# A Kunen-like model without critical continuum (Part II)

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# Recap

- In part I, we went over Silver's extension of embedding construction from a supercompact cardinal and explained why embeddings in the ground have many different extensions in a generic extension.

- We started discussing the Friedman-Magidor approach to controlling the possible extensions of an ultrapower embedding by a measure / short extender, which leads to a blueprint construction for controlling normal measures in suitable generic extensions.

- We continue describing their main arguments and then move to examine the non-normal case.

# Plan (Part II)

**Part II.1:** The Friedman-Magidor blueprint for controlling normal measures

**Part II.2:** Extending the blueprint to non-normal measures

# Part II.1

# The Friedman-Magidor blueprint for normal measures

# FM blueprint

- The Friedman-Magidor (FM) blueprint was developed to control the number of normal measures in a generic extension of a canonical inner model. We will focus on a version designed to force $2^\kappa = \kappa^{++}$ and a unique normal measure on $\kappa$.

- Given a single ultrapower embedding $j : V \to M \cong Ult(V, E)$ by a (short) extender $E$, with $cp(j) = \kappa$, ${}^\kappa M \subseteq M$, and $V_{\kappa+2} \subseteq M$, the goal is to find assumptions for an iteration poset $\mathbb{P}$ that adds $\kappa^{++}$ subsets to $\kappa$, such that for a $V$-generic $G \subseteq \mathbb{P}$ there is a unique $M$-generic $G^* \subseteq j(\mathbb{P})$ with $j``G \subseteq G^*$.

# Keys to the FM blueprint

Comparing with the standard Easton-support construction (as in Silver's work) the main ingredients of the FM-approach for a poset $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \kappa \rangle$ are

1. increase the closure rate of $\mathbb{P}$ so that $j``\mathbb{P}$ meets almost every dense open subset $D \subseteq j(\mathbb{P})$ in $M$,
2. include coding posets to make the the posets $\mathbb{Q}_\alpha$, $\alpha \leq \kappa$ rigid (i.e., have a unique generic filter)

(More details few slides below)

# $\kappa$-Fusion

**An Imprecise Definition:**

Let $\mathbb{P}$ that add subsets to $\kappa$, and for each $\alpha < \kappa$ has "up " and "down" restriction maps:

$$p \mapsto p \upharpoonright \alpha \quad (p \text{ up to } \alpha)$$

$$p \mapsto p \downharpoonright \alpha \quad (p \text{ starting from } \alpha)$$

with the domain of each being dense in $\mathbb{P}$, and a "join" operation $*$, which satisfy natural properties such that $p = p \upharpoonright \alpha * (p \downharpoonright \alpha)$ (other properties will be specified later) .

Say that a set $D \subseteq \mathbb{P}$ is dense beyond $\alpha$ if for every $p \in D$, the weaker condition $1_{\mathbb{P}} \upharpoonright (\alpha + 1) * p \downharpoonright (\alpha + 1)$ is also a member of $D$

Say that $\mathbb{P}$ has the **$\kappa$-fusion property (via restriction maps)** if for every sequence $\langle D_\alpha \mid \alpha < \kappa \rangle$ so that each $D_\alpha$ is dense beyond $\alpha$ and every $p \in \mathbb{P}$, there are $p^* \leq p$ and a club $C \subseteq \kappa$ such that for all $\alpha \in C$ the set $\{p' \in D_\alpha : p' \downharpoonright (\alpha + 1) = p^* \downharpoonright (\alpha + 1)\}$ is dense in $\mathbb{P}/p^*$.

# Remarks

- If $\mathbb{Q}$ is $\kappa^+$-closed then it has the $\kappa$-fusion property
- If $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ is an iteration poset, then we have standard restrictions maps the send $p = \langle \dot{p}_\beta \mid \beta < \kappa \rangle$ to

  $$p \restriction \alpha = \langle \dot{p}_\beta \mid \beta < \alpha \rangle \in \mathbb{P}_\alpha, \text{ and } p \downharpoonleft \alpha = \langle \dot{p}_\beta \mid \alpha \leq \beta < \kappa \rangle$$

  The $\kappa$-fusion property is then **equivalent** to the following statement about the iteration poset $\mathbb{P}$:
  For every $p \in \mathbb{P}$ and $\langle D_\alpha \mid \alpha < \kappa \rangle$ so that each $D_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name for a dense open subset of $\mathbb{P}/\mathbb{P}_{\alpha+1}$, there are $p^* \leq p$ and a club $C \subseteq \kappa$ such that

  $$\forall \alpha \in C \quad p^* \restriction (\alpha + 1) \Vdash_{\mathbb{P}_{\alpha+1}} p^* \downharpoonleft (\alpha + 1) \in D_\alpha$$

# Fusion Lemma for nonstationary support iteration of closed posets

## Lemma (0)

*Suppose that $\kappa$ is a regular cardinal and $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ is a nonstationary support iteration and $\mathbb{Q}_\alpha$ is $\alpha$-closed. Then $\mathbb{P}$ has the $\kappa$-fusion property.*

# Using fusion to extend the reach of $j``\mathbb{P}$

## Lemma (1)

*Suppose that $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa\rangle$ satisfies the assumptions of the previous lemma, and $j : V \to M \cong Ult(V, E)$ is an ultrapower map by a (short) $\kappa$-complete extender $E$. Then*

1. $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{Q}^M_\kappa * R$ *where* $R = (\mathbb{P}^M_{j(\kappa)}/\mathbb{P}^M_{\kappa+1})$ *is the tail quotient forcing of* $j(\mathbb{P}_\kappa)$ *starting stage* $\kappa + 1$.
2. *For every* $\mathbb{P}^M_{\kappa+1}$*-name of a dense open set* $D \subseteq R$ *and a condition* $p \in \mathbb{P}_\kappa$ *there is an extension* $p^* \leq p$ *such that* $j(p) \restriction (\kappa + 1) \Vdash_{\mathbb{P}^M_{\kappa+1}} j(p) \setminus (\kappa + 1) \in D$.

We sketch the proof of Friedman-Magidor theorem.

**Theorem (Friedman-Magidor 2007)**

*The existence of a model with a measurable cardinal $\kappa$ carrying a single normal measure, and $2^\kappa = \kappa^{++}$ is consistent relative to the existence to a $(\kappa, \kappa^{++})$-extender.*

- Force over a minimal model $V = L[\mathcal{E}]$ witnessing a measurable cardinal $\kappa$ carrying a $(\kappa, \kappa^{++})$-extender $E$. This means that for any other $\kappa$-complete measure/extender $F \in V = L[\mathcal{E}]$ the ultrapower embedding $j_F : V \to M_F$ satisfies $(\kappa^{++})^{M_F} << j_F(\kappa) < j_F(\kappa)^{++}$
- Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ be a nonstationary support iteration of posets

$$\mathbb{Q}_\alpha = Sacks(\alpha, \alpha^{++}) * Code_\alpha$$

- $Sacks^*(\alpha)$ is the generalized Sacks forcing consisting of $\alpha$-closed pruned trees $T \subseteq 2^{<\alpha}$, for which there is a club $C_T \subseteq \alpha$ such that a node $s \in T$ splits if and only if $len(s) \in C$ is singular.

- $Sacks^*(\alpha, \alpha^{++})$ is a $\leq \alpha$-support product of $\alpha^{++}$-many copies of $Sacks^*(\alpha)$. By a result of Friedman and Thompson, $Sacks^*(\alpha, \alpha^{++})$ is $\alpha$-closed and satisfies $\alpha$-fusion.

- $Code_\alpha$ codes the generic $Sacks^*(\alpha, \alpha^{++})$ sequence of cofinal branches $\langle s^\alpha_\tau \mid \tau < \alpha^{++} \rangle$, $s^\alpha_\tau \in 2^\alpha$ and itself.

- A standard way of coding a sequence of function $\langle f_\tau \mid \tau < \alpha^{++} \rangle \subseteq 2^\alpha$ using disjoint stationary sets $\langle S^\alpha_i \mid i < \alpha^{++} \rangle$ is by forcing a club in $\alpha^{++}$ to be disjoint from $S^\alpha_{\alpha \cdot \tau + 2\beta}$ if $f_\tau(\beta) = 0$, and forcing it to be disjoint from $S^\alpha_{\alpha \cdot \tau + 2\beta + 1}$ if $f_\tau(\beta) = 1$. This coding principle can be extended so that the generic club codes itself.

Let $j_E : V \to M_E$ be the ultrapower embedding of $V = L[\mathcal{E}]$ by $E$. To complete the proof we need the following

### Lemma (FM.1)

*If $G \subseteq \mathbb{P}$ be $V$-generic then in $V[G]$ there is a unique $M_E$-generic filter $G^* \subseteq j_E(\mathbb{P})$ so that $j``G \subseteq G^*$.*
*Moreover , the extension $j^* : V[G] \to M_E[G^*]$ satisfies that for every $x \in M_E[G^*]$ there is $f \in V[G]$, $f : \kappa \to V[G]$ such that $x = j^*(f)(\kappa)$.*

The last part of the Lemma implies that the map $j^* : V[G] \to M_E[G^*]$ is equal to the ultrapower map of $V[G]$ by the $j^*$-derived normal measure $U^* = \{X \subseteq \kappa \mid \kappa \in j^*(X)\}$.

### Lemma (FM.2)

*$U^*$ is the only normal measure on $\kappa$ in $V[G]$.*

# $j$ extends (1/2)

Write $\mathbb{P} = \mathbb{P}_\kappa * \mathbb{Q}_\kappa$ and $G = G_\kappa * g_\kappa \subseteq \mathbb{P}_\kappa * \mathbb{Q}_\kappa$.

- Use Lemma 0 to show $\mathbb{P}_\kappa$ has the $\kappa$-fusion property
- Use Lemma 1 and the $\kappa$-fusion property of $\mathbb{P}_\kappa$ to show that $G \wedge j_E``G_\kappa$ generates an $M_E$-generic set $G^*_{j(\kappa)} \subseteq j_E(\mathbb{P}_\kappa)$. Let $\bar{j} : V[G_\kappa] \to M_E[G^*_{j(\kappa)}]$ be the resulting elementary extension of $j_E$.
- Use the result of Friedman and Thompson that $\mathbb{Q}_\kappa$ has the $\kappa$-fusion property with the natural tree restriction maps, to show that $\bar{j}``g_\kappa$ generates an $M_E[G^*_{j(\kappa)}]$-generic set $g^* \subseteq \bar{j}(\mathbb{Q}_\kappa)$.

# $j$ extends (2/2)

- Let $G^* = G^*_{j(\kappa)} * g^*$ be the resulting $M_E$-generic for $j_E(\mathbb{P}_\kappa)$ and $j^* : V[G] \to V[G^*]$ the induced extension of $j_E$. Show next that every $x \in M_E[G^*]$ is of the form $j^*(f)(\kappa)$ for some function $f \in V[G]$. Since the generators of $E$ are in $[\kappa, \kappa^{++})$ it suffices to show that for every $\tau \in [\kappa, \kappa^{++})$ there is $g \in V[G]$ such that $\tau = j^*(g)(\kappa)$. For this, use the identification of $\tau$ with the $\tau$-th generic function $s^\kappa_\tau \in 2^\kappa$ form the $\mathbb{Q}_\kappa$-generic $\kappa^{++}$-sequence of functions, and observe that $s^\kappa_\tau = j^*(s^\kappa_\tau) \restriction \kappa$ is definable in $M_E[G^*]$ from the pointwise image of $j^*$ and $\kappa$.

- In $V[G]$, let $U^* = \{X \subseteq \kappa \mid \kappa \in j^*(X)\}$ be the normal measure on $\kappa$ derived from $j^*$. The last point shows that $j^* = j_{U^*} : V[G] \to M_{U^*} = M_E[G^*]$ is the ultrapower by $U^*$.

## $U^*$ is unique (1/3)

### Lemma (Friedman-Magidor)

*$U^*$ is the only normal measure on $\kappa$ in $V[G]$.*

- Let $W$ be a $\kappa$-complete ultrafilter on $\kappa$ in $V[G]$, and $j_W : V[G] \to M_W$ be its ultrapower embedding. By a theorem of Schindler, the restriction $i_W := j_W \restriction V : V \to M$ is a normal iterated ultrapower of $V = L[\mathcal{E}]$ by its extenders, and $M_W = M[G_W]$ for some $M$-generic $G_W \subseteq j_W(\mathbb{P})$.

- Since $\mathbb{P}$ is $\sigma$ closed, the iteration resulting in $M$ must be finite.

$$i_W = j_{\ell-1,\ell} \circ j_{\ell-2,\ell-2} \circ \cdots \circ j_{0,1}$$

where $\ell < \omega$ and for each $i < \ell$,

$$j_{i,i+1} = j_{F_i}^{M_i} : M_i \to M_{i+1} \cong Ult(M_i, F_i)$$

is an ultraopwer embedding by an extender $F_i \in M_i$ ($M_0 = V$) with critical point $\kappa_i$, and $\kappa = \kappa_0 < \kappa_1 < \ldots \kappa_{\ell-1}$.

# $U^*$ is unique (2/3)

- **Assume** $W$ is a normal measure . If we can show that (i) $F_0 = E$, and (ii) $\ell = 1$ then we get $j_W \restriction V = j_E$, and so the requirement $j_W``G \subseteq G_W$ translates to $j_E``G \subseteq G_W$. We can then apply the argument of Lemma FM.1 and conclude $G_W = G^*$, which implies $W = U^*$.
- (i) is an immediate consequence of the fact $\mathcal{P}(\kappa) \subseteq M_W$, which implies $(2^\kappa)^{M_W} \geq \kappa^{++}$. Since $E$ was assumed to be the only extender in $V = L[\mathcal{E}]$ to have height $\geq \kappa^{++}$, $F_0 \neq E$ would imply that $(2^\kappa)^{M[G_W]} = (\kappa^{++})^{M_W} < j_{F_0}(\kappa) < \kappa^{++}$. Absurd.

## $U^*$ is unique (3/3)

(ii) makes a critical use of the normality assumption of $W$, together with the following lemma of Friedman and Magidor, whose proof is similar to the argument for Lemma 0.

### Lemma (Friedman-Magidor)

*For every $\mathbb{P}$-name of a function $\dot{f} : \kappa \to On$ and a condition $p \in \mathbb{P}$ there are $p^* \leq p$, a club $C \subseteq \kappa$, and a function $F \in V$ with $F(\alpha) \in [On]^{\alpha^{++}}$ for all $\alpha$, such that $p^* \Vdash \forall \alpha \in C.\ \dot{f}(\alpha) \in F(\alpha)$.*

- Use the Lemma to show $\ell = 1$. Suppose otherwise, $\ell \geq 2$. The normality of the iterated ultrapower and the fact $F_0 = E$ imply $cp(j_{1,2}) > \kappa^{++}$. Since $W$ is normal in $V[G]$, there is a function $f = \dot{f}_G : \kappa \to \kappa$ such that $cp(j_{1,\ell}) = \kappa_1 = j_W(f)(\kappa)$.
- By the previous lemma, there is $F \in V$ as above such that $\kappa_1 \in j_W(F)(\kappa)$ and $j_W(F)(\kappa) \in [j_W(\kappa)]^{\kappa^{++}}$.
- Since $F \in V$, $j_W(F)(\kappa) = i_W(F)(\kappa) = j_{1,\ell}``(j_E(F)(\kappa))$, which means $\kappa_1 \in \mathrm{rng}(j_{1,\ell})$. Absurd.

# The FM-blueprint (1/2)

The Friedman-Magidor blueprint lists the requirements from the iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \kappa \rangle$ used in the proof of the last theorem.

**Definition: (FM-blueprint)**

The FM-blueprint for an iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \kappa \rangle$ and an ultrapower embedding $j : V \to M \cong Ult(V, E)$ by an extender $E$ includes the following assumptions:

(1) $\mathbb{P}$ is a nonstationary support iteration. Each $\mathbb{Q}_\alpha$ is trivial if $\alpha$ is not inaccessible, and for each inaccessible $\alpha \leq \kappa$, $\mathbb{Q}_\alpha$ is $\alpha$-closed, has size $\alpha^{++}$ and adds $\alpha^{++}$-new subsets to $\alpha$.

(2) For each inaccessible $\alpha \leq \kappa$, $\mathbb{Q}_\alpha$ self codes its generic set by destroying certain stationary sets from a sequence $\vec{S}^\alpha = \langle S^\alpha_i \mid i < \alpha^{++} \rangle$ of almost disjoint stationary subsets $S^\alpha_i \subseteq \alpha^{++} \cap \mathrm{cof}(\alpha^+)$

# The FM-blueprint (2/2)

(3) The choice of $\mathbb{Q}_\alpha$ is absolute between models that contain $H_{\alpha^{++}}$.

(4) $1_{\mathbb{P}_\kappa}$ forces that $j``\dot{G}(\mathbb{Q}_\kappa)$ generates a generic filter for $j(\mathbb{Q}_\kappa)$ over $M^{j(\mathbb{P}_\kappa)}$.

# Part I.3

# Extending the blueprint to non-normal measures

# From normal measures to non-normal measures

- The Friedman-Magidor proof shows that if $\mathbb{P} \in L[\mathcal{E}]$ satisfies the FM-blueprint then $2^\kappa = \kappa^{++}$ in a generic extension $V[G]$, $2^\kappa = \kappa^{++}$, and $\kappa$ carries a unique normal measure $U^*$.

- Finite power $W = (U^*)^\ell$ of $U^*$ are easily seen to completely described by
  1. the restriction $j_W \restriction V$, which is the $n$-iterated ultrapower by $E$ (and its images), namely $F_0 = E$ and $F_{i+1} = j_{i,i+1}(F_i)$ for all $i < \ell - 1$,
  2. the generic $G_W$ which is obtained by copying the construction of $G^*$ from $G \cup j_E``G$ $\ell$ times. Namely, $G_W = G^*_\ell$ where $G^*_0 = G$ and $G^*_{i+1} = \langle G^*_i \wedge j^*_{i,i+1}``G^*_i \rangle$.

  It is useful to note that $G_W = G^*_\ell$ contains the pointwise image by the final iteration map $i_W``G = j_{0,\ell}``G \subseteq j_{0,\ell}(\mathbb{P})$. The additional information found in $G^*_\ell$ beyond the pointwise image are the generics at critical coorinates $\kappa_0, \kappa_1, \ldots, \kappa_{\ell-1}$.

**Goal:** Modify the FM-blueprint to show that every $\kappa$-complete ultrafilter $W$ in $V[G]$ is equivalent to a finite poset $(U^*)^\ell$ for some $\ell < \omega$.

- Fix a $\kappa$-complete ultrafilter $W$ in $V[G]$ and use the notations from before (Keys for Lemma FM.2). We know $i_W = j_W \restriction V : V \to M$ is a finite iteration of some length $\ell \geq 1$, and $M_W = M[G_W]$.

- To prove that $W = (U^*)^\ell$ we need to establish

(KEY 1) $F_i = j_{0,i}(E)$ for all $i < \ell$.

(KEY 2) $G_W \restriction \mathbb{Q}^M_{\kappa_i} = G^* \restriction \mathbb{Q}^M_{\kappa_i}$ is generated by $j_{0,i-1}$“$g_\kappa$.

For functions $f, g \in {}^{\alpha}\alpha$ for a regular cardinal $\alpha$, write $f <^*_{Sing} g$ when there is a club $C \subseteq \alpha$ such that $f(\beta) < g(\beta)$ for every singular ordinal $\beta \in C$.

**Definition: (Modified blueprint)**
The modified blueprint for an iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \kappa \rangle$ includes the following changes:

1. The assumption $\mathbb{Q}_\alpha$ is $\alpha$-closed is replaced with the weaker requirement $\mathbb{Q}_\alpha$ is $\alpha$-distributive.
2. For each inaccessible $\alpha \leq \kappa$ the assumptions that $\mathbb{P}_\alpha$ satisfies $\alpha$-fusion, and $\mathbb{P}/\mathbb{P}_\alpha$ is $\alpha$-distributive are added (as they cannot be derived directly with $\mathbb{Q}_\alpha$ being only $\alpha$-distributive).
3. For each inaccessible $\alpha \leq \kappa$, $\mathbb{Q}_\alpha$ is additionally assumed to add sequence $\langle s^\alpha_\tau \mid \tau < \alpha^{++} \rangle \subseteq \alpha^\alpha$ which is $<^*_{Sing}$-increasing.
4. The stationary sets $S^\alpha_i \in \vec{S}^\alpha$ used for coding, are now assumed to be almost disjoint and nonreflecting stationary subsets of $\alpha^+ \cap cof(< \alpha)$ (i.e., have small-cofinality ordinals). .

# Modified blueprint in action

**Lemma (BN-Kaplan)**

*If $\mathbb{P}$ satisfies the modified blueprint then every $\kappa$-complete ultrafilter $W$ in a generic extension $V[G]$ by $G \subseteq \mathbb{P}$ has key properties (Key 1) and (Key 2) for every $\kappa$-complete ultrafilter $W$ in $V[G]$.*

# Constructing $\mathbb{P}$ satisfying modified blueprint (1/2)

- To add $<^*_{Sing}$-increasing sequences of functions $\langle s^\alpha_\tau \mid \tau < \alpha^{++} \rangle$, we replace the higher version of Sacks forcing with a suitable higher version of Miller forcing.

- A main challenge in finding a poset $\mathbb{P}$ which satisfies the $\kappa$-fusion under the modified blueprint assumptions, comes from the fact that coding posets which add clubs to a nonreflecting stationary subset of $\alpha^+ \cap cof(< \alpha)$ are $\alpha$-distributive but cannot be $\alpha$-closed. This is a problem since there is no general iteration theorem for distributive posets like the one for closed posets.

- For example, by a unpublished result of Adolf-BN-Schindler-Zeman, if every iteration sequence $\langle \mathbb{Q}_n \mid n < \omega \rangle$ where each $\mathbb{Q}_n$ being $\aleph_n$-distributive, has an iteration scheme that does not collapse cardinals, then PD holds.

# Constructing $\mathbb{P}$ satisfying modified blueprint (2/2)

- Gitik has constructed methods for iterating arbitrary distributive posets $\mathbb{Q}_\alpha$ using the iteration theory for Prikry forcings, but the construction requires large cardinals at finite levels of supercompacts, which does not fit the Kunen-like model framework.

- The proof of the main (Kunen-Like model) theorem solves the iteration problem by making use of the fine structure of $L[\mathcal{E}]$ to construct an iteration with the modified blueprint assumptions. The motivation for this part was a result announced by Zeman, who showed how to construct Easton-support iterations of posets that destroy nonreflecting stationary subsets, without collapsing cardinals.