Ralf Schindler

Martin Maximum^{++}:

If $\mathbb{P}$ is stationary set preserving,

if $\mathcal{D} = \{D_i : i < \omega_1\}$ is a coll of dense sets, and if

$\{\dot\tau_i : i < \omega_1\}$ is a coll. of nam for stationary subsets of ω_1,

i.e., $1\!\!1_{\mathbb{P}} \Vdash$ "$\dot\tau_i \subset \check\omega_1$ is stat.",

then $\exists$ filter $g \subset V$ s.t.

$$\left.\begin{array}{l} g \cap D_i \neq \emptyset, \\ \tau_i^g = \{\xi : \exists p \in g\ p \Vdash \check\xi \in \dot\tau_i\} \\ \quad \text{is stationary} \end{array}\right\} \text{all } i < \omega_1$$

MM^{++} is equivalent to:

For all $\mathbb{P}$ stationary set preserving, for all models $m = (M; \vec{R})$

$\le \aleph_1$ relations, functions

and for all Σ_1 formulas in $\mathcal{L}_{\in, NS_{\omega_1}}$,

if $V^{\mathbb{P}} \models \varphi(m)$, then there is in V some ~~stat~~ $\bar{m} \xrightarrow{\text{el. embedding}} m$ s.t.

typically size $\aleph_1$

$\varphi(\bar{m})$.

Ex: two def equivalent

$\Uparrow$ easy

$\Downarrow$ not that easy

Let's formulate a strong theory of MM^{++}.

<u>Definition</u> Let φ be a Σ_1 formula in $\mathcal{L}_{\dot{\in}, NS_{\omega_1}}$,

let $\mathfrak{m}$ be a model as above:

Say that $\varphi(\mathfrak{m})$ is <u>honestly consistent</u>

iff for all universally Baire functions F there is an ~~[illegible]~~ F-closed transitive model $\mathcal{U} \in V^{Col(\omega, TC(\{\mathfrak{m}\}))}$ s.t.

$$\mathcal{U} \models ZFC^- + \varphi(\mathfrak{m}),\ TC(\{\mathfrak{m}\}) \subset \mathcal{U},\ NS_{\omega_1}^{V} = NS_{\omega_1}^{\mathcal{U}} \cap V$$

"It's not just transitive model" "closed under for ex. $x^{\#}$?"

<u>$MM^{*,++}$ is ~~equivalent~~ to:</u>

for all models $\mathfrak{m} = (M; \vec{R})$ ← $\leq \aleph_1$ relations, functions

and for all Σ_1 formulas in $\mathcal{L}_{\dot{\in}, NS_{\omega_1}}$,

if $\varphi(\mathfrak{m})$ is honestly consistent,

then there is in V some $\bar{\mathfrak{m}} \xrightarrow{\text{el. embedding}} \mathfrak{m}$ st. $\varphi(\bar{\mathfrak{m}})$.

<u>$MM^{*,++}_{\lambda}$ is ~~equivalent~~ to:</u>

for all models $\mathfrak{m} = (M, \vec{R})$ of size λ ← $< \aleph_1$ relations functions

and for all Σ_1 formulas in $\mathcal{L}_{\dot{\in}, NS_{\omega_1}}$,

if $\varphi(\mathfrak{m})$ is honestly consistent,

then there is in V some $\bar{\mathfrak{m}} \xrightarrow{\text{el. embedding}} \mathfrak{m}$ s.t $\varphi(\bar{\mathfrak{m}})$.

MM^{++}_λ ~ all dense set one of size $\leq \lambda$.

$\boxed{\Gamma\text{-}MM^{*,++}_\lambda}$ is : $\Gamma \subset \Gamma^\infty_{\text{all universally Baire sets}}$

for all $A \in \Gamma$

for all models $m = (M, \vec{R})$ of size λ

$\leq \aleph_1$ relations, functions...

and for all Σ_1 formula in $\mathcal{L}_{\in, NS_{\omega_1}, \dot{A}}$

if $\varphi(m)$ is honestly consistent, then there is in V some $\bar{m} \xrightarrow{\text{elementarily}} m$ s.t. $\varphi(\bar{m})$

$MM^{++}_\lambda \Rightarrow MM^{++}(\lambda)$ (trivial)

Definition $\Gamma\text{-}BMM^{(*),++} \equiv (\Gamma\text{-})MM^{(*),++}_{\aleph_1}$

Theorem (NS_{ω_1} is ~~[illegible]~~ saturated + V is closed under $X \mapsto M^\#_\omega(X)$)

TFAE.

(1) $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})\text{-}BMM^{*,++}$ „The proof is really not very hard"

(2) $(*)$ ($\equiv$ Woodin's $\mathbb{P}_{max}$ axiom).

(3) $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})\text{-}BMM^{++}$,

$\boxed{\begin{array}{l} 1 \Rightarrow 3 \text{ trivial} \\ 2 \Rightarrow 1 \text{ Woodin} \end{array}}$

Open: Is there a reformulation of BMM^{++} as a "$(*)$-like" axiom.

"there should be"

Open: For $\lambda \geq \aleph_2$, is $\Gamma\text{-}MM^{++}_\lambda$ equivalent ~~at~~ with $\Gamma\text{-}MM^{*,++}_\lambda$?

Is $MM^{*,++}$ equivalent with MM^{++}?

"I think the answer is yes [(our proof didn't work)], I have tried but not too hard".

Open: For $\lambda \geq \aleph_3$, is $MM^{*,++}_\lambda$ consistent.

[from large cardinals]

"contradicted by Woodin something like honestly consistent (more general?)"

In my next two talks:

(1) ~~A~~ $\Pi_2^{H_{\omega_2}}$-maximality is the statement:

if φ is $\Pi_2^{H_{\omega_2}}$ and Ω-consistent then φ is true.

Give a direct proof of $\Pi_2^{H_{\omega_2}}$-maximality from MM^{++}

(2) The consistency of $MM^{*,++}_{2^{\aleph_0}}$ $(\Rightarrow 2^{\aleph_0} = \aleph_2)$

by forcing over a model D of determinacy

($D \models AD + \Theta$ is a regular limit of the Solovay sequence Σ + every set of reals is universally Baire)

$$D^{\mathbb{P}_{max} * Col(\omega_3, \omega_3)} \models ZFC + MM^{*,++}_{\aleph_2}.$$

"It's not known how to get that"

Open: Force $MM^{*,++}_{\aleph_2}$-over a ZFC-model with large cardinals.