11.4.36 $x+y+z=e^x$

$F'_x = x - e^x$

$F'_y = y - e^x$

$F'_z = z - e^x$

$\frac{\partial z}{\partial x} = -\frac{x-e^x}{z-e^x}$ $\frac{\partial z}{\partial y} = -\frac{y-e^x}{z-e^x}$

$dz = \frac{e^x - x}{z-e^x} dx + \frac{e^x-y}{z-e^x} dy$

11.4.37 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$F'_x = \frac{2x}{a^2}$

$F'_y = \frac{2y}{b^2}$

$F'_z = \frac{2z}{c^2}$

$\frac{\partial z}{\partial x} = \frac{-2xc^2}{2za^2} = -\frac{xc^2}{za^2}$

$\frac{\partial z}{\partial y} = -\frac{yc^2}{zb^2}$

$dz = -\frac{xc^2}{za^2} dx - \frac{xc^2}{za^2} dy$

11.5.1 $z = x^3 - x^2 y - y^3$

1) $\frac{\partial z}{\partial x} = 3x^2 - 2xy$; $\frac{\partial z}{\partial y} = -x^2 - 3y^2$

2) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 2xy) = 6x - 2y$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}(3x^2 - 2xy) = -2x$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}(-x^2 - 3y^2) = -2x$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(-x^2 - 3y^2) = -6y$

3) $\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial}{\partial x}(6x - 2y) = 6$

$\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial y}(6x - 2y) = -2$

$\frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial}{\partial y}(-2x) = 0$

$\frac{\partial^3 z}{\partial y^3} = \frac{\partial}{\partial y}(-6y) = -6$

11.5.2 $z = e^{xy^3}$

1) $\frac{\partial z}{\partial x} = y^3 e^{xy^3}$

$\frac{\partial^2 z}{\partial x^2} = y^6 e^{xy^3}$

$\frac{\partial^3 z}{\partial x^3} = y^9 e^{xy^3}$

$\frac{\partial^4 z}{\partial x^4} = y^{12} e^{xy^3}$

2) $\frac{\partial^4 z}{\partial x^3 \partial y} = \frac{\partial}{\partial y}\cdot\left(\frac{\partial^3 z}{\partial x^3}\right) = \frac{\partial}{\partial y}(y^9 \cdot e^{xy^3}) = 9y^8 e^{xy^3} +$

$+ 3y^{11} x e^{xy^3} = 3y^8 e^{xy^3}(y^3 x + 3)$

3) $\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial}{\partial y}\cdot(y^6 e^{xy^3}) = 6y^5 e^{xy^3} +$

$+ 3y^8 x e^{xy^3} = 3y^5 e^{xy^3}(y^3 x + 2)$

$\frac{\partial^4 z}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) = \frac{\partial}{\partial y}\cdot$

$\cdot(3y^5 e^{xy^3}(y^3 x + 2)) = 3(5y^4 e^{xy^3}(2 + y^3 x) +$

$+ 3xy^7 e^{xy^3}(2 + y^3 x) + 3y^7 x e^{xy^3}) = 3y^4 e^{xy^3}(10 +$

$+ 14xy^3 + 3x^2 y^6)$

11.5.3 1) $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = -\frac{y}{x^2+y^2}dx + \frac{x\,dy}{x^2+y^2}$

2) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = -\frac{2xy}{(x^2+y^2)^2}$

$d^2 z = \frac{2(xy\,dx^2 + (y^2-x^2)\,dx\,dy - xy\,dy^2)}{(x^2+y^2)^2}$

11.5.4 $z = \frac{xy}{x-y}$

$z'_x = \frac{\cancel{x-y}\,y + xy - xy}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$

$z'_y = \frac{x^2}{(x-y)^2}$

$z''_{x^2} = \frac{2y^2}{(x-y)^3}$

$$z''_{xy} = -\frac{2xy}{(x-y)^3}$$

$$z''_{y^2} = \frac{2x^2}{(x-y)^3}$$

$$d^2\left(\frac{xy}{x-y}\right) = z''_{x^2}dx^2 + z''_{xy}dxdy + z''_{y^2}dy^2 =$$

$$= \frac{2(y^2dx - 2xy\,dx\,dy + x^2dy^2)}{(x-y)^3}$$

$$z''_{x^2} + z''_{xy} + z''_{y^2} = \frac{2y^2 - 4xy + 2x^2}{(x-y)^3} = \frac{2(x-y)^2}{(x-y)^3} =$$

$$= \frac{2}{x-y}$$

11.5.5 $z = \frac{xy}{x+y}$

1) $z'_x = \frac{y^2}{(x+y)^2}$

$z'_y = \frac{x^2}{(x+y)^2}$

2) $z''_{x^2} = \frac{0 - 2(x+y)y^2}{(x+y)^4} = -\frac{2y^2}{(x+y)^3}$

$z''_{xy} = \frac{2y(x+y)^2 - 2(x+y)y^2}{(x+y)^4} = \frac{2xy}{(x+y)^3}$

$z''_{y^2} = \frac{0 - 2(x+y)x^2}{(x+y)^4} = -\frac{2x^2}{(x+y)^3}$

3) $d^2z = -\frac{2y^2}{(x+y)^3}dx^2 + 4\frac{xy}{(x+y)^3}dx\,dy -$

$- \frac{2x^2}{(x+y)^3}dy^2 = \frac{2(y^2dx^2 - 2xy\,dx\,dy + x^2dy^2)}{(x+y)^3} =$

$= -2\frac{(y\,dx - x\,dy)^2}{(x+y)^3}$

4) $d^3z = z'''_{x^3}dx^3 + 3z'''_{x^2y}dx^2dy + 3z'''_{xy^2}dxdy^2 +$

$+ z'''_{y^3}dy^3$

$d^3z = \frac{6}{(x+y)^4}(y^2dx^3 - (2xy - y^2)dx^2dy - (2xy - x^2)\cdot$

$\cdot dxdy^2 + x^2dy^3)$

11.5.6 $d^2z = ?$, $z = \ln(x^2+y^2)$

$d(dz) = d^2z \Rightarrow dz = z'd = 2\,\frac{x\,dx + y\,dy}{x^2+y^2}$

$d^2z = 2\left(\frac{\partial}{\partial x}\cdot\left(\frac{x\,dx+y\,dy}{x^2+y^2}\right)dx + 2\frac{\partial}{\partial y}\cdot\right.$

$\cdot\left(\frac{x\,dx+y\,dy}{x^2+y^2}\right)dy = 2\,\frac{(x^2+y^2)\,dx - 2x(x\,dx+y\,dy)}{(y^2+x^2)^2}$

$\cdot dx + 2\cdot\frac{(x^2+y^2)\,dy - 2y(x\,dx+y\,dy)}{(x^2+y^2)^2}\,dy =$

$= 2\,\frac{(y^2-x^2)\,dx^2 - 4xy\,dx\,dy + (x^2-y^2)\,dy^2}{(x^2+y^2)^2}$

11.5.7 $z = \sin x \sin y$, d^2z

1) $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$\frac{\partial z}{\partial x} = \sin y \cos x$

$\frac{\partial z}{\partial y} = \sin x \cos y$

$dz = \sin y \cos x\,dx + \sin x \cos y\,dy$

2) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\sin y \cos x) = -\sin x \overset{\sin}{\cancel{\cos}} x$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}(\sin y \cos x) = \cos x \cos y$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\sin x \cos y) = -\sin x \sin y$

$d^2z = -\sin x \overset{\sin y}{\cancel{\cos x}}\,dx^2 + \cos x \cos y\,dx\,dy -$

$- \sin x \sin y\,dy^2$