Logic and Proof Practice: Validity and Proof Strategy

Gosha Polstvin

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1 Argument Validity and Counterexamples

1.1 Birds, Insects, and Flying Species (Valid)

Given:

$$\forall x(B(x) \to \exists y(I(y) \land E(x,y))), \quad \forall y(I(y) \to F(y)).$$

To Prove:

$$\forall x(B(x) \to \exists y(F(y) \land E(x,y))).$$

Proof Sketch. Let a be arbitrary such that $B(a)$. From the first premise, $\exists y(I(y) \land E(a,y))$. Instantiate to $I(c) \land E(a,c)$. By the second premise, $I(c) \to F(c)$, so $F(c)$ holds. Hence $F(c) \land E(a,c)$, therefore $\exists y(F(y) \land E(a,y))$. By universal generalization, $\forall x(B(x) \to \exists y(F(y) \land E(x,y)))$. $\square$

Conclusion: Argument is **valid.**

1.2 Smart People, Money, and Big Houses (Invalid)

Premises:

$$\exists x(S(x) \land M(x)), \quad \exists x(M(x) \land H(x)).$$

Conclusion:

$$\exists x(S(x) \land H(x)).$$

Proof Sketch. Consider the domain $\{a, b\}$. Let $S(a)$ and $M(a)$ be true, but $H(a)$ false. Let $M(b)$ and $H(b)$ be true, but $S(b)$ false. Then:

- $\exists x(S(x) \wedge M(x))$ true for $x = a$;
- $\exists x(M(x) \wedge H(x))$ true for $x = b$;
- but $\exists x(S(x) \wedge H(x))$ is false.

Thus, premises true and conclusion false. $\square$
Conclusion: Argument is **invalid.**

1.3 Knowing, Singing, and Liking Songs (Valid)

Given:

1. $\forall x(P(x) \to \exists y(S(y) \wedge K(x,y)))$
2. $\forall y(S(y) \to \exists z(P(z) \wedge \mathrm{Sing}(z,y)))$
3. $\forall z \forall y((P(z) \wedge S(y) \wedge \mathrm{Sing}(z,y)) \to L(z,y))$

To Prove:

$$\forall x(P(x) \to \exists y(S(y) \wedge K(x,y) \wedge \exists z(P(z) \wedge L(z,y)))).$$

Proof Sketch. Let a be an arbitrary person, $P(a)$. From (1), there exists a song s such that $S(s) \wedge K(a,s)$. From (2), since $S(s)$, there exists a person t such that $P(t) \wedge \mathrm{Sing}(t,s)$. From (3), using $P(t) \wedge S(s) \wedge \mathrm{Sing}(t,s)$, infer $L(t,s)$. Hence $S(s) \wedge K(a,s) \wedge \exists z(P(z) \wedge L(z,s))$. Existentially generalize and universalize to reach the desired conclusion. $\square$
Conclusion: Argument is **valid.**

2 Choosing Proof Strategies

2.1 If $x + y + z$ is odd, then at least one of x, y, z is odd

Proof Sketch. [Strategy: Contrapositive] Contrapositive: If x, y, z are all even, then $x + y + z$ is even.

If each is even, $x = 2a, y = 2b, z = 2c$ for integers a, b, c. Then $x + y + z = 2(a + b + c)$, which is even.

Hence, if $x + y + z$ is odd, not all of x, y, z can be even. Therefore, at least one is odd. $\square$

2.2 n even $\iff$ $7n + 4$ even

Proof Sketch. [Strategy: Biconditional direct proof] ($\Rightarrow$) If n is even, $n = 2k$. Then $7n + 4 = 14k + 4 = 2(7k + 2)$, even.
($\Leftarrow$) If $7n + 4$ is even, then $7n$ is even. Since 7 is odd, dividing by 7 preserves evenness, so n is even. $\square$

2.3 If $A \cap B \neq \varnothing$ and $A \subseteq C$, then $B \cap C \neq \varnothing$

Proof Sketch. [Strategy: Element-chase] Since $A \cap B \neq \varnothing$, there exists $x \in A \cap B$. Thus $x \in A$ and $x \in B$. Given $A \subseteq C$, we have $x \in C$. Hence $x \in B \cap C$, so $B \cap C \neq \varnothing$. □