OT_Copula

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X, Y random variables in $\mathbb{R}^n$ with marginals F, G and copulas U, V respectively. The optimal map between X and U is F, and the optimal map between Y and V is G. We will denote ϕ^* as the optimal map between X and Y, and as $\tilde{\phi}^*$ as the optimal map between U and V.

General

$$\begin{array}{ccc} X & \xrightarrow{\phi^*} & Y \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ U & \xrightarrow{\tilde{\phi}^*} & V \end{array}$$

$$\phi^*(x) = G^{-1}(\tilde{\phi}^*(F(x)))$$

$$\tilde{\phi}^*(u) = G(\phi^*(F^{-1}(u)))$$

$$\phi^* = \nabla\psi \ , \ \tilde{\phi}^* = \nabla\tilde{\psi} \ , \ \psi, \tilde{\psi} \text{ convex}$$

$$\Rightarrow \frac{\partial \phi^*_i}{\partial x_j} = \frac{\partial \phi^*_j}{\partial x_i} \ , \ \frac{\partial \tilde{\phi}^*_i}{\partial u_j} = \frac{\partial \tilde{\phi}^*_j}{\partial u_i}$$

$$(\Leftarrow) \ \frac{\partial^2 \psi}{\partial x_i \partial x_j} = \frac{\partial^2 \psi}{\partial x_j \partial x_i} \ , \ \frac{\partial^2 \tilde{\psi}}{\partial x_i \partial x_j} = \frac{\partial^2 \tilde{\psi}}{\partial x_j \partial x_i}$$

$$F(x) = \begin{bmatrix} F_1(x_1) \\ \vdots \\ F_n(x_n) \end{bmatrix} \qquad G^{-1}(v) = \begin{bmatrix} G_1^{-1}(v_1) \\ \vdots \\ G_n^{-1}(v_n) \end{bmatrix}$$

$$\tilde{\phi}^*(u) = \begin{bmatrix} \tilde{\phi}^*_1(u_1 \dots u_n) \\ \vdots \\ \tilde{\phi}^*_n(u_1 \dots u_n) \end{bmatrix} \qquad \phi^*(x) = \begin{bmatrix} \phi^*_1(x_1 \dots x_n) \\ \vdots \\ \phi^*_n(x_1 \dots x_n) \end{bmatrix}$$

$$\phi^*_i(x_1 \dots x_n) = G_i^{-1}\left(\tilde{\phi}^*_i(F_1(x_1), \dots, F_n(x_n))\right)$$

$$\frac{\partial \phi^*_i}{\partial x_j} = (G_i^{-1})'(\cdot/\cdot) \cdot \frac{\partial \tilde{\phi}^*_i}{\partial x_j}(F_1(x_1), \dots F_n(x_n)) \cdot F_j'(x_j)$$

$$\partial \phi^*_j / \partial x_i = (G_j^{-1})'(\cdot/\cdot) \cdot \frac{\partial \tilde{\phi}^*_j}{\partial x_i}(F_1(x_1) \dots F_n(x_n)) \cdot F_i'(x_i)$$

We can assume that Y is a copula, so $Y = V$ and G it is the identity.

w.g.l $G(y) = y \quad , \quad y = V$

$$\Rightarrow \frac{\partial \phi_i^*}{\partial x_j} = \frac{\partial \tilde{\phi}_i^*}{\partial x_j}(F_1(x_1), \dots F_n(x_n))\, F_j'(x_j)$$

$$\frac{\partial \phi_j^*}{\partial x_i} = \frac{\partial \tilde{\phi}_j^*}{\partial x_i}(F_1(x_1), \dots, F_n(x_n))\, F_i'(x_i)$$

Hav $\frac{\partial \phi_i^*}{\partial x_j} = \frac{\partial \phi_j^*}{\partial x_i}$ and $\frac{\partial \tilde{\phi}_i^*}{\partial x_j} = \frac{\partial \tilde{\phi}_j^*}{\partial x_i}$

$$\Rightarrow \boxed{F_j'(x_j) = F_i'(x_i)} \quad \forall i,j$$

So all the coordinates of F must be equals, so all the coordinates of X follow the same marginal.