CSCA67: Assignment #1

Due on September 28, 2025 at 11:59pm

Professor Anya Tafliovich Ch. 1

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Algebraic proofs for Question 1

1. $(a \to b) \land (b \to c)$ vs. $a \to c$

$$\begin{aligned}(a \to b) \land (b \to c) &\equiv (\neg a \lor b) \land (\neg b \lor c)\\ &\Rightarrow \neg a \lor c \equiv a \to c.\end{aligned}$$

Failure of converse. Take $(a, b, c) = (\mathrm{F}, \mathrm{T}, \mathrm{F})$. Then $a \to c = \mathrm{T}$, but $(a \to b) \land (b \to c) = \mathrm{F}$, so $a \to c$ does not imply $(a \to b) \land (b \to c)$.

2. $a \land (a \to b)$ vs. $a \to b$

$$a \land (a \to b) \Rightarrow a \to b,$$

since $a \to b$ is a conjunct of the left-hand formula.
Failure of converse. Take $(a, b) = (\mathrm{F}, \mathrm{F})$. Then $a \to b = \mathrm{T}$ but $a \land (a \to b) = \mathrm{F}$, so $a \to b$ does not imply $a \land (a \to b)$.

3. $(a \to b) \land (a \to c)$ vs. $a \to (b \land c)$

$$\begin{aligned}(a \to b) \land (a \to c) &\equiv (\neg a \lor b) \land (\neg a \lor c)\\ &\equiv \neg a \lor (b \land c) \equiv a \to (b \land c),\end{aligned}$$

so the formulas are equivalent.

4. $(a \to c) \land (b \to c)$ vs. $(a \lor b) \to c$

$$\begin{aligned}(a \to c) \land (b \to c) &\equiv (\neg a \lor c) \land (\neg b \lor c)\\ &\equiv (\neg a \land \neg b) \lor c \equiv \neg(a \lor b) \lor c \equiv (a \lor b) \to c,\end{aligned}$$

so they are equivalent.

5. $a \leftrightarrow b$ vs. $(a \land b) \lor (\neg a \land \neg b)$

By definition of biconditional:

$$a \leftrightarrow b \equiv (a \land b) \lor (\neg a \land \neg b),$$

so the formulas are identical.

6. $a \to (b \to (c \to d))$ vs. $((a \land b) \land c) \to d$

$$\begin{aligned}a \to (b \to (c \to d)) &\equiv \neg a \lor (\neg b \lor (\neg c \lor d))\\ &\equiv \neg(a \land b \land c) \lor d \equiv ((a \land b) \land c) \to d,\end{aligned}$$

so they are equivalent.

7. $(a \to b) \vee (b \to a)$ vs. $a \leftrightarrow b$

Left-hand side is a tautology:

$$\begin{aligned}(a \to b) \vee (b \to a) &\equiv (\neg a \vee b) \vee (\neg b \vee a)\\ &\equiv (\neg a \vee a) \vee (\neg b \vee b) \equiv \top.\end{aligned}$$

Right-hand side is not a tautology. Example: $(a, b) = (\mathrm{T}, \mathrm{F})$ gives $a \leftrightarrow b = \mathrm{F}$. Hence they are not equivalent.

8. $a \leftrightarrow b$ vs. $\neg a \leftrightarrow \neg b$

$$\begin{aligned}\neg a \leftrightarrow \neg b &\equiv (\neg a \wedge \neg b) \vee (a \wedge b)\\ &\equiv (a \wedge b) \vee (\neg a \wedge \neg b) \equiv a \leftrightarrow b,\end{aligned}$$

so they are equivalent.

9. $(a \wedge b) \to (c \wedge d)$ vs. $((a \to c) \wedge (a \to d)) \wedge ((b \to c) \wedge (b \to d))$

RHS implies LHS:

$$\begin{aligned}\mathrm{RHS} &\equiv (\neg a \vee c) \wedge (\neg a \vee d) \wedge (\neg b \vee c) \wedge (\neg b \vee d)\\ &\equiv ((\neg a \vee c) \wedge (\neg b \vee c)) \wedge ((\neg a \vee d) \wedge (\neg b \vee d))\\ &\equiv ((\neg a \wedge \neg b) \vee c) \wedge ((\neg a \wedge \neg b) \vee d)\\ &\equiv (\neg a \wedge \neg b) \vee (c \wedge d) \equiv (a \wedge b) \to (c \wedge d).\end{aligned}$$

Failure of converse. Example: $(a, b, c, d) = (\mathrm{T}, \mathrm{F}, \mathrm{F}, \mathrm{F})$ gives LHS true, RHS false.

10. $(a \vee b) \to (c \wedge d)$ vs. $((a \to c) \wedge (a \to d)) \wedge ((b \to c) \wedge (b \to d))$

$$\begin{aligned}(a \vee b) \to (c \wedge d) &\equiv (\neg a \wedge \neg b) \vee (c \wedge d),\\ \mathrm{RHS} &\equiv (\neg a \vee c) \wedge (\neg a \vee d) \wedge (\neg b \vee c) \wedge (\neg b \vee d)\\ &\equiv ((\neg a \vee c) \wedge (\neg b \vee c)) \wedge ((\neg a \vee d) \wedge (\neg b \vee d))\\ &\equiv ((\neg a \wedge \neg b) \vee c) \wedge ((\neg a \wedge \neg b) \vee d)\\ &\equiv (\neg a \wedge \neg b) \vee (c \wedge d) \equiv (a \vee b) \to (c \wedge d),\end{aligned}$$

so they are equivalent.

Question 2. Tautology / contradiction / neither

We classify each formula and give succinct rigorous arguments.

1. $(a \to b) \vee (b \to a)$.

Classification: *Tautology.*

Proof. As in Question 1.7,

$$(\neg a \vee b) \vee (\neg b \vee a) \equiv (\neg a \vee a) \vee (\neg b \vee b) \equiv \top,$$

so the formula is true under every valuation. □

2. $(((a \to b) \land (b \to c)) \land a) \land \neg c$**.**

Classification: *Contradiction* (unsatisfiable).

Proof. Assume the whole conjunction is true. Then from $(a \to b) \land (b \to c)$ and a we can derive b and then c by successive modus ponens. But the formula also contains $\neg c$ as a conjunct, hence we deduce both c and $\neg c$, contradiction. Therefore there is no valuation that makes the formula true; it is a contradiction. □

3. $(((a \to b) \land (b \to c)) \land a) \land c$**.**

Classification: *Neither* (satisfiable but not a tautology).

Proof. Satisfiable: take $a = b = c = \mathrm{T}$. Then each of $a \to b$, $b \to c$ is T, and $a \land c$ is T, so the whole conjunction is T. Not a tautology: take $a = \mathrm{F}, b = \mathrm{F}, c = \mathrm{F}$. Then the conjunct a is F, so the whole conjunction is F. Hence neither a tautology nor a contradiction. □

4. $a \to \neg a$**.**

Classification: *Neither.*

Proof. If $a = \mathrm{F}$ the implication $a \to \neg a$ is T (antecedent false). If $a = \mathrm{T}$ the implication is $\mathrm{T} \to \mathrm{F} = \mathrm{F}$. Thus the formula is satisfiable but not always true; hence neither. □

5. $(a \land (a \to b)) \to b$**.**

Classification: *Tautology.*

Proof. Suppose $a \land (a \to b)$ holds. Then a holds and $a \to b$ holds, so by modus ponens b holds. Therefore the implication from $a \land (a \to b)$ to b is always true, i.e. a tautology. Algebraically one can expand the antecedent as $a \land (\neg a \lor b)$ and simplify to see the implication is always true, but the straightforward rule-based derivation via modus ponens is immediate and rigorous. □

Question 3. Deductive reasoning (validity / proofs)

For each argument we indicate validity. If valid, we present (i) a truth-table style check covering all rows where the premises are all true, and (ii) a rules-of-inference (natural deduction) proof. If invalid, we give an explicit counterexample valuation.

(A) Truth Table

1. If the weather is good, I either go running or swimming. $W \to (R \lor S)$.
2. I don't go running and swimming at the same time. $\neg(R \land S)$.

Validity: *Valid.*

1. $W \to (R \lor S)$ Premise
2. $\neg(R \land S)$ Premise

3. From $\neg(R \land S)$ we get $R \to \neg S$ and $S \to \neg R$. (Equivalently: $\neg(R \land S) \equiv \neg R \lor \neg S$, from which each implication follows; or prove each implication by conditional proof.)

4. To prove $(W \land R) \to \neg S$: assume $W \land R$.

   (a) From $W \land R$ infer W and R.

   (b) From W and $W \to (R \lor S)$ infer $R \lor S$.

   (c) From R and $R \to \neg S$ infer $\neg S$.

   Discharging the assumption $W \land R$ yields $(W \land R) \to \neg S$.

5. The proof of $(W \land S) \to \neg R$ is symmetric.

R	S	W	$W \to (R \lor S)$	$\neg(R \land S)$	All Premises?	$(W \land R) \to \neg S$	$(W \land S) \to \neg R$
F	F	F	T	T	T	T	T
F	F	T	F	T	F	T	T
F	T	F	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	T	T	T	F	F	T	T

(B) Truth Table

Informal argument.

1. To get a good grade G it is necessary to attend lectures and do readings: $G \to (L \land R)$.

2. I attend lectures and do readings: $L \land R$.

Conclusion: G. **Validity:** *Invalid.*

Counterexample. Take the valuation $G = \text{F}$, $L = \text{T}$, $R = \text{T}$. Then $G \to (L \land R)$ is true (antecedent false) and $L \land R$ is true, but G is false. Therefore the premises can all be true while the conclusion is false; hence the argument is invalid. Intuitively the premise states "$L \land R$ is *necessary* for G", which does not imply $L \land R$ is sufficient for G. □

G	L	R	$G \to (L \land R)$	$L \land R$	All Premises?	G
F	F	F	T	F	F	F
F	F	T	T	F	F	F
F	T	F	T	F	F	F
F	T	T	T	T	T	F
T	F	F	F	F	F	T
T	F	T	F	F	F	T
T	T	F	F	F	F	T
T	T	T	T	T	T	T

(C) Truth Table

Informal argument.

1. $((L \land R \land E) \lor K) \to G$.
2. L, R, E.

Conclusion: G.
Validity: *Valid.*

1. $((L \land R \land E) \lor K) \to G$ Premise
2. L Premise
3. R Premise
4. E Premise
5. From 2–4 infer $L \land R \land E$. $\land$-Introduction
6. From 5 infer $(L \land R \land E) \lor K$. $\lor$-Introduction
7. From 1 and 6 infer G. Modus Ponens

Thus G follows; the argument is valid.

L	R	E	K	G	$(L \wedge R \wedge E) \vee K$	Premise1: antecedent$\to G$	All Premises?	G
F	F	F	F	F	F	T	F	F
F	F	F	F	T	F	T	F	T
F	F	F	T	F	T	F	F	F
F	F	F	T	T	T	T	F	T
F	F	T	F	F	F	T	F	F
F	F	T	F	T	F	T	F	T
F	F	T	T	F	T	F	F	F
F	F	T	T	T	T	T	F	T
F	T	F	F	F	F	T	F	F
F	T	F	F	T	F	T	F	T
F	T	F	T	F	T	F	F	F
F	T	F	T	T	T	T	F	T
F	T	T	F	F	F	T	F	F
F	T	T	F	T	F	T	F	T
F	T	T	T	F	T	F	F	F
F	T	T	T	T	T	T	F	T
T	F	F	F	F	F	T	F	F
T	F	F	F	T	F	T	F	T
T	F	F	T	F	T	F	F	F
T	F	F	T	T	T	T	F	T
T	F	T	F	F	F	T	F	F
T	F	T	F	T	F	T	F	T
T	F	T	T	F	T	F	F	F
T	F	T	T	T	T	T	F	T
T	T	F	F	F	F	T	F	F
T	T	F	F	T	F	T	F	T
T	T	F	T	F	T	F	F	F
T	T	F	T	T	T	T	F	T
T	T	T	F	F	T	F	F	F
T	T	T	F	T	T	T	T	T
T	T	T	T	F	T	F	F	F
T	T	T	T	T	T	T	T	T

(D) Truth Table

Informal argument.

1. $(L \to \neg B) \vee (K \to \neg C)$.
2. $B \vee C$.

Validity: *Invalid.*

Counterexample. Set

$$(B, C, L, K) = (\mathrm{T}, \mathrm{F}, \mathrm{T}, \mathrm{T}).$$

Evaluate:

$$L \to \neg B = \mathrm{T} \to \mathrm{F} = \mathrm{F}, \qquad K \to \neg C = \mathrm{T} \to \mathrm{T} = \mathrm{T},$$

so the disjunction $(L \to \neg B) \vee (K \to \neg C)$ is T. Also $B \vee C = \mathrm{T}$. But the conclusion

$$\neg L \vee \neg K = \neg \mathrm{T} \vee \neg \mathrm{T} = \mathrm{F}$$

is false. Therefore premises true and conclusion false, so the argument is invalid. □

L	K	B	C	$L \to \neg B$	$K \to \neg C$	$(L \to \neg B) \vee (K \to \neg C)$	$B \vee C$	$\neg L \vee \neg K$
F	F	F	F	T	T	T	F	T
F	F	F	T	T	T	T	T	T
F	F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	F	T
F	T	F	T	T	F	T	T	T
F	T	T	F	T	T	T	T	T
F	T	T	T	T	F	T	T	T
T	F	F	F	F	T	T	F	T
T	F	F	T	F	T	T	T	T
T	F	T	F	F	T	F	T	T
T	F	T	T	F	T	T	T	T
T	T	F	F	F	T	T	F	F
T	T	F	T	F	F	F	T	F
T	T	T	F	F	T	F	T	F
T	T	T	T	F	F	F	T	F

(E) Truth Table

Informal argument.

1. $L \to \neg B$.
2. $K \to \neg C$.
3. $B \vee C$.

Conclusion: $\neg L \vee \neg K$. **Validity:** *Valid.*

1. $L \to \neg B$ Premise
2. $K \to \neg C$ Premise
3. $B \vee C$ Premise
4. Assume, for reductio, $L \wedge K$.
   (a) From assumption infer L and K.
   (b) From 1 and L infer $\neg B$.
   (c) From 2 and K infer $\neg C$.
   (d) From 3 and $\neg B, \neg C$ conclude contradiction, since $B \vee C$ together with $\neg B \wedge \neg C$ is impossible.
5. Hence $L \wedge K$ leads to contradiction; therefore $\neg(L \wedge K)$ holds.
6. By De Morgan, $\neg(L \wedge K) \equiv \neg L \vee \neg K$, which is the desired conclusion.

This completes the proof; the argument is valid.

L	K	B	C	$L \to \neg B$	$K \to \neg C$	$B \lor C$	$\neg L \lor \neg K$
F	F	F	F	T	T	F	T
F	F	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	F	T	T	F	T
F	T	F	T	T	F	T	T
F	T	T	F	T	T	T	T
F	T	T	T	T	F	T	T
T	F	F	F	F	T	F	T
T	F	F	T	F	T	T	T
T	F	T	F	F	T	T	T
T	F	T	T	F	T	T	T
T	T	F	F	F	T	F	F
T	T	F	T	F	F	T	F
T	T	T	F	F	T	T	F
T	T	T	T	F	F	T	F