1. Gdańsk Logic Colloquium – Schindler 2 XI '23

Martin's Maximum^{++}:

If $\mathbb{P}$ is stationary set preserving, $\mathcal{D} = \{D_i : i < \omega_1\}$ is a collection of dense sets, $\{\tau_i : i < \omega_1\}$ is a collection of names for stationary subsets of ω_1, i.e.:

$$\mathbb{1} \Vdash \tau_i \subseteq \check{\omega}_1 \text{ is stationary}$$

then $\exists$ filter $\in V$, g s.t.

$$g \cap D_i \neq \emptyset$$

$$\tau_i^g = \{\xi : \exists p \in g \ p \Vdash \check{\xi} \in \tau_i\} \text{ is stationary}$$

for all $i < \omega_1$.

MM^{++} is equivalent to: For all $\mathbb{P}$ stationary set preserving, for all models $\mathcal{M} = (M, \vec{R})$ $(|\vec{R}| \leq \aleph_1)$ and for all Σ_1 flas in $\mathcal{L}_{\in, NS_{\omega_1}}$, if

$$V^{\mathbb{P}} \models \phi(\mathcal{M})$$

then: in V there ~~some~~ is some $\bar{\mathcal{M}} \longrightarrow \mathcal{M}$ s.t. $\phi(\bar{\mathcal{M}})$.

Let's formulate a strengthening of MM^{++}.

Definition Let ϕ be a Σ_1-formula in $\mathcal{L}_{\in, NS_{\omega_1}}$.

Say that $\phi(\mathcal{M})$ is honestly consistent iff for all universally Baire functions F, there is a transitive model $\mathfrak{A} \in V^{Col(\omega, tcl(\{\mathcal{M}\}))}$ s.t. $\mathfrak{A} \models ZFC^- + \phi(\mathcal{M})$, $tcl(\{\mathcal{M}\}) \in \mathfrak{A}$, $NS_{\omega_1}^V = NS_{\omega_1}^{\mathfrak{A}} \cap V$

$MM^{*,++}$

For all models $\mathcal{M} = (M, \vec{R})$ $\vec{R}$ - $\leq \aleph_1$, for all Σ_1 - formulae in $\mathcal{L}_{\dot{\in}, \dot{NS}_{\omega_1}}$ if $\phi(\mathcal{M})$ is honestly consistent (!) there exists in V some $\overline{\mathcal{M}} \to \mathcal{M}$ s.t. $\phi(\overline{\mathcal{M}})$.

$MM^{*,++}_{\lambda}$ – restricts this to models $\mathcal{M}$ of size λ.

MM^{++}_{λ} – restricts the first formulation to collections of dense sets, all of ~~whom~~ which have size $\leq \lambda$.

Γ-$MM^{*,++}_{\lambda}$: $\Gamma \subset \Gamma^{\infty}$ - universally Baire sets of reals, in the model $\mathcal{M}$ we have predicates $\dot{A}$ for all $A \in \Gamma$.

Remark

$MM^{++}_{\lambda} \Rightarrow MM^{++}(\lambda)$ | RHS: MM^{++} in the first formulation for forcings of size $\leq \lambda$. ($< \lambda$?)

Definition

$(\Gamma$-$)BMM^{(*),++} \equiv (\Gamma$-$) MM^{(*),++}_{\aleph_1}$

Theorem (NS_{ω_1} is saturated + V is closed under $x \mapsto M^{\#}_{\omega}(x)$)

TFAE: (1) $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ -$BMM^{*,++}$

(2) $(*)$ (Woodin's $\mathbb{P}_{max}$ axiom).

| Partly essentially Woodin
| Older paper Aspéro – Schindler
| Larson 2013

(3) $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ -BMM^{++}

| New Aspero – Schindler

3. Gdańsk Logic Colloquium – Schindler $\quad 2\text{ XI }'23$

Open • Is there a reformulation of BMM^{++} as a $(*)$-like axiom?

• For $\lambda \geq \aleph_2$ is Γ-MM^{++}_λ equivalent to Γ-$M^{*,++}_\lambda$?
(~~eg~~ Is $MM^{*,++}$ equivalent to MM^{++}?).

• For $\lambda \geq \aleph_3$ is $MM^{*,++}_\lambda$ consistent from large cardinals?

In my next talks:

(1) $\Pi_2^{H\omega_2}$-maximality is the statement:
if ϕ is $\Pi_2^{H\omega_2}$ and Ω-consistent, then ϕ is true.
We'll give a direct proof of $\Pi_2^{H\omega_2}$-maximality from MM^{++}.

(2) The consistency of $MM^{*,++}_{2^{\aleph_0}}$ $(\Rightarrow 2^{\aleph_0} = \aleph_2)$ by forcing over a model $\mathcal{D}$ of determinacy.
$\left(\mathcal{D} \models AD + \vartheta \text{ is a regular limit of the Solovay sequence} + \text{every set of reals is universally Baire} \right)$

$\mathcal{D}^{\mathbb{P}_{max} * Col(\omega_3, \omega_3)} \models ZFC + MM^{*,++}_{\aleph_2}$

$\downarrow$

Cohen forcing, actually

Open Force $MM^{*,++}_{\aleph_2}$ over a ZFC model with large cardinals.