# A Kunen-like model without critical continuum (Part I)

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# About

- The purpose of the talks is to present the result of a joint work with **Eyal Kaplan** on the tension between structure theory for ultrafilters, and the continuum problem.

- Kunen's work on models $L[\mu]$ for a single $\sigma$-complete ultrafilter $\mu$, is a cornerstone of inner-model theory. It provides a the simplest possible (nontrivial) behaviour of $\sigma$-complete ultrafilters in a universe of set theory:

  1. There is a unique measurable cardinal $\kappa$ with a unique normal $\kappa$-complete ultrafilter (normal measure) $U$,
  2. every other $\sigma$-complete ultrafilter is (Rudin-Keisler) isomorphic to a finite power $U^n$ of $U$.

- By a Kunen-like model, we mean a model that witnesses the same simple behaviour of ($\sigma$-complete) ultrafilters.

Theorem (BN-Kaplan)

**The existence of a Kunen-like model in which $2^\kappa > \kappa^+$ is consistent relative to the existence of a model with a $(\kappa, \kappa^{++})$-extender.**

The result can be seen as an extension of three lines of research in set theory:

1. Structure theory for ($\sigma$-complete) ultrafilters and its implications to key properties of the set theoretic universe.
2. Structure theory for $\sigma$-complete ultrafilters in forcing extensions.
3. Iterated forcing theory and its interaction with fine structure

# Plan (Part I)

**Part I.1:** Introduction

**Part I.2:** The Friedman-Magidor blueprint for controlling normal measures

# Part I.1

# Introduction

# Structure theory for ultrafilters (1/2)

- By Silver, Kunen's model $L[U]$ satisfies GCH. The simplicity properties of $L[U]$ both in terms of the structure of ($\sigma$-complete) ultrafilters, and in cardinal arithmetic (GCH), extend to other known canonical inner models of set theory.

- The Ultrapower Axiom (UA) of Gabe Goldberg isolates a structural property for ultrafilters that holds in all known canonical inner models.

## Definition (UA)

For every $\sigma$-complete ultrafilters $U_0, U_1$, with ultrapower emb. $j_i : V \to M_{U_i} \cong Ult(V, U_i)$, $i < 2$, there are $W_1 \in M_{U_0}$ and $W_0 \in M_{U_1}$ whose ult. emb. $k_i : M_{U_i} \to N \cong Ult(M_i, W_{1-i})$, $i < 2$, have the same ultrapower $N$, and $k_1 \circ j_0 = k_0 \circ j_1$.

# Structure theory for ultrafilters (2/2)

## Theorem (Goldberg)

*UA implies:*

1. *The Mitchell order is linear*
2. *The first measurable cardinal $\kappa$ carries a single normal measure $U$ and every other measure on $\kappa$ is isomorphic to $U^n$ for some $n < \omega$*
3. *if there is a supercompact cardinal $\kappa$ then $2^\lambda = \lambda^+$ for all $\lambda \geq \kappa$.*

**Question:** Does UA (with possible extension to partial ultrafilters or/and extenders) + large cardinals implies GCH?

**Local Version:** Does UA implies $2^\kappa = \kappa^+$ for every measurable cardinal $\kappa$?

**Answer:** No (witnessed by the Kunen-like model)

# Structure theory for ultrafilters in forcing extensions (1/6)

- The preservation of elementary embeddings in forcing extensions that add many new subsets plays a key role in Silver's proof for the consistency of the failure of SCH from the consistency of a supercompact cardinal.

- Given a supercompact cardinal $\kappa$, an ult. emb. $j : V \to M \cong Ult(V, W)$ by a $\kappa^{++}$-supercompact measure $W$, and a $V$-generic filter $G \subseteq \mathbb{P}_\kappa$ for an Easton support iteration of Cohen posets $Add(\alpha, \alpha^{++})$ at inaccessibles $\alpha \leq \kappa$, Silver's master sequence construction gives an $M$-generic $G^* \subseteq j(\mathbb{P}_\kappa)$ with $j``G \subseteq G^*$, and an extension $j^* : V[G] \to M[G^*]$. The derived normal measure on $\kappa$ is $U^* = \{X \subseteq \kappa \mid \kappa \in j^*(X)\}$.

- The master sequence construction is quite flexible, and gives rise to many different possible generics $G^*$, which in turn, generates many different normal measures on $\kappa$ in $V[G]$.

# Structure theory for ultrafilters in forcing extensions (2/6)

- A driving force to the pursuit for control of elementary embeddings in generic extensions was the question about the possible number of normal measures on a measurable cardinal.
- Kunen's model $L[U]$ shows it is consistent to have a single normal measure from the minimal assumption of a measurable cardinal. The Kunen-Paris forcing shows that the maximal number of $2^{2^\kappa}$ is also possible from the same assumption.
- Mitchell's construction and theory of inner models $L[\vec{U}]$ with coherent sequences of normal measures shows that any number $\lambda \in [0, \kappa^{++}]$ of normal measures on $\kappa$ is consistent, but requires a stronger large cardinal assumption (higher Mitchell order) and does not apply to the first measurable cardinal.

# Structure theory for ultrafilters in forcing extensions (3/6)

- Baldwin constructed a models with any number $\lambda < \kappa$ of normal measures on the first measurable cardinal, from an assumption of $o(\kappa) >> \lambda$.
- Apter,Cummings, and Hamkins established the consistency of $\kappa^+$ many normal measures on the first measurable cardinal from the minimal assumption.
- Leaning constructed models with any number $\lambda < \kappa^+$ of normal measures on the first measurable cardinal from an assumption weaker than $o(\kappa) = 2$.

# Structure theory for ultrafilters in forcing extensions (4/6)

- The problem regarding the number of normal measures on the first measurable cardinal was finally resolved in 2007 by Friedman and Magidor, who showed that any number $\lambda \leq \kappa^{++}$ of normal measures on the first measurable cardinal $\kappa$ is consistent from the minimal assumption.

- In their paper, they also prove a similar result for the number of normal measures on a cardinal $\kappa$ in a model of $2^\kappa = \kappa^{++}$.

# Structure theory for ultrafilters in forcing extensions (5/6)

**Theorem (Friedman-Magidor 2007)**

*The existence of a model with a measurable cardinal $\kappa$ carrying a single normal measure, and $2^\kappa = \kappa^{++}$ is consistent relative to the existence to a $(\kappa, \kappa^{++})$-extender.*

**Theorem (Apter-Cummings 2023)**

*The failure of GCH on a strong cardinal $\kappa$ in a model where the Mitchell order on normal measures on $\kappa$ is linear, is consistent relative to a strong cardinal.*

# Structure theory for ultrafilters in forcing extensions (6/6)

- The last forcing theorems show that key structural properties for normal measures are consistent with the failure of GCH at a measurable cardinal.
- The result do not apply to non-normal measures.
- Prior to the new Kunen-like model construction, it was not known whether UA is consistent in any forcing extension adding an unbounded subset to $\kappa$.

# Part I.2

# The Friedman-Magidor blueprint for normal measures

# FM blueprint

- The Friedman-Magidor (FM) blueprint was developed to control the number of normal measures in a generic extension of a canonical inner model. We will focus on a version designed to force $2^\kappa = \kappa^{++}$ and a unique normal measure on $\kappa$.
- Given a single ultrapower embedding $j : V \to M \cong Ult(V, E)$ by a (short) extender $E$, with $cp(j) = \kappa$, ${}^\kappa M \subseteq M$, and $V_{\kappa+2} \subseteq M$, the goal is to find assumptions for an iteration poset $\mathbb{P}$ that adds $\kappa^{++}$ subsets to $\kappa$, such that for a $V$-generic $G \subseteq \mathbb{P}$ there is a unique $M$-generic $G^* \subseteq j(\mathbb{P})$ with $j``G \subseteq G^*$.

# Keys to the FM blueprint

Comparing with the standard Easton-support construction (as in Silver's work) the main ingredients of the FM-approach for a poset $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \kappa \rangle$ are

1. increase the closure rate of $\mathbb{P}$ so that $j``\mathbb{P}$ meets almost every dense open subset $D \subseteq j(\mathbb{P})$ in $M$,
2. include coding posets to make the the posets $\mathbb{Q}_\alpha$, $\alpha \leq \kappa$ rigid (i.e., have a unique generic filter)

(More details next time)

# $\kappa$-Fusion

**An Imprecise Definition:**

Let $\mathbb{P}$ that add subsets to $\kappa$, and for each $\alpha < \kappa$ has "up " and "down" restriction maps:

$$p \mapsto p \restriction \alpha \quad (p \text{ up to } \alpha)$$

$$p \mapsto p \downharpoonleft \alpha \quad (p \text{ starting from } \alpha)$$

with the domain of each being dense in $\mathbb{P}$, and a "join" operation $*$, which satisfy natural properties such that $p = p \restriction \alpha * (p \downharpoonleft \alpha)$ (other properties will be specified later) .

Say that a set $D \subseteq \mathbb{P}$ is dense beyond $\alpha$ if for every $p \in D$, the weaker condition $1_{\mathbb{P}} \restriction (\alpha + 1) * p \downharpoonleft (\alpha + 1)$ is also a member of $D$

Say that $\mathbb{P}$ has the **$\kappa$-fusion property (via restriction maps)** if for every sequence $\langle D_\alpha \mid \alpha < \kappa \rangle$ so that each $D_\alpha$ is dense beyond $\alpha$ and every $p \in \mathbb{P}$, there are $p^* \leq p$ and a club $C \subseteq \kappa$ such that for all $\alpha \in C$ the set $\{p' \in D_\alpha : p' \downharpoonleft (\alpha + 1) = p^* \downharpoonleft (\alpha + 1)\}$ is dense in $\mathbb{P}/p^*$.

# Remarks

- If $\mathbb{Q}$ is $\kappa^+$-closed then it has the $\kappa$-fusion property
- If $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ is an iteration poset, then we have standard restrictions maps the send $p = \langle \dot{p}_\beta \mid \beta < \kappa \rangle$ to

  $$p \restriction \alpha = \langle \dot{p}_\beta \mid \beta < \alpha \rangle \in \mathbb{P}_\alpha, \text{ and } p \downharpoonleft \alpha = \langle \dot{p}_\beta \mid \alpha \leq \beta < \kappa \rangle$$

  The $\kappa$-fusion property is then **equivalent** to the following statement about the iteration poset $\mathbb{P}$:
  For every $p \in \mathbb{P}$ and $\langle D_\alpha \mid \alpha < \kappa \rangle$ so that each $D_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name for a dense open subset of $\mathbb{P}/\mathbb{P}_{\alpha+1}$, there are $p^* \leq p$ and a club $C \subseteq \kappa$ such that

  $$\forall \alpha \in C \quad p^* \restriction (\alpha + 1) \Vdash_{\mathbb{P}_{\alpha+1}} p^* \downharpoonleft (\alpha + 1) \in D_\alpha$$

# Fusion Lemma for nonstationary support iteration of closed posets

**Lemma (0)**

*Suppose that* $\kappa$ *is a regular cardinal and* $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ *is a nonstationary support iteration and* $\mathbb{Q}_\alpha$ *is* $\alpha$*-closed. Then* $\mathbb{P}$ *has the* $\kappa$*-fusion property.*