ULTRAFILTERS AND LOCALLY COUNTABLE STRUCTURES

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Countable Borel equivalence relations.

A Borel equivalence relation E on a Polish space X is *countable, cBer* if each E-class is countable.

Example. Let G be any countable group. Consider the shift action of G on 2^G where $g \cdot x(h) = x(g^{-1}h)$ and its orbit equivalence relation.

Example The Turing bi-reducibility on 2^ω.

Organizing cBers.

Definition. Given Borel equivalence relations X, E and Y, F, a *Borel reduction* is an injection from X/E to Y/F with Borel lifting.

The smallest nontrivial node of the Borel reducibility quasiorder is well-known:

Definition. A Borel equivalence relation is *hyperfinite* if it is a union of an increasing chain of Borel equivalence relations with finite classes.

Example. $\mathbb{E}_0$ connects $x, y \in \mathcal{P}(\omega)$ if $x \Delta y$ is finite. It is hyperfinite.

Descriptive combinatorics.

Question. If a Borel locally countable constraint of some type has a solution, does it have a Borel solution?

Fact. Let G be the *Hamming graph* on 2^ω, connecting x, y if they differ in exactly one digit. True chromatic number is 2, the Borel chromatic number is uncountable.

Theorem. (Kechris-Solecki-Todorcevic) Every Borel graph of degree $\leq n$ has Borel chromatic number $\leq n + 1$.

The role of measure and category.

Theorem. Every cBer is hyperfinite off a meager set.

Theorem. The F_2-shift equivalence relation is not hyperfinite on any non-null Borel set.

Theorem. (Conley-Miller) Every Borel graph G of chromatic number n is of Borel chromatic number $\leq 2n-1$ off a meager set.

Theorem. (Grebik) Every Borel chromatic graph of degree n has Borel edge-chromatic number $\leq n+1$ off a null set.

The problem.

Question. Is there any theory of cBers/descriptive combinatorics without

- category arguments;
- measure theoretic arguments;
- Borel determinacy arguments?

Is there such a theory when an ultrafilter on ω is present?

Ultrafilters.

Of course, there are different types of ultrafilters.

- Ramsey ultrafilter is one which contains homogeneous sets for every partition of $[\omega]^2$
- P-points of various types;
- idempotent ultrafilters of various types.

One needs to specify the type to have a chance at good answers.

Ultrafilter models I.

- $L(\mathbb{R})[U]$, where the ambient ZFC universe has large cardinals;

- $W[U]$, where W is the choiceless Solovay model derived from an inaccessible.

A particularly well studied case: U is a Ramsey ultrafilter.

- (DiPrisco-Todorcevic)The models are generic extensions of $L(\mathbb{R})$ and W using $\mathcal{P}(\omega)/\mathrm{fin}$;

- their theory does not depend on the choice of Ramsey U; it is absolute.

Ultrafilter models II.

There is a rich preexisting theory of the models $W[U]$ or $L(R)[U]$ where U is Ramsey:

- (DiPrisco-Todorcevic) every uncountable set of reals has a perfect subset;

- there is no $\mathbb{E}_0$ selector;

- (Henle-Mathias-Woodin) in $L(\mathbb{R})[U]$, ω_1 is a partition cardinal

Dobrinen-Hathaway extended this to many other ultrafilter models.

What an ultrafilter can do I

Theorem. (Paul Larson) (ZF+DC) Let E be a hypersmooth equivalence relation on a Polish space X. Then, if an ultrafilter exists, X/E is linearly ordered.

Proof. Let $E = \bigcup_n E_n$. Since E_n is smooth, X/E_n carries a Borel linear order $\leq_n$. Let $\leq = \int \leq_n \ dU(n)$.

What an ultrafilter can do II.

Theorem. (ZF+DC) Let G be a locally countable Borel graph on a Polish space X, all of whose finite subgraphs have chromatic number $\leq n$. Suppose that the connectedness equivalence relation E is hyperfinite. Then, if an ultrafilter exists, the chromatic number of G is $\leq n$.

Proof. Let $E = \bigcup_n E_n$. Let $c_n \colon X \to n$ be a $G \cap E_n$-coloring. Let $c = \int c_n \, dU(n)$.

What an ultrafilter can do III.

Theorem. (ZF+DC) Let I be a Borel ideal on ω. Let $n \geq 3$, let $G(I, n)$ be the graph on n^ω connecting x, y if the set $\{k \in \omega \colon x(k) = y(k)\} \in I$. TFAE:

- the chromatic number of $G(I, n)$ is n;
- there is an ultrafilter on ω disjoint from I.

Proof. (2)$\to$(1): Let U be the ultrafilter, and let $c(x) = \lim_U x(n)$. (1)$\to$(2): Let c be a coloring such that $c(i) = i$ for all $i \in n$; U is generated by sets $a \subset \omega$ such that there is $x \in n^\omega$ with $c(x) = 0$ and $x \restriction a = 0$.

What a (Ramsey) ultrafilter cannot do I.

Definition. Let G_n be graphs on finite sets X_n for $n \in \omega$. The *Hamming product* is the graph G on $X = \prod_n X_n$ connecting x, y if they differ at exactly one n, and $x(n)\, G_n\, y(n)$ holds.

Theorem. Suppose that $\limsup_n \chi(G_n) = \infty$. Then in $W[U]$, the chromatic number of G is uncountable.

What a (Ramsey) ultrafilter cannot do II.

Definition. Let G_n be graphs on finite sets X_n for $n \in \omega$. The *modulo finite product* is the graph G on $X = \prod_n X_n$ connecting x, y if for all but finitely many n, and $x(n)\ G_n\ y(n)$ holds.

Theorem. Suppose that $\langle \chi(G_n) \colon n \in \omega \rangle$ is a very quickly growing sequence. Then in $W[U]$, the chromatic number of G is uncountable.

What a (Ramsey) ultrafilter cannot do Iii.

Theorem. In $W[U]$, there is no ultrafilter on ω disjoint from the summable ideal.

Theorem. In $W[U]$, there is no function which to each countable set $a \subset \mathbb{R}$ assigns an ultrafilter on a.

Theorem. In $W[U]$, there is no function which to each limit countable ordinal α assigns an ultrafilter on α containing no bounded subsets of α.

What is not known I.

Task. Classify those (countable) Borel equivalence relations E, F such that $W[U] \models |E| \leq |F|$.

Conjecture. These are exactly those such that $W \models |E| \not\leq |F|$.

Status verified: $|\mathbb{E}_0| \not\leq |\mathbb{R}|$, $|\mathbb{E}_1| \not\leq |F|$ for any orbit equivalence relation F induced by a continuous Polish group action, and $|\mathbb{F}_2| \not\leq |F|$ for any pinned Borel equivalence relation F.

Every countable Borel equivalence relation case except for $\mathbb{E}_0$ and the identity is open.

What is not known II.

Task. Classify those (locally finite, or d-regular) Borel graphs G such that $W[U] \models G$ has chromatic number $\leq n$.

Conjecture. These are exactly those homomorphically Borel-embeddable into $G(I, n)$ where I is a Fubini power of the Frechet ideal.

The case where the connectivity relation is hyperfinite is resolved in the positive. The cases of Cayley graphs on shift spaces of finitely generated groups are all open unless hyperfinite.

What is not known III.

Task. Classify those countable Borel equivalence relation E such that there is a function in $W[U]$ which to each E-class a assigns an ultrafilter on a.

Conjecture. These are exactly the hyperfinite ones.

The case when E is hyperfinite is resolved. Other cases are open.

What is not known IV.

Task. Classify those (countable) Borel equivalence relations E such that there is in $W[U]$ a linear ordering of their quotient space.

Conjecture. These are the the ones obtained from the identity by Borel reducibility, increasing union, and countable product. Among the cBers, exactly the hyperfinite ones.