On the Mathematical Sublime

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A recent email exchange between myself and Boris Zilber, stimulated by a lecture he gave in Helsinki a few years prior, began by discussing his various moves to generalize the sytnax/semantics distinction. Boris's repurposingof the syntax/semantics distinction—a distinction taken more or less for granted in foundational practice—has always been interesting...

. . . but in our exchange Boris also broke out *philosophically:*

These are, I guess, two ways of how we perceive the world: the intellectual, words-based way, and the intuitive, sensory way. In mathematics, the first way requires you to write down a full proof of the fact (the ultimate explanation). The second, semantical way, is to see a picture, mental or graphical, that talks to your experience of the world. It is also what is responsible [for the] division of mathematics into Algebra and Geometry. Michael Atiyah (in his millennium lecture?) says that Geometry-Algebra is like Space-Time pairing: In geometry you see the whole at once, no time needed. In algebra you need time to read it letter-by-letter, but not space.

The words-based way and the semantical way, to wit: the mathematician is tethered to the sign, to formal correctness and to the "letter-by-letter" of proof; while on the other hand there is insight and experience, meaning and seeing the whole picture. Two poles pulling away from each other, and the mathematician caught somewhere in between.

I would like to think about the way Boris pulls the curtain back on this binary practice of the mathematician, in his rich remark to me, so full of philosophical moves. One is struck by the phrase "seeing the whole at once, no time needed"—a move toward the sublime, I suggest, an aesthetic category important in 18th century British philosophy and of renewed interest today in the form of, for example, the environmental sublime.

Gerard

Alfred Gerard: sublimity is the state in which "the mind . . . imagines itself present in every part of the scene it contemplates."

Fred Sandback

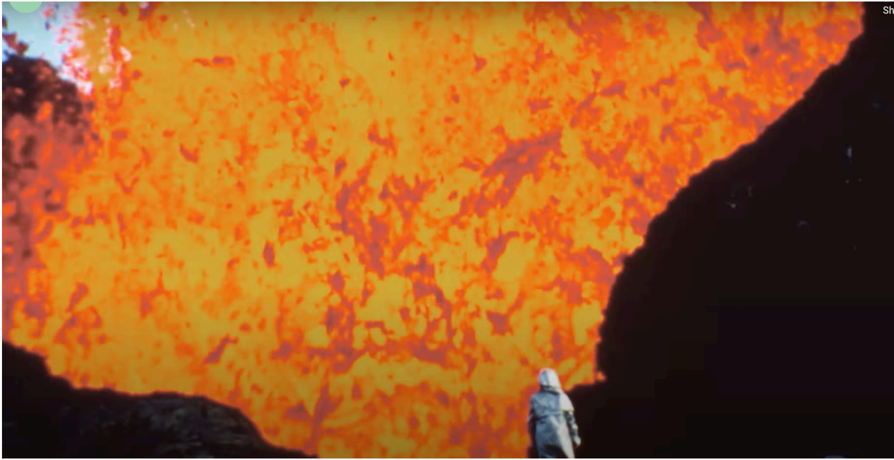
“The idea of “overall” painting was much more stimulating to me at the time than were the particular paintings.”

More commonly the sublime is thought of in the terms Kant laid out for it, namely in terms of a physical immensity, usually in nature—think of standing at the precipice of an enormous crevasse—that pitches the subject into a kind of vertigo; “doing violence to the imagination,’ as Kant put it; leaving the subject’s cognitive apparatus undone.

Emily Brady on the Kantian Sublime

The sources of the sublime response are linked to the physical properties of magnitude or power in nature but importantly also to the failure of imagination, without which it could not occur. Imagination's activity in the sublime, in contrast to the beautiful, is 'serious', where some object is "contrapurposive for our power of judgment, unsuitable for our faculty of presentation, and as it were doing violence to our imagination," but is nevertheless judged all the more sublime for that.

This “astonishment bordering on terror,” as Kant called it, involves anxiety, bordering on fear, but also, somehow, pleasure: the pleasure of being in the vicinity of danger, while at the same time being out of it; the pleasure of being in awe of something.



Kant's term for this was *negative pleasure*, while *positive* pleasure is pleasure in the beautiful, which "brings with it a promotion of life."

In its encounter with the sublime the body seizes up; Kant's point was that the mind does too—the imagination struggles to experience the scene as a phenomenal unity, but fails to do so. Interestingly enough, because pleasure is involved, sublimity is theorized by philosophers an *aesthetic* category.

The sublime response, that "shudder that seizes the human being himself at the representation of the sublime, and the horror with which nurses' tales drive children to bed late at night" is an *aesthetic* response.

Kant distinguishes the *dynamic* sublime, in which the subject is undone, so to speak, by a natural scene, from the *mathematical* sublime, in which the subject experiences a failure of the imagination, not in the face of a natural immensity but in the face of an infinite number sequence. In the encounter with the mathematical sublime the subject is thrown into a sense of unease and conceptual confusion once again, for not having a grip on the contours of the thing at hand; but also being inexorably compelled—pulled in, almost against one's will. One might call this mixture of attraction and unease the mathematician's negative pleasure.

Kant's observation was that although the senses fail to deliver a conceptual unity on their own, the sequence can nevertheless "be completely comprehended under one concept," and this is due to a faculty of "suprasensible" reason:

> *And what is most important is that to be able only to think it [the infinite: JK] as a whole indicates a faculty of mind which surpasses every standard of sense...Nevertheless, the bare capability of thinking this infinite without contradiction requires in the human mind a faculty itself suprasensible.*[1]

[1] Kant, *Critique of Judgement*.

Kant gives us what the (classical) mathematician would say is the correct outcome. Reason meets the imagination at its point of collapse, delivering the infinite object as a conceptual unity—just as reason delivers a phenomenal unity in the case of overwhelming natural phenomena. The important point here is that this faculty of reason is supra-sensible, so going beyond (sense) experience.

For Gerard, who identified sublimity with overallness, as we saw, sublime experience is cognizable not because of some special supra-sensible faculty of reason; it is rather simplicity that brings the sublime to earth:

Objects cannot possess that largeness, which is necessary for inspiring a sensation of the sublime, without simplicity. Where this is wanting, the mind contemplates, not one large, but many small objects: it is pained with the labour requisite to creep from one to another; and is disgusted with the imperfection of the idea, with which, even after all this toil, it must remain contented. But we take in, with ease, one entire conception of a simple object, however large: in consequence of this facility, we naturally account it one . . . the view of any single part suggests the whole, and enables fancy to extend and enlarge it to infinity, that it may fill the capacity of the mind.

Sublimity, then, is not an experience of defeat, or not wholly; one is able to move out of it with the help of the mind's ability to synthesize the atomized scene—to structure chaos as non-chaos, to paraphrase the artist Eva Hesse—by means of a conception.

The sublime has a moral dimension, putting us in touch, as Brady writes, with our moral capacities. The sublime tutors us in "[loving] something, even nature, without interest. . . even contrary to our (sensible) interest."

Witnessing the failure of the imagination, the failure of her imagination to comprehend the scene, the subject remains "undemeaned," as Kant put it, even so, and even has a feeling of superiority over nature, or in our case, the mathematical field, while at the same time "the human being must submit to that dominion."

Sublimity, in other words, is always connected to *power*.

Later, post-Kantian and post-Gerardian passes at the sublime by writers such as F.R. Ankersmit, would untether sublimity from awe and the idealization of nature that was characteristic of the earlier theories, so that the sublime could now be deployed in other domains, such as history, or psychoanalysis.

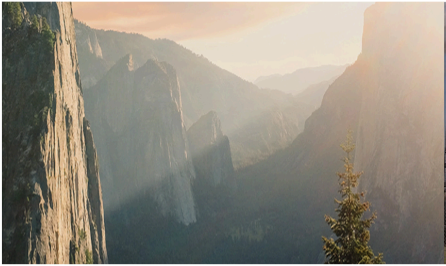
F.W. Ankersmit

The traumatic experience is too terrible to be admitted to consciousness: The experience exceeds, so to speak, our capacities to make sense of experience. Whereas normally the powers of association enable us to integrate experience into the story of our lives, the traumatic experience remains dissociated from our life's narrative since these powers of association are helpless and characteristically insufficient in the case of trauma. Characteristic of trauma is the incapacity to actually suffer from the traumatic experience itself. . . The subject of a traumatic experience is peculiarly numbed by it; he is, so to speak, put at a distance from what caused it. The traumatic experience is dissociated from one's "normal" experience of the world. . .

. . . Now, much the same can be observed for the sublime. When Burke speaks about this "tranquility tinged with terror," this tranquility is possible (as Burke emphasizes) thanks to our awareness that we are not really in danger. Hence, we have distanced ourselves from a situation of real danger—and in this way, we have dissociated ourselves from the object of experience. The sublime thus provokes a movement of derealization by which reality is robbed of its threatening potentialities. As such Burke's description of the sublime is less the pleasant thrill that is often associated, with it than a preemptive strike against the terrible.

A century later the critic Leo Marx would coin the term, the *technological sublime* to describe the conflict arising from holding the romantic (sublime) conception of the American landscape of the late 19th century, seeing that terrain as a kind of virginal paradise, while employing the rhetoric of industrial progress. And just a few years after that Hilbert would lace his oft-cited 1930 "ignorabimus" address with the language of human supremacy, expressed in terms of the technological optimism typical of the period.

The technological sublime



Ankersmit takes a *melancholic* view of sublimity, emphasising the static quality of the sublime response, the idea that the subject is locked into a back and forth cycle of attraction and repulsion.

Sublimity, in other words, is a site of conflict:

Now, aesthetics provides us with the category of the sublime for conceptualizing such a conflict of schemes without reconciliation or transcendence. Thus the Kantian sublime is not a transcendence of reason and understanding and the entry to a new and higher order reality, but can only be defined in terms of the inadequacy of both reason and understanding. . . Similarly, it is only by way of the positive numbers that we can get access to the realm of negative numbers; and gaining this access does not in the least imply the abolition or transcendence of the realm of the positive numbers, but a continuous awareness of their existence as well.

Kant's account of the role of intuition and reason in delivering conceptual coherence within sublimity is embedded in a complex theory of the mind, one drawing on specific conceptualizations of the faculties of imagination and reason.

In the philosophy of mathematics Kant's forefronting the role of intuition in cognizing the mathematically infinite was and continues to be relevant—for example, for Gödel. One might say that Kant's theory of the mathematical sublime is about our mathematical capacities überhaupt, and as such it slots easily into the contemporary conversation in the foundations of mathematics, the debates about the nature of finitary intuition, or what constitutes a genuinely constructive proof.

What rather holds my interest in thinking through Boris's beautiful remark though, are not the foundational issues per se, but the way his remark reveals that logic too is a site of conflict; a conflict that gets read into the syntax/semantics distinction; a conflict that renders logic so alive philosophically. It is astonishing that logic can even take the exact mathematical measure of that conflict, that is to say drawing out deep theorems from it, limitative results such as the Incompleteness Theorems due to Gödel, or the Undefinability of Truth due to Tarski.

Mathematics

In Ankersmit's writings the mathematical sublime is brought home to mathematics—domesticated—so that mathematical sublimity now signifies anything in the way of a mathematical unknown:

Think of the equation $f(x) = 1/3x^3 + 1/2x^2 - 12x$. *Differential calculus shows that this function will have a local maximum for* $x = -4$ *and a local minimum for* $x = 3$. *In this way differential calculus can be said to perform what, analogously, could not possibly be performed for the relationship between narrative and experience. So one might say that historical writing is in much the same situation as mathematics was before the discovery of differential calculus by Newton and Leibniz. Before this discovery there was something "sublime" about the question of where the equation* $f(x) = 1/3x^3 + 1/2x^2 - 12x$ *would attain its local optimum and minimum...*

. . . One could only hit on it experimentally (that is, by simply trying out different values for x), but no adequate explanation could be given for this. It has been Newton's and Leibniz's feat of genius to reduce what was "sublime" to what could be figured out, or to reduce what was incommensurable to what could be made commensurable thanks to the magic of differential calculus.

Ankersmit is thinking about sublime historical experience in this passage, but we can draw the moral from it that Kant's notion of the mathematical sublime (which applied only to extended objects), was too narrow. It is not just that the imagination cannot take in infinite totalities, the mathematical field is full of concepts and ideas that cause the mathematician to lose his footing. Model theory presents concepts and ideas in the face of which the mathematical sensorium fails, in the sense of, say, resisting classification, e.g. the space of all first order theories—how to find a way through that morass?

Set theory also, with its large cardinal hierarchy, is threaded with sublimity through and through.

Let us now turn to Boris's work, in particular its synthetic aspect within what one might call the *model-theoretic sublime*. As a working definition let us take "synthesis" to refer to an act of (mathematical) reason that structures some heretofore unstructured part of the mathematical field—unstructured in the sense of being untheorized, or unclassified, or simply formless.

The suggestion here is that both categoricity and classification can be viewed as synthetic acts; devices imposing structure on the mathematical field, albeit in different ways: Categoricity, a notion occupying a central place in Boris's mathematical work, by collapsing the space of all possible models (of a fixed cardinality) of an uncountably categorical theory to a single point (up to isomorphism); classifiability in virtue of being something like an organizing principle, a kind of scaffolding structure for the space of first order theories.

Logical perfection, or: Gerard redux

. . . a mathematical object of a certain "size" is logically perfect *if in a certain formal language it allows a "concise" description fully determining the object.*

Villaveces on categoricity

When faced with certain descriptions or statements, our natural reaction of disbelief can be seen as one of the roots of the search to capture, apprehend, through language, the description of a phenomenon, of a mathematical object. or an event. When faced with a statement (mathematical or not), the first natural reaction in many circumstances is usually disbelief. When in doubt, we try to seek confirmation no matter wherefrom. Leaving aside verification by authority, we can point out two main types of confirmation: by direct verification, or by a good [i.e. categorical: JK] description of the theory that supports the statement in question.

Categorical theories are logically perfect not only because they provide a compact description of a seemingly intractable field of concepts, but for enabling the possibility of regarding space as a coherent way of pasting localised versions of itself—a perfection realized, for Boris and co-authors in "On Logical Perfection," by the notion of an affine scheme due to Grothendieck.

Recall Gerard: In sublimity, the "the view of any single part suggests the whole."

Another way Boris delivers synthesis is through his classification program. Instead of a heterogeneous collection of theories (so theories this time, instead of models), and the mathematician having to creep from one theory to the next (as Gerard put it), Boris offered up the Trichotomy Conjecture, which turned out to hold of the (very ample) Zariski structures.

Trichotomy Conjecture

If X is a strongly minimal set, then exactly one of the following is true about X.

- X is trivial in the sense that algebraic closure (on a saturated model of the theory of X) defines a degenerate pregeometry (for any set $A \subseteq X$ one has $acl(A) = \bigcup\{acl(a) | a \in A\}$).
- X is essentially a vector space.
- X is bi-interpretable with an algebraically closed field.

Classification theorems in mathematics, then, as a move toward synthesis; resisting or dissolving sublimity; structuring the heretofore unstructured mathematical field as non-chaos; providing a scaffolding.

Together with Boris's work on Trichotomy one should mention Shelah's Main Gap Theorem, which is another masterpiece in the genre of classification theorems. The theorem states that the class of all first order theories falls into two categories: the tame or classifiable, and the nonclassifiable. The former have "few" models and admit a dimension-like set of geometric invariants; the latter have the maximum number of models possible, and are entangled with each other in a way that makes it difficult to tell some of them apart.

The Main Gap Theorem almost seems to be written in the language of the sublime!

Logically perfect structures admit a geometry

Perhaps the most remarkable feature of model-theoretic classification theory is that it exposes a geometric nature of some "perfect" structures. The geometric features of those structures arise from their logical definition, albeit in a highly non-trivial and initially unforeseen way... It took a while to realise the geometric character of the technical definitions and to develop a new geometric intuition around the notions. In particular, Morley rank is a very good analogue of dimension in algebraic and analytic geometry and thus we can think of "curves", "surfaces" and so on in the very general context of categorical and even stable theories.

This is a key moment in model theory, that moment when the discipline begins to transform its central concepts into geometrical ones. The development of model theory toward geometry has roots in Tarski's work, in particular in his program to "normalize" (Tarski's word) metamathematical concepts by means of what he called "the mathematical"—if not in the earlier work of the algebraic school in logic associated with figures like Peirce and Schröder.

Michael Harris:

There is an important sense in which answers to questions in number theory are widely seen as more natural or conceptual if they are seen to arise from geometric constructions. This is more a matter of habit than of any official consensus. . .

Zilber: "…in geometry you see the whole at once, no time needed."

What is this "whole" that Boris sees at once, no time needed?

Perhaps what geometry allows one to see, perhaps what geometry brings into logic, is some suggestion of place—not in any literal sense but in the sense that the architectural theorist Juhani Pallasmaa means in his writings about placeness: a site of experiential cohesion; one resonating "with the inner qualities of placeness in our minds. . . a constitutive condition for anything to exist in human consciousness."

The experience of placeness can. . . arise from countless characteristics and features, but fundamentally it is a consequence of experiential cohesion, spatial or formal singularity, communal agreement, or meaningfulness of a distinct entity in the physical world. . . Through constructions, both material and mental, useful and poetic, practical and metaphysical, we create places, existential footholds in the otherwise meaningless world.[2]

[2] From Pallasmaa's "Space, Place, and Atmosphere: Peripheral Perception in Existential Experience", in *Architectural Atmospheres: On the Experience and Politics of Architecture.*

The thought here is that through geometry and its suggestion of place, through thinking of geometry as creating the conditions for the notion of place, the mathematician is led toward the possibility of concretizing, structuring, contextualizing and internalizing mathematical ideas.

Note that we take places in at once, no time needed. As Pallasmaa puts it, "We "understand" qualities of places unconsciously before we have had any chance for intellectual evaluation or understanding."

Here geometry stands in for, in the sense of functioning as, architecture, in grounding the mathematician in the mathematical field; in enabling the possibility of lived mathematical experience.

There is also the ontological question: Is anything real in mathematics, that is not related to geometry? “Nothing is that is not placed,” as Plato reportedly said.

Going letter by letter

The words-based way—or simply, syntax traditionally construed, i.e. in Hilbert's sense of being uninterpreted strings of symbols—acts exactly against this idea of place. The mathematician "going letter-by-letter," to quote Boris, does not see "the whole," never mind seeing it all at once; he is rather drawn into a kind of dislocated, humdrum temporality—placeless, in Pallasmaa's sense of place, that is, experienced as a totality; the opposite of no time needed.

It would be remiss not to acknowledge that, at the same time, Boris speaks of the intellect in opposition to sense; of language and "ultimate explanations"; of the fact that mathematics must be put into language; that proofs must be written down.

Rather than lleaning here to some variety of the syntactic point of view, Boris simply pointing out that this is the mathematician's contract, this is what is demanded of him professionally.

Il n'y a pas de hors texte, as Derrida famously said, and this is a correct statement, at least from the professional point of view.

But really with going letter-by-letter, we are in Wittgenstein territory. Wittgenstein used the term "surveyability" to direct our attention to that moment where a readable string of symbols, or indeed the more complex phenomena facing the mathematician, tip over into illegibility. Sublimity is in the air here, in the conventional sense that unsurveyability, or unsubitizability, involves a failure of cognition.

With surveyability we are in possession of a picture:

> *When I say "the proof is a picture"—it can be thought of as a cinematographic picture. We construct the proof once and for all.*[3]

[3] Wittgenstein, PI

Intersubjectivity

J. Floyd on surveyability: "…communicability, reproducibility and intelligibility …lie at the heart, not only of Hilbert's foundational enterprise, but of the wider logico-philosophical tradition stemming from Frege, Russell and Wittgenstein."

Concluding

I would have liked to put the sublime to work here, in a way that cuts more deeply than simply calling attention to certain obvious congruences between the aesthetics literature and the impulses that seem to me to drive Boris's work.

But whatever critical artillery one brings to Boris's work, in the end the philosophical conversation must/will inevitably turn back to the mathematics.

JK: In your own work though, how is it helpful to think of the syntax/semantics distinction in the way you do?

BZ: ... here is one of my talks on the topic, attached. It is what resulted from my attempts to understand what 'non-commutative geometry' is and how it originated in Heisenberg's physics. In more detail, you can download a couple of papers from my web-page, like "The geometric semantics of algebraic quantum mechanics".

In writing model theory in the language of geometry, a hallmark of Boris's mathematical practice, the conflicted aspect of sublimity, the idea of stasis and being locked into a cycle, is set aside, and the conditions for a rapprochement between the words-based way and the semantical way are laid out, because of geometry. It is a road that opens out into freedom for the logician; it is a road that delivers the logician into the mathematical arena.

THANK YOU