Homework 1
Math 449

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1. (25pts) Assume that your uncle offers you one of the following two gifts:

   - Portfolio 1: Long one in-the-money call option with strike (current stock price - \$3) and one out-of-the money call option with strike (current stock price + \$3)
   - Portfolio 2: Long two at-the-money call options

   All options are on the same asset and have the same maturity.
   Which gift would you rather take? Why?

   Answer:

   Let, $S(t_0)$ be stock price or price of underlying asset at current time t_0 and K_x be the strike prices of the options where $K_1 > K_3 > K_2$.

   Exercising options at time t_0,

   Portfolio 1:
   - Long side to one ITM call option with strike price [$K_1 = S(t_0)$ - \$3] has intrinsic value of $\$3 * 100 = \$300$
   - Long side to one OTM call option with strike price [$K_2 = S(t_0)$ + \$3] has intrinsic value of \$0. (i.e. exercising this option is worthless at the moment)

   Portfolio 2:
   - Long sides to two ATM call options with strike price [$K_3 = S(t_0)$] have intrinsic value of \$0. (i.e. options have no value at the moment)
   Only portfolio 1 is worthy to exercise now as it can provide us a gain of \$300.

   We know that all options are on the same asset and have same maturity. However, we are not aware about the exact maturity date of the options. Let us assume that the maturity date is not near and the stock is bullish (increases over time). A bearish scenario is not important at this point because to profit from a call option we want stock price to be greater than strike price.

   Exercising in the future at time t_1,

   either $K_3 < S(t_1) < K_2$,

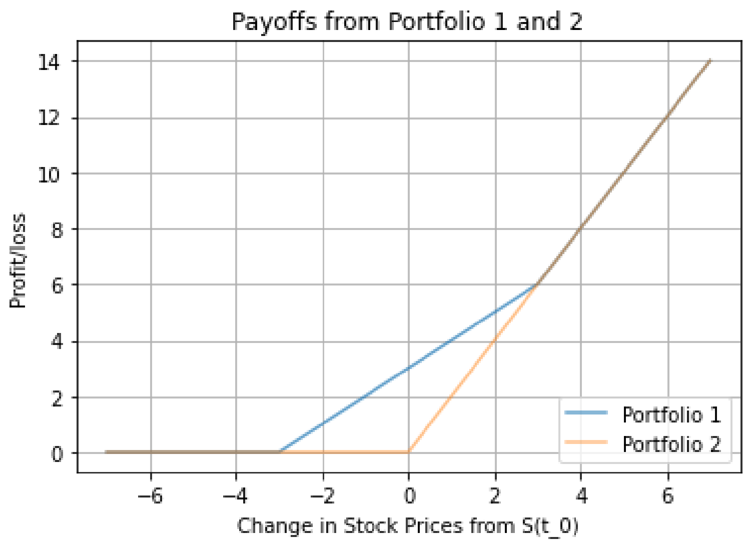
   The two ATM call options in portfolio 2 becomes ITM. However I will still earn more money from portfolio 1 as exercising the ITM option in portfolio 1 will be worth more than two ITM options in portfolio 2.

   or $K_3 < K_2 < S(t_1)$,

   All the options in portfolio 1 and 2 are ITM and both portfolios are worth the same.

$$portfolio1 = portfolio2$$
$$K_1 + K_2 = 2 * K_3$$
$$(S(t_0) - \$3) + (S(t_0) + \$3) = 2S(t)$$
$$2S(t_0) = 2S(t_0)$$

The figure below shows the payoffs from portfolio 1 and 2 as the stock price changes from $S(t_0)$:



Thus, I will take **portfolio 1** as it will never be worth less than portfolio 2 anytime today or in the future.

Note: Since I am getting calls in portfolio 1 and 2 as a gift (for free), I don't have to worry about any premium cost.

2. (25pts) When you enter one of the following four positions: (i) long a butterfly call spread, (ii) short a butterfly call spread, (iii) long a butterfly put spread, (iv) short a butterfly put spread, do you expect a credit or debit in your account?
   In each of the four cases, plot the payoff function with respect to the asset price at expiry.

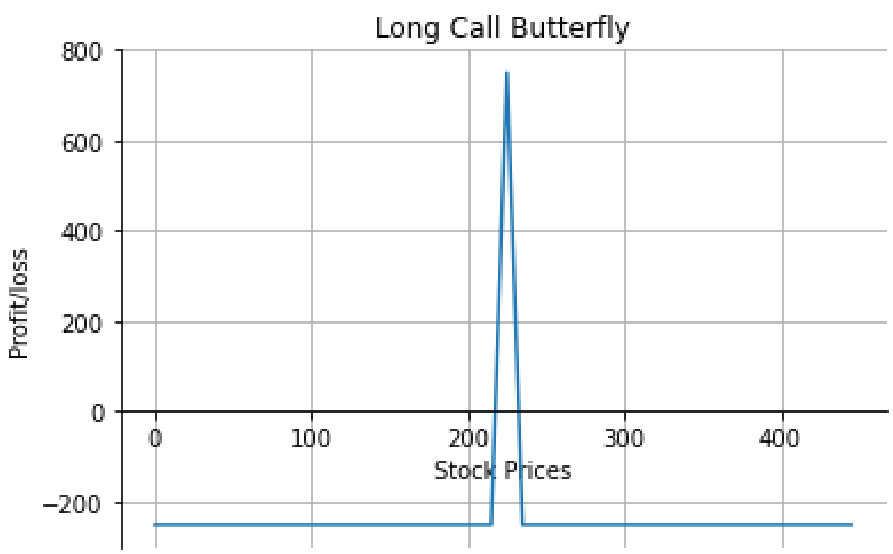
   Answer:

   I enter the butterfly spread option combination with the expectation that the current market price of underlying stock will not change much.Combinations of put or call options can be used to create different type of butterfly spread. The possible combination of options and payoff graph for each is given below:

   (a) Longing a butterfly call spread means:
   - long one in-the-money call option with a low strike price
   - short two at-the-money call options
   - long one out-of-the-money call option with a higher strike price

   Here I am paying premium for ITM and OTM call options and receiving premium for 2 ATM call options. Logically the premiums for (ITM+ATM) > premiums for (2* ATM) in any case because the payoff of one ITM and one OTM call options is always positive or equal to 2 ATM call options (See question 1 for reference).

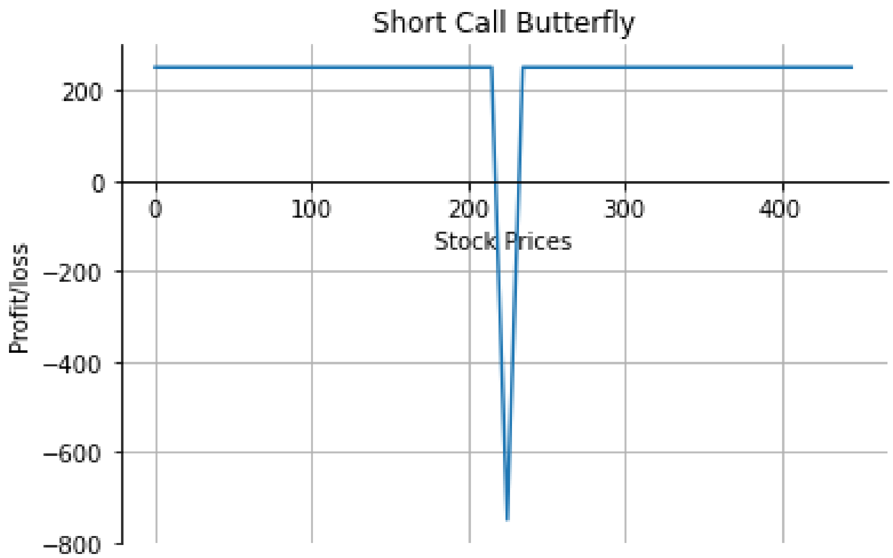
   Thus, my account will be debited when entering the trade.



   (b) Shorting a butterfly call spread means :
   - short one in-the-money call option with a lower strike price
   - long two at-the-money call options
   - short one out-of-the-money call option at a higher strike price

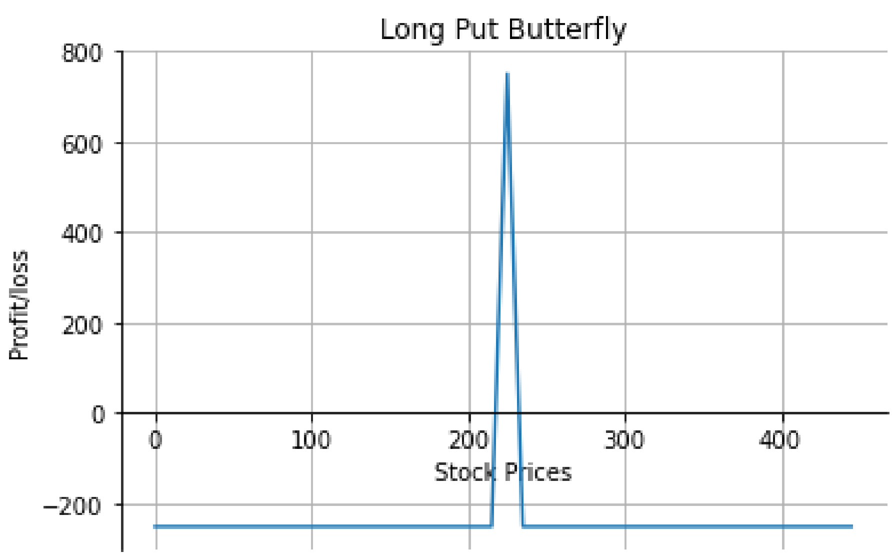
I am receiving premium for ITM and OTM call options and paying premium for 2 ATM call options. Premiums for (ITM+ATM) - premiums for (2* ATM) is always greater than 0.(same logic as above)

So, my account will be credited when entering the trade.



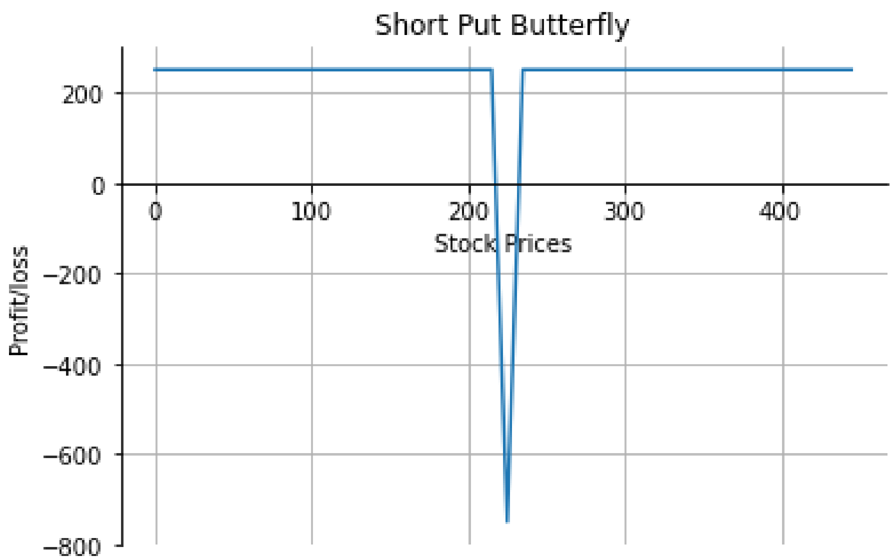
(c) Longing a butterfly put spread means:
- long one put with a lower strike price
- short two at-the-money puts
- long a put with a higher strike price

My account will be debited when entering the trade same as (a).



(d) Shorting a butterfly put spread means:
- short one out-of-the-money put option with a low strike price
- long two at-the-money puts
- short one in-the-money put option at a higher strike price

My account will be credited when entering the trade same as (b).



3. (25pts) The argument used to explain the put-call parity relation is equivalent to saying that if the relation is not satisfied, then an arbitrage opportunity exists. Make this transparent by putting yourself in the position of an arbitrage trader: explain how you would make a risk-free profit when the put-call parity is not satisfied.
Please explain your trading strategy clearly, i.e. state what to trade and when to make the trades. Make all the standard assumptions such as there are no commission fees, you can borrow money from an ideal bank and can always short-sell the asset.
Also explain why when you keep making money, the put-call parity will be satisfied, and there arbitrary opportunity will vanish.

Answer:

Put-call parity relation says that:

$$C_k + PV(k) = P_k + S \tag{1}$$

where,
K = strike price for European put and call options
C_k = price of call option with strike price K
PV(k) = present value of bond worth the strike price (K) at maturity
P_k = price of the put option with same maturity and strike price as call option
S = Current market value of the underlying asset

Possible arbitrage opportunity would result if there is a divergence between the value of calls and puts with the same strike price and expiration date.
There are two sides to the equation and consequently there are two ways to make risk free profits. I will show this with the help of an example.

(a) when $C_k + PV(K) < P_k + S$

Let, Company X be trading at S = \$60 and,
K = \$65 ; C_k =\$10 ; PV(k) = \$54 ; P_k =\$15
Since $K > S ---> C_k < P_k$

At this point I would want to sell [P_k + S] which is more expensive and buy [C_k + PV(k)] expecting that they are going to have same payoffs in the future.
So, first I short the stock S and write the put option with strike price K =\$65.

$$P_k + S = 15 + 60 = \$75$$

Then, I buy a bond at \$54 worth K =\$65 at maturity and a long call option with strike price K.

$$C_k + PV(k) = 10 + 54 = \$64$$

Profit = 75-64 = \$11

This profit is conserved no matter what happens to the stock at maturity.

Say, S decreases by \$5 at maturity which makes call option worthless [OTM]. I use \$65 to buy stock from put holder and give that to the person I initially borrowed the stock from(cover initial short position). This \$65 is covered from the bond that I have.

Say, S increases by \$5 at maturity which makes put option worthless [OTM]. I use \$65 worth of bond to exercise call option and return that stock as the borrowed stock.

Thus, everything evens out at the end and I can still keep the initial profit.

(b) when $C_k + PV(x) > P_k + S$

Let, Company Y be trading at S = \$65 too and,
K = \$62 ; C_k =\$17 ; PV(k) = \$60 ; P_k =\$8
Since $K < S - - - > C_k > P_k$

At this point I would want to buy [P_k + S] and sell [C_k + PV(k)] expecting that they are going to have same payoffs in the future.
First ,I borrow PV(k) from the bank which is worth K =\$62 at maturity and write the call option with strike price K.

$$C_k + PV(k) = 17 + 60 = \$77$$

Then, I buy a stock worth \$65 and a long put option with strike price K =\$62.

$$P_k + S = 8 + 65 = \$73$$

Profit = 77-73 = \$4

This profit is conserved no matter what happens to the stock at maturity.

Say, S decreases by \$5 at maturity which makes call option worthless [OTM]. I exercise the put option and sell stock at \$62 and use this \$62 to pay the debt I've with the bank.

Say, S increases by \$5 at maturity which makes put option worthless [OTM]. I sell the stock at \$62 to the call option holder and use this \$62 to pay the debt I've with the bank.

Thus, everything evens out at the end and I can still keep the initial profit.

However, these arbitrage opportunity doesn't last long. When there is significant more buying of put and selling of call than price of put increases due to high demand and price of call decreases due to high supply or vise versa. This will eventually bring the prices to equilibrium.

4. (25pts) We used the so-called 'static hedging strategy' to prove that early exercising a non-dividend paying call option is never optimal. The argument only says that early exercising is not in the best interest of the option holder.
   Argue that if such early exercisers exist, then an arbitrage opportunity is there for you to make a risk-free profit.
   Assume you only know that such early exercisers exist, but you do not know when they would exercise. Explain how, why and when you can make a risk-free profit. Please explain your trading strategy clearly. Make all the standard assumptions.

Answer:
Let's say, I know that there is an early exerciser. At time t_0,
- I start by shorting a call of strike price K and charge some premium. [Assume K = $50]
- I long a call with same strike price and maturity date using the premium earned from shorting the call.
This way, I am completely neutral, which is what I want for arbitrage trades.

At time t, the price of underlying asset goes to $55 and the buyer exercises it. Then,
- I keep the long call position where K =$50.
- Short 1 unit of underlying asset and give it to exerciser.
- I put $50 in bank earned from selling the stock at strike price to early exerciser which will be worth $50e^{rt}$ in the future.

This puts me into the same scenario as the "static hedging strategy" in the notes where I have 1 long call, short 1 unit of the underlying stock, and have cash K from selling the stock I borrowed.

Now, two things can happen at time T (expiration date):

1. The asset price goes up, say $60

- I exercise my long call: buy the share at $50 using $50 cash that I have in bank.
- I use this share bought from exercising the call and return that as the underlying stock I shorted at time t.
- So, I still have $\$50e^{rt} - \$50$ left in the bank.
- Final Profit = $Ke^{rt} - K$

2. the asset price goes down, say $45

- Here I won't exercise my long call.
- I use $45 cash in the bank to buy stock from the market and return that stock as the stock I borrowed at time t.
- I make a profit of $\$50e^{rt} - \$50$.

Final Profit = $Ke^{rt} – S(T)$

Note: We are only considering non dividend paying option here. Static hedging strategy is only better when the options don't pay dividend.