AGT Summative Assignment – Individual Component Report

wcrr51

2021

Exercise 1. Consider the following instance of the load balancing game where the number of tasks is equal to the number of machines, and in particular we have:

- m identical machines $M_1, M_2, \ldots, M_m$ (all of speed 1),
- m identical tasks $w_1 = w_2 = \cdots = w_m = 1$.

Consider also the mixed strategy profile A where each of the tasks is assigned to all machines equiprobably (i.e. with probability $1/m$).

(a) Calculate the ratio $cost(A)/cost(OPT)$ in the special case where $m = 2$. **[3 marks]**

Trivially, makespans of 1 and 2 have 2 assignments each. Hence, for $m = 2$, there are a total of $2^2 = 4$ possible assignments. These assignments are shown in Table 1.

	M_1	M_2	Makespan
1	1, 2	-	2
2	-	1, 2	2
3	1	2	1
4	2	1	1

Table 1: Task-machine assignments for $m = 2$ in Exercise 1

From this, $cost(A)$ can be calculated with

$$cost(A) = E[cost(B)] = \sum_{i=1}^{m} P(cost(B) = i) \cdot i \tag{1}$$

Since there is capacity for one machine per task, and this would be the optimal assignment for any positive m, hence, it holds true that

$$\forall m > 0, cost(OPT) = 1 \tag{2}$$

For $m = 2$, using (1), $cost(A) = E[cost(B)] = \frac{1}{4}(1 \cdot 2 + 2 \cdot 2) = \frac{6}{4} = \frac{3}{2} = 1.5$.
Combining this with (2), the ratio $cost(A)/cost(OPT)$ for $m = 2$ is $\frac{3}{2}/1 = \frac{3}{2} = 1.5$

(b) Calculate the ratio $cost(A)/cost(OPT)$ in the special case where $m = 3$. **[3 marks]**

For a makespan of 1, there are ${}^3P_3 = 3! = 6$ assignments, for 2, there are $3 \cdot {}^3P_2 = 18$ assignments and for 3, there are trivially 3 assignments. Hence, for $m = 3$ there are a total of $3^3 = 27$ possible assignments. These assignments are shown in Table 2.

For $m = 3$, using (1), $cost(A) = E[cost(B)] = \frac{1}{27}(1 \cdot 3 + 2 \cdot 18 + 3 \cdot 3) = \frac{51}{27} = \frac{17}{9} \approx 1.89$.
Combining this with (2), the ratio $cost(A)/cost(OPT)$ for $m = 3$ is $\frac{17}{9}/1 = \frac{17}{9} \approx 1.89$

(c) Discuss what this ratio is for arbitrary m. What does this imply about the Price of Anarchy on identical machines for mixed Nash equilibria? **[5 marks]**

There are m^m possible assignments distributed over m makespan values (from 1 to m).

As (2) holds true for all $m > 0$, the denominator of the fraction is always one.

	M_1	M_2	M_3	Makespan
1	1, 2, 3	-	-	3
2	-	1, 2, 3	-	3
3	-	-	1, 2, 3	3
4	1	2, 3	-	2
5	1	-	2, 3	2
6	-	1	2, 3	2
7	2, 3	1	-	2
8	2, 3	-	1	2
9	-	2, 3	1	2
10	2	1, 3	-	2
11	2	-	1, 3	2
12	-	2	1, 3	2
13	1, 3	2	-	2
14	1, 3	-	2	2
15	-	1, 3	2	2
16	3	1, 2	-	2
17	3	-	1, 2	2
18	-	3	1, 2	2
19	1, 2	3	-	2
20	1, 2	-	3	2
21	-	1, 2	3	2
22	1	2	3	1
23	2	1	3	1
24	2	3	1	1
25	3	2	1	1
26	3	1	2	1
27	1	3	2	1

Table 2: Task-machine assignments for $m = 3$ in Exercise 1

Exercise 2. We consider a second-price sealed-bid auction where there are n bidders who bid as follows:

- Bidders 1 up to $n-1$ bid either 1 dollar or $r > 1$ dollars equiprobably and independently of the rest.
- Bidder n bids h dollars, where $h > r$.

The seller's expected revenue R is the expectation of the second highest value.

(a) What is the value that R is approaching when n is very large? **[1 marks]**

For large n, R approaches r.

(b) Justify your answer by taking the limit. **[9 marks]**

Trivially and by the definition, R must be less than h.

Let X be a binomially distributed random variable representing the number of times r is chosen (instead of 1) from a set of $n-1$ independent trials (representing independent bidders 1 to $n-1$), each of probability $\frac{1}{2}$.

$$X \sim B\left(n-1, \frac{1}{2}\right) \tag{3}$$

$P(X=0)$ is the probability that r is chosen 0 times across the $n-1$ independent trials.

$$\begin{aligned} P(X=0) &= {}^{n-1}C_0 \cdot \frac{1}{2}^0 \cdot \frac{1}{2}^{(n-1)-0} = \frac{1}{2}^{n-1} = 2^{-(n-1)} = 2^{1-n} = 2 \cdot 2^{-n} \\ \implies \lim_{n->\infty} P(X=0) &= \lim_{n->\infty} \left(2 \cdot 2^{-n}\right) = 2 \cdot \lim_{n->\infty} 2^{-n} = 2 \cdot 0 = 0 \end{aligned} \tag{4}$$

This means that as n increases, the probability of r not being chosen approaches zero. Hence, the larger the value of n, the more likely an r will be chosen, and thus the more likely $R = r$.

Exercise 3. Mary and Alice are buying items for Sunday lunch. Mary buys either chicken (C) or beef (B) for the main course and Alice buys either juice (J) or wine (W). Both people prefer wine with beef and juice with chicken. The opposite alternatives are equally displeasing. However, Mary prefers beef over chicken, while Alice prefers chicken over beef.

We assume that Mary buys first and then tells Alice what she bought, so when Alice makes her decision, she knows if the main course is beef or chicken.

(a) Express the above preferences as payoffs by using numbers (e.g. $u_M(B, W) = 2$, $u_A(B, W) = \ldots$ etc.) **[2 marks]**

As Mary prefers B over C, and Alice prefers C over B, $u_M(B, W) > u_A(B, W)$ and $u_M(C, J) < u_A(C, J)$. As both prefer B with W and C with J, $u_i(B, W) > 0, u_i(C, J) > 0 : i \in \{M, A\}$ As both are equally displeased by B with J and C with W, $u_i(B, J) = u_i(C, W), u_i(B, J) < u_i(B, W), u_i(C, W) < u_i(B, J) : i \in \{M, A\}$
The following assignment satisfies these constraints:

$$\begin{aligned} u_M(B, W) = 2, u_A(B, W) = 1, \\ u_M(B, J) = 0, u_A(B, J) = 0, \\ u_M(C, W) = 0, u_A(C, W) = 0, \\ u_M(C, J) = 1, u_A(C, J) = 2 \end{aligned} \tag{5}$$

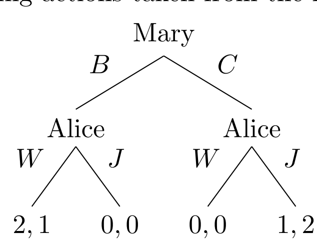
(b) Write down a bimatrix game with Mary as the row player and Alice as the column player, using your chosen payoffs. **[4 marks]**

		Alice			
		J		W	
Mary	C	1	2	0	0
	B	0	0	2	1

Table 3: Bimatrix game representation of the scenario in Exercise 3

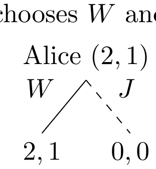
(c) Write down a game tree representing this game as an extended game. **[4 marks]**

As Mary buys first, she is at the root of the tree, the two outward edges represent the choice between B and C. Whether B or C is chosen, Alice is the next node, these each have two edges representing the choice between W and J. Finally, the leaves represent the payoffs for each player from the corresponding actions taken from the root to the leaf.

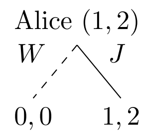


(d) Find a solution for the extended game using backward induction.
Describe your steps. **[5 marks]**

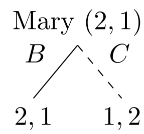
For this solution, an edge with a solid line represents the action that is chosen.
Firstly consider the left subgraph (taking action B), Alice may either choose W with reward 1 or J with reward 0. As $1 > 0$, Alice chooses W and her node takes the payoff value $2, 1$.



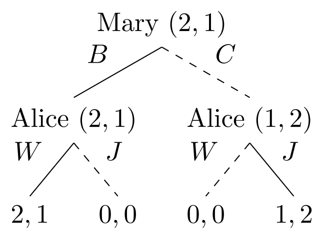
Next, return to the root and consider its right subgraph (taking action C), Alice may either choose W with reward 0 or J with reward 2. As $0 < 2$, Alice chooses W and her node takes the payoff value $1, 2$.



Finally, consider the root. Mary may either choose B with reward 2 or C with reward 1. As $2 > 1$, Alice chooses B.



The full annotated graph is as follows.



Hence, by backwards induction, Mary gets a higher payoff than Alice and they have Beef with Wine for Sunday lunch.

Exercise 4. We consider a (matching) market of k sellers and k buyers, where k is an integer, $k > 0$. Each seller sells an item and the prices of the items are initially all zero. Buyer i has valuation $k - i + 1$ for the first item and valuation 0 for every other item, as shown in the following diagram.

Buyers	Valuations (for items 1 to k)			
x_1	k,	0,	$\ldots$,	0
x_2	$k-1$,	0,	$\ldots$,	0
$\vdots$			$\vdots$	
x_k	1,	0,	$\ldots$,	0

The sellers find the market-clearing prices using the procedure discussed in the lectures.

(a) What are the prices of the sellers' items (1^{st} item, 2^{nd} item, $\ldots$, k^{th} item) when the market clears? Which buyer gets the 1^{st} item and at what price? **[3 marks]**

The 1^{st} item is sold at a price of $k - 1$. The 2^{nd} to k^{th} items are sold at a price of 0.

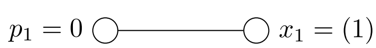
Buyer x_1 gets the 1^{st} item at a price of $k - 1$.

(b) Justify your answers to (a). **[6 marks]**

Let Algorithm 1 be the procedure for finding market-clearing prices as shown in the lectures and $x_{j,i}$ be the valuation of item i by buyer j.

Theorem 0.1. *The matching market in this case results in the market-clearing prices* $(k - 1, 0, \ldots, 0)$ *for items* $1, 2, ..., k$.

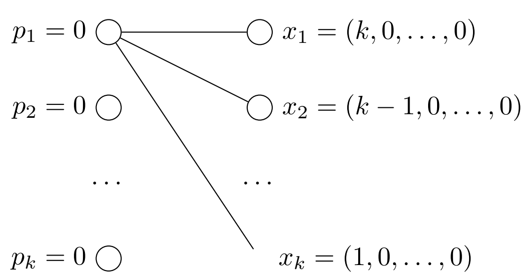
Proof. Using induction.
Base case ($k = 1$):
Applying Algorithm 1, the 1^{st} seller's price p_1 is initially 0. This trivially produces the following preferred-seller graph since buyer x_1 values the item at 1, giving them a positive payoff of $x_{1,1} - p_1 = 1 - 0 = 1$.



This is clearly a perfect bipartite matching, hence, for $k = 1$, the theorem holds.
Next, consider $k = n$ for a positive integer n.
In this case, trivially, any buyer

Lemma 0.2. *Any complete bi-partite graph* $K_{n,n}, n > 0$ *has a perfect matching.*

Proof. Trivial, connect left node i to right node i for all $1 \leq i \leq n$. □



Trivially, a single edge from item 1 to

Corollary 0.2.1. *Removing the vertices for seller* 1 *and buyer* x_1 *from the preferred-seller graph yields the complete bipartite graph* $K_{k-1,k-1}$

□

(c) Which kind of auction does the construction of market-clearing prices procedure implement in this case? **[3 marks]**

As the winner of item 1 is the buyer (x_1 in this case) who values it the highest. They pay the valuation of item 1 by the second highest buyer ($k-1$ in this case). Thus, the construction of market-clearing prices implements a second-price (Vickrey) auction for item 1 in this case.