Grishma Parajuli
HW 3
MATH 449

1 [i] Consider 1-period model and assume stock pays a proportional dividend of q at $t = 1$.

S_0 → with probability p: $uS_0 + q(uS_0) = C_u$
$= uS_0(1+q) = uQS_0$

S_0 → with probability $1-p$: $dS_0 + q(dS_0) = C_d$
$= dQS_0$

$\boxed{t=0}$ $\boxed{t=1}$ $C_1(S_1) = \text{payoffs}$.

Assume the initial price of a stock is \$ S_0 per unit. At the end of 1-period: S_0 either increases to uS_0 or decreases to dS_0 and dividend $100q\%$ is paid on the price at $t = 1$.

- No-arbitrage conditions are now $dQ < R < uQ$. This condition is explained in ~~HW~~ Week 2-3. It is similar concept, except we are considering dividend in this case.

Now, we use replicating portfolio argument to find price C_0 of the option at time $t=0$.

(*) Assuming any fractional unit of stocks can be traded, we can create a portfolio consisting of only stock and bond to replicate the return of the option.

Then, by the law of one price,

C_0 = value of replicating portfolio.

No-arbitrage opportunity exists as we are adjusting the portfolio as per the law of one price.

~~Let,~~ Then,

At time $t = 0$ we should purchase the replicating portfolio of x and y where,

$\$x$ = amount of stock &

$\$y$ = amount of bond.

such that

$$uQx + Ry = C_u \quad \text{—— (I)}$$

$$dQx + Ry = C_d \quad \text{—— (II)}$$

$(r+1 = R)$ & $(1+q = Q)$

Then, $\boxed{C_0 = x + y}$

Solving (I) and (II)

$$\begin{bmatrix} uQ & R \\ dQ & R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_u \\ C_d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} uQ & R \\ dQ & R \end{bmatrix}^{-1} \begin{bmatrix} C_u \\ C_d \end{bmatrix}$$

$$\therefore \begin{bmatrix} uQ & R \\ dQ & R \end{bmatrix}^{-1} = \frac{1}{(uQR - dQR)} \begin{bmatrix} R & -R \\ -dQ & uQ \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{QR(u-d)} \begin{bmatrix} R & -R \\ -dQ & uQ \end{bmatrix} \begin{bmatrix} C_u \\ C_d \end{bmatrix}$$

$$= \frac{1}{QR(u-d)} \begin{bmatrix} RC_u - RC_d \\ -dQC_u + uQC_d \end{bmatrix}$$

Then,

$$C_0 = \frac{R(C_u - \cancel{R}C_d)}{QR(u-d)} + \frac{uQC_d - dQC_u}{QR(u-d)}$$

$$= \frac{1}{R}\left[\frac{RQ^{-1}(C_u - C_d) - uC_d - dC_u}{(u-d)}\right]$$

$$= \frac{1}{R}\left[\frac{RQ^{-1} - d}{(u-d)} C_u + \frac{u - RQ^{-1}}{(u-d)} C_d\right]$$

If $\tilde{p} = \dfrac{RQ^{-1} - d}{(u-d)}$ Then,

$$(1 - \tilde{p}) = 1 - \frac{RQ^{-1} - d}{u-d} = \frac{u - d - RQ^{-1} + d}{u-d}$$

$$= \frac{u - RQ^{-1}}{u-d}$$

So,

$$C_0 = \frac{1}{R}\left[\tilde{p}\, C_u + (1-\tilde{p})\, C_d\right]$$

where $\tilde{p} = \dfrac{RQ^{-1} - d}{(u-d)}$

No-arbitrage condition explained: $dQ < R < uQ$

→ There is no arbitrage if and only if $d < R < u$.

(i) Suppose $R < Qd < Qu$.

Then we borrow S_0 and invest in stock at $t=0$. And at $t=1$ we payback $S_0 R$ but we have 1 unit of stock we bought at $t=0$ which is worth either $S_0 uQ$ or $S_0 dQ$. Net cash flow here is,

$(u-R)S_0$ or $(d-R)S_0$ but d and $u > R$ so, make profit in any case.

(ii) Suppose $dQ < uQ < R$:

Then we short sell one share of stock and invest proceeds in cash-account. At $t=0$, we (S_0) short sell and invest in cash account and earn $S_0 R$.

At $t=1$, we buy back the stock at either QdS_0 or QuS_0 but $R > uQ > dQ$ so we always make profit.

Important to note that who ever owns the stock gets the dividend. So, when returning we have to ~~give~~ return with dividend.

1 (ii) Now,

For multi-period binomial model we assume a proportional dividend in each period

- so dividend of $q S_i$ is paid at $t = i+1$ for each i.

Then again,

each embedded 1-period model has identical risk neutral probabilities for derivative securities priced as (i)

$$\hat{p} = \frac{R Q^{-1} - d}{u - d}, \quad R = 1+r, \; Q = 1+q$$

The no-arbitrage price of the derivative is determined by:

$$C_u = \frac{1}{R}\left[\hat{p}\, C_{uu} + (1-\hat{p})\, C_{ud}\right]$$

$$C_d = \frac{1}{R}\left[\hat{p}\, C_{du} + (1-\hat{p})\, C_{ud}\right]$$

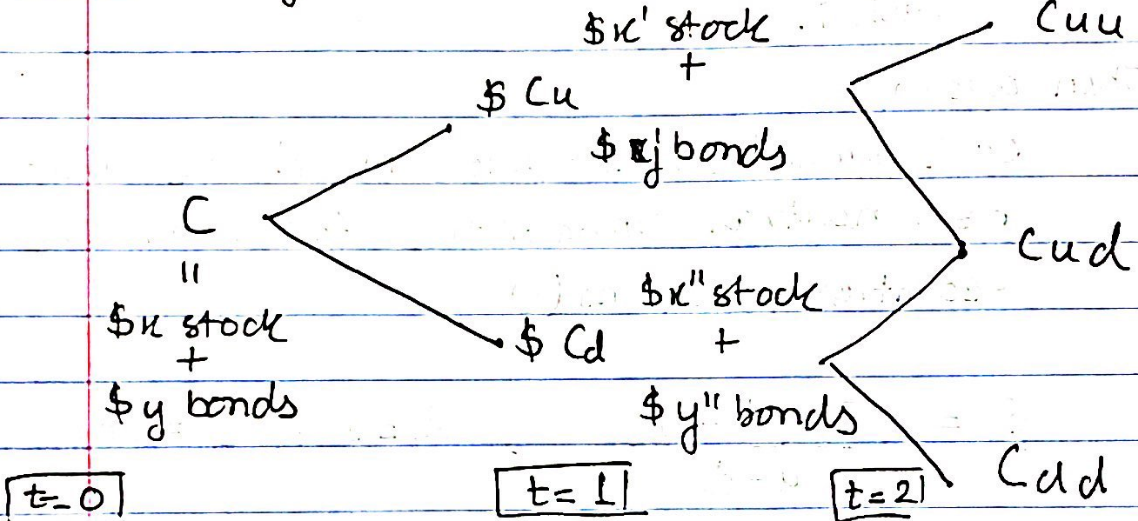
$$C_0 = \frac{1}{R}\left[\hat{p}\, C_u + (1-\hat{p})\, C_d\right]$$

$\to C_0 = x + y$ (Replicating portfolio)

Note: the derivative price is determined backward in time.

This guarantees no-arbitrage because we are using one-price model ~~to~~ discussed in HW1 to replicate the payoff of the derivatives.

This dynamically readjusted portfolio is adaptive based on the price action of the underlying. The portfolio consists of $\$x$ stock and $\$y$ bonds at time $t=0$.



The no-arbitrary price of the derivative is the price of the portfolio at time 0.

Note: It is important to understand that the amount of stock and bond of the replicating portfolio changes in all different scenarios and time setting.

Deriving as in [i],

consider moving from $t=1$ to $t=2$. By law of one-price.

- $C_u = x' + y'$
- $C_d = x'' + y''$

$\left\{ \begin{array}{l} \text{where } \forall x_i' \neq x \text{ \& } \\ \forall y' \neq y \end{array} \right\}$

Also, $\forall x'' \neq \forall x' \neq x$
$\forall x'' \neq \forall x' \neq x$.

$\left.\begin{array}{l} x' + y' \\ x'' + y'' \end{array}\right\}$ replicating portfolio

$C_u, C_d \rightarrow$ price of option at time $t=1$

Then At time $(t=1)$ we should purchase the replicating portfolio of x' and y' such that,

when $C_u \to$ $uQx' + Ry' = C_{uu}$ —— (i)

$dQx' + Ry' = C_{ud}$ —— (ii) or, x'' and y'',

when $C_d \to$ $uQx'' + Ry'' = C_{du}$ —— (iii)

$dQx'' + Ry'' = C_{dd}$ —— (iv)

We also know from 1-period binomial model that,

$$C_u = uQx + Ry = x' + y'$$

$$C_d = dQx + Ry = x'' + y''$$

and $C_0 = x + y$ (time, $t=0$)

If we solve eqn [(i) and (ii)] and [(iii) and (iv)] like in 1[i]. We get.

$$\begin{bmatrix} uQ & R \\ dQ & R \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} C_{uu} \\ C_{ud} \end{bmatrix}$$

Then,

$$C_u = x' + y' = \frac{R(C_{uu} - C_{ud})}{QR(u-d)} + \frac{uQC_{ud} - dQC_{uu}}{QR(u-d)}$$

$$\boxed{\tilde{p} = \frac{RQ^{-1} - d}{u-d}} \qquad = \frac{1}{R}\left[\frac{RQ^{-1} - d}{u-d} C_{uu} + \frac{u - RQ^{-1}}{(u-d)} C_{ud}\right]$$

similarly,

$$C_d = x'' + y'' = \frac{R(C_{du} - C_{dd})}{QR(u-d)} + \frac{uQC_{dd} - dQC_{du}}{QR(u-d)}$$

$$= \frac{1}{R}\left[\frac{RQ^{-1} - d}{(u-d)} C_{du} + \frac{u - RQ^{-1}}{(u-d)} C_{dd}\right]$$

$$= \frac{1}{R}\left[\tilde{p}\, C_{du} + (1-\tilde{p})\, C_{dd}\right]$$

And we saw from (1) that

$$C_0 = x + y = \frac{1}{R}\left[\hat{p}\, C_u + (1-\hat{p})\, C_d\right]$$

Here, we are considering a 2-period model just for explanation. But, this works for any n-period model.

From time period n-1 to n, we are replicating portfolio and the formula or algorithm we end up with always is similar. where,

$$C_m = \frac{1}{R}\left[\hat{p}\, C_a + (1-\hat{p})\, C_{\bar{a}}\right]$$

$\boxed{\text{From Week 4}}$ Considering the underlying security that doesn't pay dividends, the algorithm is very similar to the one we just derived above except for the fact that $\hat{p}$ in the non-dividend paying case is $\dfrac{R-d}{u-d}$.

And here,

$$\hat{p} = \frac{RQ^{-1} - d}{u-d}.$$

[iii] We know that,

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = u^{-1}, \quad p = \frac{1}{2}\left(1 + \frac{v}{\sigma}\sqrt{\Delta t}\right) \quad (1)$$

And, since

we are considering divident on a continuous time setting.

$$S(t+\Delta t)(1+q\Delta t) \simeq S(t+\Delta t)e^{q\Delta t}$$

The p in the binomial model and annualized growth rate v in GBM model are irrelevent for option pricing.

From (1) we can write annualized growth rate as,

$$(a) \quad v = \frac{\sigma(2p-1)}{\sqrt{\Delta t}}$$

Also,

$$(b) \quad \hat{p} = \frac{RQ^{-1} - d}{u - d} \quad \text{(proved from [i])}$$

$$= \frac{e^{r\Delta t}e^{-q\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \quad \text{(since, we are talking about continuous time setting)}$$

Here p is the discreete probability and $\hat{p}$ is risk-neutral probability which is used for the continous model. When $\hat{p}$ is replaced in (1) we can find the annualized growth rate in a continous setting.

$$\hat{V}_{cont.} = V_{discrete} = \frac{\sigma}{\sqrt{\Delta t}}\left[2\left(\frac{e^{r\Delta t}e^{-q\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}\right) - 1\right]$$

$$= \frac{\sigma}{\sqrt{\Delta t}}\left[2\left(\frac{e^{r\Delta t - q\Delta t} - e^{-\sigma\sqrt{\Delta t}} - \frac{1}{2}e^{\sigma\sqrt{\Delta t}} + \frac{1}{2}e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}\right)\right]$$

$$= \frac{2\sigma}{\sqrt{\Delta t}}\left[\frac{e^{\Delta t(r-q)} - e^{-\sigma\sqrt{\Delta t}} - \frac{1}{2}\left[e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}\right]}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}\right]$$

$$\hat{V} = \frac{2\sigma}{\sqrt{\Delta t}}\left[\frac{e^{(r-q)\Delta t} - \frac{1}{2}\left(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}\right)}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}\right]$$

Now, using Taylor's series approximation,

$$e^x \simeq 1 + x + \frac{x^2}{2} + \dots$$

↳ when $x \to 0$ then $e^x \simeq 1 + x$.

Similarly, $\Delta t \to 0$

$$\hat{V} = \frac{2\sigma}{\sqrt{\Delta t}}\left[\frac{e^{(r-q)\Delta t} - \frac{1}{2}\left(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}\right)}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}\right]$$

$$\hat{V}_{\Delta t \to 0} = \frac{2\sigma}{\sqrt{\Delta t}}\left[\frac{1 + (r-q)\Delta t - \frac{1}{2}\left(1 + \cancel{\sigma\sqrt{\Delta t}} + \frac{(\sigma\sqrt{\Delta t})^2}{2} + 1 - \cancel{\sigma\sqrt{\Delta t}} + \frac{(-\sigma\sqrt{\Delta t})^2}{2}\right)}{\cancel{1} + \sigma\sqrt{\Delta t} + \cancel{\frac{1}{2}(\sigma\sqrt{\Delta t})^2} - \cancel{1} + \sigma\sqrt{\Delta t} - \cancel{\frac{(-\sigma\sqrt{\Delta t})^2}{2}}}\right]$$

$$= \frac{2\sigma}{\sqrt{\Delta t}}\left[\frac{1 + (r-q)\Delta t - \frac{1}{2}\left(2 + (\sigma^2\Delta t)\right)}{2\sigma\sqrt{\Delta t}}\right]$$

$$= \frac{1}{\Delta t}\left[\frac{1+(r-q)\,\Delta t - \frac{1}{2}\left(2+\sigma^2(\Delta t)\right)}{1}\right]$$

$$= \frac{1}{\Delta t} \times \Delta t \left[(r-q) - \frac{1}{2}\sigma^2 \right]$$

$$\breve{v} = r - q - \frac{1}{2}\sigma^2$$

② On March 1st 2021 BABA was trading at \$241.35

Here,

Maturity date (T) = June 18, 2021 ; K = \$240

On March 1^{st}, 2021	On March 5^{th}, 2021
t_1 = March 1^{st}	t_2 = March 5^{th}
S_1 = \$241.35	S_2 = \$233.89
C_1 = \$21.25	C_2 = \$18.00
Intrinsic value (I) = \$241.35 − \$240 = \$1.35	I = max($S_{03/05/21}$ − K, 0) = \$0
Time value (M) = \$(21.25 − 1.35) = \$19.9	M = \$18.00 − \$0.00 = \$18.0

$\Delta S_{(03/01 \to 03/05)}$ = \$ −7.46

Using Black-Scholes-Formula for call option

$$C = S e^{-q(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$d_1 = \left[\ln(S/K) + \left(r + q + \frac{\sigma^2}{2}\right)(T-t)\right] / \sigma\sqrt{T-t}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$N(z)$ = c.d.f of $N(0,1)$

Calculating for March 1^{st} 2021

$S = 241.35$, $K = 240$, $r = 0.01$ or 1%

$q = 0$; $\sigma = 0.2, 0.3$ and 0.4

$$T - t = \frac{79}{252}$$

Scanned with CamScanner

$\sigma = 0.2$

$$d_1 = \left[\ln\left(\frac{241.35}{240}\right) + \left(0.01 + \frac{(0.2)^2}{2}\right)\left(\frac{79}{252}\right)\right] / 0.2\sqrt{\frac{79}{252}}$$

$$= 0.13408$$

$$d_2 = 0.13408 - 0.2 \times \sqrt{\frac{79}{252}} = 0.022096$$

$N(d_1) = 0.5533$; $N(d_2) = 0.5088$

$$C_{0.2} = 241.35 \times e^0 \times 0.5333 - 240e^{-0.01 \times \frac{79}{252}} \times 0.5088$$
$$= \underline{\underline{11.81}}$$

similarly,

$\sigma = 0.3$

$d_1 = 0.136043$; $d_2 = 0.136 - 0.3\sqrt{\frac{79}{252}} = -0.0319$

$N(d_1) = 0.5541$; $N(d_2) = 0.487264$

$C_{0.3} = \underline{\underline{17.156}}$

$\sigma = 0.4$

$d_1 = 0.151023$; $d_2 = 0.151023 - 0.4\sqrt{79/252} = -0.0729$

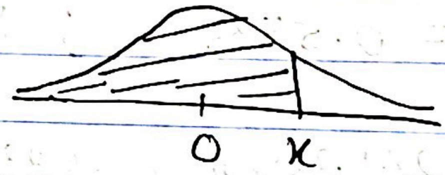
$N(d_1) = 0.560021$; $N(d_2) = 0.4709$

$C_{0.4} = \underline{\underline{22.4922}}$

In the given formula, we can see that it depends on,

- S_0 - the stock price at time zero
- K - strike price

$T-t$ - time until maturity where
T is time to maturity of option
r - the continuously compounding risk free rate (annualized)
q - the annualized dividend rate
$N(x)$ - the cumulative probability distribution function for a variable with a standard normal distribution will be less than x.



q is zero above because Alibaba doesn't pay dividend

Here, all the variables that the derivation of ~~stock~~ option price depends on is constant except for the volatility.

The higher the volatility of the underlying asset, the higher is the price for both call and put option. This happens because higher volatility increases both the up potential and down potential. The upside helps calls and downside helps put options

~~to~~ $\sigma = 0.4$ closely resembles ~~the~~ or gives the actual call option price in the market. Which could mean that at that time, BABA was trading with a volatility of $\sigma = 0.4$ //