Marginalia to a Theorem of Asperó and Schindler

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In $V^{\mathbb{H}}$, there exists a model for ϕ .

- However, in general, \mathbb{H} collapses ω_1 .
- We want to define a poset \mathbb{P}_{κ} which also adds a model for ϕ , but which is in addition stationary set preserving.

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- For all $M \in \mathcal{C}_{[0,\kappa]}$, we denote by λ_M the unique λ such that $M \in \mathcal{C}_{\lambda}$.



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 - $\bullet \delta_M = \delta_N \implies M = N,$
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- $p \le q$ iff $w_p \supseteq w_q$ and for all $N \in \mathcal{M}_q$, there exists $M \in \mathcal{M}_p$ such that $\delta_M = \delta_N$, $\lambda_M = \lambda_N$, and $M \supseteq N$.

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- For $p \in \mathbb{P}^*_{\lambda}$ and for $\bar{\lambda} \in \lambda \cap \mathcal{E}$, we define

$$p \upharpoonright \lambda := (w_p, \{M \in \mathcal{M}_p : \lambda_M < \lambda\}).$$



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- If $Q_n = (M, D)$ where $M \in \mathcal{M}_{p_{n-1}}$ and $D \in M$ is dense in \mathbb{P}_{λ_M} , then there exists $q \in D$ such that $\delta(\mathsf{Hull}(M, q)) = \delta(M)$ and $p_n \leq p_{n-1}, q$.

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- Player II wins infinite plays such that the set $\bigcup_{n<\omega} w_{p_n}$ does not contain both an atomic formula and its negation.



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Definition (De Bondt, Veličković)

Suppose that $\theta \gg \kappa$ is regular and $M \prec (H_{\theta}, \in, \kappa, \phi)$ is countable. Then M is good iff for all $p \in \mathbb{P}_{\kappa} \cap M$, there exist $q \in \mathbb{P}_{\kappa}$ and $\lambda \in \mathcal{E}$ such that

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Suppose that $\theta \gg \kappa$ is regular and $M \prec H_{\theta}$ is good. Then \mathbb{P}_{κ} is semiproper for M.

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Corollary

If there exists a local club of good models $M \prec H_{\theta}$, then \mathbb{P}_{κ} is stationary set preserving.

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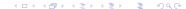
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Local club

 $C\subseteq [X]^\omega$ is a *local club* iff for weak-club many $\bar{X}\in [X]^{\omega_1}$, the set $C\cap [\bar{X}]^\omega$ contains a club.

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- **1** Let θ be large enough regular. It suffices to show that there are local-club many good models $M \prec H_{\theta}$.
- 2 Let R be a wellordering of H_{θ} and let $\mathcal{H} := (H_{\theta}, \in, R, \kappa, \phi)$. We want to show that for all $X \prec \mathcal{H}$ satisfying $\omega_1 \subseteq X$ and $|X| = \omega_1$, we have that club many countable $M \prec X$ are good.

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- 3 Let $X \prec \mathcal{H}$ satisfying $\omega_1 \subseteq X$ and $|X| = \omega_1$ be arbitrary. We denote by \mathcal{X} the structure on X inherited from \mathcal{H} .

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- 4 Let us assume towards contradiction that there exists a stationary $S \subseteq [X]^{\omega}$ such that for all $M \in S$, it holds that $M \prec \mathcal{X}$ and that there exists $p_M \in \mathbb{P}_{\kappa} \cap M$ witnessing that M is not good.

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We work in V[h].



Since ϕ is AS good, there exist elementary $\tau: V \to W$ and $\hat{\mu} \models \tau(\phi)$ such that $\mathrm{crit}(\tau) = \omega_1^V \in \tau(T)$ and for all $\psi \in \phi \downarrow$, $\hat{\mu}(\tau(\psi)) = \mu(\psi)$.

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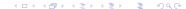
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- II It follows that in W, $\tau(p)$ does not witness that N is not good.
- Since $\delta_N = \omega_1^V \in \tau(T)$, we have that $N \in \tau(S')$.
- By definition of S' and elementarity if τ , this means that in W, $\tau(p)$ witnesses that N is not good. This is a contradiction.

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$$q := (\tau(w_p), \tau(\mathcal{M}_p) \cup \{N \downarrow \tau(\lambda)\})$$
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. Then $q \in \tau(\mathbb{P}_{\kappa})$ and $q \leq \tau(p)$.

Proof.

- The only non-trivial part is that $q \in \tau(\mathbb{P}_{\kappa})$. Let us assume otherwise.
- Then there exists a winning strategy σ for Player I in $\mathcal{G}^W_{\tau(\kappa)}(q)$. We will defeat this strategy in V[g], reaching a contradiction. (We use in this step that the game $\mathcal{G}^W_{\tau(\kappa)}(q)$ is closed for Player II.)

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- We will distinguish four possiblities for Q_n . In each, we show that Player II can make the next move and preserve conditions from c.



Proof. We have that $\hat{\mu} \models \psi$, so there exists $i \in I$ such that $\hat{\mu} \models \psi_i$. We set $p_n := (w_{p_{n-1}} \cup \{\psi_i\}, \mathcal{M}_{p_{n-1}})$.

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- **Case III.** $Q_n = (P \downarrow \tau(\lambda), D)$ where $D \in P \downarrow \tau(\lambda)$ is dense $\mathbb{P}_{\tau(\lambda)}$.
 - Let $\rho: \operatorname{Hull}(P, \tau(V_{\lambda})) \to \widehat{P \downarrow \tau(\lambda)}$ be the transitive collapse and let $D^+ := \rho^{-1}(D) \in P$. Then D^+ is dense in $\tau(\mathbb{P}_{\kappa})$.

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- Since $D^+ \in P$, there exists $\bar{D}^+ \in V$ such that $\tau(\bar{D}^+) = D^+$.
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- By the existence of the appropriate strategy for $\mathcal{G}_{\lambda}(r')$, there exists $s \in \mathbb{P}_{\lambda}$ such that $s \leq r'$ and such that there exists $t \in \bar{D}$ satisfying $s \leq t$ and $\delta(\operatorname{Hull}(\bar{M}',t)) = \delta(\bar{M}')$.



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 ${\mathbb N}$ We have $au(t) \in D$ and $p_n \le au(t)$, while

$$\begin{split} \delta(\mathsf{Hull}(M,\tau(t))) &= \delta(\tau(\mathsf{Hull}(\bar{M},t))) = \delta(\mathsf{Hull}(\bar{M},t)) = \\ &= \delta(\bar{M}) = \delta(M). \end{split}$$

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- $lacksymbol{\mathbb{M}}$ Let $P':=\mathsf{Hull}^{ au(\mathcal{H})}(\omega_1^V\cup F\cup\{ au(s)\})$ and let

$$p_n := (\tau(w_s) \cup w, \tau(\mathcal{M}_s) \cup \{P' \downarrow \tau(\lambda)\}) \in \tau(\mathbb{P}^*_{\lambda}).$$

Note that $\delta_{P'} = \omega_1^V$.

We have $\tau(t)$ ∈ D and $p_n \le \tau(t)$, while

$$\begin{split} \delta(\mathsf{Hull}(M,\tau(t))) &= \delta(\tau(\mathsf{Hull}(\bar{M},t))) = \delta(\mathsf{Hull}(\bar{M},t)) = \\ &= \delta(\bar{M}) = \delta(M). \end{split}$$

QED (Case IV)



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Since we have shown Claim 10, we conclude the proof of the main theorem.

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THANK YOU FOR YOUR ATTENTION!