Marginalia to a Theorem of Asperó and Schindler

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In $V^{\mathbb{H}}$, there exists a model for ϕ.

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- However, in general, $\mathbb{H}$ collapses ω_1.
- We want to define a poset $\mathbb{P}_\kappa$ which also adds a model for ϕ, but which is in addition stationary set preserving.

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- For all $S \subseteq \mathrm{Ord}$, we denote $\mathcal{C}_S := \bigcup_{\lambda \in S \cap \mathcal{E}} \mathcal{C}_\lambda$.
- For all $M \in \mathcal{C}_{[0,\kappa]}$, we denote by λ_M the unique λ such that $M \in \mathcal{C}_\lambda$.

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- $p \leq q$ iff $w_p \supseteq w_q$ and for all $N \in \mathcal{M}_q$, there exists $M \in \mathcal{M}_p$ such that $\delta_M = \delta_N$, $\lambda_M = \lambda_N$, and $M \supseteq N$.

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- For $p \in \mathbb{P}^*_\lambda$ and for $\bar{\lambda} \in \lambda \cap \mathcal{E}$, we define

$$p \restriction \lambda := (w_p, \{M \in \mathcal{M}_p : \lambda_M < \lambda\}).$$

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- If $Q_n = (M, D)$ where $M \in \mathcal{M}_{p_{n-1}}$ and $D \in M$ is dense in $\mathbb{P}_{\lambda_M}$, then there exists $q \in D$ such that $\delta(\mathrm{Hull}(M, q)) = \delta(M)$ and $p_n \leq p_{n-1}, q$.

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- Player II wins infinite plays such that the set $\bigcup_{n<\omega} w_{p_n}$ does not contain both an atomic formula and its negation.

Definition

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Suppose that $\theta \gg \kappa$ is regular and $M \prec (H_\theta, \in, \kappa, \phi)$ is countable. Then M is *good* iff for all $p \in \mathbb{P}_\kappa \cap M$, there exist $q \in \mathbb{P}_\kappa$ and $\lambda \in \mathcal{E}$ such that

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Local club

$C \subseteq [X]^\omega$ is a *local club* iff for weak-club many $\bar{X} \in [X]^{\omega_1}$, the set $C \cap [\bar{X}]^\omega$ contains a club.

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2. Let R be a wellordering of H_θ and let $\mathcal{H} := (H_\theta, \in, R, \kappa, \phi)$. We want to show that for all $X \prec \mathcal{H}$ satisfying $\omega_1 \subseteq X$ and $|X| = \omega_1$, we have that club many countable $M \prec X$ are good.

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4. Let us assume towards contradiction that there exists a stationary $S \subseteq [X]^\omega$ such that for all $M \in S$, it holds that $M \prec \mathcal{X}$ and that there exists $p_M \in \mathbb{P}_\kappa \cap M$ witnessing that M is not good.

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We work in $V[h]$.

8. Since ϕ is AS good, there exist elementary $\tau : V \to W$ and $\hat{\mu} \models \tau(\phi)$ such that $\mathrm{crit}(\tau) = \omega_1^V \in \tau(T)$ and for all $\psi \in \phi{\downarrow}$, $\hat{\mu}(\tau(\psi)) = \mu(\psi)$.

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9. Let $N := \mathrm{Hull}^{\tau(\mathcal{H})}(\omega_1^V \cup \{\tau(p)\}) \prec \tau(\mathcal{H})$. It holds that

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11. It follows that in W, $\tau(p)$ does not witness that N is not good.
12. Since $\delta_N = \omega_1^V \in \tau(T)$, we have that $N \in \tau(S')$.
13. By definition of S' and elementarity if τ, this means that in W, $\tau(p)$ witnesses that N is not good. This is a contradiction.

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a. The only non-trivial part is that $q \in \tau(\mathbb{P}_\kappa)$. Let us assume otherwise.

b. Then there exists a winning strategy σ for Player I in $\mathcal{G}^W_{\tau(\kappa)}(q)$. We will defeat this strategy in $V[g]$, reaching a contradiction. (We use in this step that the game $\mathcal{G}^W_{\tau(\kappa)}(q)$ is closed for Player II.)

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f We will distinguish four possiblities for Q_n. In each, we show that Player II can make the next move and preserve conditions from c.

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  Proof. We have that $\hat{\mu} \models \psi$, so there exists $i \in I$ such that $\hat{\mu} \models \psi_i$. We set $p_n := (w_{p_{n-1}} \cup \{\psi_i\}, \mathcal{M}_{p_{n-1}})$. $\square$

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- h **Case II.** $Q_n = (\psi, i) \in w_{p_{n-1}} \times I$ where $\psi \equiv \bigwedge_{i \in I} \psi_i$.

  Proof. We can take $p_n := (w_{p_{n-1}} \cup \{\psi_i\}, \mathcal{M}_{p_{n-1}})$. $\square$

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  - iii Let $\tilde{D}$ consist of all $s \in \mathbb{P}_\lambda$ satisfying that $s \leq r$ and that there exists $t \in \bar{D}$ such that $s \leq t$ and $\delta(\mathrm{Hull}(\bar{M}, t)) = \delta(\bar{M})$. We want to show that $\tilde{D}$ is dense below r.
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  - v By the existence of the appropriate strategy for $\mathcal{G}_\lambda(r')$, there exists $s \in \mathbb{P}_\lambda$ such that $s \leq r'$ and such that there exists $t \in \bar{D}$ satisfying $s \leq t$ and $\delta(\mathrm{Hull}(\bar{M}', t)) = \delta(\bar{M}')$.

- **Case IV.** $Q_n = (M, D)$ where $M \in \tau(\mathcal{M}_r)$ and $D \in M$ is dense $\mathbb{P}_{\lambda_M}$.

  Proof.

  - i. We have that $D \in P$, so there exists $\bar{D} \in V$ such that $\tau(\bar{D}) = D$.
  - ii. Let $\bar{M} := \tau^{-1}(M) \in \mathcal{M}_r$. By elementarity of τ, we have that $\bar{D} \in \bar{M} \in \mathcal{M}_r$ and that $\bar{D}$ is dense in $\mathbb{P}_{\lambda_{\bar{M}}}$.
  - iii. Let $\tilde{D}$ consist of all $s \in \mathbb{P}_\lambda$ satisfying that $s \leq r$ and that there exists $t \in \bar{D}$ such that $s \leq t$ and $\delta(\mathrm{Hull}(\bar{M}, t)) = \delta(\bar{M})$. We want to show that $\tilde{D}$ is dense below r.
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  - v. By the existence of the appropriate strategy for $\mathcal{G}_\lambda(r')$, there exists $s \in \mathbb{P}_\lambda$ such that $s \leq r'$ and such that there exists $t \in \bar{D}$ satisfying $s \leq t$ and $\delta(\mathrm{Hull}(\bar{M}', t)) = \delta(\bar{M}')$.

vi We have that

$$\delta(\bar{M}) \leq \delta(\mathrm{Hull}(\bar{M}, t)) \leq \delta(\mathrm{Hull}(\bar{M}', t)) \leq \delta(\bar{M}') = \delta(\bar{M}),$$

which shows that all of the above values are equal and consequently, $s \leq r'$ belongs to $\tilde{D}$.

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vii Since we have shown that $\tilde{D}$ is dense, we get that $g \cap \tilde{D} \neq \emptyset$. In other words, there exist $s \in g$ and $t \in \bar{D}$ such that $s \leq r, t$ and $\delta(\mathrm{Hull}(\bar{M}, t)) = \delta(\bar{M})$.

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$$p_n := (\tau(w_s) \cup w, \tau(\mathcal{M}_s) \cup \{P' {\downarrow} \tau(\lambda)\}) \in \tau(\mathbb{P}^*_\lambda).$$

Note that $\delta_{P'} = \omega_1^V$.

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QED (Case IV)

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Since we have shown Claim 10, we conclude the proof of the main theorem.

THANK YOU FOR YOUR ATTENTION!