May the successor of a singular cardinal be Jónsson?

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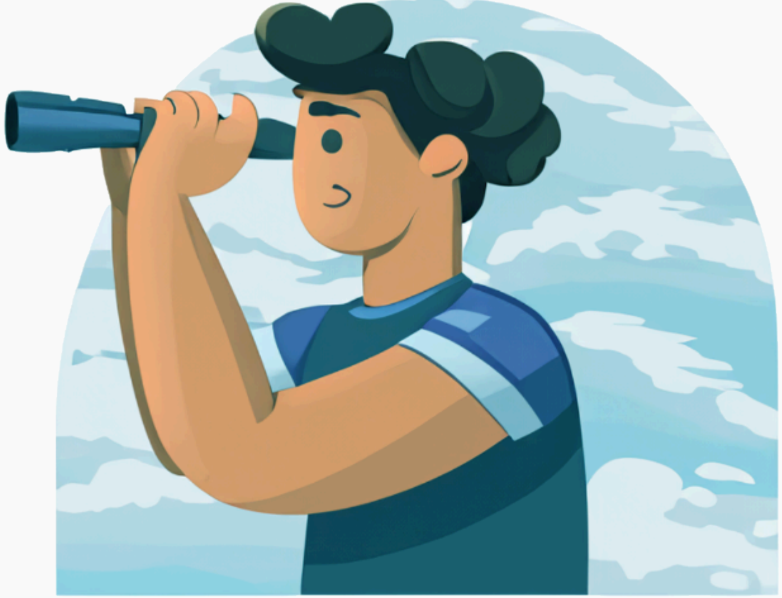
I thank the organizers for the invitation to give a *perspective* talk. Hopefully, the ten questions collected here will be picked up by the community and lead to major advances.



An unusual talk deserves unusual slides. I created this presentation using **typst**.

Background

Theorem (Ramsey, 1930). *For every 2-coloring of the unordered pairs of an infinite set X, $c : [X]^2 \to 2$, there exists an infinite subset $Y \subseteq X$ such that c is constant over $[Y]^2$.*

Theorem (Sierpiński, 1933). *There is a 2-coloring $c : [\mathbb{R}]^2 \to 2$ such that for every uncountable $Y \subseteq \mathbb{R}$, c attains both colors over $[Y]^2$.*

Definition (Erdős et al, 1965). $\kappa \nrightarrow [\kappa]^2_\theta$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ satisfying that for every $Y \subseteq \kappa$ of full size, c attains all possible colors over $[Y]^2$.

A closely related concept. κ is Jónsson iff $\kappa \to [\kappa]^{<\omega}_\kappa$ holds, i.e., for every coloring $c : [\kappa]^{<\omega} \to \kappa$ there is some $Y \subseteq \kappa$ of full size such that $c \restriction [Y]^{<\omega}$ is not surjective.

Background

Theorem (Ramsey, 1930). *For every 2-coloring of the unordered pairs of an infinite set X, $c : [X]^2 \to 2$, there exists an infinite subset $Y \subseteq X$ such that c is constant over $[Y]^2$.*

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Pump up for successor cardinals. $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ *implies* $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$.

Theorem (Sierpiński, 1932 for $\lambda = \omega$. General case by Erdős-Hajnal-Rado, 1965)
For every infinite cardinal λ such that $2^\lambda = \lambda^+$, $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ holds.

Proof. Assuming $2^\lambda = \lambda^+$, let $\langle a_\alpha \mid \alpha < \lambda^+ \rangle$ enumerate all bounded subsets of λ^+.

For each $\beta < \lambda^+$, let $\mathcal{B}_\beta$ be a disjoint refinement of $\{a_\alpha \mid \alpha < \beta\} \cap [\beta]^\lambda$.

Now, pick a coloring $c : [\lambda^+]^2 \to \lambda$ such that for every $\beta < \lambda^+$, for every $b \in \mathcal{B}_\beta$, $c[b \times \{\beta\}] = \lambda$.

This works because given $Y \subseteq \lambda^+$ of full size, we may find an $\alpha < \lambda^+$ with $a_\alpha \in [Y]^\lambda$, and then find a large enough $\beta \in Y$ to satisfy $(a_\alpha \cup \alpha) \subseteq \beta$.
Pick $b \in \mathcal{B}_\beta$ with $b \subseteq a_\alpha$. Then $\lambda = c[b \times \{\beta\}] \subseteq c\left[[Y]^2\right]$.

qed

Theorem (Sierpiński, 1932 for $\lambda = \omega$. General case by Erdős-Hajnal-Rado, 1965)
For every infinite cardinal λ such that $2^\lambda = \lambda^+$, $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ holds.

Theorem (Todorčević, 1987)
For every infinite regular cardinal λ, $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ holds.

What about successors of singulars?

Interlude: the birth of singular cardinals

At the 3rd ICM meeting in Heidelberg, 1904
all other parallel sessions were canceled
to allow everyone including Cantor and Hilbert
to attend Julius König's sensational lecture.

The birth of singular cardinals

Hausdorff's formula. $\aleph_{\alpha+1}{}^{\aleph_\beta} = \max\{\aleph_\alpha{}^{\aleph_\beta}, \aleph_{\alpha+1}\}$.

Theorem. The continuum hypothesis is false.

Proof sketch. If $2^{\aleph_0} = \aleph_1$, then $\aleph_0{}^{\aleph_0} = \aleph_1$, and then – by induction — $\aleph_\alpha{}^{\aleph_0} = \aleph_\alpha$ for all $\alpha > 0$.

However,

$$\aleph_\omega{}^{\aleph_0} = \prod_{n\in\mathbb{N}} \aleph_n > \sum_{n\in\mathbb{N}} \aleph_n = \aleph_\omega.$$

This is a contradiction.

The continuum hypothesis is false!

At the end of the lecture, Cantor said how grateful he was to have lived to see his conjecture answered, even if the answer was negative.

But König's proof had a flaw (going back to Bernstein) overlooking **singular** cardinals.

The core of König's proof is correct

Suppose that σ is a limit ordinal > 0. Given a strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < \sigma \rangle$ of regular uncountable cardinals, define a quasi-ordering $<^*$ of the product $\prod \vec{\lambda}$ by letting

$f <^* g$ iff $\{i < \sigma \mid f(i) \geq g(i)\}$ is bounded in σ.

Definition.

- $\mathfrak{b}(\vec{\lambda})$ denotes the least size of an unbounded family in $(\prod \vec{\lambda}, <^*)$.
- $\mathfrak{d}(\vec{\lambda})$ denotes the least size of a cofinal family in $(\prod \vec{\lambda}, <^*)$.

Lemma (König)

$\mathfrak{b}(\vec{\lambda})$ *is a regular cardinal greater than* $\sup(\vec{\lambda})$.

PCF theory

PCF theory

Theorem (Shelah, 1990's)
For every singular cardinal λ, there is an increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ of regular uncountable cardinals converging to λ such that $\mathfrak{d}(\vec{\lambda}) = \lambda^+$. So, by König, also $\mathfrak{b}(\vec{\lambda}) = \lambda^+$.
☞ *If λ has uncountable cofinality, may take $\lambda_i = \mu_i^+$ for some club $\langle \mu_i \mid i < \mathrm{cf}(\lambda) \rangle$* in λ.

Theorem (Todorčević, 1987)
*If $\vec{\lambda}$ is as in Shelah's theorem **and** $\lambda_i \nrightarrow [\lambda_i]^2_{\lambda_i}$ for every $i < \mathrm{cf}(\lambda)$, then $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda}$.*
☞ *In particular, the first cardinal λ to satisfy $\lambda^+ \rightarrow [\lambda^+]^2_{\lambda}$ is the CTBL limit of inaccessibles.*

Question 6.7 of Gilton's 2022 preprint PCF theory and the Tukey spectrum is equivalent to asking whether Todorčević's hypothesis of $\mathfrak{d}(\vec{\lambda}) = \lambda^+$ may be reduced to $\mathfrak{b}(\vec{\lambda}) = \lambda^+$.

Theorem (Todorčević, 1987)
Suppose $\vec{\lambda}$ *is a singular cardinal and* $\lambda = \langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ *is an increasing sequence converging to* λ with $\mathfrak{b}(\vec{\lambda}) = \lambda^+$. *If* $\lambda_i \nrightarrow [\lambda_i]^2_{\lambda_i}$ *for all* $i < \mathrm{cf}(\lambda)$, *then* $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda}$.

Modern proof. Using $\mathfrak{b}(\vec{\lambda}) = \lambda^+$, we may fix a sequence $\langle f_\alpha \mid \alpha < \lambda^+ \rangle$ of functions in $\prod \vec{\lambda}$ such that $\langle f_\alpha \mid \alpha \in Y \rangle$ is unbounded for every $Y \subseteq \lambda^+$ of full size.

For each $i < \mathrm{cf}(\lambda)$, suppose $c_i : [\lambda_i]^2 \to \lambda_i$ is a witness for $\lambda_i \nrightarrow [\lambda_i]^2_{\lambda_i}$.

By Eisworth (2013), there is a map $d : [\lambda^+]^2 \to [\lambda^+]^2 \times \mathrm{cf}(\lambda)$ satisfying that for every $X \subseteq \lambda^+$ of full size, there is $Y \subseteq \lambda^+$ of full size such that $d\left[[X]^2\right]$ covers $[Y]^2 \times \mathrm{cf}(\lambda)$.

Define $c : [\lambda^+]^2 \to \lambda$ by letting $c(\alpha, \beta) := c_i\left(f_{\gamma(i)}, f_{\delta(i)}\right)$ whenever $d(\alpha, \beta) = (\gamma, \delta, i)$. qed

Singular cardinals hypothesis: SCH_λ asserts that $2^{\mathrm{cf}(\lambda)} < \lambda$ implies $2^\lambda = \lambda^+$.

We propose the following:

Definition. *Weakly compact failure of* SCH_λ asserts that for every $\vec{\lambda} = \langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ converging to λ with $\mathfrak{b}(\vec{\lambda}) = \lambda^+$, it is the case that almost all λ_i's are weakly compact.

Note that by Solovay (1974), the *strongly compact* failure of SCH_λ is inconsistent.

Question 1. Is weakly compact failure of SCH_λ consistent say with $\mathrm{cf}(\lambda) = \omega$?

Adolf: An affirmative answer requires a Woodin cardinal.

Ben-Neria: An affirmative answer seems to emerge from Merimovich's work on supercompact extender based Prikry forcing.

Compactness and incompactness

Stationary reflection (compactness)

Theorem (Todorčević, 1987)
For a regular uncountable κ, if $\kappa \nrightarrow [\kappa]^2_\kappa$ fails, then every stationary subset of κ reflects.

Theorem (Eisworth, 2012)
If λ is a singular cardinal for which $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ fails, then every family of less than $\mathrm{cf}(\lambda)$ many stationary subsets of λ^+ reflect simultaneously.

A few years ago, a model satisfying the above form of simultaneous reflection together with $\neg\,\mathrm{SCH}_\lambda$ was obtained by Poveda, Rinot and Sinapova using iterated Prikry-type forcing, and by Ben-Neria, Hayut and Unger using iterated ultrapowers and then simplified by Gitik.

Aronszajn trees (incompactness)

Theorem (Jensen, 1972. Shore 1974)
If there is a κ-Souslin tree, then $\kappa \nrightarrow [\kappa]^2_\kappa$ holds.

Theorem (R., 2014)
If $\square(\kappa)$ holds, then so does $\kappa \nrightarrow [\kappa]^2_\kappa$.
Remains true assuming weak variants of square.

Note that both a κ-Souslin tree and $\square(\kappa)$ are particular sorts of κ-Aronszajn trees.

Aronszajn trees (incompactness)

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Remains true assuming weak variants of square.

Question 2. Suppose that λ is a singular cardinal and there exists a λ^+-Aronszajn tree. Does $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ hold?

Aronszajn trees (incompactness)

Theorem (Jensen, 1972. Shore 1974)
If there is a κ-Souslin tree, then $\kappa \not\rightarrow [\kappa]^2_\kappa$ holds.

Theorem (R., 2014)
If $\square(\kappa)$ holds, then so does $\kappa \not\rightarrow [\kappa]^2_\kappa$.
Remains true assuming weak variants of square.

Question 3. Suppose that λ is a singular cardinal and there exists a λ^+-Aronszajn tree. Does there exist a λ^+-Souslin tree? Here, I don't mind assuming the GCH.

► Results from More notions of forcing add a Souslin tree (with Brodsky, 2019) show that — in the context of GCH — singularizations of a regular λ tend to introduce λ^+-Souslin trees.

Putting it all together

The tree property: $\mathrm{TP}(\kappa)$ asserts that there are no κ-Aronszajn trees.

Theorem (Neeman, 2009)
Starting with infinitely many supercompact cardinals, it is consistent that for some singular strong limit cardinal λ of countable cofinality, SCH_λ *fails and* $\mathrm{TP}(\lambda^+)$ *holds.*

Question 4. Is the conjunction of the following consistent for some singular cardinal λ?

i) Weakly compact failure of SCH_λ;

ii) $\mathrm{TP}(\lambda^+)$;

iii) every finite family of stationary subsets of λ^+ reflect simultaneously.

Reductions and approximations

Reduction 1

Let λ denote a singular cardinal.

Theorem (Eisworth, 2013)
If $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ *holds for arbitrarily large* $\theta < \lambda$, *then* $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ *holds.*

Theorem (Shelah, 1990's)
$\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ *holds for* $\theta = \mathrm{cf}(\lambda)$.

Question 5. Does $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ hold for $\theta = \mathrm{cf}(\lambda)^+$?

Reduction 2

Let λ denote a singular cardinal.

Theorem (R., 2012)
If there are a cardinal $\mu < \lambda$ and a coloring $c : [\lambda^+]^2 \to \theta$ such that $c\left[[S]^2\right] = \theta$ for every stationary $S \subseteq \lambda^+ \cap \operatorname{cof}(> \mu)$, then $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ holds.

Question 6. Identify interesting ideals J over λ^+ for which **ZFC** proves the existence of a coloring $c : [\lambda^+]^2 \to \lambda$ satisfying $c\left[[B]^2\right] = \lambda$ for every $B \in J^+$.

Reduction 3

Given a coloring $c : [\lambda^+]^2 \to \theta$, let $\mathbb{P}_{c,\mu} := \left(\left\{x \in [\lambda^+]^{<\mu} \mid c \restriction [x]^2 \text{ is constant}\right\}, \supseteq\right)$.
This poset adds a large homogeneous set, thus ensuring c ceases to witness $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$.

The good (R., 2012)

Suppose that λ is a singular cardinal. If $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ holds, then it may be witnessed by a coloring $c : [\lambda^+]^2 \to \theta$ for which $\mathbb{P}_{c,\omega}$ has the λ^+-cc.

The bad (R.-Zhang, 2024)

Suppose that λ is a singular cardinal. Let $c : [\lambda^+]^2 \to 2$ be any coloring.

- $\mathbb{P}_{c,\lambda}$ has an antichain of size λ^+ consisting of pairwise disjoint sets;
- If λ is the limit of strongly compacts, then this is true already for $\mathbb{P}_{c,\,\mathrm{cf}(\lambda)^+}$.

Question 7. Given a coloring $c : [\lambda^+]^2 \to \theta$ witnessing $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$, is there a cofinality-preserving notion of forcing for killing c? Identify features of c that enable a YES answer.

Reduction 4 and two ZFC approximations

Theorem (Inamdar-R., 2023)
Suppose a singular cardinal λ is a strong limit or satisfies $\aleph_\lambda > \lambda$.
If there exists a coloring $c : \lambda \times \lambda^+ \to \lambda$ such that for every $Y \subseteq \lambda^+$ of full size, there is $i < \lambda$ with $c[\{i\} \times Y] = \lambda$, then $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ holds.

Theorem (Inamdar-R., 202⌛)
For every singular cardinal λ, for every $\theta < \lambda$, there is a coloring $c : \lambda \times \lambda^+ \to \theta$ such that for every $Y \subseteq \lambda^+$ of full size, there is $i < \lambda$ with $c[\{i\} \times Y] = \theta$.

One cannot get $\theta = \lambda$ in **ZFC**, as we proved it fails in a model of [GaSh:949].

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Theorem (Inamdar-R., 202⧗)
For every singular cardinal λ, there is a coloring $c : \lambda \times \lambda^+ \to \lambda$ such that for every $Y \subseteq \lambda^+$ of full size, there is $i < \lambda$ with $\mathrm{otp}(c[\{i\} \times Y]) = \lambda$.

Curiously, the analogous assertion for λ regular is equivalent to $\mathfrak{b}_\lambda = \lambda^+$.

Club guessing

Club guessing

Consider $S := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ for a given singular cardinal λ.

Suppose that $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ is a sequence such that each C_δ is a club in δ.

- $\vec{C}$ is guessing clubs iff for every club $D \subseteq \lambda^+$, there is some $\delta \in S$ with $C_\delta \subseteq D$;
-

Theorem (Shelah, 1990's)
There is a $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ that guesses clubs with $\mathrm{otp}(C_\delta) = \mathrm{cf}(\lambda)$ for all $\delta \in S$.

Club guessing

Consider $S := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ for a given singular cardinal λ.

Suppose that $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ is a sequence such that each C_δ is a club in δ.

- $\vec{C}$ is guessing clubs iff for every club $D \subseteq \lambda^+$, there is some $\delta \in S$ with $C_\delta \subseteq D$;
- $\vec{C}$ is uninhibited iff for club many $\delta \in S$, for every $\mu < \lambda$, $\sup(\mathrm{nacc}(C_\delta) \cap \mathrm{cof}(> \mu)) = \delta$.

Theorem (Eisworth-Shelah, 2009)
If λ has uncountable cofinality, then it admits an uninhibited club guessing sequence.

Question 8. What about singular cardinals of countable cofinality?

Club guessing ideals

Given a sequence of local clubs $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$, consider the following ideal:

$$J := \{A \subseteq \lambda^+ \mid \exists \text{ club } D \subseteq \lambda^+ \forall\delta \in S \exists\mu < \lambda[\sup(\mathrm{nacc}(C_\delta) \cap \mathrm{cof}(> \mu) \cap D \cap A) < \delta]\}.$$

Shelah [Sh:365] proved that if there is $B \in J^+$ with $B \subseteq \{\beta < \lambda^+ \mid \mathrm{cf}(\beta) \text{ is not Jónsson}\}$, then λ^+ is not Jónsson.

Question 9. Is $\lambda^+ \to [\lambda^+]^2_\lambda$ equivalent to the Jónsson-ness of λ^+? to $\lambda^+ \to [\lambda^+]^3_\lambda$?

Eisworth (2009) proved that assuming $\mathrm{otp}(C_\delta) < \lambda$ for all $\delta \in S$, whenever there are θ many pairwise disjoint J^+-sets (a.k.a. J is not weakly θ-saturated), $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ holds.

Club guessing ideals

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The club guessing ideal J is σ-indecomposable for every regular cardinal $\sigma \in \lambda \setminus \{\mathrm{cf}(\lambda)\}$.
$*$ An ideal is *σ-indecomposable* iff it is closed under <u>increasing</u> unions of length σ.

Club guessing ideals

Given a sequence of local clubs $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$, consider the following ideal:

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Eisworth (2009) proved that assuming $\mathrm{otp}(C_\delta) < \lambda$ for all $\delta \in S$, whenever there are θ many pairwise disjoint J^+-sets (a.k.a. J is not weakly θ-saturated), $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ holds.

The club guessing ideal J is σ-indecomposable for every regular cardinal $\sigma \in \lambda \setminus \{\mathrm{cf}(\lambda)\}$. The extent of the failure of weak saturation of indecomposable ideals is studied in Part III of our series Was Ulam Right? (joint work with Inamdar).

Proposition

Suppose λ is a singular cardinal, and $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ is a club guessing sequence such that $\mathrm{otp}(C_\delta) = \lambda$ for all $\delta \in S$. Then $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ holds.

Proof. Fix a partition $S = \biguplus_{\tau<\lambda} S_\tau$ such that $\vec{C} \restriction S_\tau$ guesses clubs for each $\tau < \lambda$.

Recall we may assume λ is the limit of inaccessibles, so $\lambda = \aleph_\lambda$ and we may find a pairwise disjoint sequence $\langle K_\tau \mid \tau < \lambda \rangle$ of cofinal subsets of $\{\mu < \lambda \mid \mathrm{cf}(\mu) = \mu\}$ of order-type $\mathrm{cf}(\lambda)$.

For all $\tau < \lambda$ and $\delta \in S_\tau$, let $D_\delta := \{C_\delta(i) \mid i \in \mathrm{cl}(K_\tau)\}$. Consider the corresponding ideal:

$$J := \{A \subseteq \lambda^+ \mid \exists \text{ club } D \subseteq \lambda^+ \forall \delta \in S \exists \mu < \lambda [\sup(\mathrm{nacc}(D_\delta) \cap \mathrm{cof}(> \mu) \cap D \cap A) < \delta]\}.$$

For every $\tau < \lambda$, $B_\tau := \{\beta < \lambda^+ \mid \mathrm{cf}(\beta) \in K_\tau\}$ is in J^+. If $\tau \neq \tau'$, then $B_\tau \cap B_{\tau'} = \emptyset$.

Guessing with large order-type

Proposition
Suppose λ is a singular cardinal, and $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ is a club guessing sequence such that $\mathrm{otp}(C_\delta) = \lambda$ for all $\delta \in S$. Then $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ holds.

Proof. Fix a partition $S = \dot{\bigcup}_{\tau<\lambda} S_\tau$ such that $\vec{C} \restriction S_\tau$ guesses clubs for each $\tau < \lambda$.

Recall we may assume λ is the limit of inaccessibles, so $\lambda = \aleph_\lambda$ and we may find a pairwise disjoint sequence $\langle K_\tau \mid \tau < \lambda\rangle$ of cofinal subsets of $\{\mu < \lambda \mid \mathrm{cf}(\mu) = \mu\}$ of order-type $\mathrm{cf}(\lambda)$.

For all $\tau < \lambda$ and $\delta \in S_\tau$, let $D_\delta := \{C_\delta(i) \mid i \in \mathrm{cl}(K_\tau)\}$. Consider the corresponding ideal:

$$J := \{A \subseteq \lambda^+ \mid \exists \text{ club } D \subseteq \lambda^+ \forall \delta \in S \exists \mu < \lambda[\sup(\mathrm{nacc}(D_\delta) \cap \mathrm{cof}(> \mu) \cap D \cap A) < \delta]\}.$$

So J admits λ many pairwise disjoint positive sets, and hence $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ holds. qed

Consider $S := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ for a given singular cardinal λ.

Question 10. Is there a club guessing sequence $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ such that $\mathrm{otp}(C_\delta) = \lambda$ for all $\delta \in S$?

► By Abraham and Shelah [AbSh:182] it is consistent for a regular λ to have λ^{++} many clubs in λ^+ such that the intersection of any λ^+ many of them has size $< \lambda$.
► See [Sh:186] , [Sh:667] and Gitik-R. (2012) for related work (consistency results on the failure of diamond at successors of singulars).

Guessing with large order-type (cont.)

Consider $S := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ for a given singular cardinal λ.

Question 10. Is there a club guessing sequence $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ such that $\mathrm{otp}(C_\delta) = \lambda$ for all $\delta \in S$? Does $\square_\lambda$ entail such a sequence that is moreover coherent?

- By Abraham and Shelah [AbSh:182] it is consistent for a regular λ to have λ^{++} many clubs in λ^+ such that the intersection of any λ^+ many of them has size $< \lambda$.
- See [Sh:186] , [Sh:667] and Gitik-R. (2012) for related work (consistency results on the failure of diamond at successors of singulars).
- An affirmative answer to the 2nd part was shown to follow from $2^\lambda = \lambda^+$ in R. (2015).