume now that II have a winning strategy τ, then we let

$$A_\alpha = \{\tau(a)_n : a \in {}^\omega\omega, \forall i < n, a_i \in \mathrm{WO}_{<\alpha}\}$$

be a set o reals coding wellorderings occurring as a τ-response to a partial play where only reals coding ordinals $<\alpha$ were previously involved. Note that A_α is analytic. By the Boundedness Lemma, there is the smallest b_α such that for each $b \in A_\alpha$, $|b| < |b_\alpha|$, let $f : \omega_1 \to \omega_1 : \alpha \to |b_\alpha|$, it is obviously unbounded and hence the set of its fixed point C is a club. It follows if I plays any sequence from C the result end up in C, hence $C \subset A$. Similarly, if I has a winning strategy, there is a club disjoint from A. □

Using AC_ω we can easily obtain the completeness of the filter.

Fact 39. *The club filter on* ω_1 *is normal.*

Proof. Let $f : \omega_1 \to \omega_1$ be a regressive function and assume that filter is not normal, thus there is a sequence of measure one sets $\{A_\xi : \xi < \omega_1\}$ with $A_\xi = \{\alpha < \omega_1 : f(\alpha) \neq \xi\}$ for each $\xi < \omega_1$. We aim to define a sequence of countable ordinals η_i for all $i < \omega$ such that $\sup_{i<\omega} \eta_i = \eta$, a sequence of collection of strategies $\{X_i : i < \omega\}$ such that X_i contains a strategy for $G(A_\xi)$ for each $\xi \in [\eta_{i-1}, \eta_i)$, where $\eta_{-1} = 0$ and a and $\alpha_i \in \bigcap_{\beta<\alpha_i} A_\beta$, thus $\alpha \in \bigcap A_\alpha$. Let

$$Z = \{(\sigma, x) : \sigma \text{ is a w.s. in the game } G(A_{|x|})\} \subset \mathrm{WO} \times {}^\omega\omega$$

By The Coding Lemma, there is a choice subset $Z' \subset Z$ of complexity $\mathbf{\Delta}^1_2$. In fact, by The Basic Coding lemma, there is $X \subset {}^\omega\omega \times {}^\omega\omega$ with is $\mathbf{\Sigma}^1_1$ and $X \cap (\mathrm{WO} \times {}^\omega\omega) = Z'$, note that $X_\alpha = X \cap (\mathrm{WO}_{<\alpha} \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ as well. The following argument does not require AD anymore, but requires DC, thus we can pick a real t such that X is $\Sigma^1_1(t)$, then $X \cap L[t] = X' \in L[t]$ and by Σ^1_1-correcteness.

Let η_0 be some ordinal and let

$$Z_0 = \{\tau : \tau \text{ is a winning strategy against } A_{|x|}, x \in \mathrm{WO}_{<|\eta|}\}$$

is $\mathbf{\Sigma}^1_1$ as basically being the projection of $X_{|\eta_0|}$. Hence, there is a bound η_1 for the set $\{(\tau(a))_0 : \tau \in Z_0 \wedge a \in {}^\omega\omega\} \subset \mathrm{WO}$, proceeding inductively we get a sequence of η_i which played against II in the games $G(A_\beta)$ for $\beta < \eta$ witnesses that $\eta \in A_\beta$ for all $\beta < \eta$. We get:

$$\begin{array}{lll} \eta_0 < \sup\{(|\tau_\xi(a)_0|\}_{a\in{}^\omega\omega,\xi<\eta_0} < & \eta_1 < \sup\{|\tau_\xi(a)_1|\}_{a\in{}^\omega\omega,\xi<\eta_0} < & \eta_2 < \sup\{|\tau_\xi(a)_2|\}_{a\in{}^\omega\omega,\xi<\eta_0} \cdots \\ & \eta_1 < \sup\{|\tau_\xi(a)_0|\}_{a\in{}^\omega\omega,\eta_0\leq\xi<\eta_1} < & \eta_2 < \sup\{|\tau_\xi(a)_1|\}_{a\in{}^\omega\omega,\eta_0\leq\xi<\eta_1} \cdots \\ & & \eta_2 < \sup\{|\tau_\xi(a)_0|\}_{a\in{}^\omega\omega,\eta_1\leq\xi<\eta_2} \cdots \\ & & \cdots \end{array}$$

Let $\{\eta_{i+1}, \eta_{i+2}, \dots\}$ be a play of I, then it is a legal play against τ_ξ for any $\eta_i < \xi < \eta_{i+1}$, hence $\eta = \sup\{\eta_i\}_{i<\omega} \in \bigcap_{\xi<\eta} A_\xi$. □

Fact 40. *For all $n < \omega$, $\boldsymbol{\delta}^1_n$ is measurable*

Proof. Let $W \subset ({}^\omega\omega)^3$ be $\boldsymbol{\Sigma}^1_n$ universal and let $S = \{a : W_a$ is a wf binary relation$\}$. Let $|a| = lh(W_\alpha)$. Thus, $\boldsymbol{\delta}^1_n = \sup\{|a| : a \in S\}$. For $A \subset \boldsymbol{\delta}^1_n$ consider the game G^A. Player I plays a and II plays b and II wins if and only if:

$$[\exists i(\alpha_i \notin S \vee b_i \notin S) \wedge (i_0 \text{ is least such, then } a_{i_0} \notin S)] \vee$$

or

$$[\forall i(a_i \in S \wedge b_i \in S) \wedge \sup\{a_0, b_0, a_1, b_1, \dots\} \in A]$$

We say that $A \in U$ iff II has a winning strategy in G^A.

Claim 41. *U is upward closed.*

Proof. Trivial. □

Claim 42. *U is closed under intersections.*

Proof. Assume II has winning strategies for games A, B, then II can play $b' \oplus b''$. □

Claim 43. *U contains no bounded sets.*

Proof. Otherwise I could easily win. □

Claim 44. *U is an ultrafilter.*

Proof. One can see that if I has a winning strategy for A, then it is a winning strategy for II for the complement of A. □

Claim 45. *U is $\boldsymbol{\delta}^1_n$-additive.*

Proof. Let $\{A_\xi : \xi < \eta\}$ be a sequence of $< \boldsymbol{\delta}^1_n$ members of U, it suffices to show that the intersection is nonempty.

Let $\leq$ be a $\boldsymbol{\Delta}^1_n$ prewellordering of ${}^\omega\omega$ of length η with assisicated norm ϕ for $\xi < \eta$ we let

$$f(\xi) = \{\tau : \tau \text{ is a winning strategy for } A_\xi\}$$

Let g be a $\Sigma^1_n(y)$(-coded) choice subfunction.

Claim 46. *For each $m < \omega$ there is a function $f_m : {}^{m+1}S \to S$ such that for all $a^0, \dots, a^m \in S$ for all a with $a_i = a^i$ if $i \leq m$, and for all $\tau \in \bigcup_{\xi<\eta} g(\xi)$*

$$|f_m(a_0, \dots, a^m)| \geq |(\tau[a])_m|$$

Proof. Given $a^0, \dots a^m \in S$, sonsider the following wf relation:

$$\langle a, x, \tau, z \rangle \prec \langle a', x', \tau', z' \rangle \iff a = a', x = x', \tau = \tau'$$
$$\forall i \leq m(a_i = a_i) \wedge (x, \tau) \in CODE(g; \leq) \wedge (z, z') \in W_{\tau[a]_m}$$

informally:

$$a = a', \xi = \xi', \tau \in g(\xi), z <_{\tau[a]i_m} z'$$

then we let $f_m(a^0, \dots, a^m) =\prec_{a^0,\dots,a^m}$. □

Now for an $a^0 \in S$ we let $\theta = \sup\{a^0, f_0(a^0), f_1(a, f_0(a)) \dots\}$. We claim that θ in the intersection. Indeed, let I play a with $a_i = a^i$ and let II play according to $\tau \in g(\xi)$, then the resulting real is $\theta \in A_\xi$. □

□

□

References

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