ON FORCING WITH SIDE CONDITIONS

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1. Introduction

Our starting point is the contention that MM^{++} is a very successful axiom (for $H(\omega_2)$).[1]

(1) (**Maximal forcing axiom**) MM^{++} is a consistent (relative to a supercompact cardinal), provably maximal forcing axiom relative to collections of $\aleph_1$-many dense sets.
(2) (**Completeness modulo forcing**) If MM^{++} holds, then $Th(H(\omega_2)^V) = Th(H(\omega_2)^{V^{\mathcal{P}}})$ for every forcing $\mathcal{P}$ such that $\Vdash_{\mathcal{P}} \mathsf{MM}^{++}$ (since $\mathsf{MM}^{++} \Rightarrow (*)$ (A.–Schindler)).
(3) (Π_2 **maximality**) If MM^{++} holds, then $(H(\omega_2); \in, \mathrm{NS}_{\omega_1}) \models \sigma$ whenever σ is a Π_2 sentence such that $(H(\omega_2); \in, \mathrm{NS}_{\omega_1}) \models \sigma$ is forcible (again, since $\mathsf{MM}^{++} \Rightarrow (*)$); in fact, tinkering a bit with the proof that $\mathsf{MM}^{++} \Rightarrow (*)$ one can show that already MM is Π_2 maximal for the theory of $(H(\omega_2); \in)$ (A.–Schindler)).

The general question we will address is the following: Are there competitors for MM^{++} higher up? In other words, are there axioms approximating any of (1)–(3) for $H(\omega_3)$, or $H(\kappa)$ for some higher κ?

1.1. MM^{++} and completeness for $H(\omega_3)$.

The completeness provided by $(*)$ for the theory of $H(\omega_2)$ certainly doesn't extend to $H(\omega_3)$: Force $\square_{\omega_1}$ by $<\omega_2$-distributive forcing, hence preserving $(*)$.

How about MM^{++}? Does MM^{++} provide a complete theory, modulo forcing, for $H(\omega_3)$?

The answer of course is No, but it's not so straightforward to find examples. In fact, MM^{++} is surprisingly efficient at deciding natural combinatorial questions at $H(\omega_3)$. Here are some examples:

- (Todorčević) PFA implies $\neg\square_{\omega_1}$.
- (Sakai) MM implies partial square on $S^{\omega_2}_{\omega_1}$.
- PFA implies $2^{\aleph_1} = \aleph_2$ (Todorčević, Veličković), so it implies $\diamondsuit(S^{\omega_2}_{\omega})$ (Shelah).
- (Baumgartner) PFA implies $\diamondsuit(S^{\omega_2}_{\omega_1})$.

Given a cardinal κ of uncountable cofinality and a stationary set $S \subseteq \kappa$, *Strong Club Guessing at* S, $\mathsf{SCG}(S)$, is the following statement:

There is a sequence $(C_\delta : \delta \in S)$ such that

- for every $\delta \in S$, C_δ is a club of δ, and
- for every club $D \subseteq \kappa$ there are club-many $\delta \in D$ such that if $\delta \in S$, then $C_\delta \setminus \alpha \subseteq D$ for some $\alpha < \delta$.

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[1] This is an, at some points slightly expanded, version of the talk I gave at the Perspectives in Set Theory conference (IMPAN, Warsaw) in November 2023. I thank the organizers for inviting me and for organizing this particularly stimulating event.

Theorem 1.1. $\mathrm{Add}(\omega_2, \omega_3)$ *forces* $\neg\mathsf{SCG}(S)$ *for every stationary* $S \subseteq S^{\omega_2}_{\omega}$. *Hence, if* MM^{++} *holds, then forcing with* $\mathrm{Add}(\omega_2, \omega_3)$ *yields a model of* $\mathsf{MM}^{++} + \neg\mathsf{SCG}(S)$ *for every stationary* $S \subseteq S^{\omega_2}_{\omega}$.

Theorem 1.2. *Let* κ *be a supercompact cardinal and let* $\mathcal{P}$ *be the standard RCS-iteration of length* κ *forcing* MM^{++}. *Let* $S = (S^{\omega_2}_{\omega})^V$. *Then* $\mathcal{P} * \dot{\mathcal{Q}}(S)$ *forces* $\mathsf{MM}^{++} + \mathsf{SCG}(S)$. *Here,* $\dot{\mathcal{Q}}(S)$ *is a natural* $\aleph_1$*-support iteration of length* ω_3 *for adding some club-sequence* $(\dot{C}_\delta : \delta \in S)$ *and then shooting clubs through*

$$\{\delta \in \omega_2 : \delta \in S \Rightarrow \dot{C}_\delta \setminus \alpha \subseteq \dot{D}_\alpha \text{ for some } \alpha < \delta\},$$

where $\dot{D}_\alpha$ *is a club of* ω_2.

Question 1.3. Is there any forcible Σ_2 axiom A deciding the theory of $H(\omega_3)$ modulo forcing?

1.2. **Limitations on completeness.**

Theorem 1.4. *(Woodin) Suppose the* Ω *conjecture and the* AD^+*-conjecture are true in all set-generic extensions. Then there is no forcible* Σ_2 *axiom* A *such that* A *provides, modulo forcing, a complete theory for* Σ^2_3 *sentences.*

Theorem 1.5. *(Woodin) Suppose the* Ω *conjecture holds and there is a proper class of Woodin cardinal. Then there is no forcible* Σ_2 *axiom* A *such that* A *provides, modulo forcing, a complete theory for* $H(\delta_0^+)$, *where* δ_0 *is the first Woodin cardinal.*

1.3. **High Π_2 maximality?** Π_2 forcing maximality for the theory $H(\omega_3)$ is false, at least in the presence of a Mahlo cardinal:

Both $\square_{\omega_1}$ and $\neg\square_{\omega_1}$ can be forced, and $\square_{\omega_1}$ is $\Sigma_1(\omega_2)$ over $H(\omega_3)$.

Question 1.6. Does ZFC prove that Π_2 forcing maximality for the theory $H(\omega_3)$ is false? Does it in fact prove that there is a $\Sigma_1(\omega_2)$ sentence σ such that both σ and $\neg\sigma$ are forcible?

A vague question:

Question 1.7. Can there (still) be any reasonable successful analogue of MM^{++}, as forcing axiom, for $H(\omega_3)$ or higher up?

- Such an analogue of MM^{++}, if it extends $\mathrm{FA}_{\omega_2}(\{\mathrm{Cohen}\})$, should presumably imply $2^{\aleph_0} = \aleph_3$.
- Alternatively, we could instead focus, in the context of CH, on interesting classes Γ of countably closed forcings.

In what follows, we will be addressing this question. As is well-known, classical forcing preservation arguments are not suitable for proving preservation of sensible notions of high properness.[2] The right approach is therefore to use the method of forcing with side conditions. Side conditions tend to go hand-in-hand with the notion of strong properness. The present focus will be therefore to give a "bottom-up" presentation of the possible landscape taking the notion of strong properness as guiding idea.

2. Strong properness and side conditions

Definition 2.1. (Mitchell) A partial order $\mathcal{P}$ is *strongly proper* iff for every large enough cardinal θ, every countable $M \preccurlyeq H(\theta)$ such that $\mathcal{P} \in M$, and every $p \in \mathcal{P} \cap M$ there is some $q \leq_{\mathcal{P}} p$ which is *strongly* $(M, \mathcal{P})$*-generic*, i.e., for every $q' \leq_{\mathcal{P}} q$ there is some $\pi_M(q') \in \mathcal{P} \cap M$ such that every $r \in \mathcal{P} \cap M$ with $r \leq_{\mathcal{P}} \pi_M(q')$ is compatible with q'.

[2]For example, any countable support iteration of nontrivial forcings of length of uncountable cofinality forces $2^{\aleph_0} \leq \aleph_2$.

It is easy to see that both Cohen forcing and Baumgartner's forcing for adding a club of ω_1 with finite conditions are strongly proper.[3]

Some basic facts.

Fact 2.2. *If $\mathcal{P}$ is strongly proper, $M \preccurlyeq H(\theta)$ is countable, $\mathcal{P} \in M$, q is strongly $(M, \mathcal{P})$-generic, $G \subseteq \mathcal{P}$ is generic over V, and $q \in G$, then $G \cap M$ is $\mathcal{P} \cap M$-generic over V.*

Corollary 2.3. *Every ω-sequence of ordinals added by a strongly proper forcing notion is in a generic extension of V by Cohen forcing.*

Lemma 2.4. *(Neeman) Suppose $\mathcal{P}$ is strongly proper. Then $\mathcal{P}$ does not add new $ht(T)$-branches through trees T such that $\mathrm{cf}(ht(T)) \geq \omega_1$.*

Thus, strong properness is a natural condition to look at if we are interested in things like the tree property.

2.1. **Some pure side condition forcings (chains).** The simplest side condition forcings involve chains, or chain-like structures, of models. These forcings are aimed at preserving low cardinals (typically ω_1, or perhaps ω_1 and ω_2) while possibly collapsing other cardinals. The main classical examples are the following.

(1) (Todorčević) $\mathbb{C}_1$: conditions are chains $\mathcal{C} = \{M_0, \ldots, M_n\}$ with $M_i \preccurlyeq H(\theta)$, $|M_i| = \aleph_0$, $M_i \in M_{i+1}$ for all i.
   - $\mathbb{C}_1$ is strongly proper for countable models.
   - $\mathbb{C}_1$ covers $H(\theta)^V$ by an $\in$-chain of length ω_1 of countable models in V.

(2) (Neeman) $\mathbb{C}_2$: conditions are $\mathcal{C} = \{Q_0, \ldots, Q_n\}$, where
   (a) for all i, Q_i is either a countable $M \preccurlyeq (\theta)$ or $N \preccurlyeq H(\theta)$ such that $|N| = \aleph_1$ and N internally club (IC).
   (b) $Q_i \in Q_{i+1}$ for all $i < n$.
   (c) If N, $M \in \mathcal{N}$, $N \in M$, $|N| = \aleph_1$, $|M| = \aleph_0$, then $N \cap M \in \mathcal{C}$.
   - $\mathbb{C}_2$ is strongly proper for countable models and IC models of size $\aleph_1$.
   - $\mathbb{C}_2$ covers $H(\theta)^V$ by an $\in$-chain of length ω_1 of $\aleph_1$-sized models in V.

The following fundamental limitation was observed by Veličković (s. [2]).

Fact 2.5. *(Veličković) The natural pure side condition forcing $\mathbb{C}_3$ for three types of models (say countable, size $\aleph_1$ IC, and size $\aleph_2$ IC) doesn't work. In fact, this forcing necessarily collapses some cardinal $\leq\aleph_2$.*

2.2. **More pure side condition forcings (symmetric systems).** We might instead want to preserve all cardinals, while relaxing the requirement that the models being added form an $\in$-like structure. We are then naturally led to considering "matrices" of models.

(3) (Todorčević, A.–Mota, ...) $\mathbb{S}_1$: conditions are finite collections $\mathcal{N}$ of countable $M \preccurlyeq H(\theta)$ such that
   (a) For all M_0, $M_1 \in \mathcal{N}$, if $\delta_{M_0} = \delta_{M_1}$ ($\delta_M = M \cap \omega_1$), then $M_0 \cong M_1$ and the isomorphism
$$\Psi_{M_0,M_1} : M_0 \to M_1$$
   is the identity on $M_0 \cap M_1$.
   (b) For all M_0, $M_1 \in \mathcal{N}$, if $\delta_{M_0} = \delta_{M_1}$, then $\Psi_{M_0,M_1}``\mathcal{N} \cap M_0 = \mathcal{N} \cap M_1$.
   (c) For all M_0, $M_1 \in \mathcal{N}$, if $\delta_{M_0} < \delta_{M_1}$, then there is some $M_1' \in \mathcal{N}$ such that $M_0 \in M_1'$ and $\delta_{M_1'} = \delta_{M_1}$.

   ($\mathcal{N}$ is a *symmetric system*)
   - $\mathbb{S}_1$ is strongly proper for countable models.

[3]ccc does not imply strongly proper. In fact, most ccc forcings are not strongly proper.

- (CH) $\mathbb{S}_1$ has the $\aleph_2$-c.c. and preserves CH.

(4) (Gallart, Hoseini Naveh) $\mathbb{S}_2$: conditions are *symmetric systems $\mathcal{N}$ of models of two types* (countable and IC of size $\aleph_1$).
  (a) This is a natural combination of Neeman's notion of two-type chain of models ($\mathbb{C}_2$) and the notion of symmetric system ($\mathbb{S}_1$).
  (b) Given two models M_0, $M_1 \in \mathcal{N}$ of the same height ϵ_M ($= \sup(M \cap \omega_2)$), we ask that in fact

$$(\mathrm{Hull}(M_0, \omega_1); \in, M_0) \cong (\mathrm{Hull}(M_1, \omega_1); \in, M_1)$$

- $\mathbb{S}_2$ is strongly proper for countable models and for $\aleph_1$-sized IC models.
- ($2^{\aleph_1} = \aleph_2$) $\mathbb{S}_2$ has the $\aleph_3$-c.c. and preserves $2^{\aleph_1} = \aleph_2$.

2.3. An application of $\mathbb{S}_2$.

Definition 2.6. A *strong ω_3-chain of subsets of* ω_1 is a sequence $(X_i : i < \omega_3)$ of subsets of ω_1 such that for all $i_0 < i_1$,

- $X_{i_0} \setminus X_{i_1}$ is finite and
- $|X_{i_1} \setminus X_{i_0}| = \aleph_1$.

Theorem 2.7. *(A.–Gallart [1]) (GCH) There is a forcing notion $\mathcal{P}$ with the following properties.*

(1) *$\mathcal{P}$ is proper for countable models and for IC models of size $\aleph_1$.*
(2) *$\mathcal{P}$ has the $\aleph_3$-chain condition.*
(3) *$\mathcal{P}$ forces the existence of a strong ω_3-chain of subsets of ω_1.*

$\mathcal{P}$ uses side conditions from $\mathbb{S}_2$ in a crucial way.

This result is optimal:

Theorem 2.8. *(Inamdar [9]) There is no strong ω_3-chain of subsets of ω_2.*

A *strong ω_3-chain of functions from ω_1 into ω_1* is a sequence $(h_i : i < \omega_3)$ of functions $h_i : \omega_1 \to \omega_1$ such that for all $i_0 < i_1 < \omega_3$,

$$\{\tau \in \omega_1 : h_{i_1}(\tau) \leq h_{i_0}(\tau)\}$$

is finite.

Question 2.9. Is it consistent to have a strong ω_3-chain of functions from ω_1 into ω_1?

3. Extending strong properness to $\kappa > \omega$

The notion of strong properness can be naturally extended to higher cardinals:

Suppose κ is an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. A partial order $\mathcal{P}$ is *κ-strongly proper* iff for every $M \preccurlyeq H(\theta)$ such that $\mathcal{P} \in M$ and such that

- $|M| = \kappa$, and
- ${}^{<\kappa}M \subseteq M$,

every $\mathcal{P}$-condition in M can be extended to a strongly $(M, \mathcal{P})$-generic condition.

We will need the following closure property: Given an infinite regular cardinal κ, a partial order $\mathcal{P}$ is *$<\kappa$-directed closed with greatest lower bounds* in case every directed subset X of $\mathcal{P}$ (i.e., every finite subset of X has a lower bound in $\mathcal{P}$) such that $|X| < \kappa$ has a greatest lower bound in $\mathcal{P}$.

We will also say that *$\mathcal{P}$ is κ-lattice.*

All facts about strongly proper (i.e., ω-strongly proper) forcing we have seen extend naturally to κ-strongly proper forcing notions which are κ-lattice (assuming $\kappa^{<\kappa} = \kappa$).

For example, every κ-sequence of ordinals added by a forcing in this class belongs to a generic extension by adding a Cohen subset of κ.

Lemma 3.1. *(Reflection Lemma) Let κ be an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. Suppose $\mathcal{P}$ is a κ-lattice and κ-strongly proper forcing. If θ is large enough and $(M_i)_{i<\kappa^+}$ is a $\subseteq$-continuous $\in$-chain of elementary submodels of $H(\theta)$ such that $\mathcal{P} \in M_i$, $|M_i| = \kappa$, and ${}^{<\kappa}M_i \subseteq M_i$ for all $i \in S^{\kappa^+}_{\kappa}$, then $\mathcal{P} \cap N$ is κ-lattice and κ-strongly proper, for $N = \bigcup_{i<\kappa^+} M_i$.*

Proof. Let χ large enough and $M^* \preccurlyeq H(\chi)$ such that $\mathcal{P}$, $(M_i)_{i<\kappa^+} \in M^*$, $|M^*| = \kappa$ and ${}^{<\kappa}M^* \subseteq M^*$. Then $M^* \cap N = M_\delta \in N$ for $\delta = M^* \cap \kappa^+$. But every strongly $(M_\delta, \mathcal{P})$-generic is strongly $(M^*, \mathcal{P} \cap N)$-generic. $\square$

Compare the above reflection property with the reflection of κ-c.c. forcing to substructures M such that ${}^{<\kappa}M \subseteq M$.

Theorem 3.2. *(A.–Cox–Karagila–Weiss [3]) Assume* GCH *and let κ be infinite regular cardinal. Then there is a κ-lattice and κ-strongly proper forcing $\mathcal{P}$ which forces $2^\kappa = \kappa^{++}$ together with the κ-*Str PFA *(= FA_{κ^+}(κ-lattice + κ-strongly proper)).*

Proof sketch: Let $\theta = \kappa^{++}$. By first forcing with $\mathrm{Coll}(\kappa^+, {<}\theta)$, we may assume that $\diamondsuit(S^{\theta}_{\kappa^+})$ holds.

Our forcing $\mathcal{P}$ is $\mathcal{P}_\theta$, where $(\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta, :\ \alpha \in E \cup \{\theta\},\ \beta \in E)$, $E \subseteq S^{\theta}_{\kappa^{++}}$, is a ${<}\kappa$-support iteration à la Neeman with side conditions from $\mathbb{C}_2(\mathcal{S}, \mathcal{T})$, for

$$\mathcal{S} = \{M \,:\, |M| = \kappa,\ {}^{<\kappa}M \subseteq M\}$$

and

$$\mathcal{T} = \{N_\alpha \,:\, \alpha \in E\},$$

where $(N_\alpha \,:\, \alpha \in E)$ is some filtration of $H(\theta)$.

Condition are $p = (w_p, \mathcal{C}_p)$, where

- $\mathrm{dom}(w_p) \in [\theta]^{<\kappa}$;
- $\mathcal{C}_p \in \mathbb{C}_2(\mathcal{S}, \mathcal{T})$;
- for all $\alpha \in \mathrm{dom}(w_p)$, $N_\alpha \in \mathcal{C}_p$ and

$$(w_p \restriction \alpha, \mathcal{N}_p \cap N_\alpha) \Vdash_{\mathcal{P}_\alpha} \text{“}w_p(\alpha) \text{ is strongly } (M[\dot{G}_\alpha], \dot{\mathcal{Q}}_\alpha)\text{-generic”}$$

  for all $M \in \mathcal{C}_p \cap \mathcal{S}$ with $\alpha \in M$.

At stage α, if our diamond feeds us a $\mathcal{P}_\alpha$-name $\dot{Q}_\alpha$ for a κ-lattice κ-strongly proper forcing, then we let $\dot{\mathcal{Q}}_\alpha = \dot{Q}_\alpha$.

The Reflection Property is used to show that our construction captures κ-strongly proper forcings of arbitrary size.

The proof uses the fact that every κ-sequence of ordinals is in a κ-Cohen extension since each $\mathcal{P}_\alpha$ is κ-lattice and κ-strongly proper, which enables a typical model $N_\alpha \in \mathcal{T}$ to have access to the relevant $\mathcal{P}_\alpha$-names for κ-sized elementary submodels M (so the relevant $\dot{Q}_\alpha$'s are in fact such that $\Vdash_{\mathcal{P}_\alpha} \dot{Q}_\alpha$ is κ-strongly proper).

Also: The proof crucially uses the fact that our forcings are κ-lattice (it would not work if we just assumed ${<}\kappa$-directed closedness). $\square$

κ-Str PFA does not decide 2^κ. In fact:

Theorem 3.3. *Assume* GCH*, and let $\kappa < \kappa^+ < \kappa^{++} \leq \theta$ be infinite regular cardinals. Suppose $\diamondsuit(S^{\kappa^{++}}_{\kappa^+})$ holds. Then there is a κ-lattice and κ-strongly proper forcing $\mathcal{P}$ which forces $2^\kappa = \theta$ together with κ-*Str PFA.

Proof sketch: We build an iteration

$$(\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta \,:\, \alpha \in E \cup \{\kappa^{++}\}, \beta \in E)$$

as before, except that at each stage $\alpha \in E$ now we look at whether our diamond feeds us a $\mathcal{P}_\alpha \times \mathrm{Add}(\kappa, \kappa^+)$-name $\dot{Q}_\alpha$ for a κ-lattice and κ-strongly proper poset.[4] If so we let $\dot{\mathcal{Q}}_\alpha = \mathrm{Add}(\kappa, \kappa^+) * \dot{Q}_\alpha$.

The forcing witnessing the theorem is

$$\mathcal{P} = \mathcal{P}_{\kappa^{++}} \times \mathrm{Add}(\kappa, \theta)$$

To see this, take a suitable forcing in the extension via $\mathcal{P}$. By the Reflection Property it reflects to a forcing of size κ^{++}. Let $\dot{Q}$ be a $\mathcal{P}$-name for the corresponding forcing.

By κ^{++}-c.c. of $\mathcal{P}$ we may identify $\dot{Q}$ with a $\mathcal{P}_{\kappa^{++}} \times \mathrm{Add}(\kappa, \kappa^{++})$-name, which we may code by a subset of κ^{++}. Now we use our diamond to capture $\dot{Q}$ as in the proof of the previous theorem. $\square$

As far as I know this is the first example of a forcing axiom $\mathrm{FA}_{\kappa^+}(\Gamma)$ such that $\mathrm{FA}_{\kappa^{++}}(\Gamma)$ is false but nevertheless $\mathrm{FA}_{\kappa^+}(\Gamma)$ is compatible with 2^κ arbitrarily large. To see that $\mathrm{FA}_{\kappa^{++}}(\kappa\text{-lattice} + \kappa\text{-strongly proper})$ is false, one only needs to look at the forcing $\mathbb{P}$ of $<\kappa$-length $\in$-chains of suitable models $N \prec H(\kappa^{++})$ of size κ (this is $\mathbb{C}_1$ in this context). An application of $\mathrm{FA}_{\kappa^{++}}(\{\mathbb{P}\})$ would cover κ^{++} with a κ^+-chain of models of size κ.

κ-Str PFA does not seem to have many applications. It does imply $\mathfrak{d}(\kappa) > \kappa^+$, that the covering number of natural meagre ideals is $> \kappa^+$, and weak failures of Club-Guessing at κ, but not much more than that.

3.1. **Relaxing strongness or of g.l.b.'s?** Let us say that a forcing $\mathcal{P}$ is κ-MRP-*strongly proper* if for every large enough θ, every $M \prec H(\theta)$ of size κ such that ${}^{<\kappa}M \subseteq M$ and $\mathcal{P} \in M$, and every $p \in M \cap \mathcal{P}$ there is $q \leq_{\mathcal{P}} p$ such that for every $q' \leq_{\mathcal{P}} q$,

$$\mathcal{X}_{q'} = \{X \in [M]^\kappa \,:\, \exists \pi_X(q') \in \mathcal{P} \cap X \, \forall r \leq_{\mathcal{P}} \pi_X(q'), r \in X \longrightarrow r ||_{\mathcal{P}} q'\}$$

is M-stationary (i.e., for every club $E \in M$ there is some $X \in E \cap \mathcal{X}_{q'} \cap M$).

The reason we are using MRP in the notation above is of course that this weak form of strong properness is enjoyed by the standard forcing for adding an MRP-reflecting sequence with side conditions (in the classical case, i.e., when $\kappa = \omega$). More generally, $\mathrm{FA}_{\kappa^+}(\{\mathcal{P} \,:\, \mathcal{P}\ \kappa\text{-lattice and } \kappa\text{-}\mathsf{MRP}\text{-strongly proper}\})$ implies a natural high analogue of MRP which in turn implies $2^{\kappa^+} = \kappa^{++}$. Unfortunately, it also implies too much:

Theorem 3.4. *Suppose $\kappa \geq \omega_1$ is a regular cardinal and $\kappa^{<\kappa} = \kappa$. Then*

$$\mathrm{FA}_{\kappa^+}(\{\mathcal{P} \,:\, \mathcal{P}\ \kappa\text{-lattice, } \kappa^+\text{-c.c., and } \kappa\text{-}\mathsf{MRP}\text{-strongly proper}\})$$

is false.

This theorem can be proved using the inconsistent uniformization principle highlighted in the following result.

Theorem 3.5. *(Shelah) Let $\kappa \geq \omega_1$ be a regular cardinal and let $\langle C_\alpha \,:\, \alpha \in S^{\kappa^+}_\kappa \rangle$ be a club-sequence. Then there is a sequence*

$$\langle f_\alpha \,:\, \alpha \in S^{\kappa^+}_\kappa \rangle$$

of colourings, with $f_\alpha : C_\alpha \to 2$ for all α, for which there is no function

$$G : \kappa^+ \to 2$$

such that for all $\alpha \in S^{\kappa^+}_\kappa$,

$$G(\xi) = f_\alpha(\xi)$$

for club-many $\xi \in C_\alpha$.

[4] For technical reasons, we actually consider forcings belonging to a slightly larger class (referring to the original ground model V for its definition).

Now let $\langle C_\alpha : \alpha \in S^{\kappa^+}_\kappa \rangle$ be a club-sequence and $\langle f_\alpha : \alpha \in S^{\kappa^+}_\kappa \rangle$ be a sequence of colourings which cannot be club-uniformized. Let $\mathcal{P}$ be the forcing consisting of $<\kappa$-sized functions p with $\mathrm{dom}(p) \subseteq S^{\kappa^+}_\kappa$ such that

- for all $\alpha \in \mathrm{dom}(p)$, $p(\alpha) < \alpha$, and
- for all $\alpha_0 < \alpha_1$ in $\mathrm{dom}(p)$, if $\xi \in (C_{\alpha_0} \setminus p(\alpha_0)) \cap (C_{\alpha_1} \setminus p(\alpha_1))$, then $f_{\alpha_0}(\xi) = f_{\alpha_1}(\xi)$.

Then $\mathcal{P}$ is κ^+-c.c., κ-lattice, and κ-MRP-strongly proper, so an application of $\mathrm{FA}_{\kappa^+}(\{\mathcal{P}\})$ gives us a function $G : \kappa^+ \to \{0,1\}$ which in fact uniformizes the sequence of colourings $\langle f_\alpha : \alpha \in S^{\kappa^+}_\kappa \rangle$ modulo co-bounded sets — for each $\alpha \in S^{\kappa^+}_\kappa$ there is $p(\alpha) < \alpha$ such that $G(\xi) = f_\alpha(\xi)$ for all $\xi \in C_\alpha \setminus p(\alpha)$. $\square$

Existence of greatest lower bounds cannot be relaxed either:

Theorem 3.6. *(Shelah) Suppose $\kappa \geq \omega_1$ is a regular cardinal and $\kappa^{<\kappa} = \kappa$. Then*

$$\mathrm{FA}_{\kappa^+}(\{\mathcal{P} : \mathcal{P} \ <\kappa\text{-directed closed, } \kappa^+\text{-c.c., and } \kappa\text{-strongly proper}\})$$

is false.

This is similar to the previous proof, with a natural forcing for adding $G : \kappa^+ \to \{0,1\}$ and clubs $D_\alpha \subseteq C_\alpha$ (for $\alpha \in S^{\kappa^+}_\kappa$) such that $G(\xi) = f_\alpha(\xi)$ for all α and all $\xi \in D_\alpha$.

4. κ-strong semiproperness

Let κ be an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. Let us say that a forcing notion $\mathcal{P}$ is *κ-strongly semiproper* if and only if for every large enough θ and every $M \prec H(\theta)$ such that $\mathcal{P} \in M$, $|M| = \kappa$, and ${}^{<\kappa}M \subseteq M$, every $p \in \mathcal{P} \cap M$ can be extended to some $q \in \mathcal{P}$ which is *κ-strongly $(M, \mathcal{P})$-semigeneric*, i.e., there is some $\sigma \in [H(\theta)]^{\leq\kappa}$ such that

(1) $\mathrm{Hull}(M, \sigma) \cap \kappa^+ = M \cap \kappa^+$, and
(2) q is strongly $(\mathrm{Hull}(M, \sigma), \mathcal{P})$-generic.

Given an infinite regular κ, let the *κ-Strongly Semiproper Forcing Axiom* be

$$\mathrm{FA}_{\kappa^+}(\{\mathcal{P} : \mathcal{P} \ \kappa\text{-lattice and } \kappa\text{-strongly semiproper}\})$$

We will now consider a family of high analogues of the Strong Reflection Principle. Given an infinite regular κ and a cardinal $\mu \leq \kappa$, let $\mathsf{SRP}(\kappa^+, \mu)$ be the following reflection principle: Suppose X is a set and $\mathcal{S} \subseteq [X]^\kappa$. If θ is such that $X \in H(\theta)$, there is a $\subseteq$-continuous $\in$-chain $(M_i)_{i<\kappa^+}$ such that for each $i < \kappa^+$, $M_i \prec H(\theta)$ and $|M_i| = \kappa$, and if $\mathrm{cf}(i) = \kappa$:

- $M_i \cap X \notin \mathcal{S}$ if and only if there is no $\sigma \in [X]^{\leq\mu}$ such that
  (a) $\mathrm{Hull}(M_i \cup \sigma)$ is a κ^+-end-extension of M (i.e., $\mathrm{Hull}(M_i \cup \sigma) \cap \kappa^+ = M_i \cap \kappa^+$), and
  (b) $\mathrm{Hull}(M_i \cup \sigma) \cap X \in \mathcal{S}$.

It is easy to see that the κ-Strongly Semiproper Forcing Axiom implies $\mathsf{SRP}(\kappa^+, \kappa)$. But, again, this axiom implies too much:

Theorem 4.1. *For every $\kappa \geq \omega_1$, $\mathsf{SRP}(\kappa^+, \omega)$ is false. In particular, the κ-Strongly Semiproper Forcing Axiom is false.*

Proof. Let $\mathcal{S}$ be the collection of $X \in [\kappa^{++}]^\kappa$ such that $\mathrm{cf}(X) = \omega$. By an application of $\mathsf{SRP}(\kappa^+, \omega)$ to $\mathcal{S}$ there is a $\subseteq$-continuous $\in$-chain $(M_i)_{i<\kappa^+}$ of models of size κ such that for each $i < \kappa^+$ such that $\mathrm{cf}(i) = \kappa$, if $\mathrm{cf}(M_i \cap \kappa^{++}) \neq \omega$, then there is no countable $\sigma \subseteq \kappa^{++}$ such that

- $\mathrm{Hull}(M_i \cup \sigma) \cap \kappa^+ = M_i \cap \kappa^+$ and
- $\mathrm{cf}(\mathrm{Hull}(M_i \cup \sigma) \cap \kappa^{++}) = \omega$.

Claim 4.2. *The set*

$$S = \{i \in S^{\kappa^+}_\kappa \ :\ \text{there is no countable } \sigma \subseteq \kappa^{++} \text{ as above for } M_i\}$$

cannot be stationary.

Proof. Suppose S is stationary. Let $\alpha \in \kappa^{++}$, $\mathrm{cf}(\alpha) = \omega$, such that $F``[\alpha]^{<\omega} \cap \kappa^{++} \subseteq \alpha$ for some $F : [H(\lambda)]^{<\omega} \to H(\lambda)$ generating a club of elementary submodels R such that $(M_i)_{i<\kappa^+} \in R$.

Now we can easily find $X \subseteq \alpha$ cofinal in α such that $R = F``[X]^{<\omega}$ is such that $|R| = \kappa$ and $i := R \cap \kappa^+ \in S$. Let $\sigma \subseteq X$ be countable and cofinal in X. But then R is a κ^+-end-extension of M_i and $\mathrm{cf}(R \cap \kappa^{++}) = \omega$, and so σ witnesses that $M_i \notin S$. Contradiction. $\square$

Now we get club-many i such that if $\mathrm{cf}(i) = \kappa$, then $\mathrm{cf}(M_i \cap \kappa^{++}) = \omega$. But this is impossible since $(\sup(M_i \cap \kappa^{++}))\ :\ i < \kappa^+)$ is strictly increasing and continuous and therefore $\mathrm{cf}(M_i \cap \kappa^{++}) = \kappa > \omega$ if $\mathrm{cf}(i) = \kappa$. $\square$

Recall that, given an infinite regular κ and a stationary $S \subseteq \kappa^+$, $\mathrm{NS}_{\kappa^+} \restriction S$ *is saturated* iff every collection $\mathcal{A}$ of stationary subsets of S such that $S_0 \cap S_1$ is nonstationary for all $S_0 \neq S_1$ in $\mathcal{A}$ is such that $|\mathcal{A}| \leq \kappa^+$.

The standard argument for deriving the saturation of NS_{ω_1} from the classical SRP shows in fact the following.

Fact 4.3. *If κ is an infinite regular cardinal,* $\mathsf{SRP}(\kappa^+, 1)$ *implies that* $\mathrm{NS}_{\kappa^+} \restriction S^{\kappa^+}_\kappa$ *is saturated.*

Let us call a forcing $\mathcal{P}$ is *κ-strongly 1-semiproper* iff it satisfies the definition of 'κ-strongly semiproper' replacing $\mathrm{Hull}(M, \sigma)$, for $|\sigma| \leq \kappa$, with $\mathrm{Hull}(M, \sigma)$, for $|\sigma| \leq 1$.

κ-strong 1-semiproperness is the least demanding excursion of κ-strong properness into the realm of semiproperness.

$$\mathrm{FA}_{\kappa^+}(\{\mathcal{P}\ :\ \mathcal{P}\ \kappa\text{-lattice, } \kappa\text{-strongly 1-semiproper}\})$$

implies $\mathsf{SRP}(\kappa^+, 1)$ and therefore the saturation of $\mathrm{NS}_{\kappa^+} \restriction S^{\kappa^+}_\kappa$.

Question 4.4. Is $\mathrm{FA}_{\kappa^+}(\{\mathcal{P}\ :\ \mathcal{P}\ \kappa\text{-lattice, } \kappa\text{-strongly 1-semiproper}\})$ consistent for any $\kappa \geq \omega_1$?

Question 4.5. Suppose $\kappa \geq \omega_1$ is regular and $\mathrm{NS}_{\kappa^+} \restriction S^{\kappa^+}_\kappa$ is saturated. Does it follow that GCH cannot hold below κ?

5. On high properness when adding reals

Neeman considers side conditions consisting of *nodes* of either of the following types.

(1) (Countable type elementary) These are models $M \prec H(\theta)$ such that $|M| = \aleph_0$.
(2) (Type ω_1) These are IC models $N \prec H(\theta)$ such that $|N| = \aleph_1$.
(3) (Countable type tower.) These are countable $\in$-chains $\mathcal{T}$ of nodes of type ω_1 such that $\mathcal{T} \cap N \in N$ for all $N \in \mathcal{T}$.

Definition 5.1. (Neeman) A *two-size side condition* is a finite set $\mathcal{N}$ of nodes of the above types which is $\in$-increasing (i.e., every node belongs to the next), and closed under intersection in the sense that:

- If N, $M \in \mathcal{N}$, $N \in M$, N of type ω_1, and M countable elementary, then $M \cap N \in \mathcal{N}$.
- If N, $\mathcal{T} \in \mathcal{N}$, $N \in \mathcal{T}$, $\mathcal{T}$ of type tower, and $\mathcal{T} \cap N \neq \emptyset$, then there is a tower $\mathcal{T}' \supseteq \mathcal{T} \cap N$ occurring in $\mathcal{N}$ before N.

Definition 5.2. (Neeman) A partial order $\mathcal{P}$ is *two-size proper* if for every large enough θ there is a function $f : [H(\theta)]^{<\omega} \to H(\theta)$ such that for every two-size side condition $\mathcal{N}$ with all models involved closed under f, every $Q \in \mathcal{N}$, and every $p \in \mathcal{P} \cap Q$, if p is $(R, \mathcal{P})$-generic for every $R \in \mathcal{N} \cap Q$, then there is $q \leq_{\mathcal{P}} p$ which is $(R, \mathcal{P})$-generic for all $R \in \mathcal{N}$. (If $\mathcal{T}$ is a tower, a condition is $(\mathcal{T}, \mathcal{P})$-generic iff it is $(N, \mathcal{P})$-generic for all $N \in \mathcal{T}$.)

Theorem 5.3. *(Neeman) If κ is a supercompact cardinal, then there is a partial order $\mathcal{P} \subseteq V_\kappa$ forcing* $\mathrm{FA}_{\aleph_2}(\{\mathcal{P} : \mathcal{P}$ *two-size proper*$\})$.

We can now make the following definition.

Definition 5.4. A partial order $\mathcal{P}$ is *two-size strongly semiproper* if for every large enough θ there is a function $f : [H(\theta)]^{<\omega} \to H(\theta)$ such that for every two-size side condition $\mathcal{N}$ with all models involved closed under f, every $Q \in \mathcal{N}$, and every $p \in \mathcal{P} \cap Q$, if p is $(R, \mathcal{P})$-strongly ω_2-semigeneric for every $R \in \mathcal{N} \cap Q$, then there is $q \leq_{\mathcal{P}} p$ which is $(R, \mathcal{P})$-strongly ω_2-semigeneric for all $R \in \mathcal{N}$.

Theorem 5.5. $\mathrm{FA}_{\aleph_2}(\{\mathcal{P} : \mathcal{P}$ *two-size strongly semiproper*$\})$ *implies* $\mathsf{SRP}(\omega_2, \omega)$ *and therefore it is inconsistent.*

Two-size strong 1-semiproperness is the least demanding excursion of two-size properness into the realm of semiproperness. And

$$\mathrm{FA}_{\aleph_2}(\{\mathcal{P} : \mathcal{P} \text{ two-size strongly 1-semiproper}\})$$

implies $\mathsf{SRP}(\omega_2, 1)$.

Question 5.6. Is $\mathrm{FA}_{\aleph_2}(\{\mathcal{P} : \mathcal{P}$ two-size strongly 1-semiproper$\})$ consistent?

In joint work with Veličković, and using forcing with virtual models with generators, we do get consistency of a shadow of $\mathsf{SRP}(\omega_2, 1)$ but which unfortunately doesn't seem to be enough to get saturation of $\mathrm{NS}_{\omega_2} \restriction S^{\omega_2}_{\omega_1}$.

5.1. **On high stationary reflection and $2^{\aleph_0}$.** Regarding the connection between reflection principles following from strong forcing axioms and cardinal arithmetic, we recall that the classical WRP implies $2^{\aleph_0} \leq \aleph_2$ and is consistent with both $2^{\aleph_0} = \aleph_1$ (just Lévy collapse a supercompact cardinal to become ω_2) and $2^{\aleph_0} = \aleph_2$ (WRP follows from MM). We now very briefly consider the prospect of having similar phenomena at higher cardinals as this is a feature we would like strong high forcing axioms to have. In this respect we have:

Theorem 5.7. *(Sakai)*

(1) $\mathsf{WRP}_{\omega_1} \restriction IA_\omega$ *implies* $2^{\aleph_0} \leq \aleph_3$.[5]
(2) *If κ is supercompact, then the $\aleph_1$-support iteration of length κ with mixed support for collapsing α to ω_2 (for $\alpha < \kappa$) with conditions of size $\aleph_1$ while also adding Cohen reals forces* $\mathsf{WRP}_{\omega_1} \restriction IA_\omega + 2^{\aleph_0} = \aleph_3$.

While suitable high reflection principles both imply the bound $2^{\aleph_0} \leq \aleph_3$ and are compatible with $2^{\aleph_0} = \aleph_3$, we do not know of any such principle which actually decides $2^{\aleph_0}$.

Question 5.8. Is there any consistent high analogue R^* of any reflection principle R following from MM^{++} such that R^* implies $2^{\aleph_0} = \aleph_3$?

[5] We are not defining $\mathsf{WRP}_{\omega_1} \restriction IA_\omega$ but just point out that it holds after Lévy collapse a supercompact cardinal to become ω_3.

5.2. **(Strong high) bounded forcing axioms: final questions.** Strong classical forcing axioms, like MM or even BPFA, are known to not only imply $2^{\aleph_0} = \aleph_2$, but in fact to entail the truth of Π_2 sentences over $H(\omega_2)$ which in turn yield the existence of well-orders of $H(\omega_2)$ simply definable over $H(\omega_2)$ from some parameter (for example MM implies ψ_{AC}, which is such a statement, and another one, also following from Moore's MRP, is implied by BPFA). Maximization of Π_2 truth being of course in the spirit of forcing axioms, it is natural to ask if anything like this applies to high forcing axioms (replacing $H(\omega_2)$ with $H(\omega_3)$). More generally, one can ask the following.

Question 5.9. Is there any Π_2 sentence σ such that the following holds?

(1) ZFC proves that if $H(\omega_3) \models \sigma$, then $2^{\aleph_0} = \aleph_3$.
(2) For some reasonable large cardinal axiom LC, ZFC+ LC proves that it is forcible that $H(\omega_3) \models \sigma$.

Let us now go back down to $H(\omega_2)$ and let us note that all known proofs of $2^{\aleph_0} = \aleph_2$ from BPFA involve codings of reals by ordinals that one gets from MRP, and therefore use forcing which is very badly non-ω-proper.[6] It is then natural to enquire whether similar codings can be carried out using just ω-proper forcing.

Conjecture 5.1. $\mathsf{BFA}(\{\mathcal{Q} : \mathcal{Q}\ \omega\text{–}proper\})$ *implies* $2^{\aleph_0} = \aleph_2$.

We are stating this as a conjecture as building a model of $\mathsf{BFA}(\{\mathcal{Q} : \mathcal{Q}\ \omega\text{–proper}\})$ with large continuum looks hopeless. But if the conjecture is true, that suggests that there is a whole family of coding techniques in this context waiting beyond MRP.

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[6] Although the original proofs of $2^{\aleph_0} = \aleph_2$ from PFA use only ω-proper forcing, and in fact forcing of the form σ-closed $*$ c.c.c.