CSCA67H3 - Discrete Mathematics
Week 1 Tutorial Questions

Gosha Polstvin

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Solutions

Conditionals

If Alice is at the party, then so is Bob

$$A \to B$$

Converse: $B \to A$. If Bob is at the party, then Alice is at the party.
Contrapositive: $\neg B \to \neg A$. If Bob is not at the party, then Alice is not at the party.

Charlie is at the party, only if both Alice and Bob are

$$C \to (A \wedge B)$$

Converse: $(A \wedge B) \to C$. If both Alice and Bob are at the party, then Charlie is at the party.
Contrapositive: $\neg(A \wedge B) \to \neg C$. If at least one of Alice or Bob is not at the party, then Charlie is not at the party.

David is not at the party, if Alice is

$$A \to \neg D$$

Converse: $\neg D \to A$. If David is not at the party, then Alice is at the party.
Contrapositive: $D \to \neg A$. If David is at the party, then Alice is not at the party.

If Bob is not at the party, then Alice is

$$\neg B \to A$$

Converse: $A \to \neg B$. If Alice is at the party, then Bob is not at the party.
Contrapositive: $\neg A \to B$. If Alice is not at the party, then Bob is at the party.

If Bob is not at the party, then neither is Alice

$$\neg B \to \neg A$$

Converse: $\neg A \to \neg B$. If Alice is not at the party, then Bob is not at the party.
Contrapositive: $A \to B$. If Alice is at the party, then Bob is at the party.

Alice is not at the party, unless Bob is

$$\neg B \to \neg A$$

Converse: $\neg A \to \neg B$. If Alice is not at the party, then Bob is not at the party.
Contrapositive: $A \to B$. If Alice is at the party, then Bob is at the party.

Neither Alice nor Bob being at the party is a sufficient condition for Charlie

$$(\neg A \land \neg B) \to C$$

Converse: $C \to (\neg A \land \neg B)$. If Charlie is at the party, then neither Alice nor Bob is at the party.
Contrapositive: $\neg C \to (A \lor B)$. If Charlie is not at the party, then at least one of Alice or Bob is at the party.

Both Alice and Bob being at the party is a necessary condition for Charlie

$$C \to (A \land B)$$

Converse: $(A \land B) \to C$. If both Alice and Bob are at the party, then Charlie is at the party.
Contrapositive: $\neg(A \land B) \to \neg C$. If it is not the case that both Alice and Bob are at the party, then Charlie is not at the party.

Logical Equivalences

$\neg(a \to b)$ and $\neg a \land b$

We know:

$$a \to b \equiv (\neg a \lor b).$$

Therefore:

$$\neg(a \to b) \equiv \neg(\neg a \lor b) \equiv a \land \neg b.$$

Hence $\neg(a \to b)$ is **not equivalent** to $\neg a \land b$ (counterexample: $a = F, b = T$).

$\neg(a \to b)$ and $a \land \neg b$

$$\begin{aligned} \neg(a \to b) &\equiv \neg(\neg a \lor b) \quad \text{(Conditional Law)} \\ &\equiv \neg\neg a \land \neg b \quad \text{(De Morgan's)} \\ &\equiv a \land \neg b. \end{aligned}$$

So they are **equivalent**.

$a \leftrightarrow \neg b$ and $(a \land \neg b) \lor (\neg a \land b)$

By definition:

$$a \leftrightarrow \neg b \equiv (a \land \neg b) \lor (\neg a \land b).$$

Therefore they are **equivalent**.

Logical Inference

Problem 1: Hockey, soreness, whirlpool

Proof (Proof). Let $H =$ "play hockey", $S =$ "sore", and $W =$ "use whirlpool".

1. $H \to S$ (premise).
2. $S \to W$ (premise).
3. $\neg W$ (premise).
4. $\neg S$ (2,3 Modus Tollens).
5. $\neg H$ (1,4 Modus Tollens).

Therefore, I did not play hockey. □

Problem 2: Dreaming or hallucinating

Proof (Proof). Let $D =$ "dreaming", $H =$ "hallucinating", $E =$ "see elephants".

1. $D \lor H$ (premise).
2. $\neg D$ (premise).
3. H (1,2 Disjunctive Syllogism).
4. $H \to E$ (premise).
5. E (3,4 Modus Ponens).

Therefore, I see elephants running down the road. □

Problem 3: Running, swimming, sunburn

Proof (Proof). Let R = "running", Sw = "swimming", S = "stay in the sun too long", B = "sunburn".

1. $R \to S$ (premise).
2. $Sw \to S$ (premise).
3. $S \to B$ (premise).
4. $\neg B$ (premise).
5. $\neg S$ (3,4 Modus Tollens).
6. $\neg R$ (1,5 Modus Tollens).
7. $\neg Sw$ (2,5 Modus Tollens).
8. $\neg R \wedge \neg Sw$ (6,7 Conjunction).

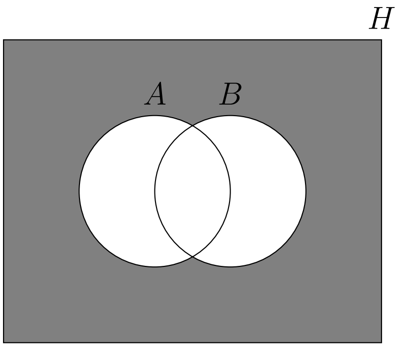
Therefore, I neither went running nor swimming. $\square$

Variables and Sets

1. $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Proof (element wise, using $\forall x$):

$$\begin{aligned}
\forall x\colon\ x \in \overline{A \cup B} &\Leftrightarrow x \notin (A \cup B) \\
&\Leftrightarrow \neg\big(x \in A \ \vee\ x \in B\big) \\
&\Leftrightarrow \big(\neg(x \in A)\big) \wedge \big(\neg(x \in B)\big) \\
&\Leftrightarrow (x \in \overline{A}) \wedge (x \in \overline{B}) \\
&\Leftrightarrow x \in (\overline{A} \cap \overline{B}).
\end{aligned}$$



So for every x membership matches, hence the sets are equal.

2. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof (element wise):

$$\begin{aligned}
\forall x\colon\ x \in \overline{A \cap B} &\Leftrightarrow x \notin (A \cap B) \\
&\Leftrightarrow \neg\Big(x \in A \ \wedge\ x \in B\Big) \\
&\Leftrightarrow \Big(\neg(x \in A)\Big) \vee \Big(\neg(x \in B)\Big) \\
&\Leftrightarrow (x \in \overline{A}) \vee (x \in \overline{B}) \\
&\Leftrightarrow x \in (\overline{A} \cup \overline{B}).
\end{aligned}$$

So they match for every x, hence equal.