University of Zurich[UZH]

ROBOTICS & PERCEPTION GROUP

Vision Algorithms for Mobile Robotics

Lecture 09
Multiple View Geometry 3

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Lab Exercise 7 – Today

Implement the P3P algorithm and RANSAC

Outline

- Robust Structure from Motion
- Bundle Adjustment

Robust Estimation

- Matched points are usually contaminated by **outliers** (i.e., wrong image matches).

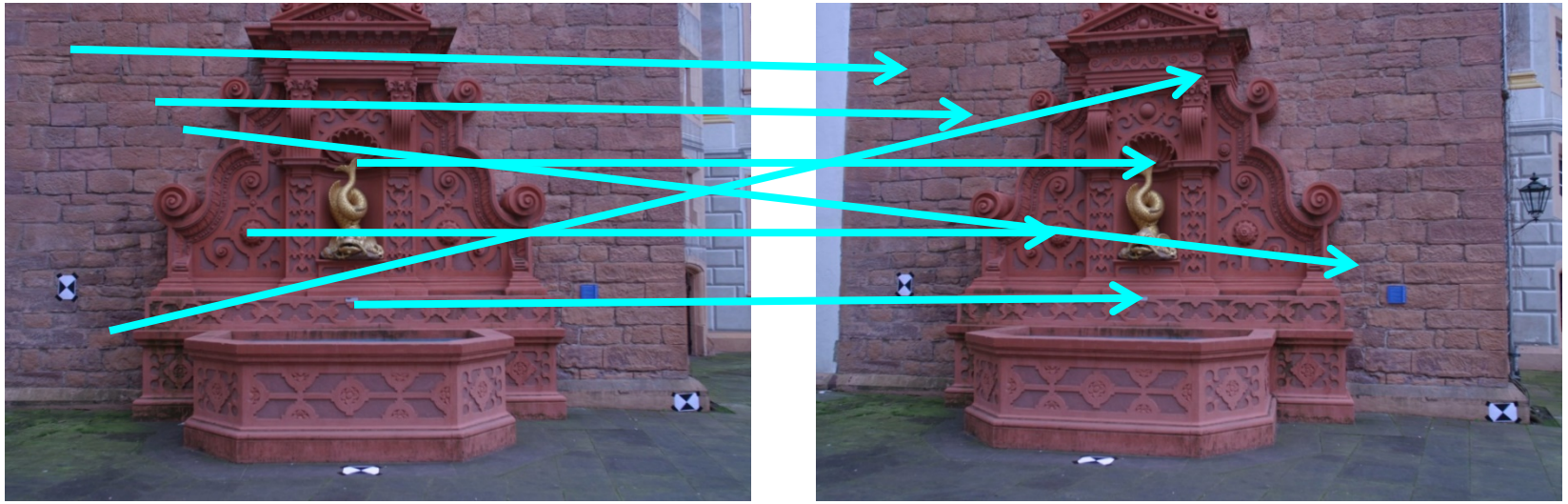


Image 1 Image 2

Robust Estimation

- Matched points are usually contaminated by **outliers** (i.e., wrong image matches). Causes of outliers are:
  - Repetitive features
  - changes in view point (including scale) and illumination
  - image noise
  - Occlusions
  - Moving objects
  - blur
- For reliable and accurate visual odometry, outliers must be removed
- This is the task of **Robust Estimation**

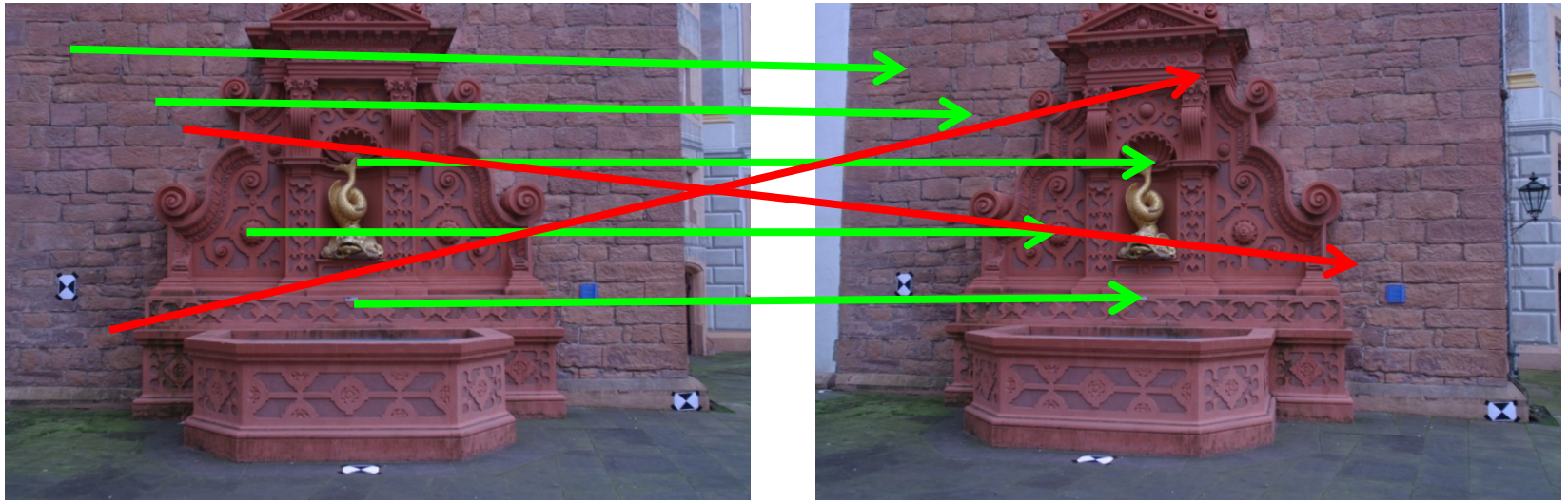
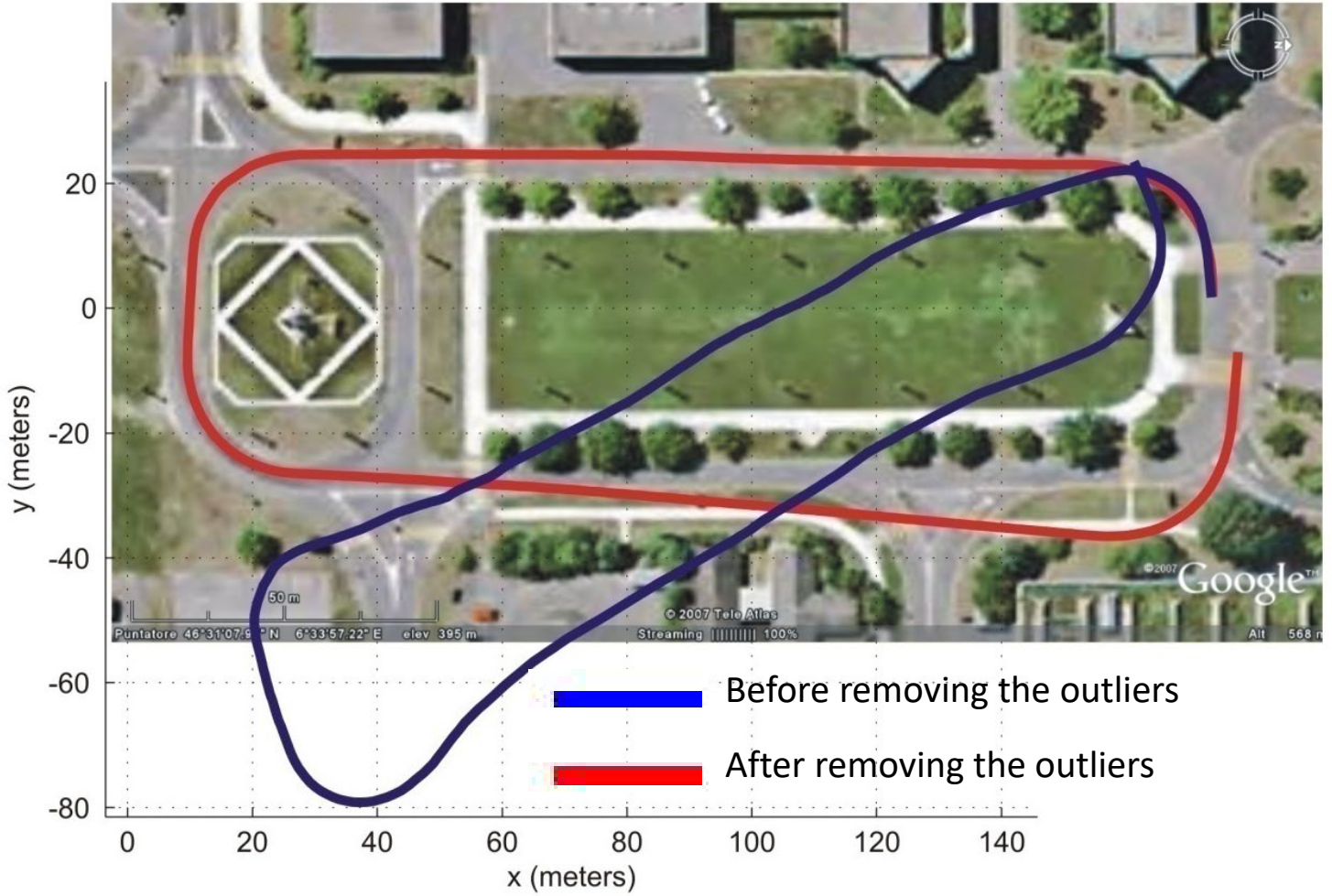


Image 1 Image 2

Effect of Outliers on Visual Odometry



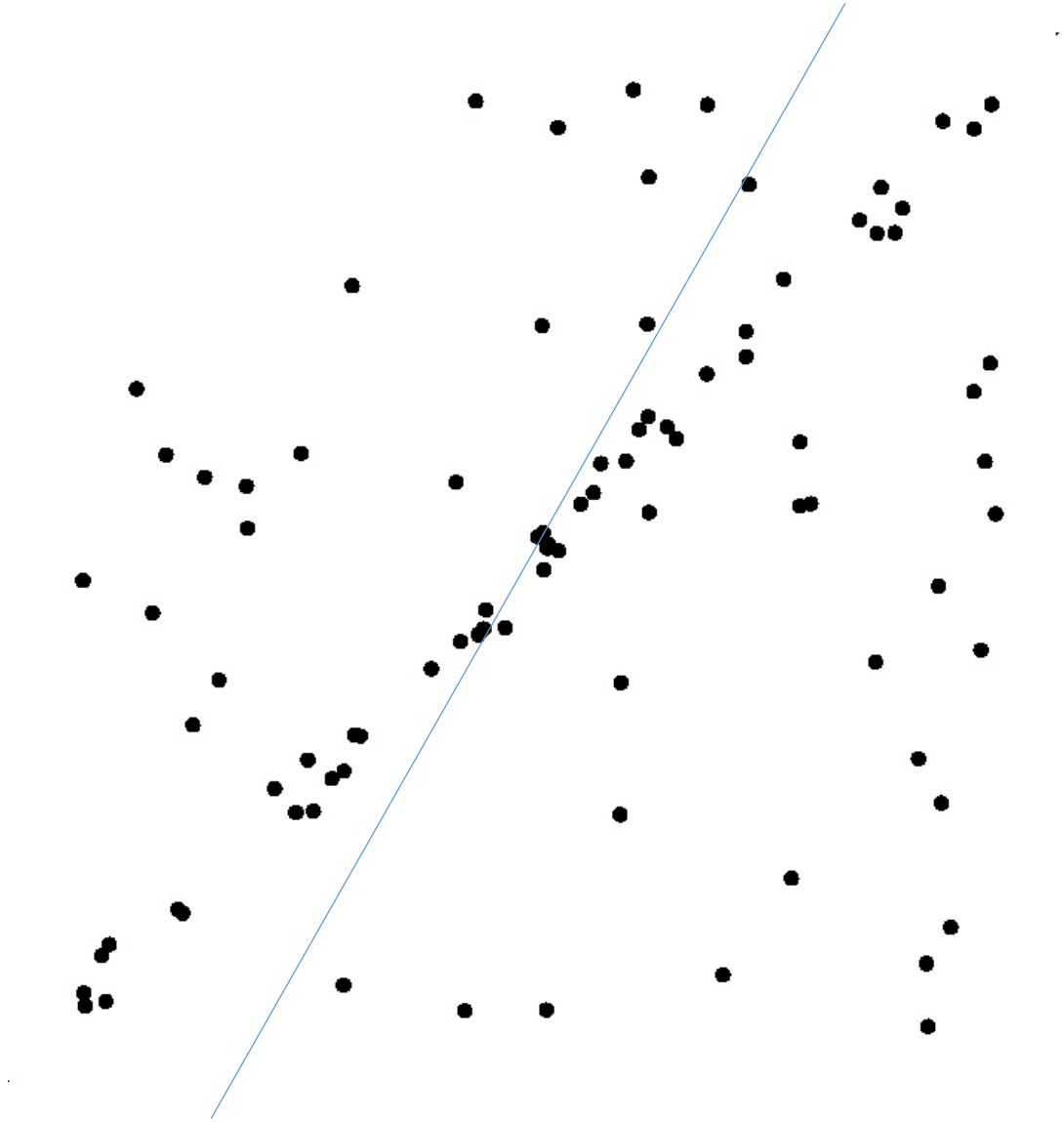
Expectation Maximization (EM) algorithm

- EM is a simple **method for model fitting in the presence of outliers** (very noisy points or wrong data)
- It can be applied to all sorts of problems where the goal is to **estimate the parameters of a model from the data** (e.g., camera calibration, Structure from Motion, DLT, PnP, P3P, Homography, etc.)
- Let's review EM applied to the line fitting problem

[1] Dellaert, The *expectation maximization algorithm,* Georgia Institute of Technology, 2002. PDF (explains the original papers below)
[2] Hartley, *Maximum likelihood estimation from incomplete data*, Biometrics, 1958.
[3] Dempster, Laird, Rubin, *Maximum likelihood from incomplete data via the EM algorithm*, Journal of the Royal Statistical Society, 1977.

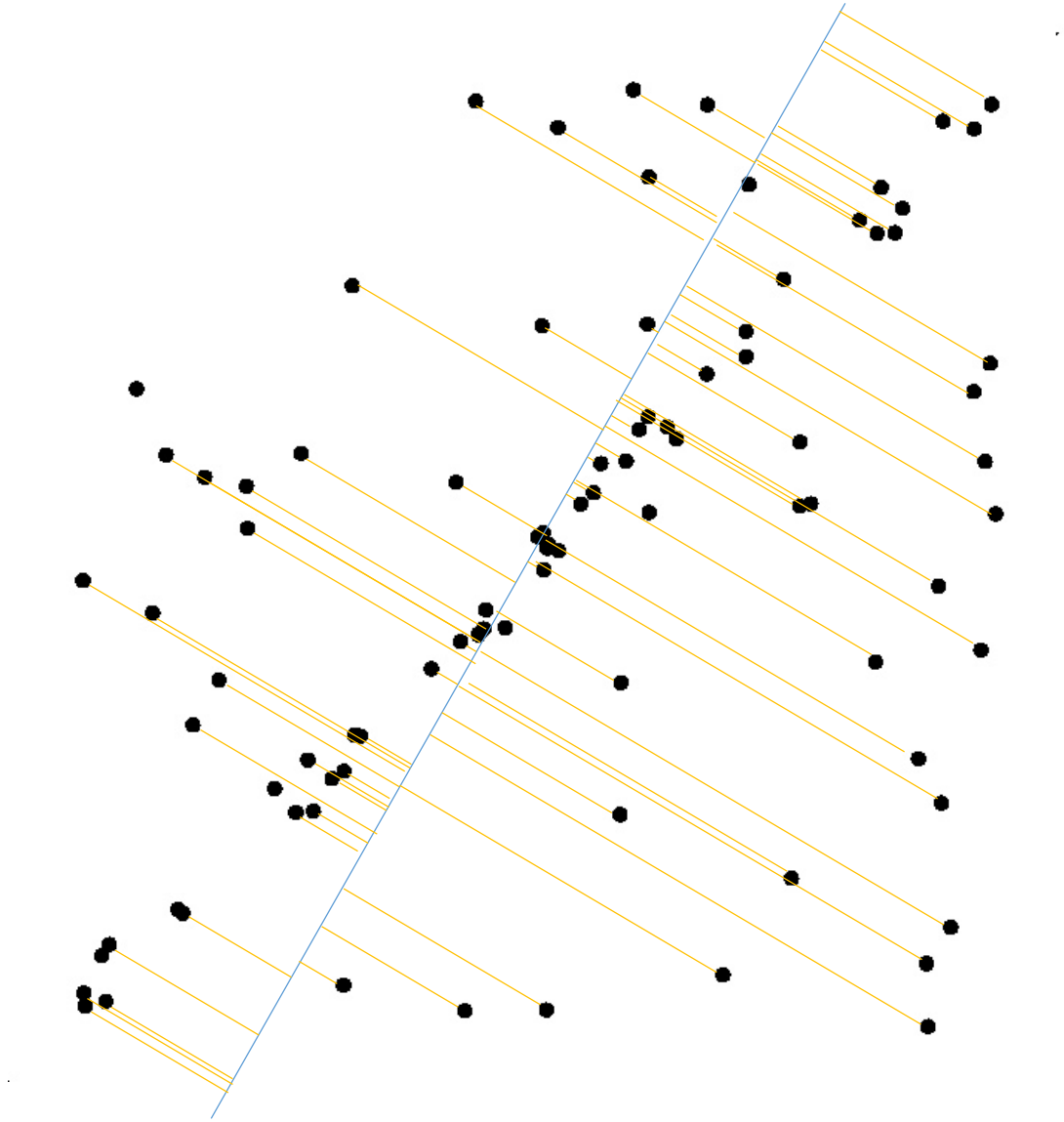
EM applied to line fitting

EM applied to line fitting



1. **Estimate line parameters that fit all data points (e.g., using least-square: $\min \sum r_i^2$, where r_i is the point-to-line distance)**

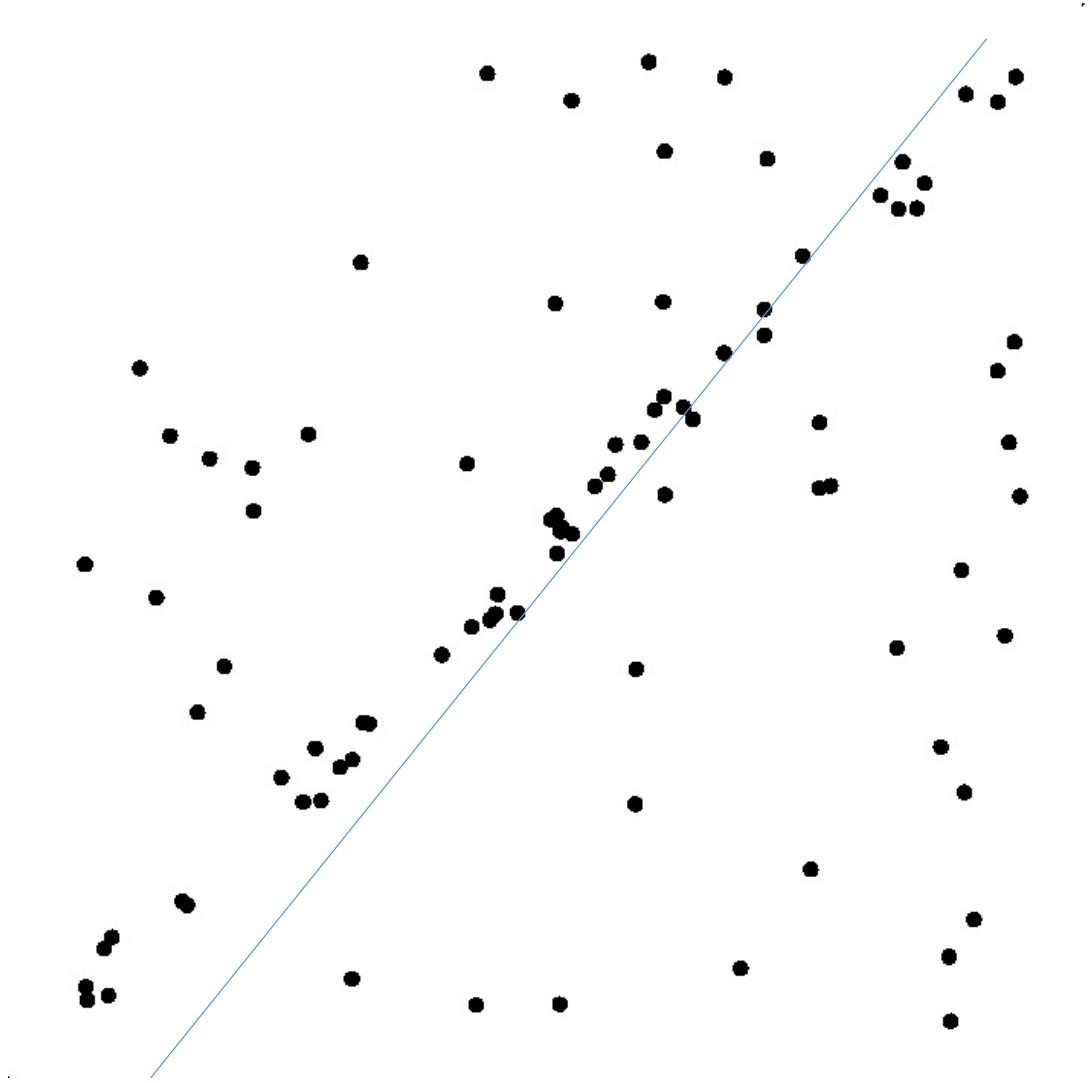
EM applied to line fitting



1. Estimate line parameters that fit all data points (e.g., using least-square: $\min \sum r_i^2$, where r_i is the point-to-line distance)

2. Calculate residual error r_i for each data point and assign it a weight (e.g., $w_i = e^{-r_i^2}$ representing the likelihood that such assignment is correct (estimates the **Expectation**)

EM applied to line fitting



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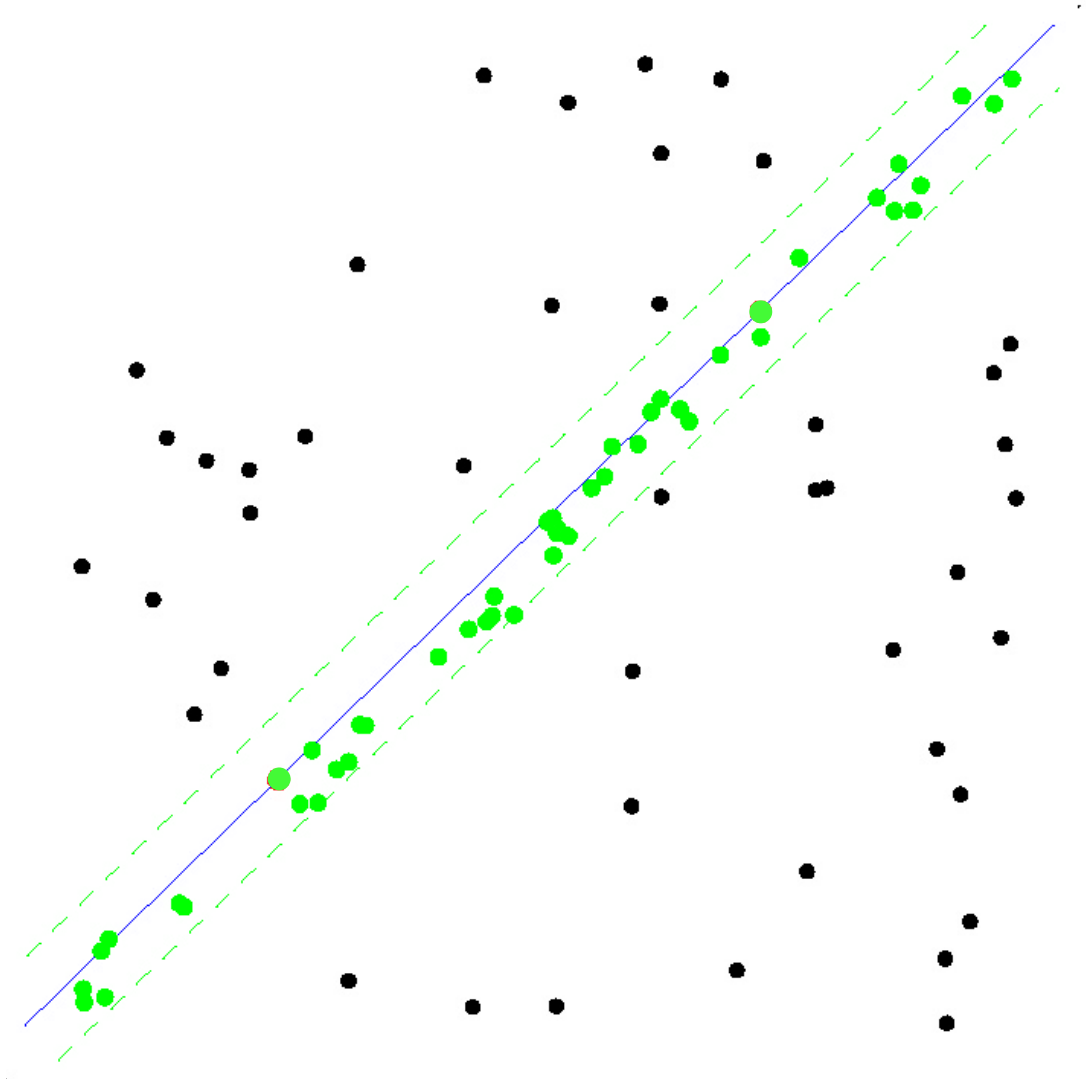
3. Re-estimate line parameters (e.g., using weighted least-squares: $\min \sum w_i r_i^2$) **(Maximization Step)**

EM applied to line fitting



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3. Re-estimate line parameters (e.g., using weighted least-squares: $\min \sum w_i r_i^2$) **(Maximization Step)**

4. Iterate 2 and 3 till convergence

5. Select as **inliers the** data points with weight higher than a threshold

Problem of EM algorithm

Very sensitive to initial condition:

- This is because EM selects the initial condition by minimizing the sum of squared residuals $\sum r_i^2$.
- While this is a convex function, the result is strongly influenced by a few large error values (e.g., outliers).
- Thus, EM converges to the wrong solution if initial condition is far from the true one

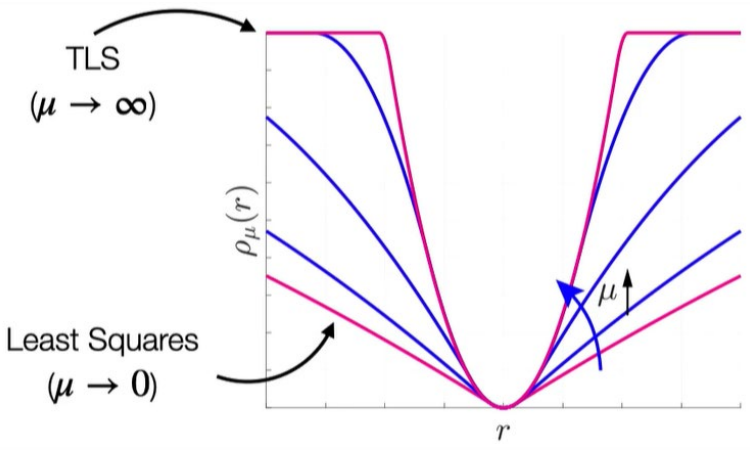
- Alternative options:
  - GNC algorithm
  - RANSAC algorithm

Graduated Non-Convexity algorithm (GNC)

Won't be asked at the exam ☺

Idea: optimize a surrogate function $\sum \rho_\mu(r_i)$, where μ controls the amount of non-convexity.

- Start by solving the non-robust convex optimization function (μ → 0, i.e., least squares)
- At each iteration, gradually increase non-convexity (μ → ∞) and recompute weights w_i till we achieve the desired level of robustness.
- It is shown in [1] to be robust up to 90% of outliers with five times fewer iterations than RANSAC.
- However, RANSAC can cope with even more than 90% outliers.



[1] Yang, Antonante, Tzoumas, Carlone, *Graduated Non-Convexity for Robust Spatial Perception: From Non-Minimal Solvers to Global Outlier Rejection*, International Conference on Robotics and Automation **(ICRA), 2020. Best paper award in Robot Vision**. PDF. Code.
[2] Blake, Zisserman, *Visual Reconstruction*. MIT Press, Cambridge, Massachusetts, 1987.

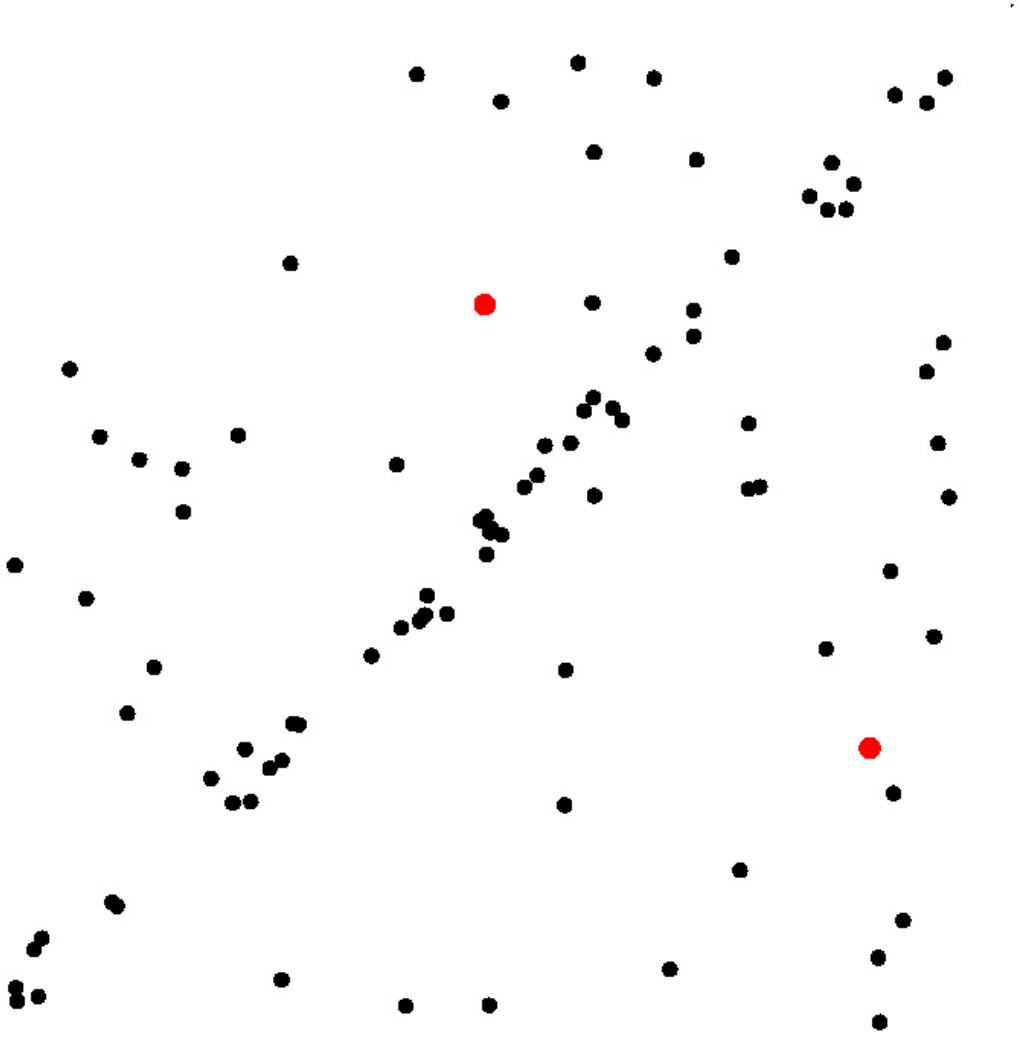
RANSAC (RAndom SAmple Consensus)

- RANSAC is the **standard method for model fitting in the presence of outliers** (very noisy points or wrong data)
- It is **non-deterministic**: you get a different result everytime you run it
- It is not sensitive to the initial condition, and does not get stuck in local maxima
- It can be applied to all sorts of problems where the goal is to **estimate the parameters of a model from the data** (e.g., camera calibration, Structure from Motion, DLT, PnP, P3P, Homography, etc.)
- Let's review RANSAC for line fitting and see how we can use it to do Structure from Motion

M. A.Fischler and R. C.Bolles. Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography. Graphics and Image Processing, 1981. PDF.

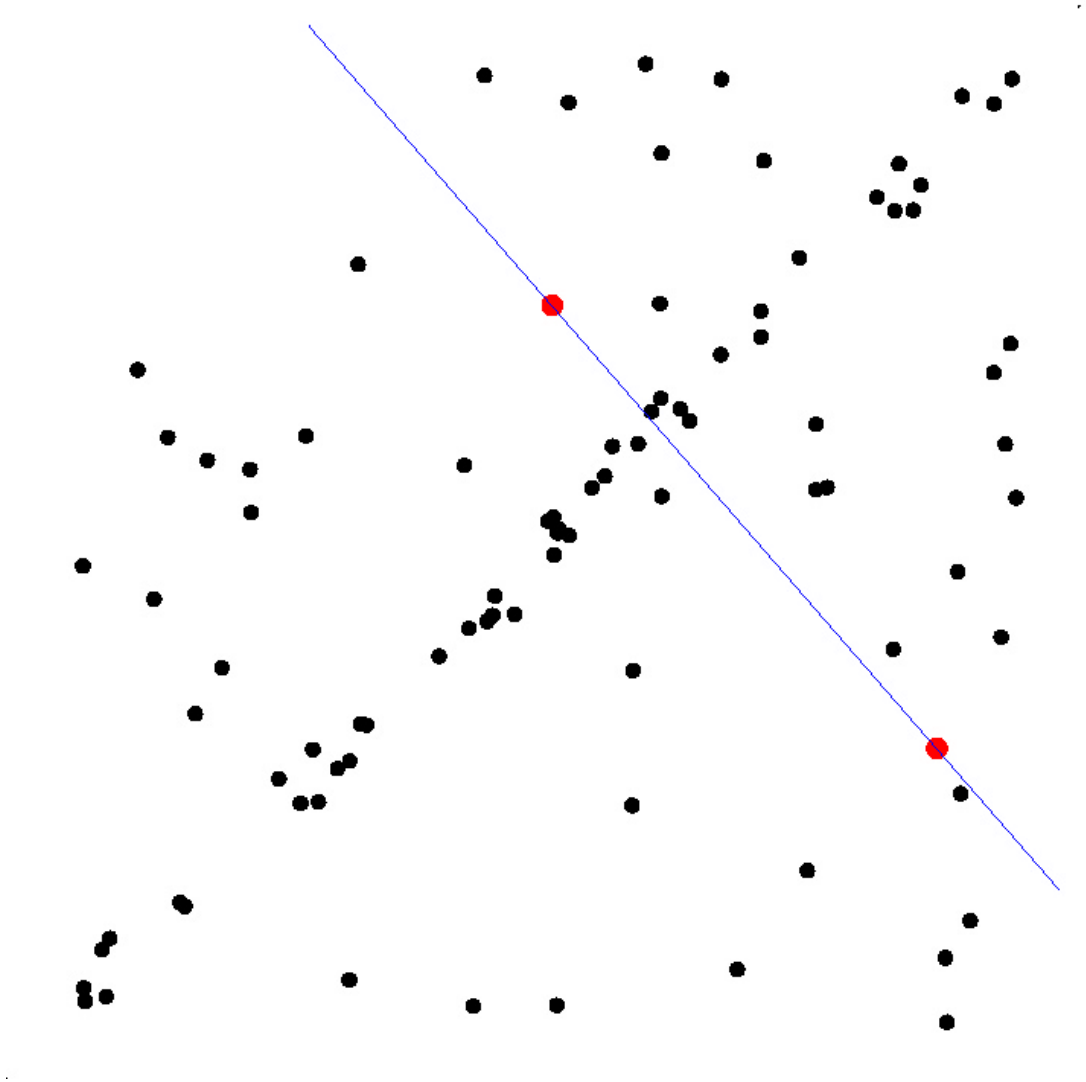
RANSAC

RANSAC



1. **Select a sample of 2 points at *random***

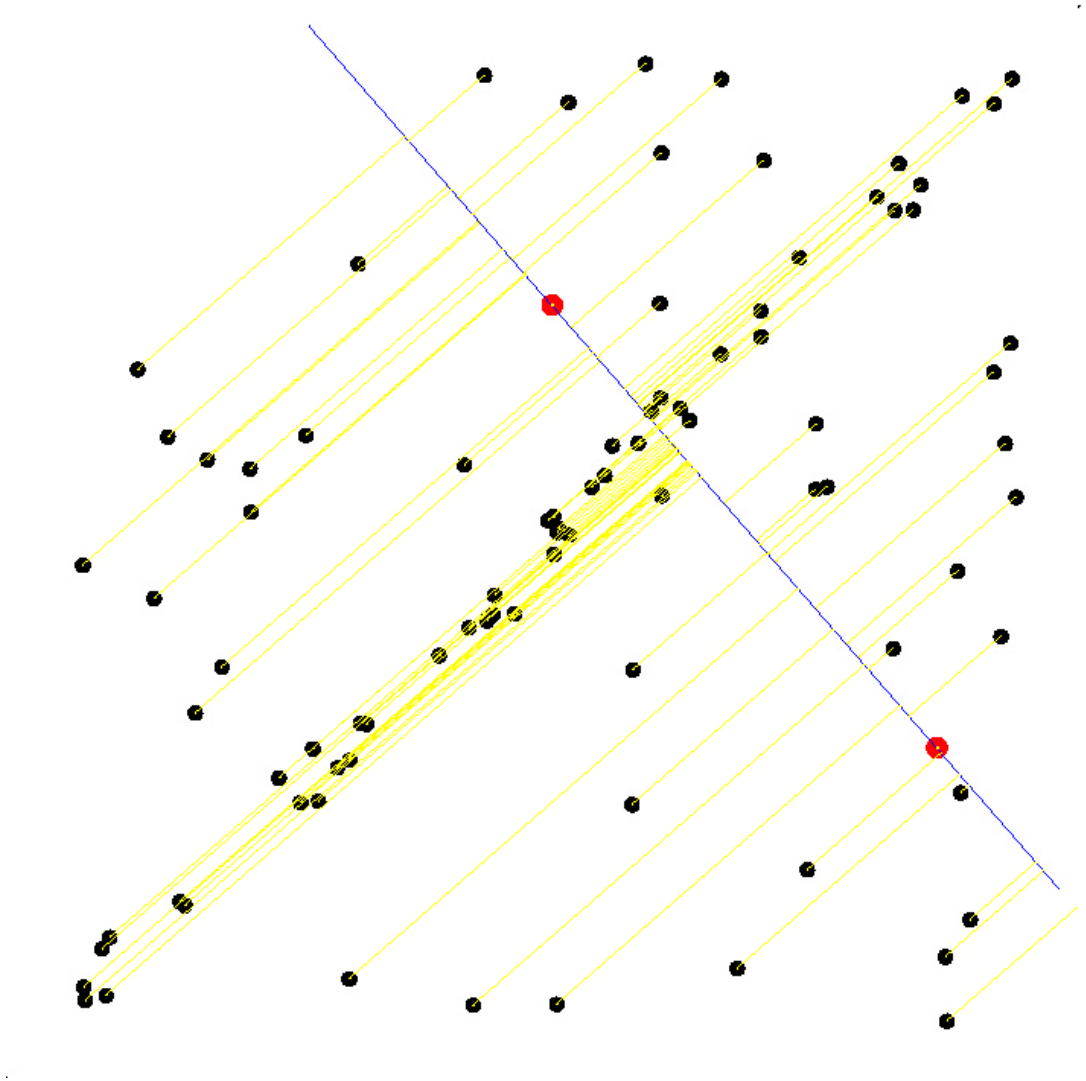
RANSAC



1. Select a sample of 2 points at *random*

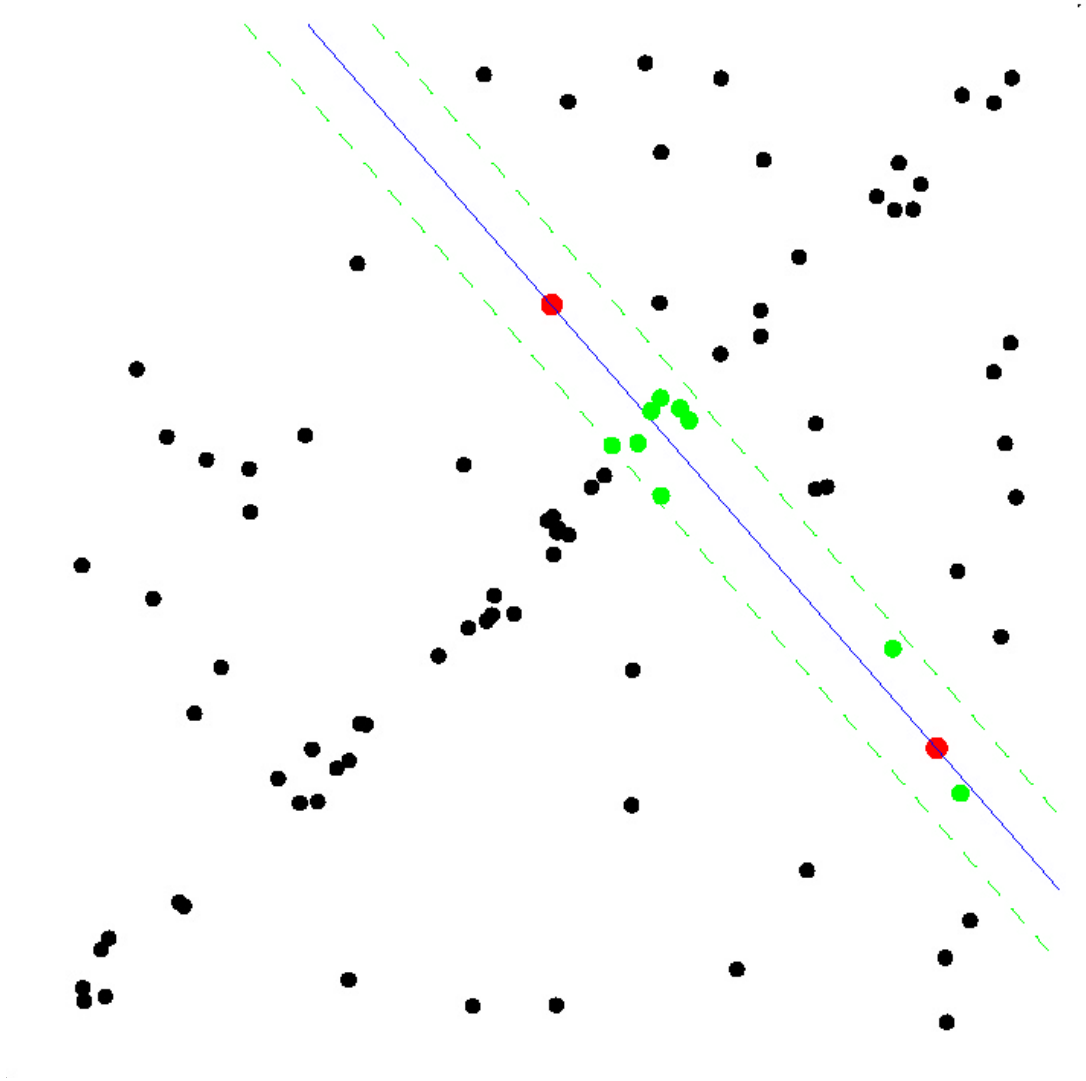
2. **Calculate model parameters that fit the data in the sample**

RANSAC



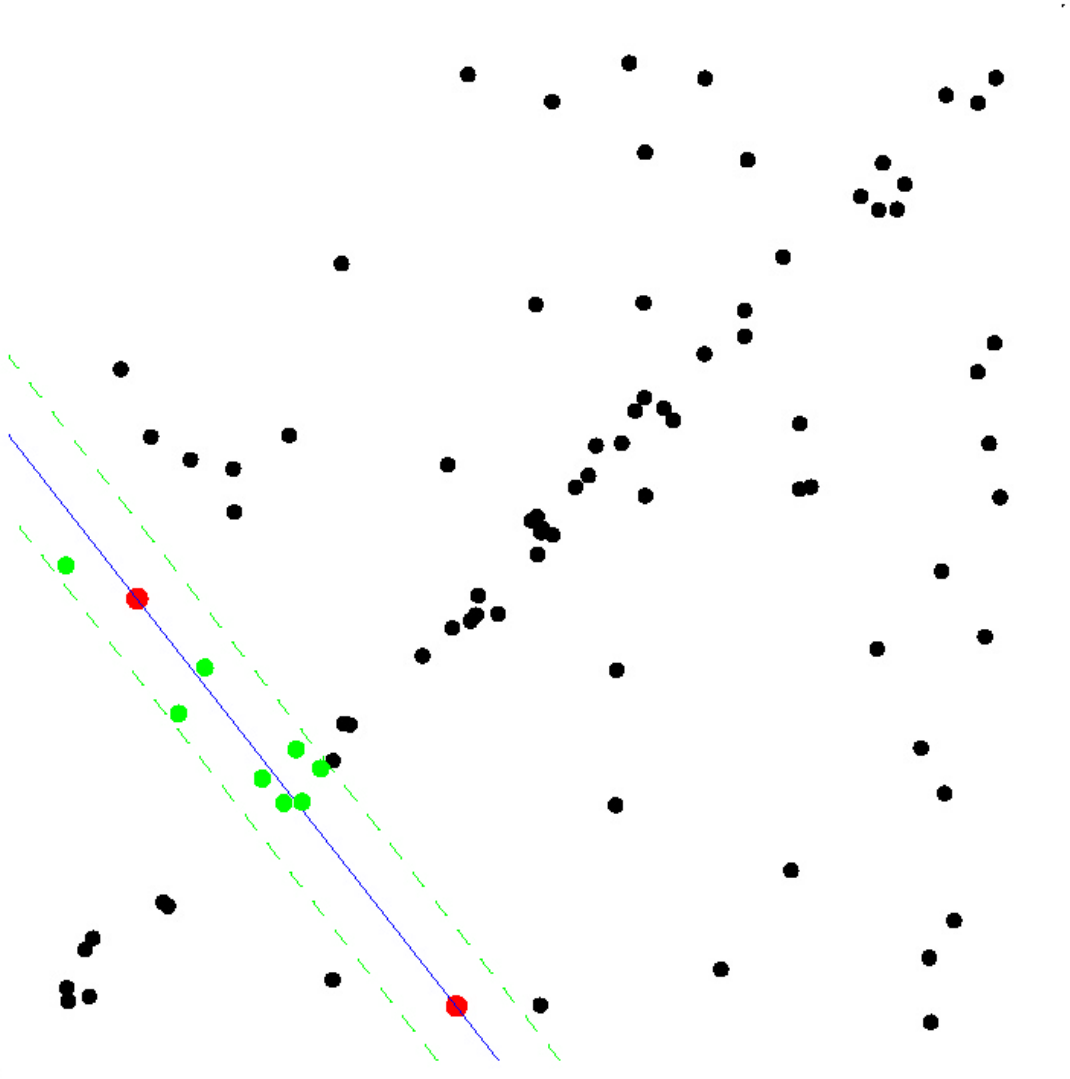
1. Select a sample of 2 points at *random*
2. Calculate model parameters that fit the data in the sample
3. **Calculate the residual error for each data point**

RANSAC



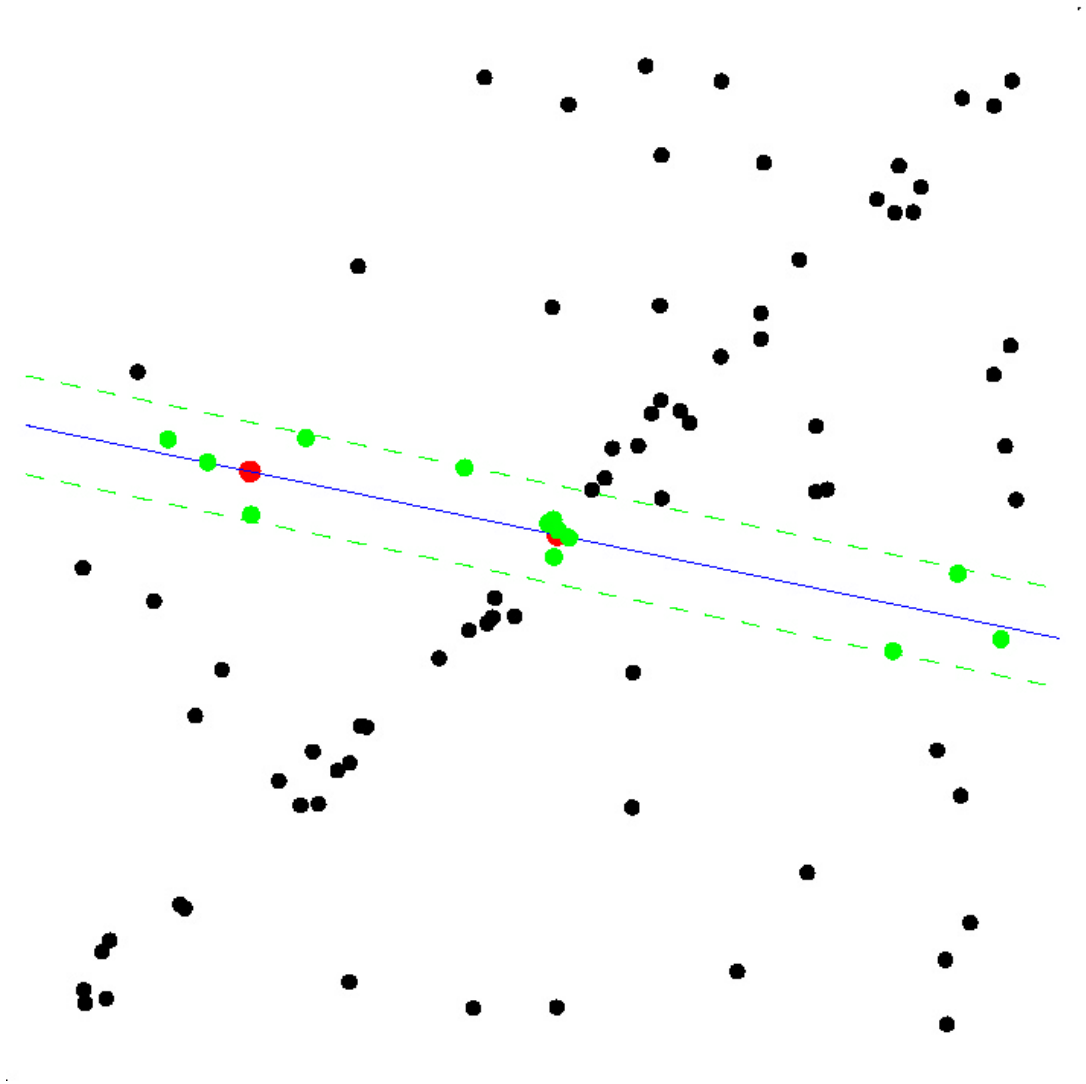
1. Select a sample of 2 points at *random*
2. Calculate model parameters that fit the data in the sample
3. Calculate the residual error for each data point
4. **Select data that support current hypothesis**

RANSAC



1. Select a sample of 2 points at *random*
2. Calculate model parameters that fit the data in the sample
3. Calculate the residual error for each data point
4. Select data that support current hypothesis
5. **Repeat from step 1 for k times**

RANSAC



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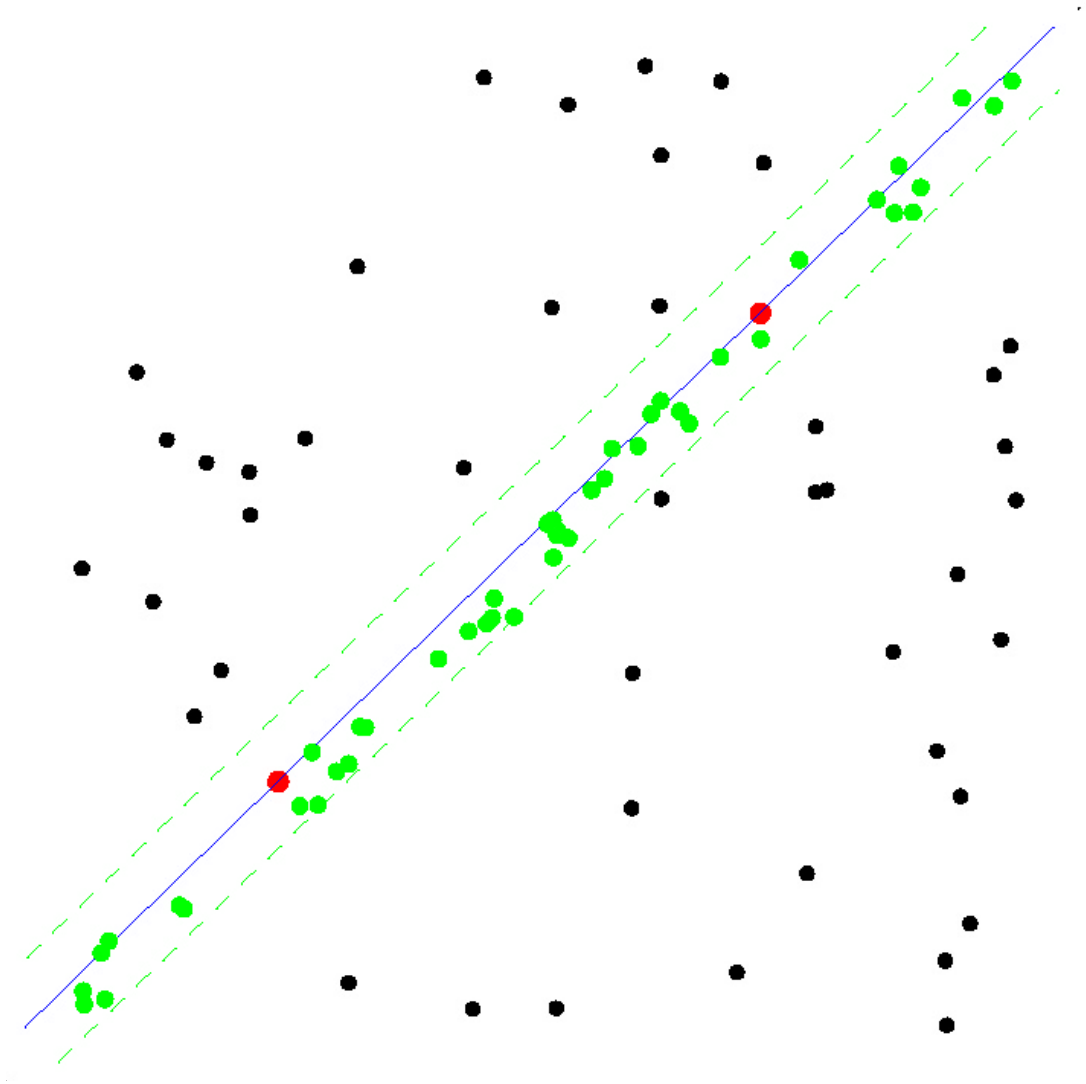
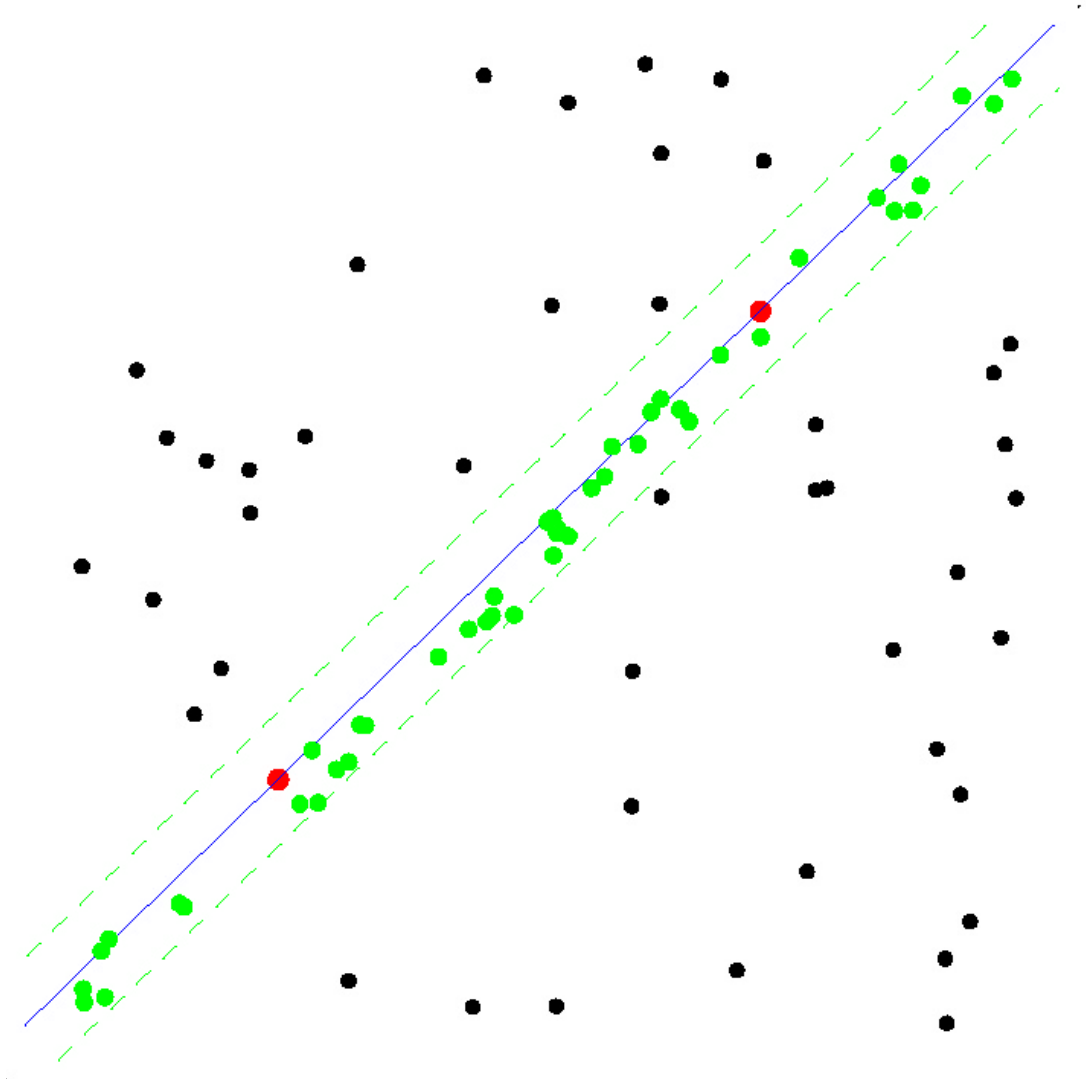


1. Select a sample of 2 points at random
2. Calculate model parameters that fit the data in the sample
3. Calculate the residual error for each data point
4. Select data that support current hypothesis
5. Repeat from step 1 for k times
6. **Select the set with the maximum number of inliers obtained within k iterations**

RANSAC



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NB: RANSAC is non deterministic: every time you run it you get a different result (due to the random hypotheses' generation process). Conversely, **EM and GNC** are **deterministic**

RANSAC

- How many iterations does RANSAC need?
- Ideally: check all possible combinations of 2 points in a dataset of N points.
- Number of all pairwise combinations: $\frac{N(N-1)}{2}$
  - computationally unfeasible if N is too large.
    Example, for 1000 points you need to check all 1000×999/2 ≅ 500'000 possibilities!

- Do we really need to check all possibilities or can we stop RANSAC after some iterations?
  - We will see that it is **enough to check a subset of all combinations if we have** a rough **estimate of the percentage of inliers** in our dataset
  - This can be done in a **probabilistic way**

RANSAC

- How many iterations does RANSAC need?

- $\boldsymbol{w}$:= number of inliers $/N$
  $\boldsymbol{N}$:= total number of data points
  $\rightarrow \boldsymbol{w}$: fraction of inliers in the dataset $\rightarrow \boldsymbol{w} =$ P(selecting an inlier-point out of the dataset)
- Assumption: the 2 points necessary to estimate a line are selected independently
  - $\rightarrow \boldsymbol{w}^{\boldsymbol{2}} = P$(both selected points are inliers)
  - $\rightarrow \boldsymbol{1 - w^2} = P$(at least one of these two points is an outlier)
- Let $\boldsymbol{k}$ be the number of RANSAC iterations executed so far
- $\rightarrow (\boldsymbol{1 - w^2})^{\boldsymbol{k}} = P$(RANSAC never selected two points that are both inliers)
- Let $\boldsymbol{p}$:= Probability of success
- $\rightarrow \boldsymbol{1 - p} = (\boldsymbol{1 - w^2})^{\boldsymbol{k}}$ and therefore:

$$k = \frac{\log(1-p)}{\log(1-w^2)}$$

RANSAC

- How many iterations does RANSAC need?

$$k = \frac{\log(1-p)}{\log(1-w^2)}$$

$\rightarrow$ knowing the fraction of inliers $\boldsymbol{w}$, after $\boldsymbol{k}$ RANSAC iterations we will have a probability $\boldsymbol{p}$ of finding a set of points free of outliers

- Example: if we want a probability of success $\boldsymbol{p} = \boldsymbol{99}\%$ and we know that $\boldsymbol{w} = \boldsymbol{50}\% \rightarrow \boldsymbol{k} = 16$ iterations
  - these are **significantly fewer** than the number of **all possible combinations**!
  - **Notice: number of points does not influence minimum number of iterations $\boldsymbol{k}$, only $\boldsymbol{w}$ does!**

- In practice we only need a rough estimate of $\boldsymbol{w}$. More advanced variants of RANSAC estimate the fraction of inliers and adaptively update it at every iteration (how?)

RANSAC applied to Line Fitting

1. Initial: let *A* be a set of *N* points
2. **repeat**
3. Randomly select a sample of **2** points from *A*
4. **Fit a line** through the **2** points
5. Compute the **distances** of all other points **from this line**
6. Construct the inlier set (i.e. count the number of points whose distance $<$ d)
7. Store these inliers
8. **until** maximum number of iterations $\boldsymbol{k}$ reached
9. The set with the maximum number of inliers is chosen as a solution to the problem

$$k = \frac{\log(1-p)}{\log(1-w^2)}$$

RANSAC applied to General Model Fitting

1. Initial: let *A* be a set of *N* points
2. **repeat**
3. Randomly select a sample of $\boldsymbol{s}$ points from *A*
4. **Fit a model** from the $\boldsymbol{s}$ points
5. Compute the **distances** of all other points **from this model**
6. Construct the inlier set (i.e. count the number of points whose distance $<$ d)
7. Store these inliers
8. **until** maximum number of iterations $\boldsymbol{k}$ reached
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$$k = \frac{\log(1-p)}{\log(1-w^s)}$$

RANSAC applied to General Model Fitting

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9. The set with the maximum number of inliers is chosen as a solution to the problem

$$k = \frac{\log(1-p)}{\log(1-(1-\varepsilon)^s)}$$

NB: The formula is more commonly written as a function of the **fraction of outliers $\boldsymbol{\varepsilon}$**

The Three Key Ingredients of RANSAC

In order to implement RANSAC for Structure From Motion (SFM), we need three key ingredients:

1. What's the **model** in SFM?
2. What's the **minimum number of points** to estimate the model?
3. How do we compute the distance of a point from the model? In other words, can we define a **distance metric** that measures how well a point fits the model?

Answers

1. **What's the model** in SFM?
    - The **Essential Matrix** (for calibrated cameras) or the **Fundamental Matrix** (for uncalibrated cameras)
    - Alternatively, **R** and **T**

2. What's the **minimum number of points** to estimate the model?
    1. We know that 5 points is the theoretical minimum number of points for calibrated cameras
    2. However, if we use the *8-point algorithm*, then **8** is the minimum (for both calibrated or uncalibrated cameras)

3. How do we compute the **distance** of a point from the model?
    1. Algebraic error
    2. Directional error
    3. Epipolar line distance
    4. Reprojection error

Example: 8-point RANSAC applied to SFM

- Let's consider the following image pair and its image correspondences (e.g., Harris, SIFT, etc.), denoted by arrows

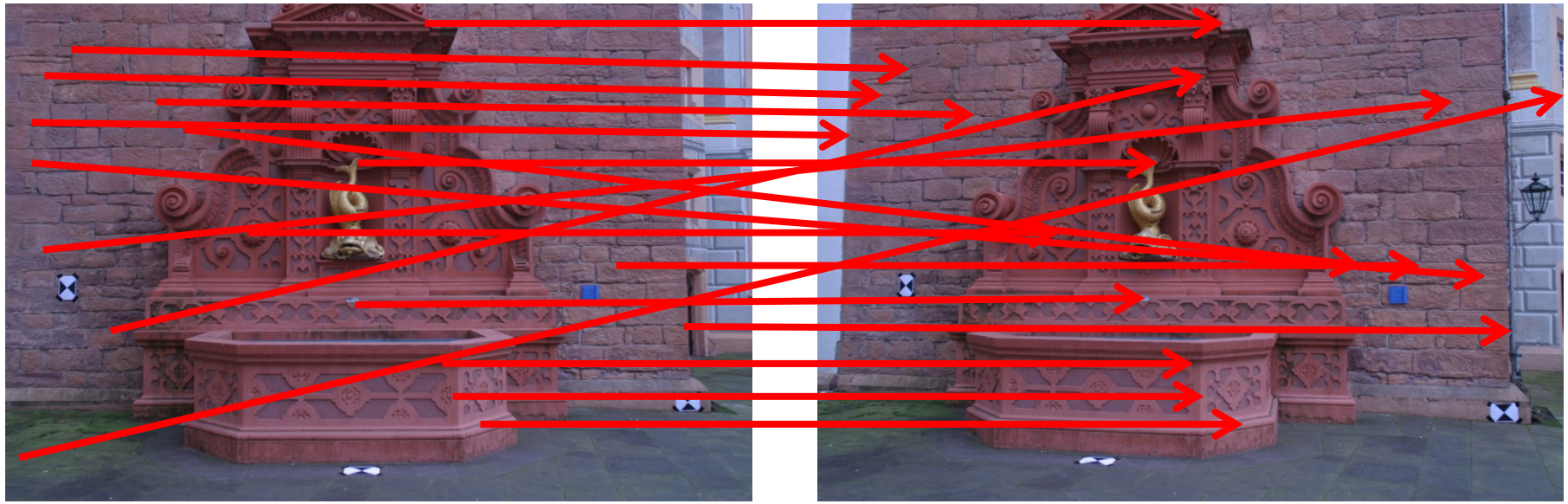
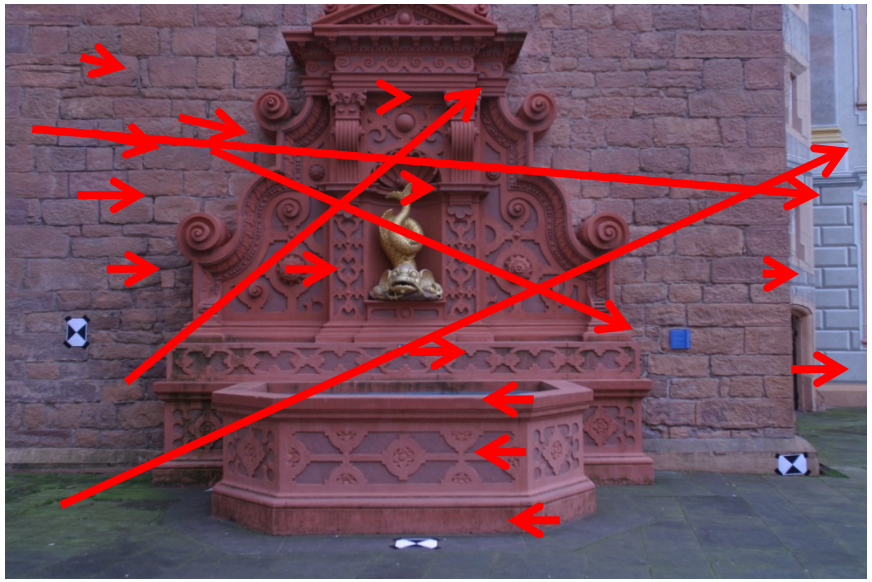


Image 1

Image 2

Example: 8-point RANSAC applied to SFM

- Let's consider the following image pair and its image correspondences (e.g., Harris, SIFT, etc.), denoted by arrows
- For convenience, we overlay the features of the second image on the first image and use arrows to denote the *motion vectors* of the features



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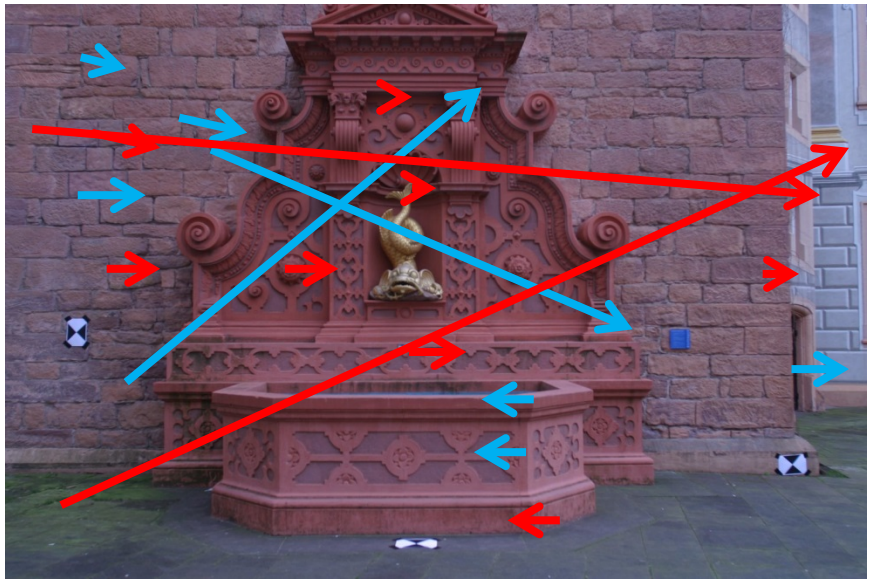
1. Randomly select 8 point correspondences and compute the model



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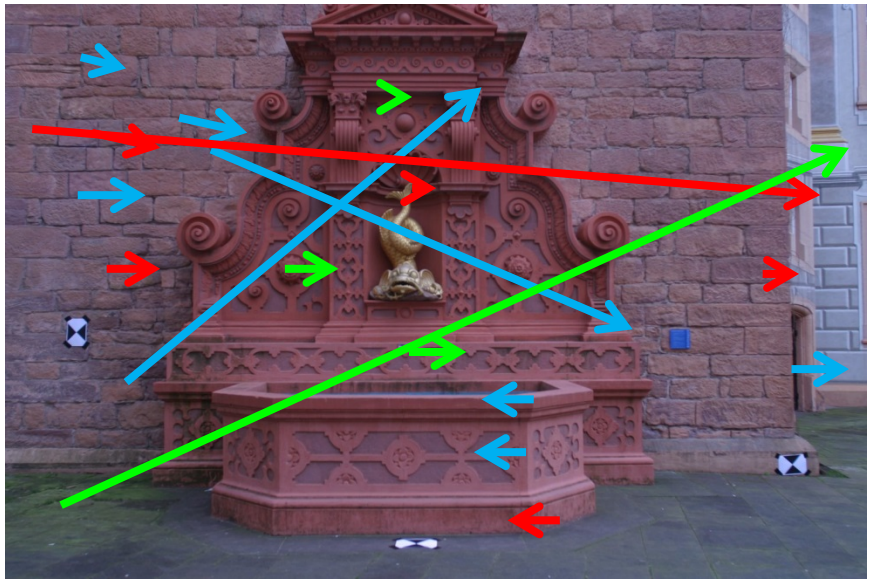
2. Compute distance of all other points from this model and count the inliers



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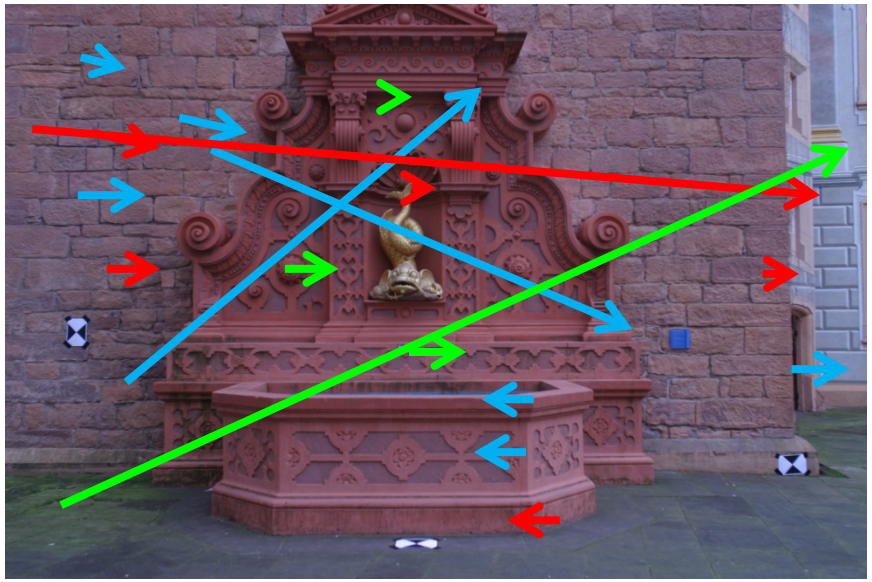
2. Compute distance of all other points from this model and count the inliers
3. Repeat from 1



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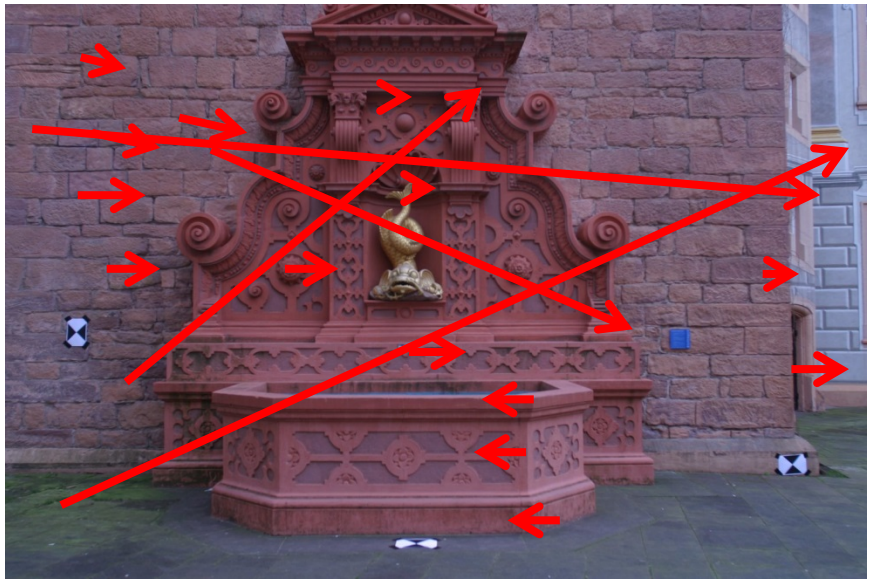

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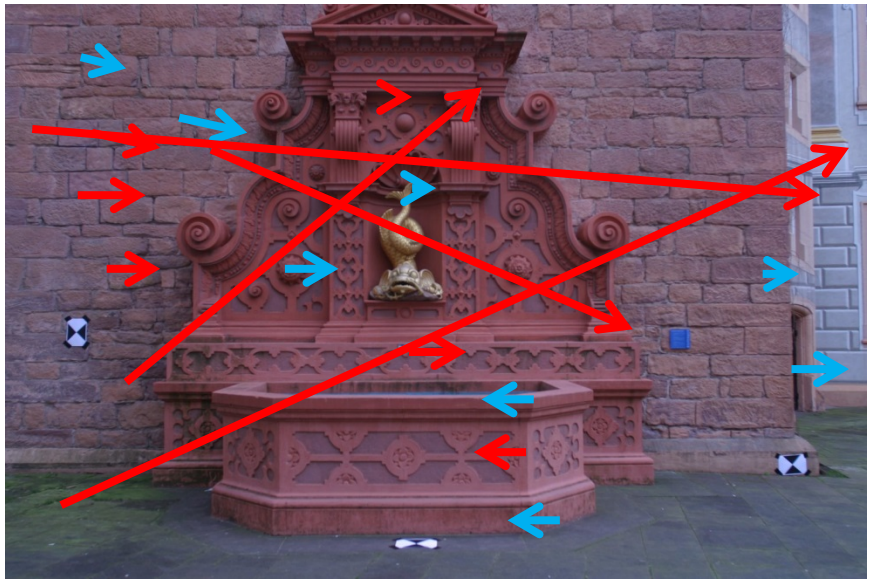

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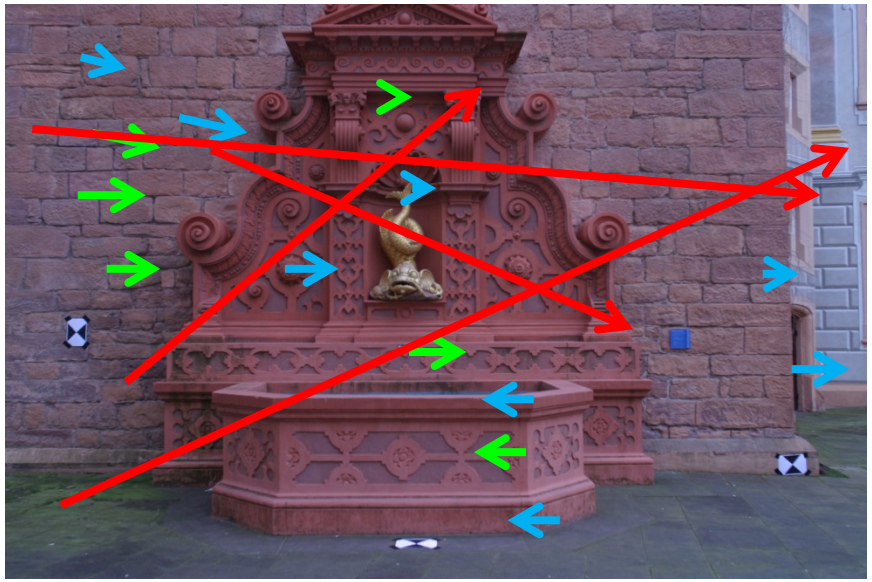

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- Let's consider the following image pair and its image correspondences (e.g., Harris, SIFT, etc.), denoted by arrows
- For convenience, we overlay the features of the second image on the first image and use arrows to denote the *motion vectors* of the features

1. Randomly select 8 point correspondences and compute the model
2. Compute distance of all other points from this model and count the inliers
3. Repeat from 1 for $\boldsymbol{k}$ times

$$k = \frac{\log(1-p)}{\log(1-(1-\varepsilon)^8)}$$

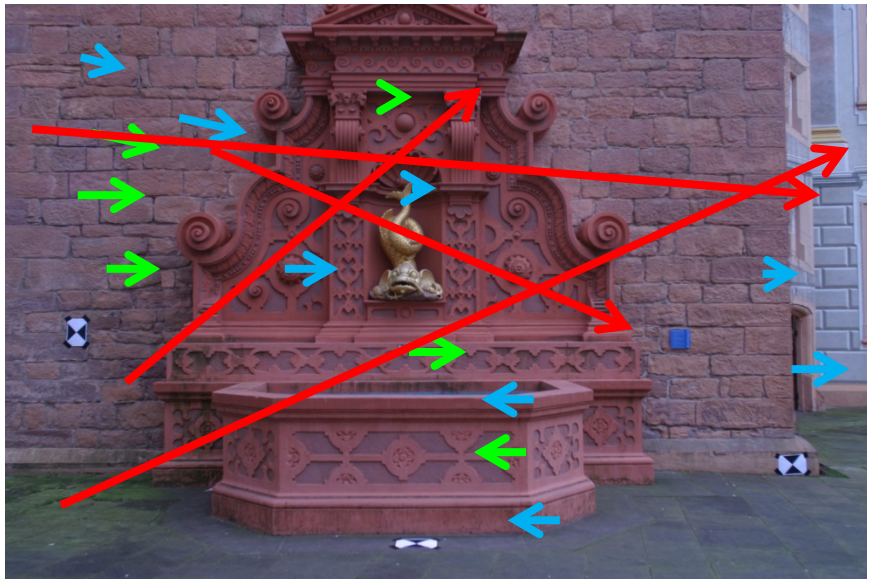


Image 1

RANSAC iterations k vs. s

k increases exponentially with the number of points s estimate the model

Let's assume p = 99% and ε = 50% (fraction of outliers):

- **8-point RANSAC**
  - s = 8 points (8-point algorithm)

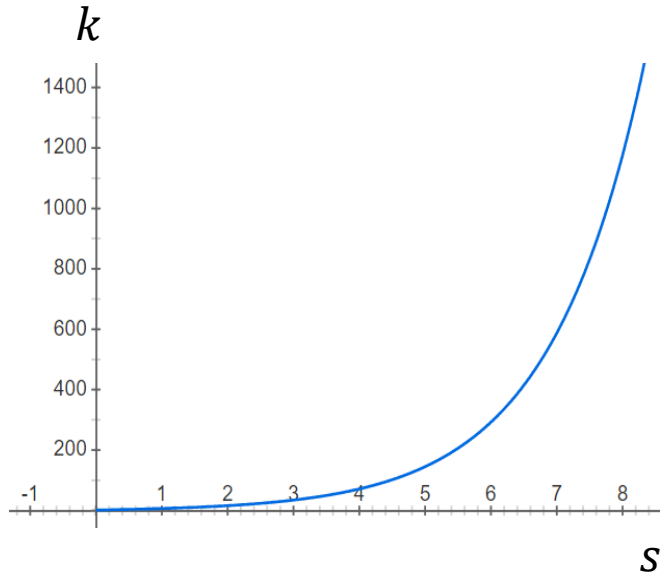
$$k = \frac{\log(1-p)}{\log(1-(1-\varepsilon)^8)} = 1177 \ \ iterations$$

- **5-point RANSAC**
  - s = 5 points (5-point algorithm)

$$k = \frac{\log(1-p)}{\log(1-(1-\varepsilon)^5)} = 145 \ \ iterations$$

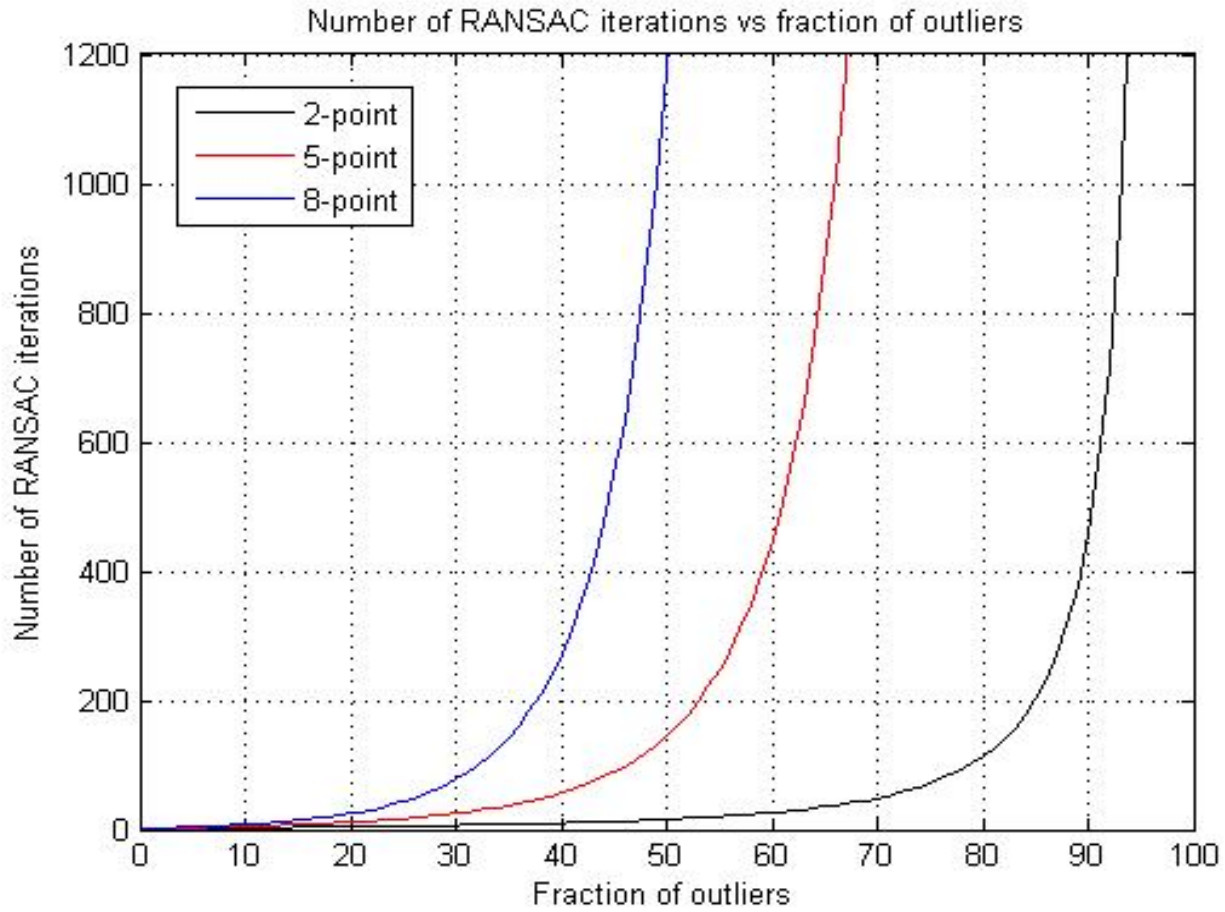
- **2-point RANSAC (e.g., line fitting)**
  - s = 2 points

$$k = \frac{\log(1-p)}{\log(1-(1-\varepsilon)^2)} = 16 \ \ iterations$$



RANSAC iterations k vs. ε

k is increases exponentially with the fraction of outliers ε:



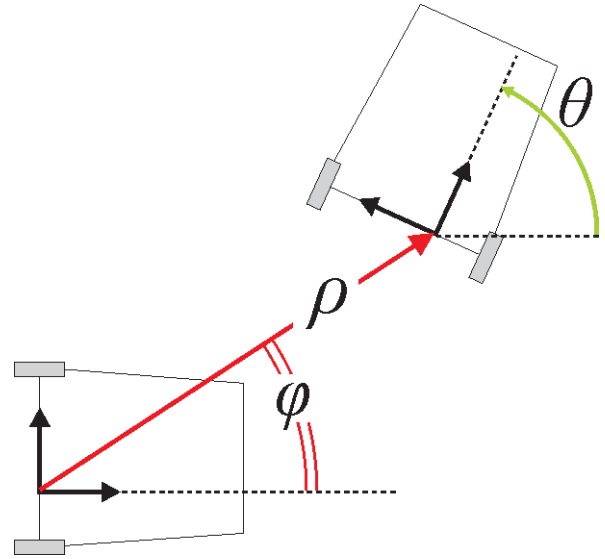
RANSAC iterations

- As observed, $\boldsymbol{k}$ is exponential with the number of points $\boldsymbol{s}$ necessary to estimate the model
- The **8-point algorithm** is extremely simple and was very successful; however, it requires more than **1177 iterations**
- Because of this, there has been a large interest by the research community in **using smaller motion parameterizations** (i.e., smaller $\boldsymbol{s}$)
- The first efficient solution to the minimal-case solution (5-point algorithm) took almost a century (Kruppa 1913 → Nister 2004)
- The **5-point RANSAC** (Nister 2004) only requires **145 iterations**; however:
  - The **5-point algorithm** can return **up to 10 solutions of E (worst case scenario)**
  - The **8-point algorithm** only returns a **unique solution of E**

Can we use less than 5 points?

Yes, if you use motion constraints!

Planar Motion



Planar motion is described by three parameters: ϑ, φ, ρ

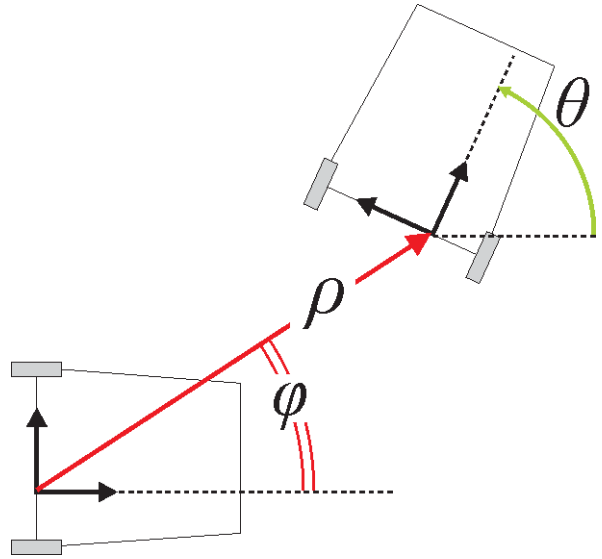
$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T = \begin{bmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ 0 \end{bmatrix}$$

Let's compute the Epipolar Geometry

$$\mathrm{E} = [T_{\times}]R$$ Essential matrix

$$\overline{p}_2^T \; E \; \overline{p}_1 = 0$$ Epipolar constraint

Planar Motion



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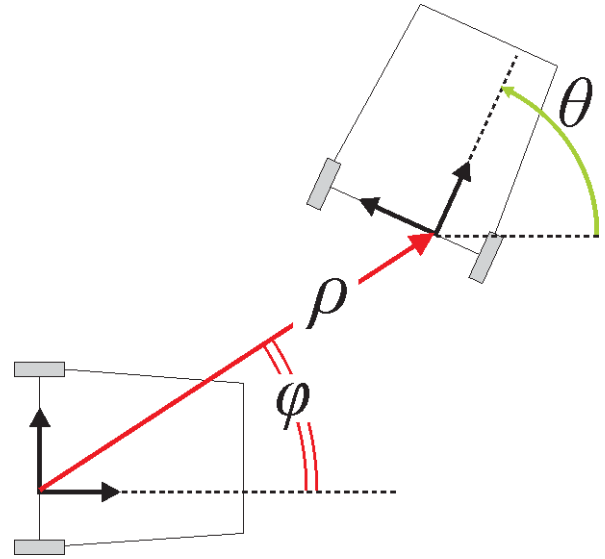
Let's compute the Epipolar Geometry

$$[T_\times] = \begin{bmatrix} 0 & 0 & \rho\sin\varphi \\ 0 & 0 & -\rho\cos\varphi \\ -\rho\sin\varphi & \rho\cos\varphi & 0 \end{bmatrix}$$

Planar Motion



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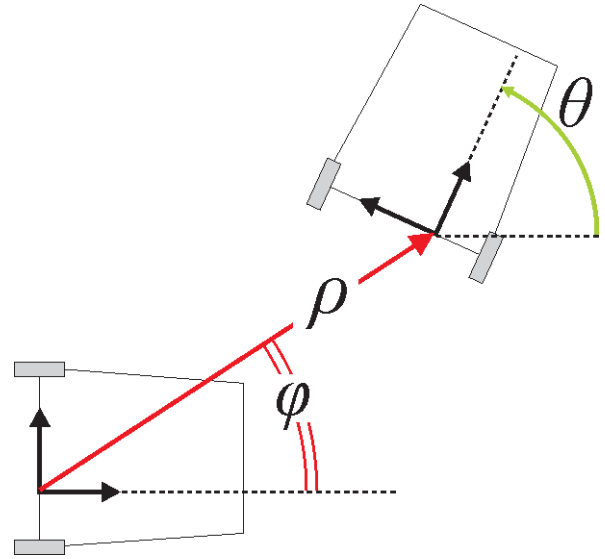
Let's compute the Epipolar Geometry

$$E = [T_\times]R = \begin{bmatrix} 0 & 0 & \rho\sin\varphi \\ 0 & 0 & -\rho\cos\varphi \\ -\rho\sin\varphi & \rho\cos\varphi & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Planar Motion



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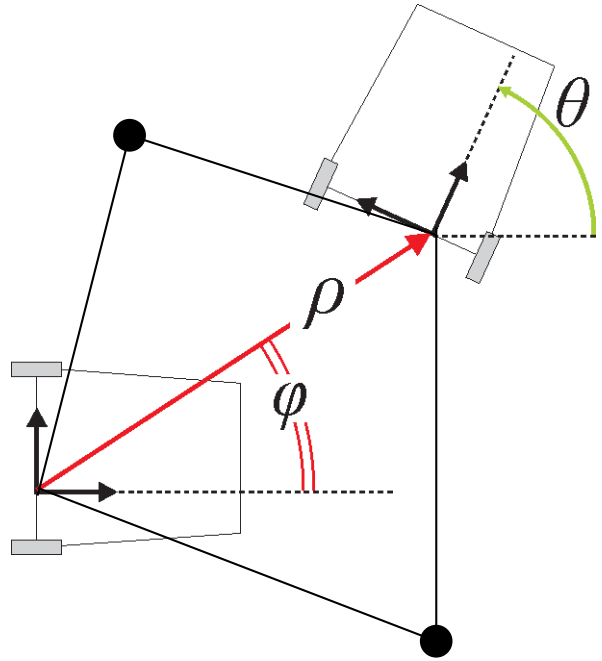
Observe that E has 2DoF (θ, φ, because ρ is the scale factor); thus, 2 correspondences are sufficient to estimate θ and φ [Ortin, 2001]

$$E = [T_\times]R = \begin{bmatrix} 0 & 0 & \rho\sin(\varphi) \\ 0 & 0 & -\rho\cos(\varphi) \\ -\rho\sin(\varphi-\theta) & \rho\cos(\varphi-\theta) & 0 \end{bmatrix}$$

"2-Point RANSAC", Ortin & Montiel, Indoor robot motion based on monocular images, Robotica, 2001. PDF.

Planar Motion



Planar motion is described by three parameters: ϑ, φ, ρ

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T = \begin{bmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ 0 \end{bmatrix}$$

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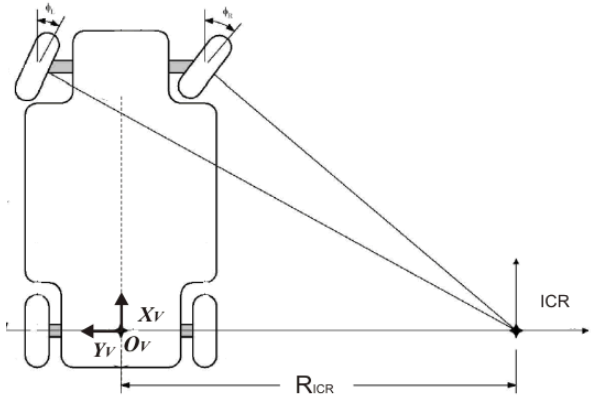
"2-Point RANSAC", Ortin & Montiel, Indoor robot motion based on monocular images, Robotica, 2001. PDF.

Less than 2 points?

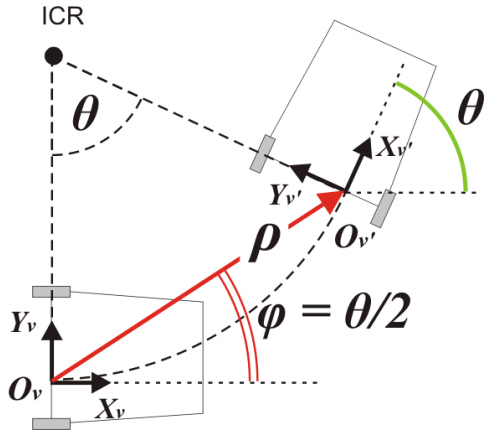
- Can we use less than 2 point correspondences?
  - Yes, if we exploit wheeled vehicles with **non-holonomic** constraints

Planar & Circular Motion (e.g., cars)

Wheeled vehicles, like cars, follow locally-planar circular motion about the Instantaneous Center of Rotation (ICR)



Example of Ackerman steering principle



Locally-planar circular motion

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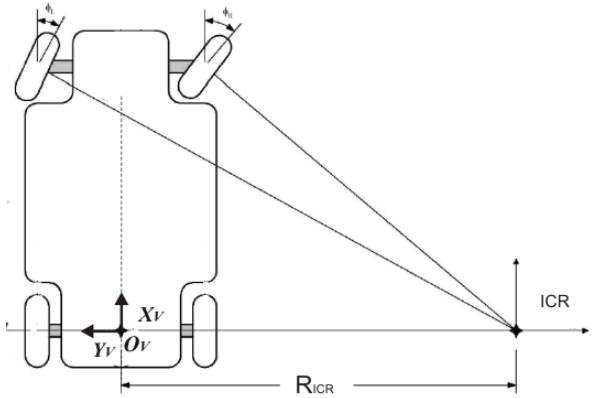

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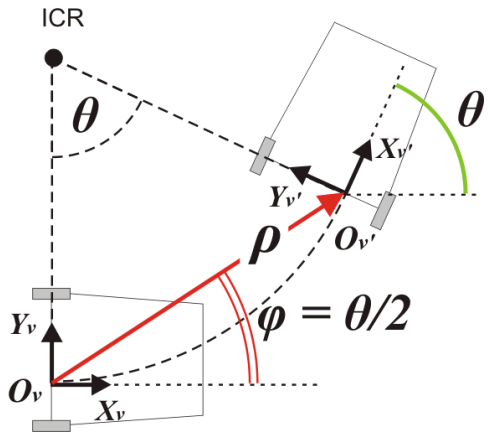


Locally-planar circular motion

$\varphi = \theta/2$ => only 1 DoF (θ); thus, **only 1 point correspondence** is sufficient [Scaramuzza, 2011]

This is the smallest parameterization possible and results in the most efficient algorithm for removing outliers

Scaramuzza, 1-Point-RANSAC Structure from Motion for Vehicle-Mounted Cameras by Exploiting Non-Holonomic Constraints, International Journal of Computer Vision, 2011. PDF.

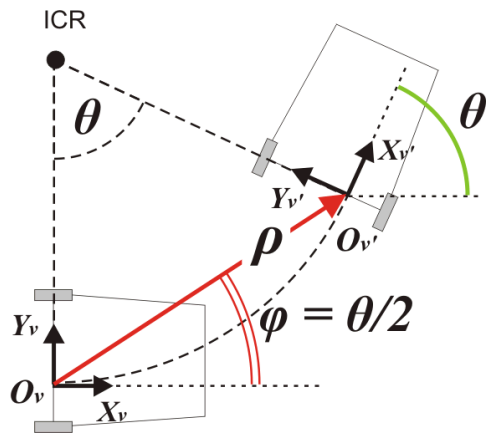
Planar & Circular Motion (e.g., cars)

Let's compute the Epipolar Geometry

$\mathrm{E} = [T_{\times}]R$ *Essential matrix*

$\overline{p}_2^T \; E \; \overline{p}_1 = 0$ *Epipolar constraint*

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} \rho\cos\frac{\theta}{2} \\ \rho\sin\frac{\theta}{2} \\ 0 \end{bmatrix}$$



Locally-planar circular motion

Scaramuzza, 1-Point-RANSAC Structure from Motion for Vehicle-Mounted Cameras by Exploiting Non-Holonomic Constraints, International Journal of Computer Vision, 2011. PDF.

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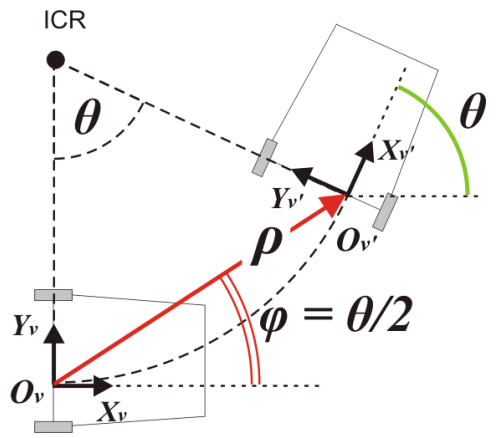


Locally-planar circular motion

$$\mathrm{E} = [T_\times]R = \begin{bmatrix} 0 & 0 & \rho\sin\frac{\theta}{2} \\ 0 & 0 & -\rho\cos\frac{\theta}{2} \\ -\rho\sin\frac{\theta}{2} & \rho\cos\frac{\theta}{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \rho\sin\frac{\theta}{2} \\ 0 & 0 & \rho\cos\frac{\theta}{2} \\ \rho\sin\frac{\theta}{2} & -\rho\cos\frac{\theta}{2} & 0 \end{bmatrix}$$

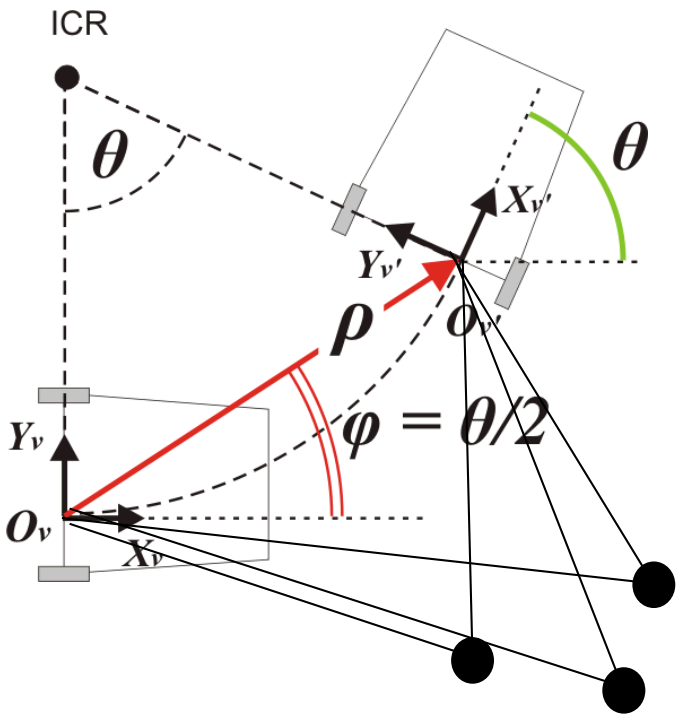
Notice that ρ can be cancelled out

$$p_2^T \; E \; p_1 = 0 \quad \Rightarrow \quad \sin\left(\frac{\theta}{2}\right)\cdot(u_2 + u_1) + \cos\left(\frac{\theta}{2}\right)\cdot(v_2 - v_1) = 0$$

$$\theta = -2\tan^{-1}\left(\frac{v_2 - v_1}{u_2 + u_1}\right)$$

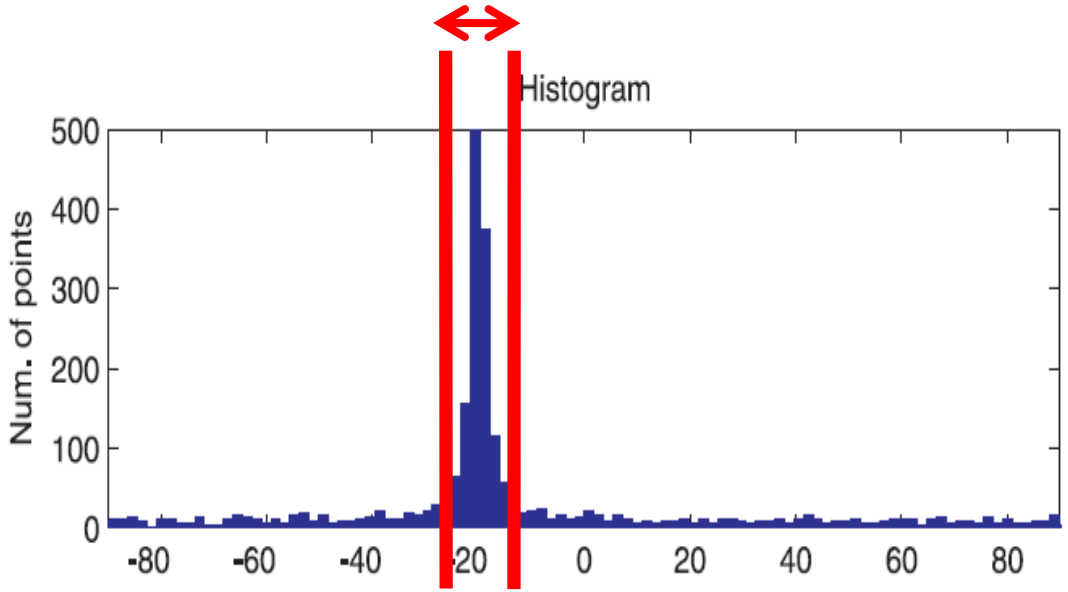
Scaramuzza, 1-Point-RANSAC Structure from Motion for Vehicle-Mounted Cameras by Exploiting Non-Holonomic Constraints, International Journal of Computer Vision, 2011. PDF.

1-Point RANSAC Algorithm



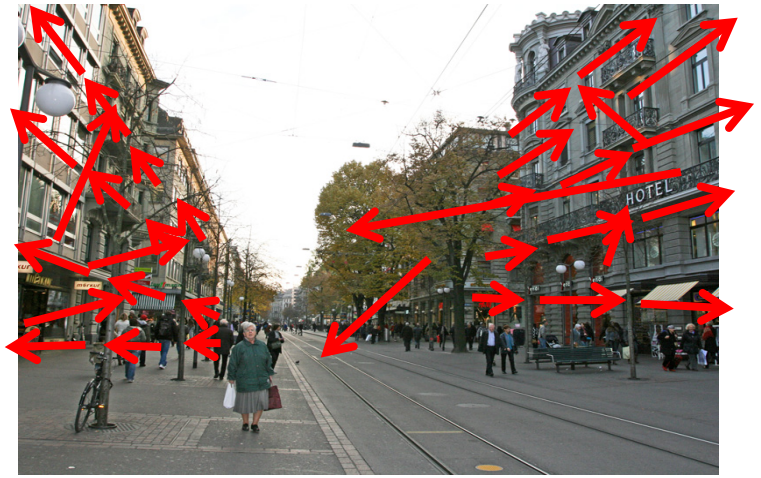
Compute ϑ for every point correspondence

$$\theta = -2\tan^{-1}\left(\frac{v_2 - v_1}{u_2 + u_1}\right)$$



Only 1 iteration!

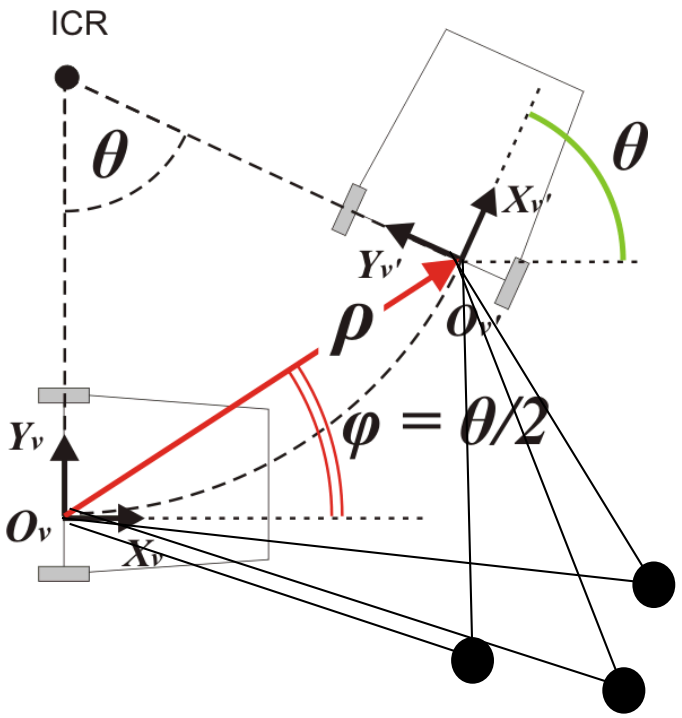
The most efficient algorithm for removing outliers (<1ms)



1-Point RANSAC is ONLY used to find the inliers.

Motion is then estimated from them in 6DOF

1-Point RANSAC Algorithm



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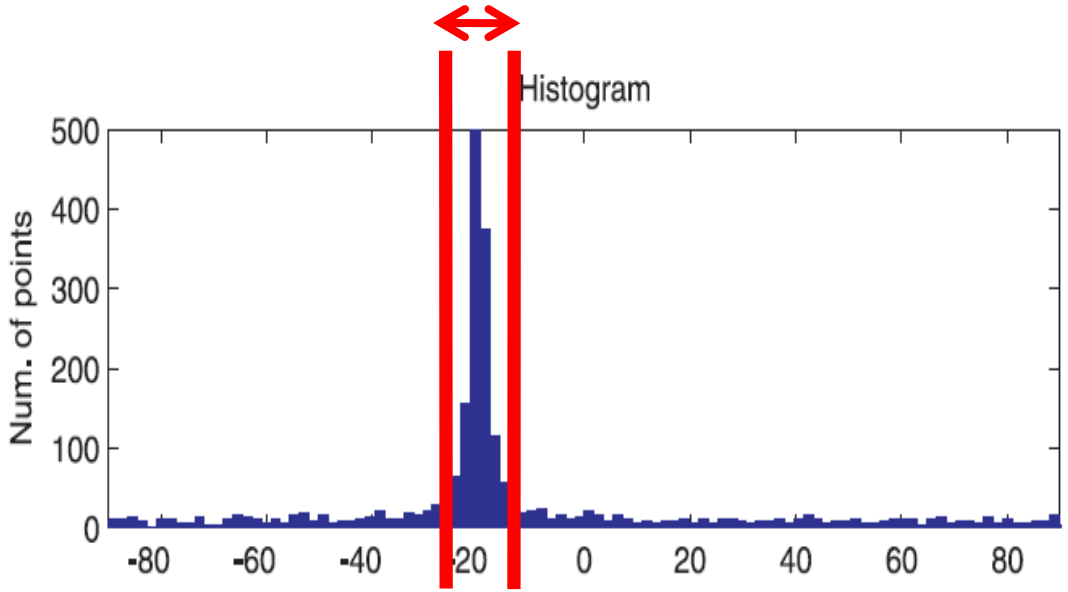

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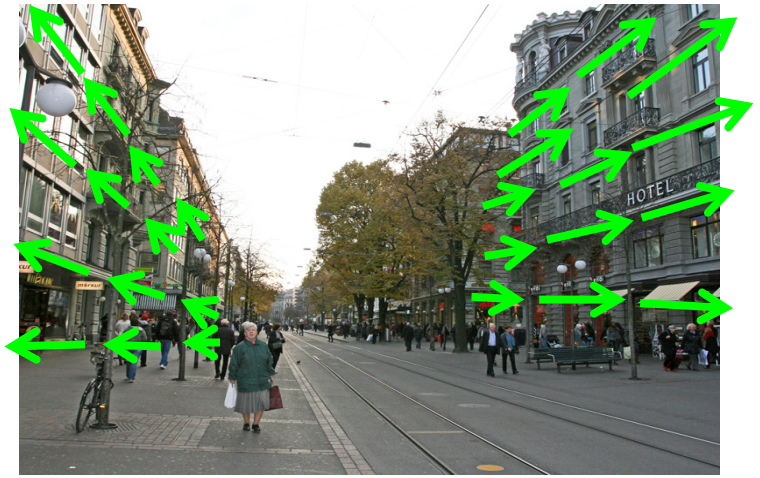

1-Point RANSAC is ONLY used to find the inliers.

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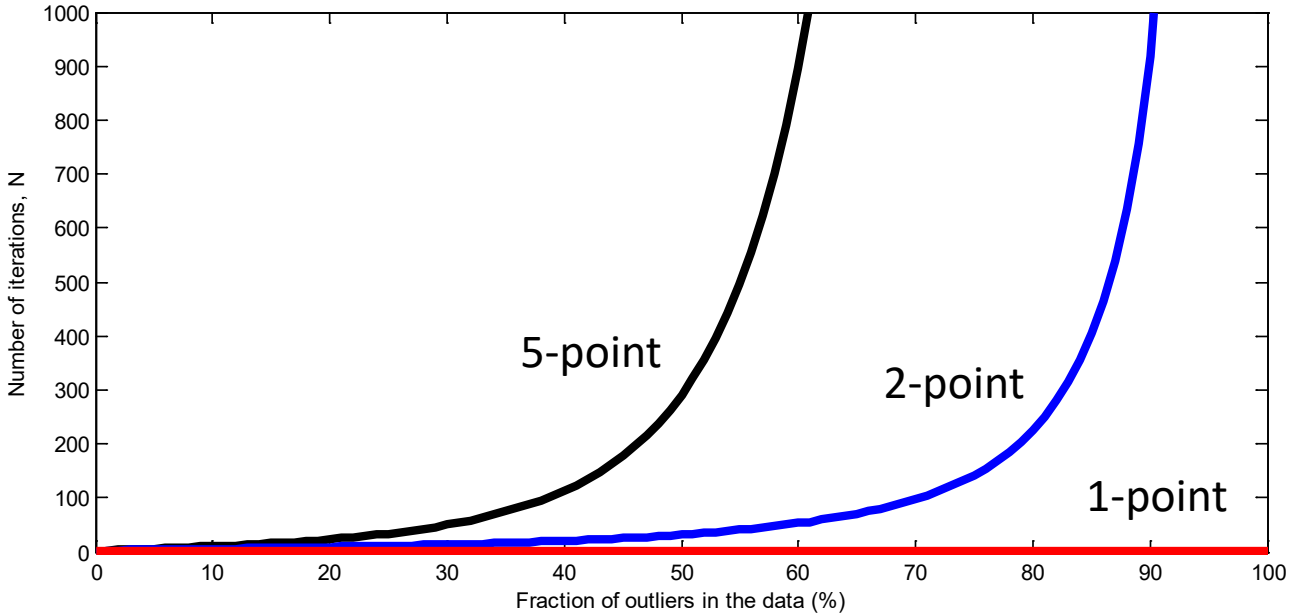
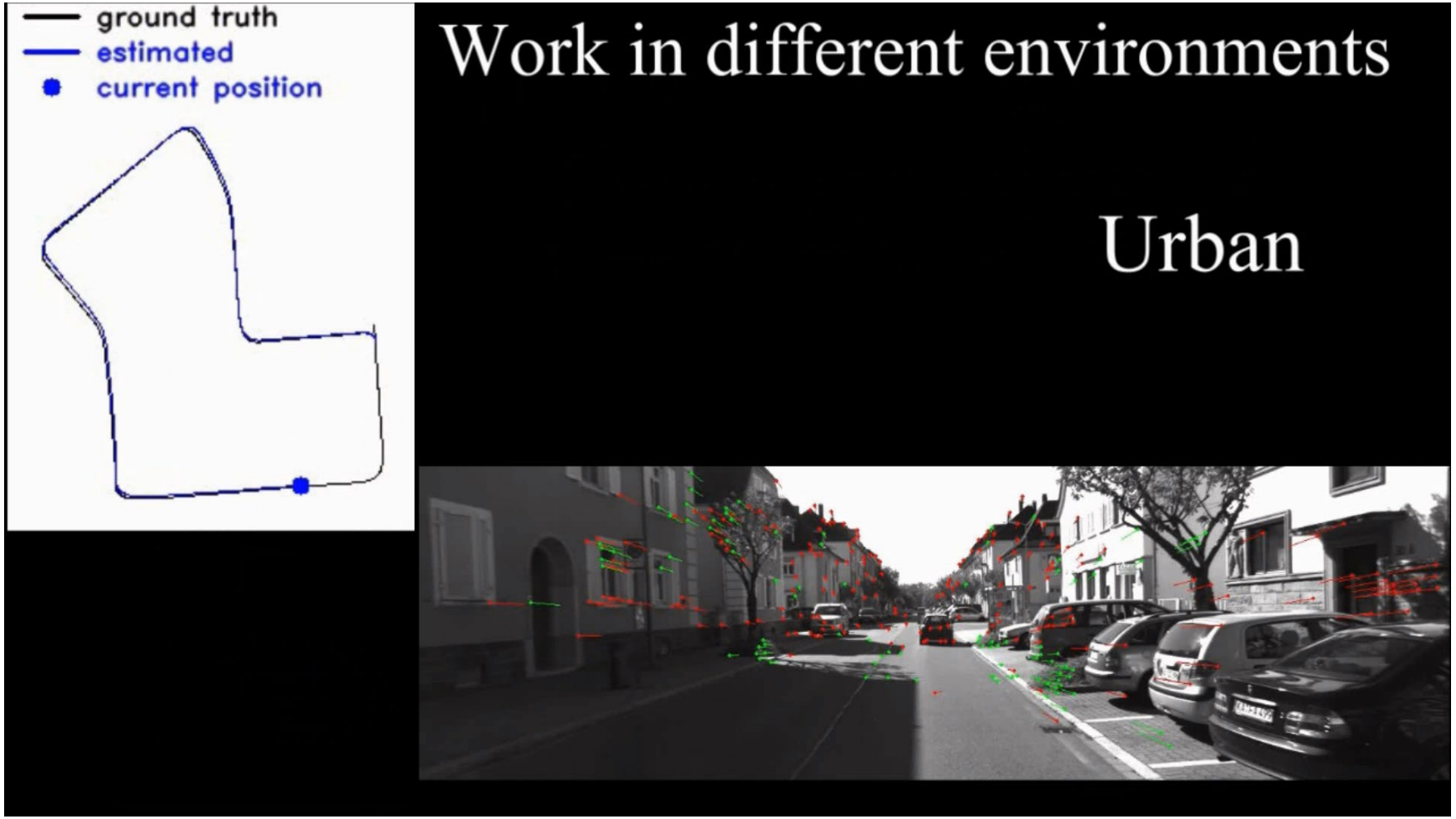
Comparison of RANSAC algorithms



$$N = \frac{\log(1-p)}{\log(1-(1-\varepsilon)^s)} \quad \text{where we typically use } p = 99\%$$

	8-Point RANSAC [Longuet-Higgins'81]	5-Point RANSAC [Nister'04]	2-Point RANSAC [Ortin'01]	1-Point RANSAC [Scaramuzza'11]
Numb. of iterations	**> 1177**	**>145**	**>16**	**=1**

Visual Odometry with 1-Point RANSAC



Scaramuzza, 1-Point-RANSAC Structure from Motion for Vehicle-Mounted Cameras by Exploiting Non-Holonomic Constraints, International Journal of Computer Vision, 2011. PDF.

Latest and Greatest ☺

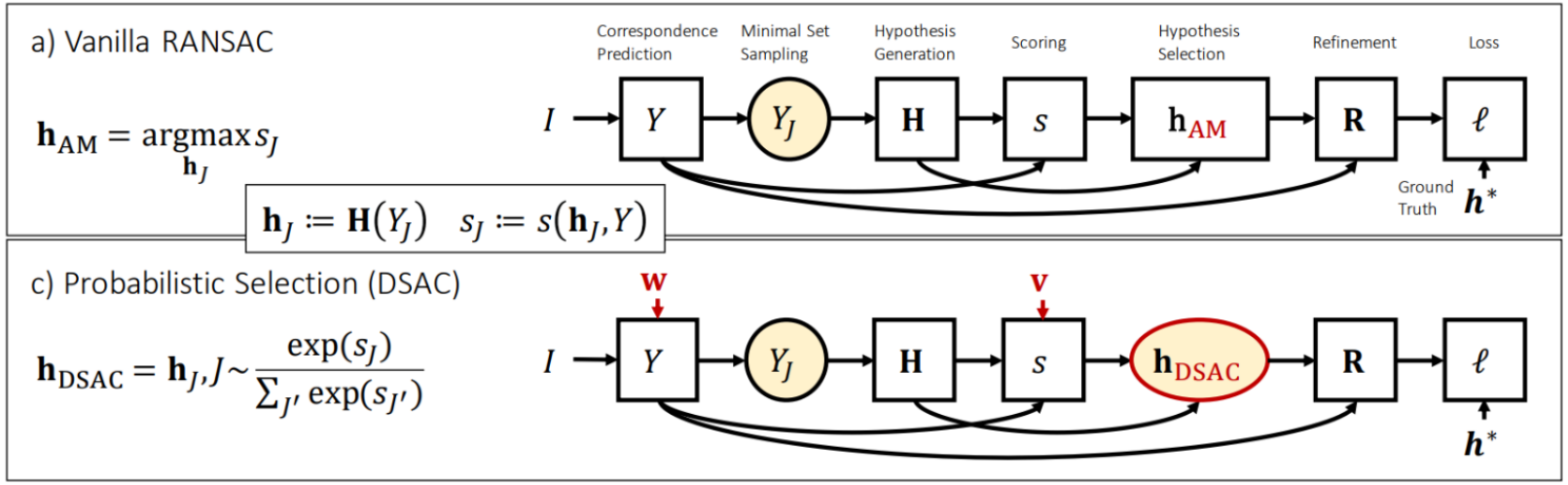
Differentiable RANSAC

This will be covered in Lecture 12c

- RANSAC is not differentiable since it relies on selecting a hypothesis based on maximizing the number of inliers (i.e., argmax).
- DSAC shows how sample consensus can be used in a differentiable way
- This enables the use of sample consensus in a variety of learning tasks.

Choose the best based on score

Randomly sample based on score

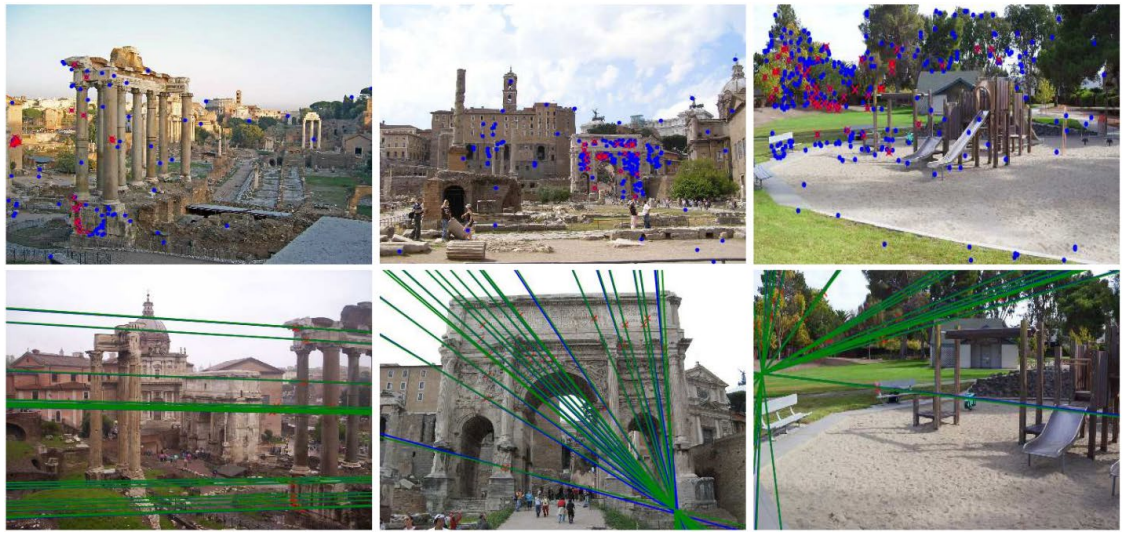


E. Brachmann et al., DSAC - Differentiable RANSAC for Camera Localization, International Conference on Computer Vision and Pattern Recognition (CVPR), 2017. PDF. Video.

Deep Fundamental Matrix Estimation

This will be covered in Lecture 12c

- **Input**: two sets of noisy local features (coordinates + descriptors) contaminated by outliers
- **Output**: fundamental matrix
- **Idea**: solve a weighted homogeneous least-squares problem, where robust weights are estimated using deep networks
- **Robust**: handles extreme wide-baseline image pairs



Top-bottom as image-pair

Red: inlier correspondences
Blue: outlier correspondences

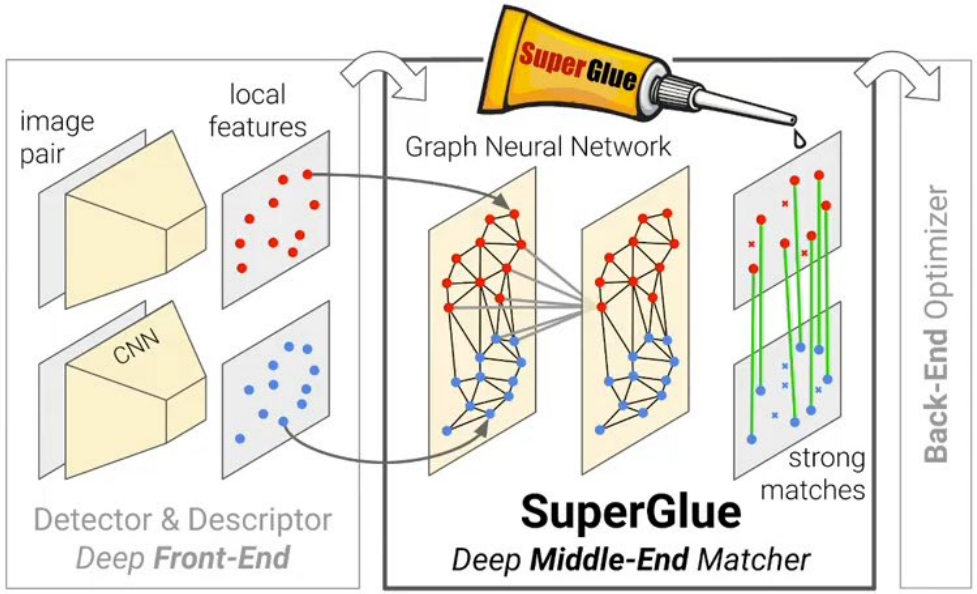
Epipolar lines

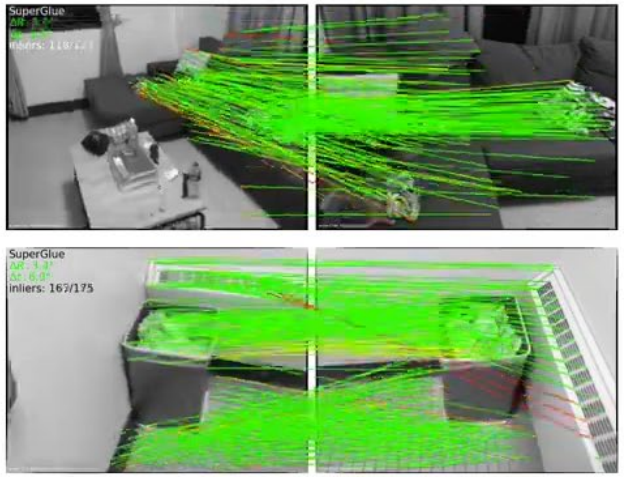
Green: estimated
Blue: ground-truth

Ranftl, Koltun, *Deep Fundamental Matrix Estimation*, European Conference on Computer Vision (ECCV), 2018. PDF.

SuperGlue: Learning Feature Matching with Graph Neural Networks

- **Input**: two sets of noisy local features (coordinates + descriptors) contaminated by outliers
- **Output**: strong & outlier-free matches
- **Combines deep learning with classical optimization** (Graph Neural Networks, Attention, Optimal Transport
- **Robust**: handles extreme wide-baseline image pairs





Sarlin, DeTone, Malisiewicz, Rabinovich, *SuperGlue: Learning Feature Matching with Graph Neural Networks*, International Conference on Computer Vision and Pattern Recognition (CVPR), 2020. PDF. Code.

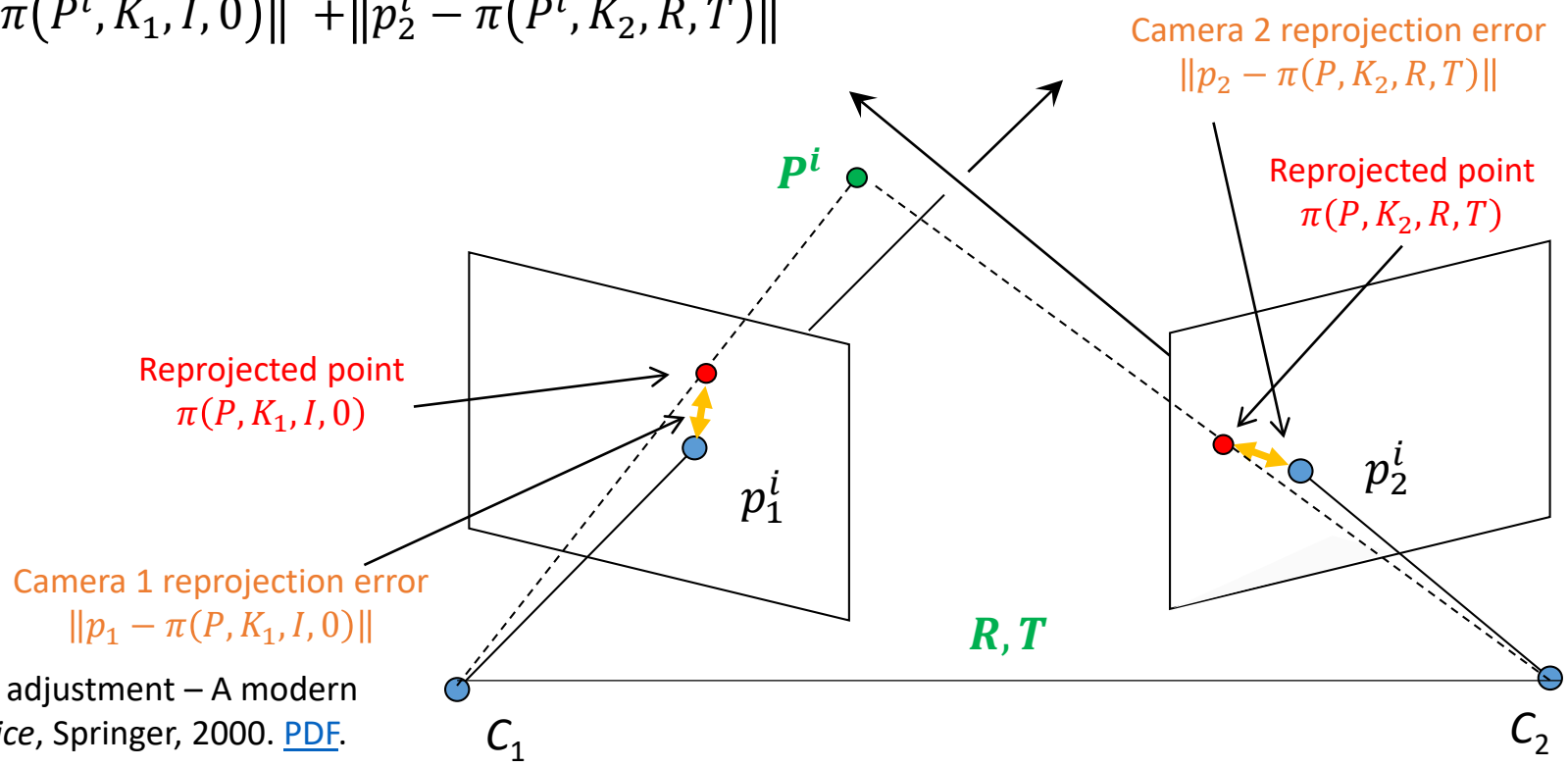
Outline

- Robust Structure from Motion
- Bundle Adjustment

2-View Bundle Adjustment (BA)

- **Non-linear, joint optimization of structure, P^i, and motion R, T**
- Commonly used after least square estimation of R and T (e.g., after 8- or 5-point algorithm)
- Optimizes P^i, R, T by minimizing the **Sum of Squared Reprojection Errors**:

$$P^i, R, T = argmin_{P^i,R,T} \sum_{i=1}^{N} \left\| p_1^i - \pi(P^i, K_1, I, 0) \right\|^2 + \left\| p_2^i - \pi(P^i, K_2, R, T) \right\|^2$$



Triggs, McLauchlan, Hartley, Fitzgibbon, Bundle adjustment – A modern synthesis, *Vision Algorithms: Theory and Practice*, Springer, 2000. PDF.

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Good to know:

- Like in the formula, we typically assume the first camera as the world frame, but it's arbitrary
- Occasionally, the residual terms are weighted
- In order to not get stuck in local minima, the **initial values of P^i, R, T should be close to the optimum**
- Can be minimized using **Levenberg–Marquardt** (more robust than Gauss-Newton to local minima)
- **Can be modified to also optimize the intrinsic parameters**
- Implementation details in **Exercise 9**

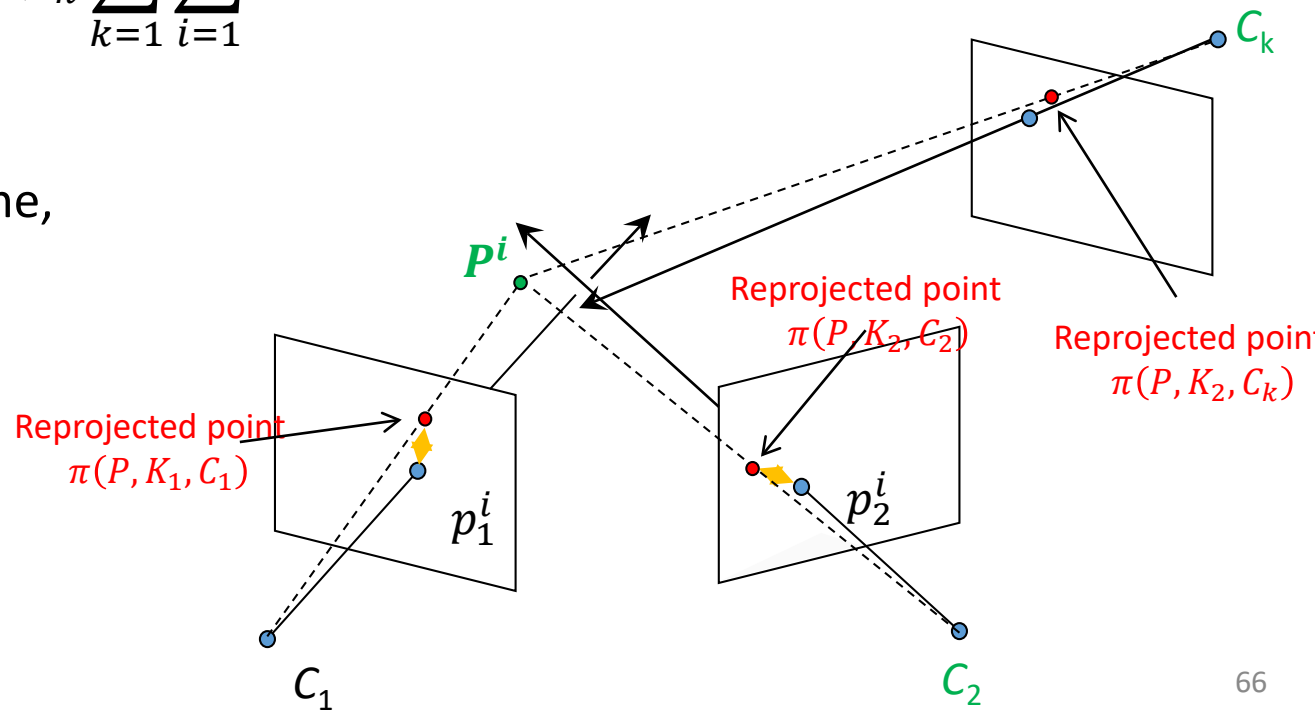
What is the key difference with the reprojection error minimization seen in previous lectures (03 and 07)?

n-View Bundle Adjustment (BA)

- **Non-linear, joint optimization of structure, P^i, and camera poses $C_1 = [I, 0], \dots, C_k = [R_k, T_k]$**
- Minimizes the Sum of Squared Reprojection Errors **across all views**

$$P^i, C_2, \dots, C_n = argmin_{P^i, C_2, \dots, C_n} \sum_{k=1}^{n} \sum_{i=1}^{N} \left\| p_k^i - \pi\left(P^i, K_k, C_k\right) \right\|^2$$

- **NB**: we assume the first camera as the world frame, that's why $C_1 = [I, 0]$



Triggs, McLauchlan, Hartley, Fitzgibbon, Bundle adjustment – A modern synthesis, *Vision Algorithms: Theory and Practice*, Springer, 2000. PDF.

Huber and Tukey Norms

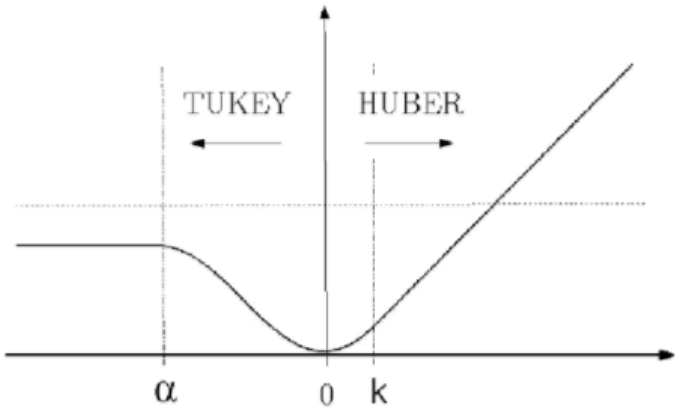
- To prevent that large reprojection errors can negatively impact the optimization, a more robust norm ρ( ) is used instead of the L_2:

$$P^i, C_2, \dots, C_n = argmin_{P^i, C_2, \dots, C_n} \sum_{k=1}^{n} \sum_{i=1}^{N} \rho\left(p_k^i - \pi\left(P^i, K_k, C_k\right)\right)$$

- ρ( ) is a robust cost function (**Huber or Tukey**) to alleviate the contribution of wrong matches:

- **Huber norm**: $\rho(x) = \begin{cases} x^2 & \text{if } |x| \le k \\ k(2|x| - k) & \text{if } |x| \ge k \end{cases}$

- **Tukey norm**: $\rho(x) = \begin{cases} \alpha^2 & \text{if } |x| \ge \alpha \\ \alpha^2 \left(1 - \left(1 - \left(\frac{x}{\alpha}\right)^2\right)^3\right) & \text{if } |x| \le \alpha \end{cases}$



These formulas are not asked at the exam but their plots and meaning is asked ☺

Things to remember

- EM algorithm
- RANSAC algorithm and its application to SFM
- 8 vs 5 vs 1 point RANSAC, pros and cons
- Bundle Adjustment

Reading

- CH. 8.1.4, 8.3.1, 11.3 of Szeliski book, 2nd edition
- Ch. 14.2 of Corke book

Understanding Check

Are you able to answer the following questions?

- What are the causes of outliers?
- What effects may outliers have on VO?
- How does EM work? What are the issues?
- Why do we need RANSAC?
- What is the theoretical maximum number of combinations to explore?
- After how many iterations can RANSAC be stopped to guarantee a given success probability?
- What is the trend of RANSAC vs. iterations, vs. the fraction of outliers, vs. the number of points to estimate the model?
- How do we apply RANSAC to the 8-point algorithm, DLT, P3P?
- How can we reduce the number of RANSAC iterations for the SFM problem? (1- and 2-point RANSAC)
- Bundle Adjustment. Mathematical expression and illustration. Tukey and Huber norms.