**Theorem** $MM \Rightarrow \Pi_2^{H\omega_2}$ maximality

**Definition** Let $\phi$ be a statement. We say $\phi$ is 1-$\Omega$-consistent iff, for all $x \in \mathbb{R}$ there is a transitive model $\mathcal{M} = (M, \in)$ s.t.:

- $\mathcal{M} \models ZFC + \phi$
- $x \in \mathcal{M}$, $\mathcal{M}$ is closed under $y \mapsto M_1^\#(y)$ | for any $y \in \mathcal{M}$, not just reals.

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**Definition** Let $\phi$ be a statement and let $A \in V$. We say $\phi(A)$ is 1-honestly-consistent iff there is a transitive model $\mathcal{M} = (M, \in)$ (in $V^{Col(\omega, 2^{\aleph_1})}$) s.t.:

- $\mathcal{M} \models ZFC + \phi(A)$
- $H_{\omega_2}^V \in M$
- $M$ is closed under $y \mapsto M_1^\#(y)$ | for all $y \in \mathcal{M}$.

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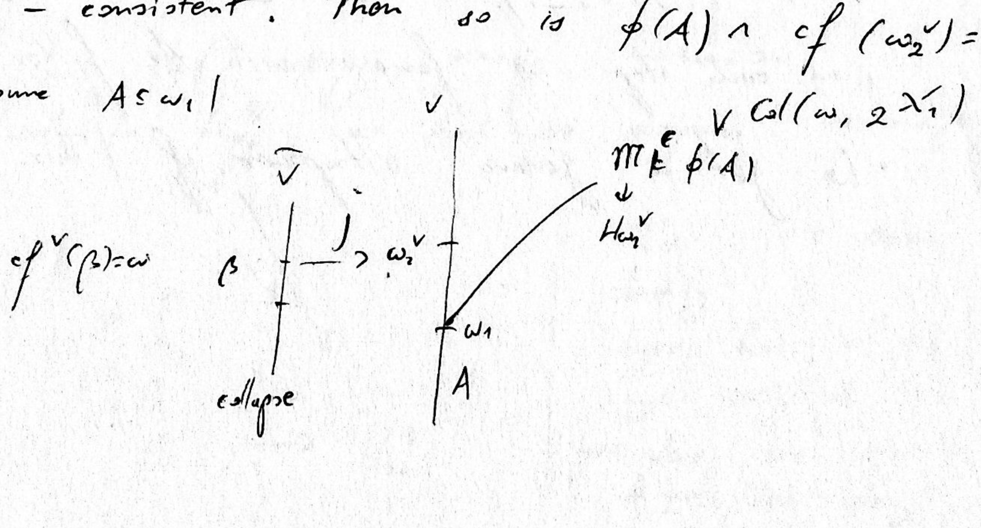
$\Pi_2^{H\omega_2}$ maximality means: if $\phi$ is $\Pi_2^{H\omega_2}$ and if $\phi$ is 1-$\Omega$-consistent, then $\phi$ is true.

**Lemma** Let $\phi$ be $\Sigma_1$, let $A \in H_{\omega_2}$. Suppose $\phi(A)$ to be 1-honestly-consistent. Then so is $\phi(A) \wedge cf(\omega_2^V) = \omega$.

**Proof** | Assume $A \subseteq \omega_1$ |



2 Now, $\mathfrak{M} \models ZFC \wedge \phi(A)$

$H_{\omega_2}^{\bar{V}} \in \mathfrak{M}$ (since $H_{\omega_2}^{V} \in M$)

$\mathfrak{M} \models cf(\omega_2^{\bar{V}}) = \omega$.

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Such an $\mathfrak{M}$ exists in $\bar{V}^{Col(\omega_1, (2^{\aleph_1})^{\bar{V}})}$, so by Schoenfield |something also to do with closure under sharps to get $\Sigma^1_2$-correctness? | (?) |Unclear fragment|

You get this $\mathfrak{M}$ in $\bar{V}^{Col(\omega, (2^{\aleph_1})^{\bar{V}})}$, so you push it by $j$ to $V^{Col(\omega, (2^{\aleph_1})^{V})}$

--- (?)

∎

<u>Lemma</u> Suppose $NS_{\omega_1}$ is saturated (in $V$). Suppose $\phi$ is $\Sigma_1$, $A \in H_{\omega_2}$, $(\phi(A) \wedge cf(\omega_2^V) = \omega)$ is 1-honestly-consistent as witnessed by $\mathfrak{M} \in V^{Col(\omega_1, 2^{\aleph_1})}$. Then in $\mathfrak{M}$, there's a generic iteration $(M_i, \pi_i, i \leq j \leq \omega_1)$ s.t. $M_0$ is ctbl, and $M_{\omega_1} = (H_{\omega_2}^V, \in, NS_{\omega_1}^V)$

By a <u>generic iteration</u> in this context we mean:

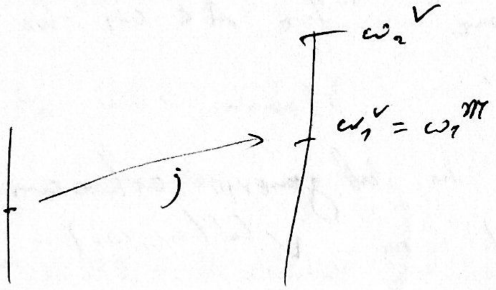
- $M_i = (M_i, \in, I_i)$, where $I_i$ is $NS_{\omega_1}^{M_i}$
- in each step, we force with the filter dual to $I$
- We get a generic ultrapower. |This is $M_{i+1}$

Proof

Note $\omega_1^V = \omega_1^{M}$, since by def. of 1-h-con:

$$NS_{\omega_1}^{M} \cap V = NS_{\omega_1}^V.$$

$(M_0; \in, NS_{\omega_1}^{M_0}) = M_0$



$$g = \{ S \in P(\omega_1^{M_0}) \cap M_0 : \omega_1^{M_0} \in j(S) \}.$$

Key fact: $g$ is generic for $\underbrace{(NS_{\omega_1}^{M_0})^+}$ over $M_0$.

- positive sets of this ideal
- what gives the generic ultrapower

Why: Let $A \in M_0$ be a max antichain in $NS_{\omega_1}^{M_0}$.

Let $j(A) := (S_i : i < \omega_1)$. Let $C$ be a club s.t.:

$\forall \alpha \in C \quad S_i \in \alpha$ for some $i < \alpha$.

w.l.o.g. $C \in \mathrm{im}\, j$. So (?) $\alpha_0 = \mathrm{crit}\, j_0 = \omega_1^{M_0} \in C$

$\Rightarrow \alpha_0 \in j(\underbrace{S_i \cap \alpha_0}_{\in A})$

$\alpha_0 \in S_i$ for some $i \in \alpha_0$

(?) Unclear. Apparently standard.

Now, this shows that every generic extension is realised in $H_{\omega_2}^V$ so that the diagram commutes:

$$\begin{tikzcd}
M_{\omega_1} \arrow[drr] & & \\
\vdots \arrow[u] \arrow[dr] & & \\
M_1 \arrow[u, "\text{gen}"'] \arrow[r] & & \\
M_0 \arrow[u, "\text{gen}"] \arrow[rr, "j"'] & & H_{\omega_2}^V
\end{tikzcd}$$

The bit arrow is actually identity:
- you have all the ordinals – you get everything up to $\omega_1^V$, since you have $\omega_1$ sequence of growing crit
- you have everything between $\omega_2^V$ and $\omega_1^V$ since in $\mathrm{cof}(\omega_2^V) = \omega$ and you include it in $\mathrm{im}\, j(M_0)$

4. <u>Now</u>

$MM \Rightarrow \Pi_2^{H_{\omega_2}}$ -maximality

~~Fix~~ <u>Proof</u> Fix $\phi := \forall X \in H_{\omega_2}\ \exists Y \in H_{\omega_2}\ \bar{\phi}(X, Y)$.

Assume $\phi$ is $1$-$\Omega$-consistent.

We want to see $\phi$ is true. Fix $A \subset \omega_1$. We want to see:

$\exists Y\ \bar{\phi}(A, Y)$ is true.

$\phi$ is $1$-honestly-consistent in all generic extensions. In particular, $\phi$ is $1$-honestly-consistent in $V^{Col(\omega, \omega_2)}$.

Now ~~fM~~ $((H_{\omega_2})^V, \in, NS_{\omega_1}^V) \in \mathcal{M}$ wlog, since it is ctbl, where $\mathcal{M}$ witnesses $1$-$\Omega$-consistency of $\phi$.

Iterate $((H_{\omega_2})^V, \in, NS_{\omega_1}^V)$ ~~ω₂~~ inside $\mathcal{M}$ with length $\omega_1^{\mathcal{M}}$ generically

$$\downarrow j$$
$$M_{\omega_1^{\mathcal{M}}}$$

By general $\mathbb{P}_{max}$ theory, ~~positive sets~~ we can arrange:

$$NS_{\omega_1}^V \underset{j}{\longmapsto} NS_{\omega_1}^{\mathcal{M}} \cap M_{\omega_1^{\mathcal{M}}}.$$

Now, $\mathcal{M} \models \exists Y\ \bar{\phi}(j(A), Y)$ | since $j(A) \in (H_{\omega_2})^{\mathcal{M}}$

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Notice: $H_{\omega_2}^V$ is rank initial segment of $V$, so we can lift the iteration to an iteration of $V$ | by saturation antichain in $NS_{\omega_1}^V$ are in ~~M~~ $H_{\omega_2}^V$ | Call the iterate $V'$

| There is a model $\mathcal{M}$ witnessing $1$-honest cons. of $\phi(A)$ in $V^{Col(\omega, \omega_2)} \supset V'$,
By absoluteness, there is such an $\mathcal{M}$ for $V_{..}$ (?) | Unclear.

You gave me an arbitrary $A \subset \omega_1$. We showed $\exists Y\ \bar{\phi}(A, Y)$ is honestly consistent.

<u>Key Lemma</u> If $\exists Y\ \phi(A, Y)$ is $1$-honestly consistent, then there is a stationary set preserving $\mathbb{P}$ s.t. $V^{\mathbb{P}} \models \exists Y\ \phi(A, Y)$.