Introduction to Discrete Structures

Mark R. Budden

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Preface

These notes are intended to support Math 310 (Discrete Structures) during Spring 2024 at Western Carolina University. The course textbook (*Discrete Mathematics with Ducks,* by Sarah-Marie Belcastro) will serve as a general reference, but these notes will be self-contained and all assigned homework will come from the Exercises in these notes. Discrete Structures essentially serves as an introduction to combinatorics: the study of counting.

Enumeration will be the focus of Chapter 1. We will discuss basic counting methods, considering each method based on what conditions are placed on order and repetition. Combinations will lead us to the Binomial Theorem and its related identities. Generating functions will prove to be a useful tool for more complicated counting problems and we will investigate their use in solving recurrence relations.

Graph Theory is a branch of combinatorics that has become prominent in the last century. Roughly two-thirds of the semester will be spent developing the theory of graphs, often using earlier enumeration methods to study the graphs at hand. Chapter 2 will include an introduction to the theory of graphs: terminology, substructures, chromatic numbers and other forms of enumeration. Chapter 2 will also develop the significant role that trees (minimally connected graphs) have on the subject and their various applications.

Finally, Chapter 3 will provide an excursion into extremal combinatorics, in which one is able to precisely determine how large a structure must be to guarantee the presence of certain properties. It is unlikely that all of the topics in Chapter 3 can be covered in a single semester, but the topics here have been chosen to provide options based on student interest.

Of course, good proof-writing skills will be fundamental to learning the material contained in these notes and will be expected in the successful completion of the exercises contained here. Along with the exercises, there will occasionally be “bonus exercises” which will not be required for the course, but will serve the motivated student to learn the subject beyond the standard course content.

The main prerequisites for these notes include the topics usually taught in an introductory course on proof and logic. Such background includes set theory, logical statements, functions, and basic methods of proof-writing, including induction.

Mark R. Budden

CHAPTER 1

Counting Methods

1.1. Preliminaries

In the beginning of the 1900's, there was a shift in mathematics in which rigor became expected and the need for universal notations led to the development of set theory. Naïvely, a *set* is a (well-defined) collection of elements. Of course, in most branches of mathematics, we are concerned with more than just the elements in a set. We study the ways in which those elements interact. Number systems are examples of such sets that involve operations, and sometimes ordered relationships. We start by discussing the main number systems (and their symbols) that typically arise in undergraduate mathematics.

Perhaps the first number system utilized by humans was the set of *natural numbers* $\mathbb{N} := \{1, 2, 3, \dots\}$, as it serves as the basis for counting. Elements in $\mathbb{N}$ are ordered and the system is closed under the operations of addition and multiplication. That is, adding or multiplying any two elements in $\mathbb{N}$ always results in another element in $\mathbb{N}$. Commutativity, associativity, and the distributive properties are also satisfied with regard to addition and multiplication in $\mathbb{N}$, but algebraic issues begin to arise when one seeks out an additive identity element and additive inverses.

The algebraic issues with addition in $\mathbb{N}$ are resolved by introducing the number 0 and the negatives of the natural numbers. The result is the set of *integers*

$$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Here, the symbol $\mathbb{Z}$ comes from the word "Zahlen," which translates to "numbers" in German. Additively, $\mathbb{Z}$ forms what is known as a "group" in abstract algebra, but its multiplicative structure is much more complicated. The branch of mathematics known as number theory focuses primarily on studying the building blocks of the multiplicative structure of $\mathbb{Z}$ (that is, the prime numbers).

One algebraic issue that arises in $\mathbb{Z}$ is the lack of multiplicative inverses for all nonzero elements. This issue is resolved with the introduction of fractions. The *rational numbers* are defined by

$$\mathbb{Q} := \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

Here, the symbol $\mathbb{Q}$ comes from the word "quotient." Algebraically, the rational numbers form a "field," and hence, satisfy most desirable algebraic properties.

When one begins to look at the analytic structure of $\mathbb{Q}$, limits of sequences of such numbers do not always lie within $\mathbb{Q}$. Addressing this issue leads to the creation of the *real numbers* $\mathbb{R}$. With limits as a tool, the theory of calculus can be developed in this setting, but not without one major algebraic issue: not all polynomials with coefficients in $\mathbb{R}$ can be factored into linear factors whose coefficients are also in $\mathbb{R}$. Addressing this final issue results in the *complex numbers*, defined by

$$\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}.$$

Each of the number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ have their role in various branches of mathematics. In discrete mathematics, we focus on mathematical structures that are distinct and separable. While the limitation is not exclusive, this means we primarily use the number systems $\mathbb{N}$ and $\mathbb{Z}$ as two distinct numbers selected from either of these systems are separated by minimum distances.

In all areas of mathematics, one defines the objects of interest, then assumes some basic axioms upon which to build a theory. The Well-Ordering Principle is one such axiom, and as such, cannot be proved with only the usual assumed properties of the number systems $\mathbb{N}$ and $\mathbb{Z}$.

AXIOM 1.1 (Well-Ordering Principle). *Every nonempty subset of* $\mathbb{N}$ *contains a least element.*

Note that it is not much of a stretch to extend this axiom to all nonempty subsets of $\{n \in \mathbb{Z} \mid n \geq n_0\}$ for a fixed $n_0 \in \mathbb{Z}$, and we will often use this variation. It will serve as the reason why many of the graphical parameters considered in Chapters 2 and 3 are well-defined. It is also worth noting that the Well-Ordering Principle is logically equivalent to the Principle of Mathematical Induction (i.e., assuming either one of them, it is possible to prove the other one). By stating the Well-Ordering Principle as an axiom, we may use induction in our proofs.

The first result that we will prove is one of the most useful principles of counting: the Pigeonhole Principle. Unlike an axiom, it can be proved, so we state it as a theorem.

THEOREM 1.1 (Pigeonhole Principle). *Let* $n, k \in \mathbb{N}$ *satisfy* $n > k$. *If we distribute* n *objects (pigeons) into* k *compartments (pigeonholes), then there will be at least one compartment that contains at least two objects.*

PROOF. We prove this statement using proof by contradiction. Suppose false, then each of the k compartments has at most one object in it. In this case, the total number of objects is at most $k < n$, giving a contradiction. Thus, at least one compartment contains more than one object. $\square$

EXAMPLE 1.2. A chess tournament has $n \geq 2$ participants and any two players play exactly one game against each other. Show that at any point in time, there are two players that have finished the same number of games.

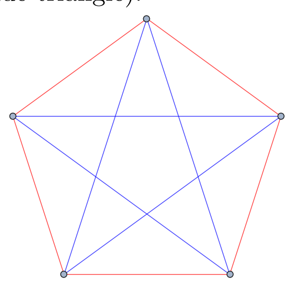
Solution: At any given time, a player has completed $0, 1, 2, \ldots, n-1$ games (there are n possibilities). If any player has completed $n-1$, then it is not possible that another player has completed 0 games. In this case, there are only $n-1$ possibilities for the number of games completed by each player. Since there are n players, the Pigeonhole Principle guarantees that at least two players have completed the same number of games. Otherwise, no player has completed $n-1$ games, and again, there are only $n-1$ possibilities for the number of games each player has completed. The same argument applies and we find that in all cases, there exists at least two players that have completed the same number of games.

THEOREM 1.3 (General Pigeonhole Principle). *Let $n, m, r \in \mathbb{N}$ satisfy $n > rm$. If we distribute n objects into m compartments, then there is at least one compartment that contains at least $r + 1$ objects.*

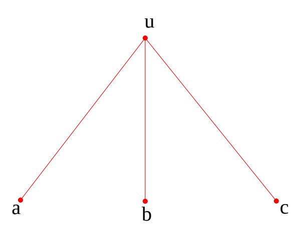
The proof of the General Pigeonhole Principle follows the same approach as the proof in the simpler version and we leave it as an exercise (see Exercise 1.1.1). As an example, we turn to a standard problem in Ramsey theory.

EXAMPLE 1.4. What is the minimum number of people that must be at a gathering to guarantee that there are either three mutual acquaintances or three mutual strangers? Note that we are assuming that the property of "knowing someone" is symmetric. That is, Alice knows Bob if and only if Bob knows Alice.

Solution: This problem becomes easier to visualize if we represent a gathering using a graph: people are represented with vertices and edges between two people are colored red if the corresponding people are acquaintances and blue if they are strangers. For example, the following graph demonstrates a scenario in which a gathering of five people lacks both three mutual acquaintances (a red triangle) and three mutual strangers (a blue triangle).



So, we find that five people are not enough! What about six? In a gathering of six people, a fixed person (which we represent with vertex u) is adjacent with five other people via either red or blue edges. By the general version of the Pigeonhole Principle, at least three of the edges incident with u must be the same color (we are placing five edges (pigeons) into two colors (pigeon holes)). Without loss of generality, assume that u is incident with at least three red edges, as pictured below.



If edge ab is red, then uab forms a red triangle. Similarly, if either bc or ac are red, we obtain a red triangle. The only remaining possibility is that all three edges ab, bc, and ac are blue, forming a blue triangle. Hence, the answer to the original question is six, since we have shown that every gathering of six (or more) people must contain either three mutual acquaintances or three mutual strangers.

We conclude this section with two important counting principles: the Addition and Multiplication Principles.

The Addition Principle *Let $S_1, S_2, \ldots, S_n$ be a finite collection of pairwise disjoint sets. Then the total number of elements in $S_1 \cup S_2 \cup \cdots \cup S_n$ is given by the sum of the cardinalities of the underlying sets:*

$$|S_1 \cup S_2 \cup \cdots \cup S_n| = |S_1| + |S_2| + \cdots + |S_n|.$$

Recall that if $S_1, S_2, \ldots, S_n$ is a finite list of sets, then the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$ is the set of ordered n-tuples

$$S_1 \times S_2 \times \cdots \times S_n := \{(s_1, s_2, \ldots, s_n) \mid s_i \in S_i \text{ for each } 1 \leq i \leq n\}.$$

When counting the elements in a Cartesian product, the choice of an element from a given S_i is independent of the elements chosen from other underlying sets, resulting the following general counting principle.

The Multiplication Principle *Let $S_1, S_2, \ldots, S_n$ be a finite collection of sets. Then the number of elements in $S_1 \times S_2 \times \cdots \times S_n$ is given by*

$$|S_1 \times S_2 \times \cdots \times S_n| = |S_1| \cdot |S_2| \cdot \cdots \cdot |S_n|.$$

Example 1.5. A shelf contains 18 books: 8 different algebra books, 6 different biology books, and 4 different chemistry books.

(a) How many ways are there to select three books, one from each subject?
Solution: Let A denote the set of algebra books, B denote the set of biology books, and C denote the set of Chemistry books. Then selecting three books, one from each subject, is equivalent to selecting a single element from $A \times B \times C$. By the Multiplication Principle, there are

$$|A \times B \times C| = |A| \cdot |B| \cdot |C| = 8 \cdot 6 \cdot 4 = 192$$

ways to make such a choice.

(b) How many ways are there to make a row of three books in which exactly one subject is missing? (Note that the order of the books matters.)
Solution: Let S_A be the set of rows of three books that lack algebra (but contain both biology and chemistry). Define S_B and S_C in a similar manner. Note that these three sets are disjoint. Hence, by the Addition Principle, the number of rows of three books that lack a single subject is given by

$$|S_A| + |S_B| + |S_C|.$$

To determine the cardinality of S_A, note that the order of the books matters and it is possible to have one biology book and two chemistry books, or to have two biology books and one chemistry book. So, the following arrangements are possible:

$$BCC,\ CBC,\ CCB,\ CBB,\ BCB,\ BBC.$$

Also, any arrangement that falls into one of these categories does not fall into the others (the type of arrangements are disjoint sets and we can again apply the Addition Principle). Counting the number of arrangements that fall into each category then makes use of the Multiplication Principle. Thus, there are

$$3 \cdot (6 \cdot 4 \cdot 3) + 3 \cdot (4 \cdot 6 \cdot 5) = 576$$

such arrangements. Using these same ideas, we find that

$$|S_B| = 960 \quad \text{and} \quad |S_C| = 1728,$$

giving a total of 3264 rows.

The key to applying the Multiplication Principle is that the choice made when selecting an element within a given S_i is independent of the choice made when selecting an element from S_j $(i \neq j)$. The following example demonstrates the importance of independence.

EXAMPLE 1.6. How many odd 3-digit natural numbers are there that do not have any repeated digits?

Solution: In trying to count the total number of such natural numbers, we can count the number of options for each digit, then as long as our choices are independent, the Multiplication Principle will apply. Let's begin by working left-to-right (starting with the hundreds position). In order to have a 3-digit natural number, note that we cannot start with 0. So, there are 9 possible digits that can be selected for the hundreds position. When counting the number of possibilities for the tens position, we cannot select the digit that has already been used for the hundreds position. Still, we have independence since regardless of what digit was chosen for the hundreds position, there will be 9 choices for the tens position. Finally, when counting the possibilities for the ones position, we run into a problem. For our 3-digit number to be odd, only the digits $\{1, 3, 5, 7, 9\}$ are possible. Depending on our previous choices, there may be 3, 4, or 5 possibilities. So, we do not have independence among our choices and the Multiplication Principle does not apply.

$$\underbrace{\overline{\qquad}}_{9} \quad \underbrace{\overline{\qquad}}_{9} \quad \underbrace{\overline{\qquad}}_{?}$$

Rather than work from left-to-right, let's count the number of possibilities for each position, starting with the most restricted and working towards the least restricted. The ones place is the most restricted and there are 5 possible digits that can be used. After we have made this choice, the next most restricted position is the hundreds place. For this position, we must avoid the digit that has been chosen for the ones place and we must avoid 0, leaving 8 possibilities. Finally, the tens place must be chosen to be different from the two distinct digits that we have already selected: 8 possibilities.

$$\underbrace{\overline{\qquad}}_{8\ (2^{nd})} \quad \underbrace{\overline{\qquad}}_{8\ (3^{rd})} \quad \underbrace{\overline{\qquad}}_{5\ (1^{st})}$$

As our choices have been made independently, by the Multiplication Principle, there are

$$5 \cdot 8 \cdot 8 = 320$$

odd 3-digit odd natural numbers that do not have repeated digits.

Exercises for Section 1.1

EXERCISE 1.1.1. Prove the General Pigeonhole Principle (Theorem 1.3).

EXERCISE 1.1.2. Ten points are randomly given within a unit square. Prove that two of them are within 0.48 units of one another.

Exercise 1.1.3. Ten points are randomly given within a unit square. Prove that at least three of them can be covered by a disc of radius 0.5.

Exercise 1.1.4. Suppose that we select $n+1$ distinct integers from the set $\{1, 2, \ldots, 2n\}$. Prove that there will always be at least two among the selected integers whose greatest common divisor is 1.

Exercise 1.1.5. How many five letter "words" (a sequence of any five letters, with repetition allowed) are there? How many five letter words are there that do not have any repeated letters?

Exercise 1.1.6. How many ways are there to roll two distinct 6-sided dice to yield an even sum?

Exercise 1.1.7. How many k-digit natural numbers are there?

Exercise 1.1.8. How many even 3-digit natural numbers are there that do not have any repeated digits?

1.2. Permutations and Combinations

An arrangement of distinct objects in a linear order is called a **permutation**. When counting the number of permutations of a collection of distinct objects, it is important to remember that order is important (switching the order of two objects gives a different permutation) and repetition is not allowed. It will be convenient to use factorial notation when determining the number of permutations of a collection of objects. For $n \in \mathbb{N}$, define n **factorial**, denoted $n!$, to be the product

$$n! = n(n-1)\cdots 2 \cdot 1.$$

By convention, we usually define $0! = 1$.

Example 1.7. In how many ways can one arrange 5 different books on a shelf (ie., how many permutations of 5 distinct objects are there)?

Solution: There are 5 choices for the first book in the arrangement. Regardless of which book is chosen to go first, there are 4 choices for a book to be in the second position (independence!). Then there are 3 possibilities for the third book, 2 possibilities for the second book, and 1 choice for the last book.

$$\underbrace{\overline{\qquad}}_{5} \quad \underbrace{\overline{\qquad}}_{4} \quad \underbrace{\overline{\qquad}}_{3} \quad \underbrace{\overline{\qquad}}_{2} \quad \underbrace{\overline{\qquad}}_{1}$$

By the Multiplication Principle, there are a total of

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

arrangements.

From this example, one can apply inductive reasoning to find that the total number of arrangements of n distinct objects is given by $n!$. More generally, a k**-permutation** of n distinct objects ($0 \le k \le n$) is an arrangement of k of the n distinct objects. We denote by $P(n,k)$ the number of k-permutations of n distinct objects. Thus, $P(n,n) = n!$ and using the same approach as in the previous example, we find that

$$P(n,k) = n(n-1)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}.$$

EXAMPLE 1.8. A gathering consists of 4 (different) boys and 4 (different) girls.

(a) How many ways are there to seat everyone along one side of a table with 8 seats?
Solution: The fact that there are 4 boys and 4 girls is irrelevant. We are arranging 8 different people. Thus, there are $P(8,8) = 8! = 40,320$ ways to seat them.
(b) How many ways are there to seat them if we require boys and girls to alternate?
Solution: We must count separately the arrangements depending upon whether we we start with a boy or start with a girl. As these two sets of arrangements are disjoint, the Addition Principle will be needed to get a final answer. The number of arrangements that start with a boy is 4!4! The number of arrangements that start with a girl is also 4!4!. Thus, the total number of such arrangements is given by

$$4!4! + 4!4! = 2(4!)^2 = 1152.$$

When counting permutations, the order of elements is important. Now we turn to selections of elements where order is ignored. A **k-combination** of n distinct objects is an unordered selection of k of the n objects. In other words, it is a k-element subset of an n-element set. We denote by $C(n,k)$ the number of k-combinations of n distinct objects. This is sometimes read "n choose k" and written $\binom{n}{k}$. To obtain a formula for $C(n,k)$, note that we can determine $P(n,k)$ by first choosing a k-element subset of elements to arrange. Once these elements are chosen, there exists $P(k,k) = k!$ ways to arrange them. Hence, by the Multiplication Principle,

$$P(n,k) = C(n,k) \cdot k!.$$

Thus,

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}.$$

A simple consequence of this formula is that $C(n,k) = C(n,n-k)$. One can also give a combinatorial proof of this claim by noting that choosing k elements to include in a subset of an n element set is equivalent to choosing $n-k$ elements not to include.

EXAMPLE 1.9. How many ways can a committee be formed from 4 (different) men and 6 (different) women in which:

(a) there are at least 2 men and at least twice as many women as men?
Solution: Given the assumptions, we count the following disjoint sets:
2 men and 4 women: $C(4,2) \cdot C(6,4) = 6 \cdot 15 = 90$.
2 men and 5 women: $C(4,2) \cdot C(6,5) = 6 \cdot 6 = 36$.
2 men and 6 women: $C(4,2) \cdot C(6,6) = 6 \cdot 1 = 6$.
3 men and 6 women: $C(4,3) \cdot C(6,6) = 4 \cdot 1 = 4$.
By the Addition Principle, there are $90 + 36 + 6 + 4 = 136$ ways to form such a committee.
(b) the committee contains between 3 and 5 people (inclusively) and Bob is excluded?

Solution: Since Bob is excluded, there are a total of 9 people to choose from when forming a committee. We count the 3, 4, and 5 person committees separately:

$$C(9,3) + C(9,4) + C(9,5) = 84 + 126 + 126 = 336.$$

(c) the committee consists of 4 members, at least 2 of whom are women, and Mr. and Mrs. Smith cannot both be chosen?
Solution: First, we separate the possible committees based on the number of men/women. Then we count the number of committees based on whether or not Mr. Smith is included.
2 women and 2 men with Mr. Smith: $C(3,1) \cdot C(5,2) = 3 \cdot 10 = 30$.
2 women and 2 men without Mr. Smith: $C(3,2) \cdot C(6,2) = 3 \cdot 15 = 45$.
3 women and 1 man with Mr. Smith: $C(5,3) = 10$.
3 women and 1 man without Mr. Smith: $C(6,3) \cdot C(3,1) = 20 \cdot 3 = 60$.
4 women and 0 men: $C(6,4) = 15$.
Thus, there are $30 + 45 + 10 + 60 + 15 = 160$ such committees by the Addition Principle.

Exercises for Section 1.2

Exercise 1.2.1. How many ways are there to arrange 3 different math books and 5 different biology books if the first book must be a math book?

Exercise 1.2.2. How many ways are there to partition 12 people into 3 groups, with sizes 2, 4, and 6?

Exercise 1.2.3. A university organization consists of 10 students and 5 faculty members. How many ways are there to form a committee of 6 people if at most 3 can be faculty?

Exercise 1.2.4. A class consists of 20 math majors and 15 computer science majors (assume these sets are disjoint). How many ways are there to form a 4-person study group if:

(a) exactly one member is a math major?
(b) at least two members are computer science majors?

Exercise 1.2.5. A man has n friends and invites a different subset of 4 of them to his house every night for one year (365 days). How large must n be? Note that subsets are different if they are not equal.

1.3. Allowing Repetition

In the previous section, we counted the number of permutations and combinations when repetition was not allowed. In this section, we allow for the possibility of repetition and find that such permutations can be counted using combinations and vice-versa!

EXAMPLE 1.10. In how many ways can the letters in BANANA be arranged?

Solution: Among the six positions, start by selecting positions for the 2 N's: $C(6,2) = 15$. Then select 3 positions for the A's: $C(4,3) = 4$. Finally, there is one position left for the B to occupy. The Multiplication Principle implies that there are $15 \cdot 4 = 60$ such arrangements.

Note that in Example 1.10 the solution can be written as

$$\begin{aligned} C(6,2) \cdot C(4,3) \cdot C(1,1) &= \frac{6!}{2!\ 4!} \cdot \frac{4!}{3!\ 1!} \cdot \frac{1!}{1!\ 0!} \\ &= \frac{6!}{2!\ 3!\ 1!}, \end{aligned}$$

where the terms in the denominator correspond to the number of letters of each type. In fact, we will find that this is indeed the case. In general, suppose that $n, k, a_1, a_2, \ldots, a_k$ are nonnegative integers satisfying

$$n = a_1 + a_2 + \cdots + a_k.$$

Consider a collection of objects in which there are exactly a_i objects of type i (for each $i \in \{1, 2, \ldots, k\}$). Denote the number of ways to arrange these objects by $P(n; a_1, a_2, \ldots, a_k)$. Using the same approach as above, we find that

$$\begin{aligned} P(n; a_1, a_2, \ldots, a_k) &= C(n, a_1) \cdot C(n - a_1, a_2) \cdots C(n - a_1 - a_2 - \cdots - a_{k-1}, a_k) \\ &= \frac{n!}{a_1!\ (n-a_1)!} \cdot \frac{(n-a_1)!}{a_2!\ (n-a_1-a_2)!} \cdots \frac{(n-a_1-a_2-\cdots-a_{k-1})!}{a_k!\ (n-a_1-a_2-\cdots-a_k)!} \\ &= \frac{n!}{a_1!\ a_2!\ \cdots a_k!}. \end{aligned}$$

EXAMPLE 1.11. Again, we consider arrangements of BANANA, but this time, we will place conditions on those arrangements.

(a) How many arrangements are there in which the B is immediately followed by an A?
Solution: View BA as a single letter. Then we are considering arrangements of the letters BA, N, A, N, A. The number of such arrangements is given by

$$P(5; 1, 2, 2) = \frac{5!}{1!\ 2!\ 2!} = 30.$$

(b) How many arrangements are there in which the pattern BNN never occurs?
Solution: From Example 1.10, we know that there are a total of 60 arrangements when no conditions are imposed. We just need to subtract off the arrangements that have BNN. As in part (a), view BNN as a single letter. There are

$$P(4; 1, 3) = \frac{4!}{1!\ 3!} = 4$$

arrangements that have the pattern BNN. Hence, there are $60 - 4 = 56$ arrangements that lack the pattern BNN.

(c) How many arrangements are there in which the B occurs before all of the A's (but not necessarily immediately before)?
Solution: The relative order of the B and A's is B-A-A-A. In this problem, it helps if we think of B, A, A, A as if they are four copies of the same letter. The total number of such arrangements is

$$P(6; 4, 2) = \frac{6!}{4!\ 2!} = 15.$$

Now we turn our attention to selections when repetition is allowed. As with arrangements, we start with an example that will help us to understand the general approach.

Example 1.12. How many ways are there to select 7 apples from 3 different varieties: Gala, Fuji, and Red Delicious?

Solution: It helps to set up this problem if you approach it using an "order form" as if taking an order from 7 people. The table below shows several possible "orders."

Gala	Fuji	Red Del.
$**$	$*$	$****$
$***$	$***$	$*$
$*$		$******$
$****$	$*$	$**$
$\vdots$	$\vdots$	$\vdots$

Note that each row can be viewed as a sequence of 7 $*$'s and 2 $|$'s. For example, the first row corresponds to the sequence $**\,|\,*\,|\,****$. The total number of such arrangements is given by

$$P(9; 7, 2) = \frac{9!}{7!\ 2!} = 36.$$

From the previous example, we see that if we wish to select n objects from k different types of object, this corresponds to filling out an order form that contains k columns (separated by $k-1$ $|$'s). So, counting the number of selections is equivalent to counting the number of arrangements of n $*$'s and $k-1$ $|$'s. Thus, it is given by

$$P(n + k - 1; n, k - 1) = \frac{(n + k - 1)!}{n!\ (k - 1)!} = C(n + k - 1, n).$$

Selections in which repetition is allowed can also be used to solve counting problems involving distributions, as the following example demonstrates.

Example 1.13. A teacher has 20 identical pencils that she wishes to give to her top 4 students. In how many ways can she distribute the pencils if:

(a) there are no restrictions?
Solution: This problem is equivalent to selecting 20 objects from 4 different types of object. Thus, there are

$$C(20 + 4 - 1, 20) = \frac{23!}{20!\ 3!} = 1771$$

ways to distribute the pencils.

(b) each child must receive at least 2 of the pencils?
Solution: As with selections that involve repetition, we can use the approach of creating an order form. However, before we begin filling it in, we start with two $*$'s in each column:

Child 1	Child 2	Child 3	Child 4
$**$	$**$	$**$	$**$

There is only one way to put two $*$'s in each column. So, our answer is determined by the number of ways of distributing the remaining 12 pencils. There are

$$C(12+4-1,12) = \frac{15!}{12!\,3!} = 455$$

ways to complete this distribution.

Finally, we complete this section by considering a problem in which some objects are identical and some are distinct.

Example 1.14. How many ways are there to arrange 10 identical apples and 5 different oranges in a row such that no two oranges appear side-by-side?

Solution: Even though this problem involves an arrangement, we can think of the different oranges as distinguishing between the types of objects in a distribution. Begin by counting the number of ways of choosing a relative order for the oranges. There are $5! = 120$ ways to arrange them. Now consider an order form in which the lines separating the columns represent the 5 oranges. Since no two oranges can be side-by-side, the 4 middle columns must all contain at least one $*$ (apple).

	$*$	$*$	$*$	$*$	

Among the 6 categories, we must now distribute the remaining 6 apples. There are

$$C(6+6-1,6) = \frac{11!}{6!\,5!} = 77$$

ways to complete this distribution, giving a total of $120 \cdot 77 = 9240$ ways to arrange the apples and oranges.

Exercises for Section 1.3

Exercise 1.3.1. How many 7-digit numbers can be formed that have three 3's, two 5's, and two 7's?

Exercise 1.3.2. Consider arrangements of the the letters in the word COMBINATION.

(a) How many arrangements are there without any conditions?
(b) How many arrangements are there in which the C is immediately followed by an O?
(c) How many arrangements are there in which the O's are not consecutive?
(d) How many arrangements are there in which the T occurs before both of the N's (but not necessarily immediately before)?

Exercise 1.3.3. A group of 9 people buy sandwiches in a deli. 3 of the people always order veggie, 2 always order ham, 2 always order turkey, and 2 order any of the 3 types of sandwich.

(a) If their order is lined up across the counter, how many different sequences of sandwich are possible?
(b) How many different (unordered) collections of sandwich are possible?

Exercise 1.3.4. If 3 identical 6-sided dice are rolled, how many different outcomes are possible?

Exercise 1.3.5. How many ways are there to arrange 4 different algebra books and 7 identical calculus books on a shelf if not all 4 algebra books are consecutive?

Exercise 1.3.6. How many ways are there to distribute 6 identical apples and 5 identical pears to 3 people if each person gets at least one pear?

Exercise 1.3.7. How many ways are there to distribute 6 different apples and 5 identical pears to 4 people?

Exercise 1.3.8. A nursery sells 4 different varieties of fruit tree: apple, cherry, pear, and peach. In how many ways can a landscaper buy a selection of 15 trees if he wants to have at least 3 varieties and each type of tree requires a pollinator of the same variety (ie., if he buys a particular variety, then he must buy at least 2 of that variety)?

1.4. Generating Functions

Generating functions provide a means of using properties of polynomials and power series to solve a variety of combinatorial problems. If a_k is the number of ways of selecting k objects in a certain procedure, then one defines the **generating function** for a_k to be the expansion

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_kx^k + \ldots.$$

Note that if this expansion is finite, then $g(x)$ is a polynomial. Otherwise, it is a power series. Consider the following example.

Example 1.15. Suppose that a drawer contains 12 beads: 3 red, 4 blue, and 5 green. Assume that beads of the same color are indistinguishable. In how many ways can 6 beads be selected?

Solution: We could go through and break up the possible ways in which 6 beads can be selected into cases, based upon the number of beads selected in each color. In a sense, generating functions allow us to count selections without having to apply the Addition and Multiplication Principles explicitly. We identify the polynomial $1 + x + x^2 + x^3$ with the selection of red beads since it is possible that either 0, 1, 2, or 3 red beads may be chosen. Similarly, we identify $1 + x + x^2 + x^3 + x^4$ with the selection of blue beads and $1+x+x^2+x^3+x^4+x^5$ with the selection of green beads. Thus, for the procedure of selecting beads from the drawer, we obtain the generating function

$$\begin{aligned} g(x) &= (1+x+x^2+x^3)(1+x+x^2+x^3+x^4)(1+x+x^2+x^3+x^4+x^5) \\ &= 1+3x+6x^2+10x^3+14x^4+17x^5+18x^6+17x^7+14x^8+10x^9+6x^{10}+3x^{11}+x^{12}. \end{aligned}$$

So, we find that there are 18 ways to select 6 beads (the coefficient of x^6 in $g(x)$) as this coefficient counts all of the ways in which

$$e_R + e_B + e_G = 6,$$

where e_R is the number of red beads selected, e_B is the number of blue beads selected, and e_G is the number of green beads selected (corresponding to the powers in the polynomials). Of course, this function also provides us with much more information. Namely, we are able to determine the number of ways of selecting k beads (the coefficient a_k in $g(x)$) for all $k \geq 0$.

Some selections allow for the possibility of choosing any number of a given object, and hence, require power series. The **geometric series** will be particularly useful to us. Recall that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots . \tag{1.1}$$

Of course, this series only converges when $|x| < 1$, but we aren't so concerned with convergence here. Rather, we wish to use such power series as formal objects whose algebraic manipulation enables us to keep track of various selections. Before we move on to more examples, there are a few properties of polynomials and power series that we must develop.

Consider the expansion of

$$\frac{1}{(1-x)^n} = (1 + x + x^2 + x^3 + \dots)^n.$$

Here, the coefficient of x^k comes from the number of formal products

$$x^{e_1} x^{e_2} \cdots x^{e_n} = x^k,$$

where $e_i \geq 0$ represents the number of selections of type i (coming from the i^{th} copy of $\frac{1}{1-x}$). Equivalently, the coefficient counts the number of selections such that

$$e_1 + e_2 + \cdots + e_n = k,$$

which is equivalent to selecting k objects from among n types: $C(k+n-1, k)$. Thus, we find that

$$\frac{1}{(1-x)^n} = 1 + C(1+n-1,1)x + C(2+n-1,2)x^2 + C(3+n-1,3)x^3 + \cdots . \tag{1.2}$$

With regard to polynomials, it is worth noting that

$$(1-x)(1 + x + x^2 + x^3 + \cdots + x^{m-1}) = 1 - x^m,$$

from which it follows that

$$\frac{1-x^m}{1-x} = 1 + x + x^2 + \cdots + x^{m-1}. \tag{1.3}$$

We also note that if $h(x) = f(x)g(x)$ where

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \quad \text{and} \quad g(x) = b_b + b_1 x + b_2 x^2 + \cdots ,$$

then

$$h(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots .$$

That is, the coefficient of x_k in $h(x)$ is given by

$$\sum_{i=0}^{k} a_i b_{k-i}. \tag{1.4}$$

EXAMPLE 1.16. Bob goes to the doughnut shop and finds that they sell 3 varieties: chocolate, glazed, and strawberry.

(a) What is the generating function for a_k, the number of ways that Bob can select k doughnuts, assuming that he must select at least two of each variety and at most 5 chocolate?

Solution: Since at least two of each type of doughnut must be selected and there are at most 5 chocolate doughnuts chosen, the polynomial that we identify with the selection of chocolate doughnuts is $x^2 + x^3 + x^4 + x^5$. Since no maximum number of glazed or strawberry doughnuts is mentioned, the power series that we identify with these selections are each

$$x^2 + x^3 + x^4 + \cdots = x^2(1 + x + x^2 + x^3 + \ldots) = x^2 \left(\frac{1}{1-x}\right).$$

Thus, the generating function for this selection is

$$g(x) = (x^2 + x^3 + x^4 + x^5)x^4 \left(\frac{1}{1-x}\right)^2 = (x^6 + x^7 + x^8 + x^9)\left(\frac{1}{1-x}\right)^2$$

(b) For the generating function found in part (a), in how many ways can Bob select 13 doughnuts?

Solution: This corresponds to finding the coefficient of x^{13} in $g(x)$. Letting

$$f_1(x) = x^6 + x^7 + x^8 + x^9$$

(denoting the i^{th} coefficient in $f_1(x)$ by a_i) and

$$f_2(x) = \left(\frac{1}{1-x}\right)^2 = b_0 + b_1 x + b_2 x^2 + \cdots,$$

and applying property (1.4), we find that the coefficient of x^{13} in $g(x)$ is given by

$$\begin{aligned} &a_6b_7 + a_7b_6 + a_8b_5 + a_9b_4 \\ &\qquad = C(7+2-1,7) + C(6+2-1,6) + C(5+2-1,5) + C(4+2-1,4) \\ &\qquad = 8 + 7 + 6 + 5 = 26. \end{aligned}$$

We now turn our attention to using generating functions to solve recurrence relations. In general, if

$$a_0, a_1, a_2, a_3, \ldots$$

is a sequence of numbers, then we say that this sequence satisfies a **recurrence relation** if a_n can be described in terms of an expression involving $a_0, a_1, a_2, \ldots, a_{n-1}$. A solution to such a recurrence relation is an expression for a_n that does not depend on the previous values in the sequence (although it will usually still depend upon n).

EXAMPLE 1.17. Suppose that the frog population of an infinitely-large lake grows fourfold each year. In an attempt to control the population, on the last day of each year, 100 frogs are taken from the lake and relocated to another lake. Assuming there were 50 frogs in the lake initially, how many frogs will there be after n years? You may assume the frogs are immortal.

Solution: Let a_n denote the number of frogs in the lake after n years. The population follows the following recurrence:

$$\begin{aligned}
a_0 &= 50 \\
a_1 &= 4 \cdot 50 - 100 = 100 \\
a_2 &= 4 \cdot 100 - 100 = 300 \\
a_3 &= 4 \cdot 300 - 100 = 1100 \\
&\vdots \\
a_n &= 4 \cdot a_{n-1} - 100
\end{aligned}$$

While this description of the population is useful for determining a_n when n is small, it is not as useful for larger values of n. Let

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function for this sequence. Beginning with the recurrence $a_n = 4 \cdot a_{n-1} - 100$, multiply both sides of the equation by x^n, then sum over all $n \geq 1$:

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 4a_{n-1} x^n - \sum_{n=1}^{\infty} 100 x^n.$$

The sum on the left-hand side of the equation is $g(x) - a_0$ and the right-hand side can be algebraically manipulated to obtain:

$$\begin{aligned}
g(x) - a_0 &= 4x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} - 100x \sum_{n=1}^{\infty} x^{n-1} \\
&= 4x \sum_{n=0}^{\infty} a_n x^n - 100x \left(\frac{1}{1-x} \right) \\
&= 4x g(x) - \frac{100x}{1-x}.
\end{aligned}$$

We now solve this equation for $g(x)$ and plug in $a_0 = 50$:

$$g(x) = \frac{50}{1-4x} - \frac{100x}{(1-4x)(1-x)}.$$

After finding the partial fraction expansion of the rational function on the right and using the geometric series (1.1), we find that

$$\begin{aligned} g(x) &= \frac{50}{1-4x} - \left(\frac{-100/3}{1-x} + \frac{100/3}{1-4x}\right) \\ &= 50\sum_{n=0}^{\infty}(4x)^n + \frac{100}{3}\left(\sum_{n=0}^{\infty}x^n - \sum_{n=0}^{\infty}(4x)^n\right) \\ &= 50\sum_{n=0}^{\infty}4^n x^n + \frac{100}{3}\sum_{n=0}^{\infty}(1-4^n)x^n \\ &= \sum_{n=0}^{\infty}\left(50\cdot 4^n + \frac{100}{3}(1-4^n)\right)x^n. \end{aligned}$$

It follows that

$$a_n = 50\cdot 4^n + \frac{100}{3}(1-4^n)$$

is a closed-form solution.

Exercises for Section 1.4

Exercise 1.4.1. For each of the following expressions, determine the coefficient of x^5.

(a) $(1+x+x^2)(1+x)^2$
(b) $(1+x+x^2+x^3+x^4)^2$
(c) $(1+x)(1+x+x^2+x^3+\cdots)^5$
(d) $(1+x+x^2+x^3+\cdots)^3(x^3+x^4+x^5+\cdots)^2$

Exercise 1.4.2. Suppose that we wish to collect a selection of balls from among 3 piles: red, blue, and green. For our purposes, assume that the piles have more than enough balls for any selection that we consider (ie., assume infinitely-many balls are in any one of the piles). Also assume that all balls of a given color are identical. Use generating functions to answer the following questions. In how many ways can one select 10 balls such that:

(a) the selection has at least 2 balls of each color?
(b) the selection has at most 2 red balls?
(c) the selection has an even number of green balls?

Exercise 1.4.3. Suppose that 10 distinct 6-sided dice are rolled. What is the generating function for the number of ways to roll a sum of k?

Exercise 1.4.4. Use generating functions to determine the number of ways to distribute r identical apples among 5 children if:

(a) each child gets at least 2 apples (you may assume $r \geq 10$).
(b) each child gets an even number of apples (you may assume r is even).

Exercise 1.4.5. A population of sloths is reintroduced to an island where they previously went extinct. Initially, 100 sloths are released on the island. Assume that their population doubles each year and 10 sloths die each year. Let a_n denote the sloth population on the island after n years.

(a) Give a recurrence relation for a_n.

(b) Find a closed-form solution for a_n.
(c) What is the population of sloths on the island after 30 years?

1.5. The Binomial Theorem

The combination $C(n,r)$ is often read as "n choose r" and is written as $\binom{n}{r}$. In this context, it is referred to as a *binomial coefficient* due to the role it plays in the Binomial Theorem.

THEOREM 1.18 (The Binomial Theorem). *For all nonnegative $n \in \mathbb{Z}$,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

PROOF. In the expansion of

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n\ terms},$$

each product takes a single x or y from each term $(x+y)$. Thus, one obtains the product $x^k y^{n-k}$ each time x is selected from exactly k of the terms (and y is selected from the remaining $n-k$ terms). This occurs in exactly $\binom{n}{k}$ ways, from which the proof follows. □

Plugging in different values for x and y results in many binomial identities. For example, when $n \geq 1$, plugging $x = -1$ and $y = 1$ into the Binomial Theorem yields the following:

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

This identity is equivalent to

$$\sum_{\substack{0 \leq k \leq n \\ k\ even}} \binom{n}{k} = \sum_{\substack{0 \leq k \leq n \\ k\ odd}} \binom{n}{k},$$

from which it follows that nonempty every set has the same number of even and odd subsets.

Plugging $x = 1 = y$ into the Binomial Theorem gives

$$2^n = \sum_{k=0}^{n} \binom{n}{k}. \tag{1.5}$$

From this identity, we observe that the number of subsets of an n-element set is 2^n. This is due to the fact that the sum on the right-hand side of (1.5) counts all possible subsets of $\{1, 2, \ldots, n\}$. Now, we turn to an important identity concerning binomial coefficients, which we will prove in multiple ways to distinguish between algebraic and combinatorial proofs.

THEOREM 1.19. *For all $n \in \mathbb{N}$ and $0 \le k < n$,*

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

ALGEBRAIC PROOF. Back in Section 1.2, we showed that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. It follows that

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-(k+1))!} \\
&= \frac{n!(k+1) + n!(n-k)}{(k+1)!(n-k)!} \\
&= \frac{n!(n+1)}{(k+1)!(n-k)!} \\
&= \frac{(n+1)!}{(k+1)!(n+1-(k+1))!} = \binom{n+1}{k+1},
\end{aligned}$$

proving the given identity. $\square$

Now we provide a combinatorial proof of Theorem 1.19. That is, we will count the same quantity in two different ways, resulting in equal values.

COMBINATORIAL PROOF. Consider the problem of selecting a $k+1$-person committee from a group of $n+1$ people. There are a total of $\binom{n+1}{k+1}$ ways to make such a selection. On the other hand, suppose that Bob is one of the people. Then the number of such committees that contain Bob is $\binom{n}{k}$. The number of such committees that do not contain Bob is $\binom{n}{k+1}$. As every committee either contains Bob or does not contain Bob and these are disjoint sets that we have counted, the Addition Principle applies and we find that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1},$$

completing the proof. $\square$

Theorem 1.19 is easy to remember in the form of Pascal's Triangle:

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Using Theorem 1.19, we now offer an additional proof of (1.5).

PROOF OF (1.5). We proceed by (weak) induction on n. When $n = 1$, we have

$$2 = 1 + 1 = \binom{1}{0} + \binom{1}{1} = \sum_{k=0}^{1} \binom{n}{k}.$$

Now suppose that the identity holds for $n = t$:

$$2^t = \sum_{k=0}^{t} \binom{t}{k}.$$

Consider the sum:

$$\begin{aligned}
&\sum_{k=0}^{t+1} \binom{t+1}{k} \\
&\quad = \binom{t+1}{0} + \binom{t+1}{1} + \cdots + \binom{t+1}{t} + \binom{t+1}{t+1} \\
&\quad = \binom{t+1}{0} + \left(\binom{t}{0} + \binom{t}{1}\right) + \cdots + \left(\binom{t}{t-1} + \binom{t}{t}\right) + \binom{t+1}{t+1} \\
&\quad = \binom{t}{0} + \left(\binom{t}{0} + \binom{t}{1}\right) + \cdots + \left(\binom{t}{t-1} + \binom{t}{t}\right) + \binom{t}{t} \\
&\quad = 2\left(\sum_{k=0}^{t} \binom{t}{k}\right) = 2 \cdot 2^t = 2^{t+1},
\end{aligned}$$

by the inductive hypothesis. $\square$

Additional identities can be found to follow from the Binomial Theorem by successively plugging in values for y and taking derivatives/antiderivatives. For example, plug in $y = 1$:

$$(x + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{1.6}$$

Differentiating both sides of this equation with respect to x gives

$$n(x + 1)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} kx^{k-1}.$$

Now plugging in $x = 1$ yields the identity

$$n2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k.$$

Alternately, we could have taken the antiderivative of both sides of (1.6):

$$\frac{(x + 1)^{n+1}}{n + 1} = c + \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k + 1}.$$

To determine the value of c that satisfies the identity, plug in $x = 0$. Hence, it follows that $c = \frac{1}{n+1}$.

Exercises for Section 1.5

EXERCISE 1.5.1. Prove that for all natural numbers $n \geq 2$,

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$$

(a) using a combinatorial proof (ie., count the same selection in two distinct ways).
(b) using an algebraic proof.

EXERCISE 1.5.2. Prove that for all natural numbers $n \geq k \geq m$,

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

(a) using a combinatorial proof.
(b) using an algebraic proof.

EXERCISE 1.5.3. Prove the identity

$$n3^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k2^{k-1}.$$

EXERCISE 1.5.4. Prove the identity

$$n(n-1)2^{n-2} = \sum_{k=2}^{n} \binom{n}{k} (k-1)k.$$

CHAPTER 2

Basic Graph Theory

2.1. Definitions and Terminology

Graph theory is a branch of combinatorics in which one studies finite collections of objects and their relationships with one another. They are used as models in many scientific disciplines, but we will study them abstractly. A **graph** $G = (V, E)$ consists of a nonempty finite set V (whose elements are called **vertices**) and a set E of different unordered pairs of distinct vertices from V. Elements of E are called **edges**. When emphasizing the specific graph G being considered, it is customary to write $V(G)$ and $E(G)$ in place of V and E, respectively.

One typically represents a graph in a plane by drawing points in place of vertices and connecting two vertices with a segment or curve if the pair forms an edge in E. For example, suppose that G consists of vertex set $V = \{a, b, c, d, e\}$ and edge set

$$E(G) = \{ab, ad, bc, bd, be, ce\}$$

(see Figure 1). Note that we denote the unordered pair consisting of vertices a and b by ab, but we could also denote this edge by ba. Unless otherwise stated, the graphs

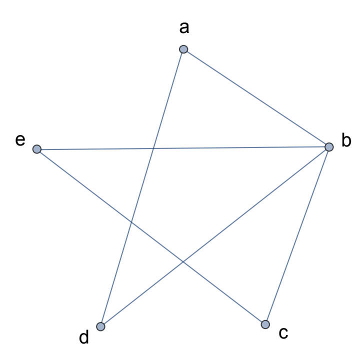


FIGURE 1. A representation of a graph.

we consider are assumed to be **simple**. That is, they do not contain **multiedges** (numerous edges connecting the same pair of distinct vertices) or **loops** (edges in which the vertices are not distinct). Figure 2 shows examples of graphs containing multiedges and loops.

The cardinality $|V(G)|$ is called the **order** of G and the cardinality $|E(G)|$ is called the **size** of G. Two vertices $a, b \in V(G)$ are called **adjacent** if $ab \in E(G)$. Two edges are called **adjacent** if they have a vertex in common. If $ab \in E(G)$,

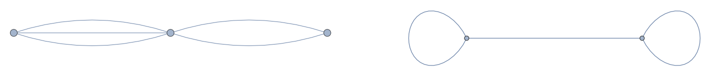


FIGURE 2. A graph containing multiedges and a graph containing loops.

then we say that a and ab are **incident** (b and ab are also incident). The **degree of a vertex** a is the number of edges incident with a, and is denoted $\deg(a)$ (or $\deg_G(a)$ when we wish to emphasize that a is in the graph G). For example, in Figure 1, $\deg(a) = 2$ and $\deg(b) = 4$. The following theorem relates the degrees of the vertices of a graph to the graph's size. As the first theorem in graph theory, this result is credited to Euler [**8**].

THEOREM 2.1 ([**8**]). *In any graph G,*

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

That is, the sum of the degrees of the vertices is twice the number of edges.

PROOF. Summing the degrees of the vertices counts the number of instances of some vertex being incident with an edge. Since each edge is incident with exactly two vertices, the total number of edge-vertex incidences is twice the number of edges. □

In particular, note that the sum of the degrees of the vertices in a graph is always even. If $U = \{u_1, u_2, \ldots, u_m\}$ denotes the subset of $V(G)$ consisting of vertices of even degree and $W = \{w_1, w_2, \ldots, w_m\}$ denotes the subset of $V(G)$ consisting of vertices of odd degree, then

$$\sum_{u \in U} \deg(u) + \sum_{w \in W} \deg(w) = 2|E(G)|.$$

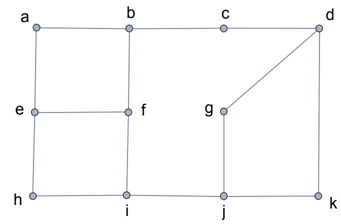
Since $\sum\limits_{u \in U} \deg(u)$ is even, it follows that $\sum\limits_{w \in W} \deg(w)$ is also even. This observation implies the following corollary.

COROLLARY 2.2. *In any graph, the number of vertices of odd degree is even.*

Now we turn our attention to an example of a real-world problem that can be modeled (and solved!) using graphs.

EXAMPLE 2.3. Suppose that a company has several committees consisting of its employees, each of which meets for one hour each week. One would like to have a schedule of committee meetings that minimizes the total number of hours of meetings, but such that two committees with overlapping membership do not meet at the same time.

Solution: We can model this situation by representing the committees as vertices and making two vertices adjacent if and only if they have overlapping membership. Suppose that the graph below represents such a model. In order to find the minimal number of hours needed for committees to meet, we seek to have as many



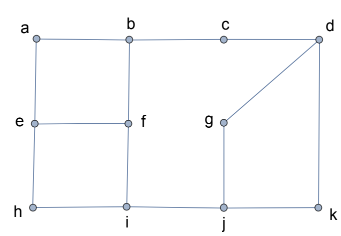
committees meet at one time as possible. This corresponds to finding the largest subset of vertices such that no two vertices are adjacent. A subset of the vertex set of a graph in which no vertices are adjacent is called an **independent set**. So, we would like to identify an independent set of maximal cardinality. In this case, $\{a, c, f, g, h, k\}$ is such a set. The committees corresponding to these vertices can all meet at the same time. As the remaining vertices do not form an independent set, we find that at least 3 hours will be needed for all of the committees to meet. During the second hour, $\{b, e, d, j\}$ can all meet. Then, during the third hour, $\{i\}$ can meet.

Before we move on to another example, we will define the concept of an edge cover and relate it back to independent sets. A subset $C \subseteq V(G)$ is called an **edge cover** if every edge in a graph G is incident with some vertex in C. The following theorem relates edge covers to independent sets.

THEOREM 2.4. *A subset $I \subseteq V(G)$ is an independent set if and only if $V(G) - I$ is an edge cover.*

PROOF. As this is a biconditional statement, we must prove each direction separately.
($\Longrightarrow$) Let I be an independent set of vertices. Then no edges in G have both end vertices in I. In other words, every edge in G is incident with at least one vertex from $V(G) - I$. So, $V(G) - I$ is an edge cover.
($\Longleftarrow$) Let C be an edge cover. So, every edge in G is incident with some vertex in C. Hence, there are no edges joining two vertices from $V(G) - C$, making $V(G) - C$ an independent set. Letting $I = V(G) - C$, and noting that $V(G) - I = V(G) - (V(G) - C) = C$, we obtain the statement of the theorem. $\square$

EXAMPLE 2.5. For this example, we use the same graph that was used in Example 2.3. This time, suppose the graph represents a section of streets in a city. We wish



to position police officers at corners (vertices) so that they can keep every street (edge) under surveillance. Of course, we wish to minimize the number of officers. What is the smallest number of police officers needed?

Solution: This problem is equivalent to trying to find a minimal edge cover for the given graph. While we could approach this problem by inspection, Theorem 2.4 described a nice relationship between edge covers and independent sets. In fact, if I is a maximal independent set, then $V(G) - I$ will be a minimal edge cover. In Example 2.3, we found that $\{a, c, f, g, h, k\}$ was a maximal independent set, so $\{b, d, e, i, j\}$ must be a minimal edge cover. Hence, 5 officers are needed.

For $n \in \mathbb{N}$, a graph with n vertices in which every vertex is adjacent to every other vertex is called a **complete graph of order** n and is denoted K_n. Figure 3 shows several complete graphs. As every pair of vertices in K_n are adjacent, it

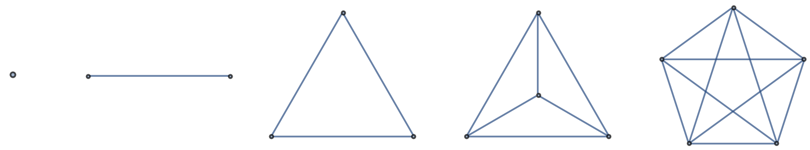


FIGURE 3. The complete graphs K_1, K_2, K_3, K_4, and K_5.

follows that the size of K_n is given by

$$|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}.$$

A graph is called **bipartite** if its vertex set can be partitioned into two sets V_1 an V_2 such that every edge in the graph joins a vertex in V_1 with a vertex in V_2. For example, see Figure 2.9. For $m, n \in \mathbb{N}$, the **complete bipartite graph** $K_{m,n}$

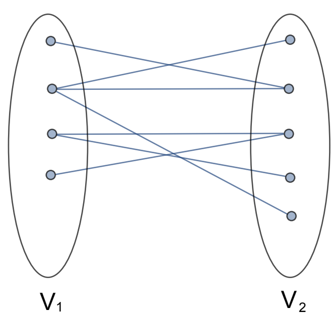


FIGURE 4. A bipartite graph.

consists of sets V_1 and V_2 having cardinalities m and n, respectively, in which every vertex in V_1 is adjacent to every vertex in V_2. For example, Figure 5 shows the bipartite graphs $K_{3,3}$ and $K_{2,5}$. When trying to determine the size of $K_{m,n}$, note that a selected vertex in V_1 has degree equal to n. As there are m vertices in V_1, there are a total of mn edges:

$$|E(K_{m,n})| = mn.$$

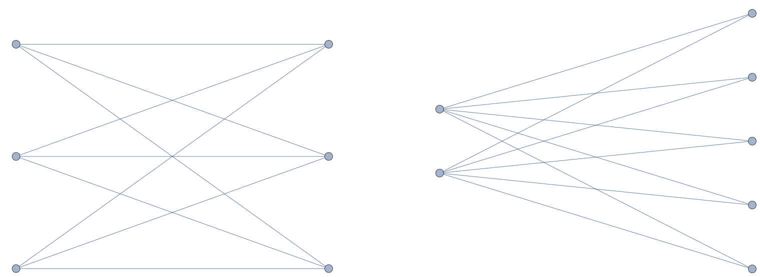


FIGURE 5. The complete bipartite graphs $K_{3,3}$ and $K_{2,5}$.

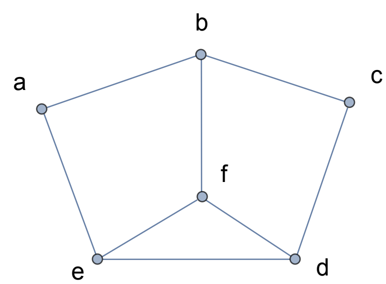
Exercises for Section 2.1

EXERCISE 2.1.1. How many vertices will the following graphs have if they contain:

(a) 12 edges and all vertices of degree 2.
(b) 21 edges, three vertices of degree 4, and the other vertices of degree 3.
(c) 28 edges and all vertices of the same degree.

EXERCISE 2.1.2. What is the largest possible number of vertices in a graph with 19 edges and all vertices of degree at least 3?

EXERCISE 2.1.3. Find a minimal edge cover and a maximal independent set in the following graph.



EXERCISE 2.1.4. Is the graph in Exercise 2.1.3 bipartite? Explain your answer.

EXERCISE 2.1.5. If a complete graph contains exactly 55 edges, then what is its order? What is the degree of each of its vertices?

EXERCISE 2.1.6. A graph is called **regular** if all of its vertices have the same degree. Prove that if G is regular, bipartite, and has size at least 1, then $|V_1| = |V_2|$.

EXERCISE 2.1.7. Is it possible that K_n is bipartite? Explain your answer.

EXERCISE 2.1.8. Let G be a graph with order $n \geq 2$. Prove that there exists at least two vertices in G that have the same degree.

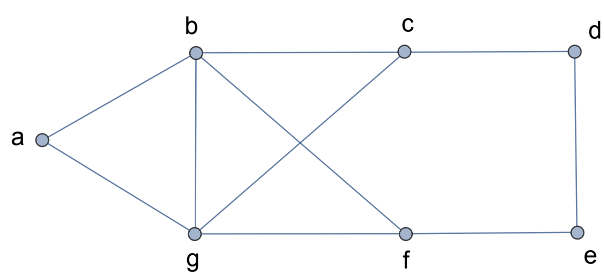
2.2. Paths and Cycles

Before we turn our attention to subgraphs, we will focus on defining several basic graph structures and investigating their properties. In the broadest sense, a **walk**

$$W := x_1x_2\cdots x_n$$

is a (finite) sequence of vertices (not necessarily distinct) in which each consecutive pair of vertices are adjacent. Here, vertices may be repeated, but not consecutively since we are not allowing for loops. Edges may also be repeated. A **trail** is a walk in which edges are not allowed to be repeated (although vertices are still not assumed to be distinct). A trail $T = x_1x_2\cdots x_n$ is a **closed trail** if $x_1 = x_n$. A **path** is a trail in which no vertex is repeated. If $x_1x_2\ldots x_n$ is a path and x_1x_n is also an edge, then we say that $x_1x_2\cdots x_nx_1$ forms a **cycle** . Notice that in a cycle, edges are not repeated and vertices are not repeated, with the exception that the initial vertex is the terminal vertex.

Example 2.6. Consider the following graph. We identity a walk that is not a trail.



Consider the walk

$$W = a\ b\ g\ c\ d\ e\ f\ b\ g\ f.$$

In this walk, there are several vertices that get repeated (b, f, and g)and the edge bg gets repeated (preventing it from satisfying the definition of a trail. The trail

$$T = a\ b\ g\ f\ b\ c\ d\ e$$

repeats the vertex b, but all edges are distinct. So, T is a trail that is not a path. An example of a path is the sequence

$$P = a\ b\ f\ e\ d\ c\ g.$$

Adding in the edge ga forms the cycle

$$C = a\ b\ f\ e\ d\ c\ g\ a.$$

Here, no edges are repeated and no vertices are repeated, with the exception of a being both the initial and terminal vertex.

Given a graph G, if for every pair of distinct vertices $a, b \in V(G)$, there exists a path from a to b, then we say that G is **connected**. Otherwise, there exists some pair of vertices for which no such path exists and we say that G is **disconnected**. If G is disconnected and $n \in \mathbb{N}$ is the least number for which G can be viewed as the union of n connected graphs, then we say that G has n **connected components**. Two vertices a and b are said to be in the same component if a and b are joined by some path in G. For example, the graph in Figure 6 is disconnected since no path exists connecting a and b. In fact, the graph decomposes into two disjoint cycles on 4 vertices ($a\ c\ e\ g\ a$ and $b\ d\ f\ h\ b$). If G is a graph, then the **complement** of

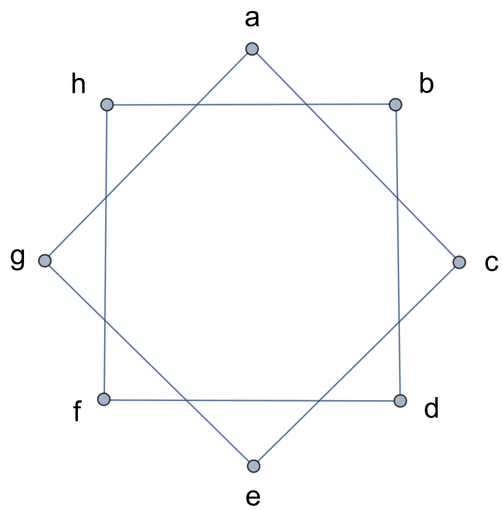


FIGURE 6. A disconnected graph consisting of 2 connected components.

G, denoted $\overline{G}$, is the graph with vertex set $V(\overline{G}) = V(G)$ and edge set

$$E(\overline{G}) = \{ab \mid ab \notin E(G)\}.$$

The following result is usually attributed to Paul Erdős and Richard Rado (e.g., see [**3**] and [**9**]) and we leave its proof as an exercise (see Exercise 6).

THEOREM 2.7 (Erdős and Rado). *If a graph G is disconnected, then $\overline{G}$ is connected.*

Now we turn to a problem that many consider to be one of the first problems in graph theory. The Prussian city of Königsberg was located along the Pregel River. Two islands were located in the river and were connected to the banks by bridges as in Figure 7. Inhabitants of the town would take walks in which they attempted

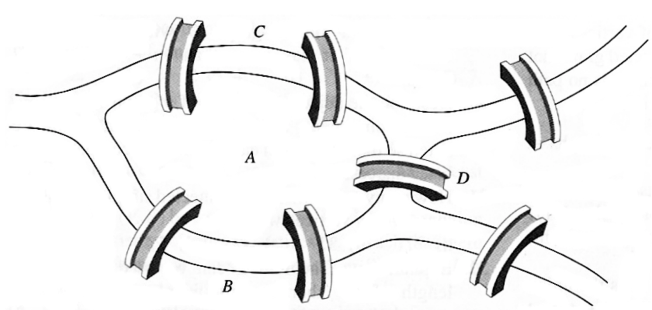


FIGURE 7. The bridges of Königsberg.

to cross every bridge exactly once, returning to their original location at the end. No one seemed able to take a walk without repeating at least one of the seven bridges, so the town's residents wondered if such a walk was possible. Leonard Euler provided a solution to this problem in 1735. He did so by representing the layout of the bridges using a multigraph (in which multiedges are allowed) as in Figure 8.

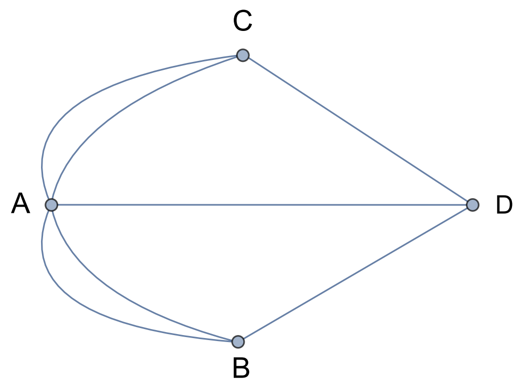


FIGURE 8. Multigraph representation of the bridges of Königsberg.

Using this multigraph, we can rephrase the Königsberg bridge problem in terms of trails. This is because vertices (locations) are allowed to be repeated, but edges are not. If a trail utilizes all of the edges in a graph, then it is called an **Eulerian trail**. So, the Königsberg bridge problem is equivalent to asking if the multigraph in Figure 8 contains a closed Eulerian trail. The key to solving this problem comes from observing the role that the degrees of the vertices play in a closed Eulerian trail. Namely, one must be able to leave any vertex that they enter. Notice that all four vertices have odd degree, and thus, at some point, one enters one of these vertices, but cannot leave it. More precisely, the following theorem shows that the condition of all vertices having even degree is both necessary and sufficient for guaranteeing the existence of a closed Eulerian trail.

THEOREM 2.8. *A connected multigraph G contains a closed Eulerian trail if and only if all vertices of G have even degree.*

PROOF. ($\Longrightarrow$) We begin by assuming that a multigraph G has a closed Eulerian trail. For such a trail to exist, G must be connected and we must be able to leave any vertex that we enter. Hence, it is necessary for all vertices to be of even degree. ($\Longleftarrow$) Now suppose that a multigraph G is connected and all of its vertices have even degree. We will argue that it contains a closed Eulerian trail by actually constructing one. Begin by picking a vertex a and begin tracing out a trail starting with a. Since all vertices have even degree, we are never forced to stop at any vertex that we enter, with the exception that we may not be able to leave a once we return to it. When we are eventually forced to stop, we will have traced out a closed trail C_1. Denote by G_1 the multigraph formed by taking G and removing the edges in

C_1. G_1 may not be connected, but each connected component of G_1 must have some vertex in common with C_1 (otherwise, G would not have been connected). Let b be such a vertex and form a closed trail C_2 beginning and ending at b, as we did with C_1. Now let G_2 be the multigraph formed by removing the edges of C_2 from G_1. We continue in this manner until all of the edges of G have been used up. To form a single closed Eulerian trail, we begin with the vertex a and begin tracing out the trail C_1. when we come to vertex b, we take a detour and follow C_2. Each time we come to an initial vertex of one of our closed trails, we trace out the closed trail that was formed. Putting the closed trails together in this way, we obtain a closed Eulerian trail. □

The strength of the previous proof is that it outlines a means of constructing a closed Eulerian trail in any connected multigraph in which all of the vertices have even degree. Consider the graph in Figure 9, which is connected and in which all vertices have even degree. As in the proof of Theorem 2.8, begin with vertex a and

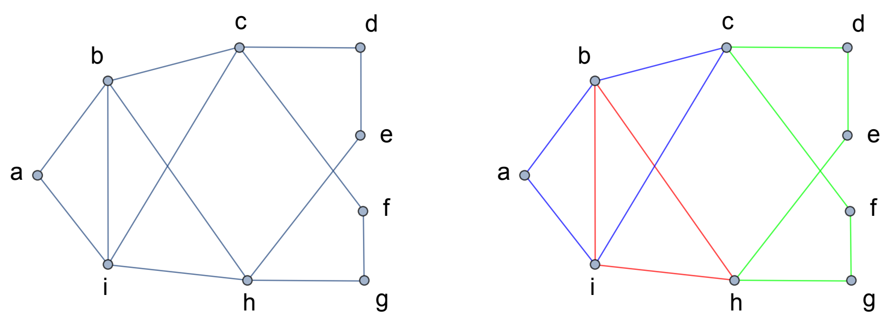


FIGURE 9. A connected graph that contains a closed Eulerian trail.

trace out a closed trail, say $a\ b\ c\ i\ a$, which is given by the blue edges in Figure 9. Removing these edges from the graph, we find that vertex b still has positive degree, so we begin tracing out a closed trail with initial vertex b, say $b\ h\ i\ b$, which is given color red. Finally, we trace out a closed trail with initial vertex c, say $c\ d\ e\ h\ g\ f\ c$, in green. These three closed trails can then be combined to form the closed Eulerian trail

$$a\ b\ h\ i\ b\ c\ d\ e\ h\ g\ f\ c\ i\ a.$$

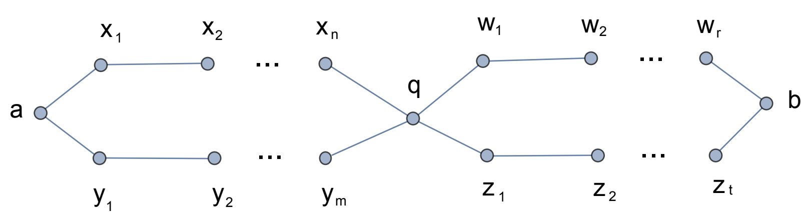
The **length** of a path or cycle is the number of edges it contains. So, if $x_1x_2\cdots x_n$ is a path, then its length is $n-1$. If $x_1x_2\cdots x_nx_1$ is a cycle, then its length is n. The following theorem provides necessary and sufficient conditions for a graph to be bipartite.

THEOREM 2.9. *A graph G is bipartite if and only if every cycle in G has even length.*

Proof. First note that it is sufficient to prove the theorem for connected bipartite graphs. This follows from the observation that a graph is bipartite if and only if all of its connected components are bipartite. Also, all cycles in a graph have even length if and only if all cycles in any given connected component have even length.
($\Longrightarrow$) Assume G is a connected bipartite graph. Then any cycle $C = x_1x_2 \cdots x_nx_1$ has vertices that alternate between the partite sets V_1 and V_2. Without loss of generality, suppose that the vertices x_i that are in V_1 have odd indices and those in V_2 have even indices. Then x_n must be in V_2 since $x_1 \in V_1$. Hence, n is even, from which it follows that C has even length.
($\Longleftarrow$) Now assume that G is a connected graph in which all cycles have even length. Pick a vertex a and place it in V_1. Place all vertices adjacent with a in V_2. Next, place all vertices that are two edges away from a in V_2, and continue in this manner. In general, if there is a path of odd length from a to a vertex x, place x in V_2, if there is a path of even length from a to x, place x in V_1. At this point, we claim that there is no vertex b such that there exist paths P and P' from a to b such that P has even length and P' has odd length. If such a vertex b exists, let P and P' be such a pair of even and odd lengths, respectively. If P and P' do not have any vertices in common, other than a and b, then together they form a cycle of odd length. So, assume that P and P' have some other vertex in common. In tracing out these paths from a to b, suppose that q is the first vertex common to both paths. Writing

$$P = a\ x_1\ x_2 \cdots x_n\ q\ w_1\ w_2 \cdots w_r\ b \qquad \text{and} \qquad P' = a\ y_1\ y_2 \cdots y_m\ q\ z_1\ z_2 \cdots z_t\ b,$$

we find that the x_i are distinct from the y_j. This is not necessarily true of the w_k and z_ℓ. Here, the length of P is $n + r + 2$ (even) and the length of P' is $m + t + 2$



(odd). Thus, $n + r + m + t$ must be odd. Since

$$a\ x_1\ x_2 \cdots x_n\ q\ y_m \cdots y_2\ y_1\ a$$

is a cycle of length $n + m + 2$, which is assumed to be even, it follows that

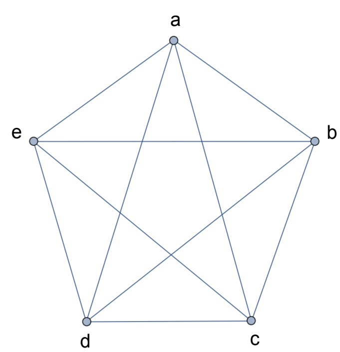
$$q\ w_1\ w_2 \cdots w_r\ b\ z_t \cdots z_2\ z_2\ q$$

is a closed walk of length $r + t + 2$, which must be odd. So, one of the paths

$$q\ w_1w_2 \cdots w_r\ b \qquad \text{and} \qquad q\ z_1\ z_2 \cdots z_t\ b$$

from q to b must be even and the other odd. Let q' be the first vertex common to both of these paths as we trace them out from q to b. We repeat this process until all of the vertices common to P and P' have been exhausted, resulting in a cycle of odd length, giving us a contradiction. Thus, we have affirmed our claim that there is no vertex b such that there exist paths P and P' from a to b such that P has even length and P' has odd length. Now, without loss of generality, suppose that two vertices $c, d \in V_1$ are adjacent. Then there exist paths Q and Q' of even length joining each of these vertices to a, respectively. Then the path Q followed by cd produces a path of odd length from a to d. However, we have argued that it is not possible to have paths of both even length (Q') and odd length (Q with cd) from a to another vertex d, giving us a contradiction. Therefore, no two vertices in V_1 are adjacent. A similar argument can be used to show that no two vertices in V_2 are adjacent. □

We conclude this section with a type of cycle that arises in truck routing problems. In such a problem, one wishes to optimize a route so that each destination (vertex) is visited exactly once, returning the truck to its starting point. Define a **Hamiltonian cycle** of a graph G to be a cycle that visits each vertex of the graph exactly once. For example, consider the complete graph K_5. Observe that several

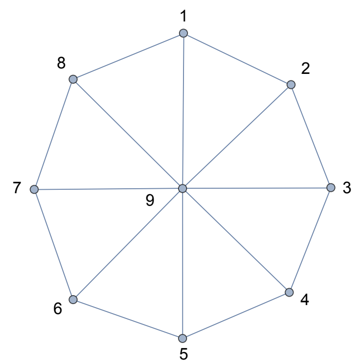


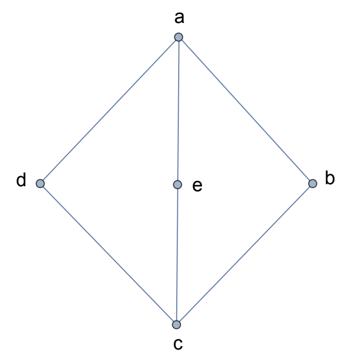
Hamiltonian cycles can be found. One can trace out the cycle on the exterior: $a\ b\ c\ d\ e\ a$. One can be formed using the "star" in the interior: $a\ c\ e\ b\ d\ a$. A Hamiltonian cycle can also be formed using a combination of the edges from these two examples: $a\ b\ c\ e\ d\ a$.

While we will not be able to determine precise conditions for a Hamiltonian cycle to exist in any given graph, we make a few observations about necessary conditions for one to exist. If a graph G has a Hamiltonian cycle, then:

(1) All vertices must have degree ≥ 2.
(2) If a vertex has degree 2, then both edges incident with that vertex must be included in the Hamiltonian cycle.
(3) In the process of building a Hamiltonian cycle, no proper subcycles (not containing all of the vertices) can be formed.
(4) In building a Hamiltonian cycle, once two edges incident with a given vertex are used, all other edges incident with that vertex are removed from consideration.

For example, consider the following two graphs. In both cases, all vertices have degree at least 2. In the first graph, one can begin tracing out a Hamiltonian cycle by working through the vertices in increasing order, starting with 1. When one reaches vertex 8, it is then necessary to move along the interior of the graph to include vertex 9 before returning to 1. This gives the Hamiltonian cycle 1 2 3 4 5 6 7 8 9 1. In the second graph, note that vertices b and d both have degree



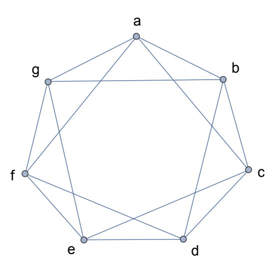


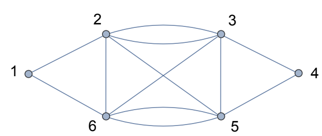
two. This means that if a Hamiltonian cycle exists, it must include edges ad, cd, ab, and bc. However, using these four edges excludes ae and ce from consideration (as we could have used two edges incident with a and two edges incident with c. Thus, there is no way to include vertex e, so a Hamiltonian cycle does not exist.

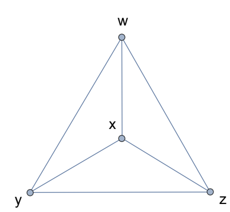
Exercises for Section 2.2

Exercise 2.2.1. Does the graph in Figure 6 contain a closed Eulerian trail? Explain your answer.

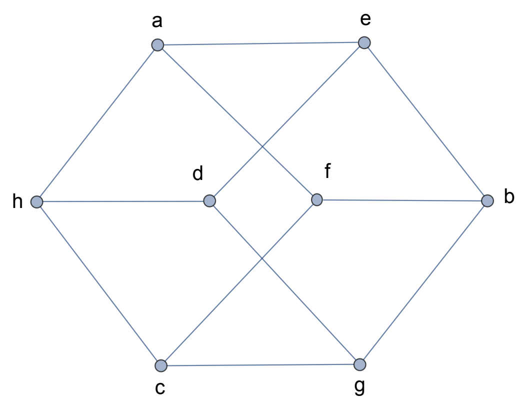
Exercise 2.2.2. Consider the following (multi)graphs.







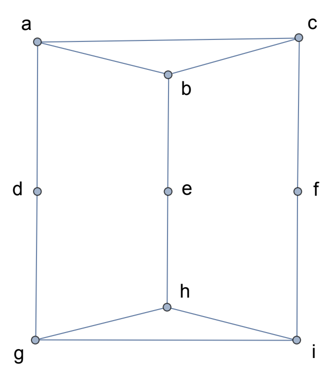
(a) For each of these (multi)graphs, give an example of a walk that is not a trail.
(b) For each of these (multi)graphs, give an example of a trail that is not a path.
(c) For each of these (multi)graphs, give an example of a path.
(d) Do these (multi)graphs contain closed Eulerian trails? If so, give an example of such a trail. If not, explain how you know that such a closed trail does not exist.
(e) Are these graphs bipartite? Explain.



Exercise 2.2.3. Consider the following graph.

(a) Is the graph bipartite? If so, determine which vertices go into the partite sets V_1 and V_2. If not, explain why.
(b) Does the graph have a Hamiltonian cycle? If it does, what is it? If not, prove that such a cycle does not exist.

Exercise 2.2.4. Consider the following graph.



(a) Does the graph contain a closed Eulerian trail? If so, what is it? If not, explain your answer.
(b) Is the graph bipartitie? Explain your answer.
(c) Does the graph have a Hamiltonian cycle? If it does, what is it? If not, prove that such a cycle does not exist.

Exercise 2.2.5. Prove that if a connected bipartite graph has a Hamiltonian cycle, then the cardinalities of its partite sets V_1 and V_2 must be equal.

Exercise 2.2.6. Prove Theorem 2.7. That is, prove that if G is a disconnected graph, then $\overline{G}$ is connected.

2.3. Isomorphisms and Subgraphs

With any mathematical object, one considers the functions (morphisms) that preserve the object's structure. In this section, we will use such maps to determine when two representations correspond to the same graph (perhaps with a different labeling of its vertices. Consider the graphs in Figure 10. While these two graphs

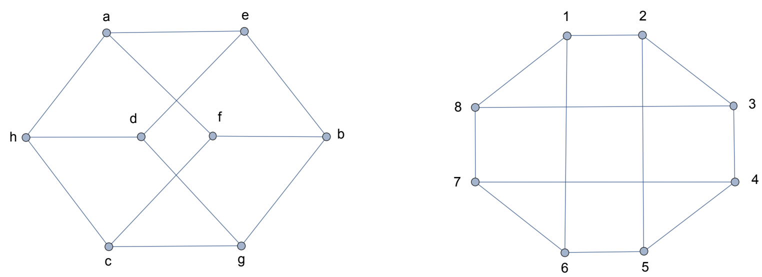


FIGURE 10. Two isomorphic graphs.

appear different, we claim that they are actually the same graph. In particular, if one sends the vertices in the left graph to the right graph according to

$$\begin{array}{cccc} a \mapsto 1 & f \mapsto 2 & c \mapsto 5 & h \mapsto 6 \\ e \mapsto 8 & b \mapsto 3 & g \mapsto 4 & d \mapsto 7 \end{array}$$

then we find that all adjacencies are preserved. In other words, two vertices are adjacent in the first graph if and only if their images are adjacent in the second graph.

More formally, we say that two graphs G and G' are **isomorphic**, and we write $G \cong G'$ if there exists a one-to-one correspondence (a bijection) $\phi : V(G) \longrightarrow V(G')$ such that $xy \in E(G)$ if and only if $\phi(x)\phi(y) \in E(G')$, for all $x, y \in V(G)$. If such a map exists, then it is called an **isomorphism**. Since isomorphic graphs are really the same graph, we note that degrees are preserved. Hence, one simple way of arguing that two graphs are not isomorphic is to note that they contain a different number of vertices of a given degree. Of course, it is possible that two graphs can have the same numbers of vertices of any given degree, but still not be isomorphic.

Another method we can use for showing that two graphs are not isomorphic is to show that one graph contains a particular subgraph, but the other does not. If G is a graph, then a **subgraph** of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Since an isomorphism is just a relabeling of the vertices of a graph, subgraphs must be preserved. Consider the graph in Figure 11. Like the graphs in Figure 10, this graph contains 8 vertices, all of which have degree 3. However it is not isomorphic to the graphs in Figure 10 as it can be observed that this graph contains a cycle of length 5, but the graphs in Figure 10 do not.

Complete subgraphs are another common subgraph that can be used to argue that two graphs are not isomorphic. Consider the graphs in Figure 12. They both have 10 vertices, all of which have degree 3. However, the first graph contains triangles (subgraphs isomorphic to K_3), but the second graph does not, so no isomorphism can exist between the two graphs.

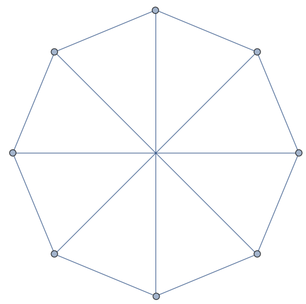


FIGURE 11. Another graph of order 8 in which all vertices have degree 3.

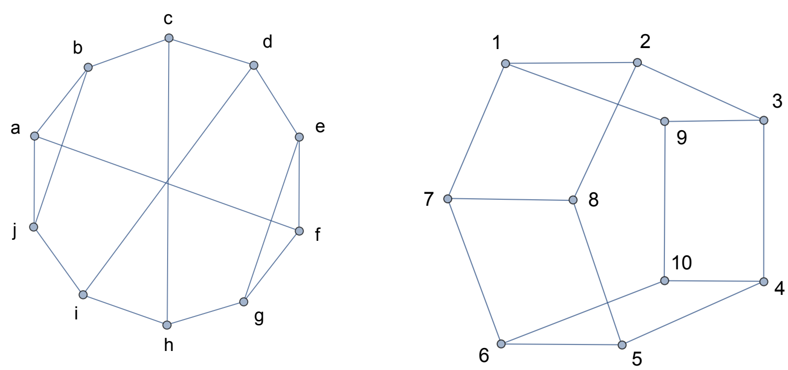
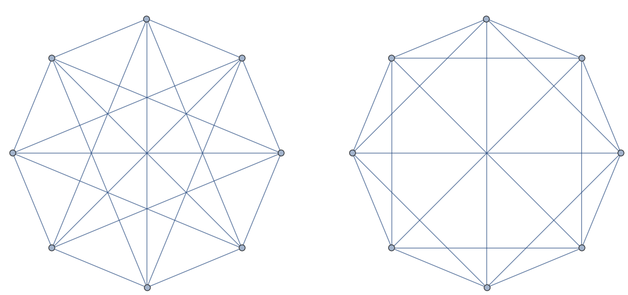


FIGURE 12. Two nonisomorphic graphs.

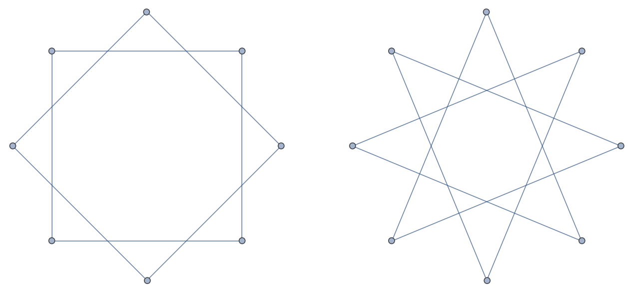
Recall that we denote the complement of a graph G by $\overline{G}$. Note that the union of G and $\overline{G}$ is a complete graph, but the edge sets are disjoint. It is easily observed from the definition that two graphs G and G' are isomorphic if and only if $\overline{G}$ and $\overline{G'}$ are isomorphic. A graph G is called **self-complementary** if $G \cong \overline{G}$.

When trying to determine whether or not two graphs are isomorphic where the graphs contain a large number of edges, it may be easier to consider the graphs' complements. For example, the following two graphs have order 8 and all vertices have degree 5. They both contain subgraphs isomorphic to K_3 and subgraphs



isomorphic to K_4, so that isn't much help (unless they happen to be isomorphic).

The large number of edges makes it challenging to determine if they are isomorphic, so instead, consider their complements. Now, it is easily observed that the first

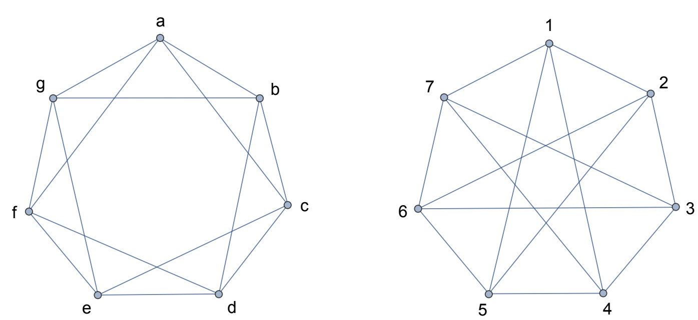


graph is disconnected, while the second is connected. More specifically, the first graph is the disjoint union of two cycles of length 4 while the second graph is a single cycle of length 8. Thus, no isomorphism exists between the two original graphs since their complements are not isomorphic.

We've taken a bit of time to discuss ways in which two graphs can be shown to not be isomorphic. If they are isomorphic, all of these methods will fail and justifying the existence of an isomorphism comes down to describing explicitly where each vertex is mapped under an adjacency-preserving bijection. Finally, we conclude this section by defining the concept of an induced subgraph. Let G be a graph and $S \subseteq V(G)$. Then the **subgraph of G induced by S**, denoted $G[S]$, is the subgraph of G with vertex set S and edge set

$$E(G[S]) = \{ab \in E(G) \mid a, b \in S\}.$$

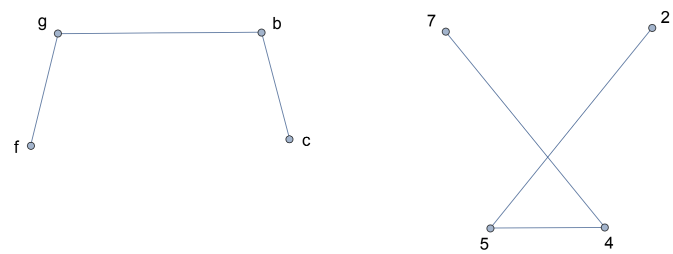
Example 2.10. Are the two graphs below isomorphic? If so, give an isomorphism. If not, justify your answer. Solution: Suppose that the two graphs (call the one



on the left G and the one on the right G') are isomorphic. Then by symmetry, there must be some isomorphism $\phi : V(G) \longrightarrow V(G')$ that maps $a \mapsto 1$. Since isomorphisms preserve adjacency, it follows that the neighbors of a (the vertices adjacent to a) must map to the neighbors of 1. That is

$$\{b, c, f, g\} \mapsto \{2, 4, 5, 7\}.$$

Consider the subgraphs induced by these two sets of vertices. The following graphs are $G[\{b, c, f, g\}]$ and $G'[\{2, 4, 5, 7\}]$, respectively. We see that these two subgraphs are indeed isomorphic, and we can gain more information about the map ϕ. Namely,



$$\{b, g\} \mapsto \{4, 5\} \quad \text{and} \quad \{c, f\} \mapsto \{2, 7\}.$$

At this point, it seems like there is a good chance that G and G' are isomorphic, so let's pick one of the two options for images of the vertex b and see if we can finish constructing an isomorphism. Without loss of generality, suppose that $\phi(b) = 5$. Then it follows that $\phi(c) = 2$, $\phi(f) = 7$, and $\phi(g) = 4$. The only vertex adjacent to both c and g in G, other than a and b, is e, which must map to 3 (the only vertex adjacent to both 2 and 4 in G', other than 1 and 5). We continue in this way, finding that ϕ is given by

$$a \mapsto 1 \quad b \mapsto 5 \quad c \mapsto 2 \quad d \mapsto 6 \quad e \mapsto 3 \quad f \mapsto 7 \quad g \mapsto 4,$$

showing that $G \cong G'$.

Exercises for Section 2.3

Exercise 2.3.1. List all nonisomorphic graphs of order 3 (note that they may not be connected).

Exercise 2.3.2. List all nonisomorphic graphs of order 4 (note that they may not be connected).

Exercise 2.3.3. Draw two nonisomorphic graphs of order 6 and size 10. Explain how you know they are not isomorphic.

Exercise 2.3.4. Are the two graphs in Figure 13 isomorphic? If so, give an isomorphism. If not, justify your answer.

Exercise 2.3.5. Are the two graphs in Figure 14 isomorphic? If so, give an isomorphism. If not, justify your answer.

Exercise 2.3.6. Prove that if a graph G is self-complementary, then $|V(G)| \equiv 0, 1 \pmod 4$. That is, prove that 4 divides $|V(G)|$ or 4 divides $|V(G)| - 1$.

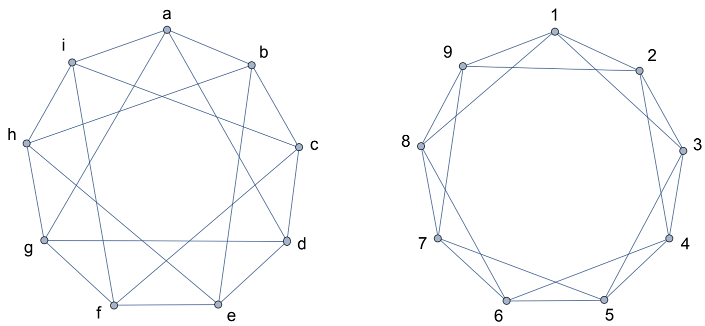


Figure 13. Two graphs of order 9.

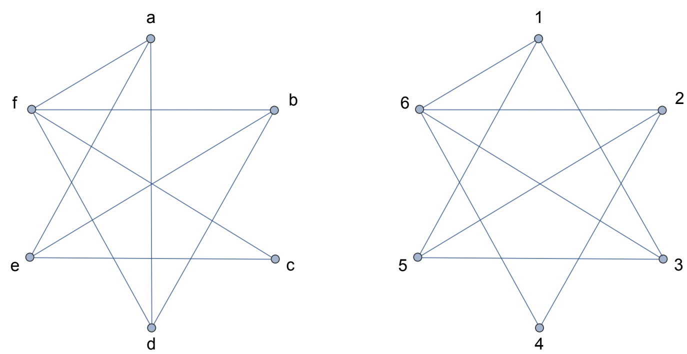
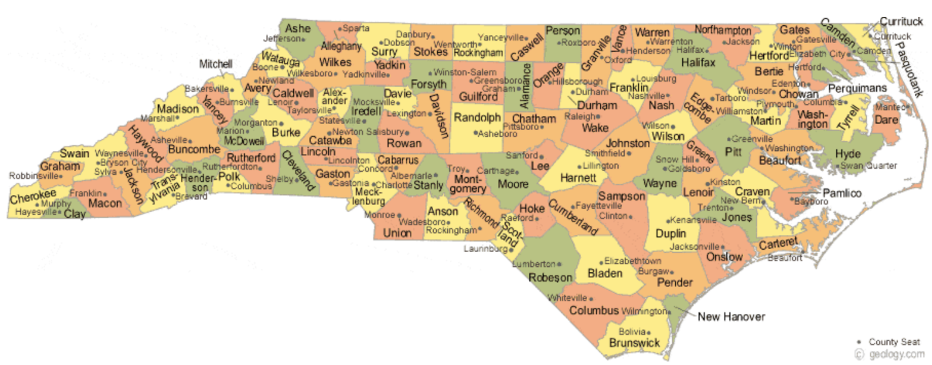


Figure 14. Two graphs of order 6.

2.4. Vertex Colorings and Planar Graphs

Consider the following map of the counties of North Carolina. Map-makers have

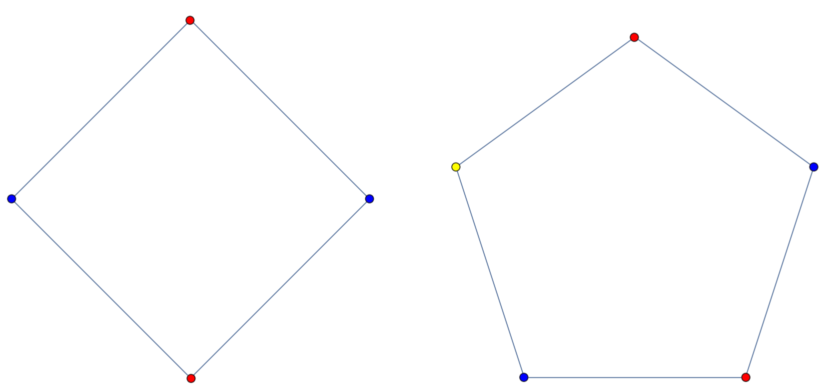


long observed that regardless of the map, only four colors seem to be necessary to color the regions (states, countries, counties, etc.) such that regions that share a border receive different colors. In this example, only the colors red, yellow, orange, and green are needed. This problem can be rephrased in graph theory by identifying each region with a vertex and two vertices are adjacent if and only if they share a border. Then, we are trying to color the vertices of the graph such that adjacent vertices receive different colors. Of course, it is not possible to obtain every graph in this way. For example, the complete graph K_5 requires five colors to guarantee that adjacent vertices receive different colors, but we claim that no map will ever result in a subgraph isomorphic to K_5. This will lead us to the concept of planar graphs.

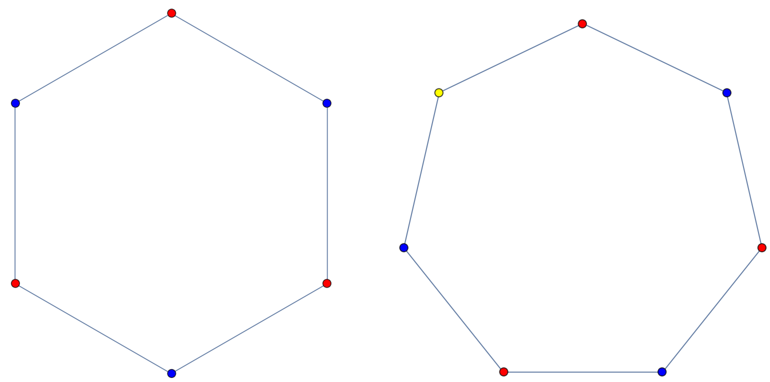
In 1976, Appel and Haken proved the 4-Color Theorem (that every planar graph can be properly 4-colored). The proof they offered was a first in mathematics! Namely, they reduced the problem down to a couple of thousand base cases, then programed computers to meticulously check each case. While certainly not a "proof from the book" like Paul Erdös would like, the proof was confirmed by Appel and Haken's peers. We will not attempt to prove the 4-Color Theorem, but the 5-Color Theorem will be proved later in this section. First, we offer some definitions concerning the coloring of graphs.

A **proper (vertex) coloring** of a graph G is an assignment of a collection of colors to the vertices of G such that no two adjacent vertices receive the same color. Equivalently, we can think of a proper coloring as corresponding to a map $f : V(G) \longrightarrow \mathbb{N}$, where each natural number corresponds to a different color. The minimum number of colors needed to properly color a graph G is called the **chromatic number** of G, and is denoted $\chi(G)$. Since every vertex in K_n is adjacent to every other vertex, it follows that $\chi(K_n) = n$. In the complete bipartite graph $K_{m,n}$ all vertices in V_1 can receive the same color, as can all vertices in V_2. Hence, $\chi(K_{m,n}) = 2$ (assuming $m \geq 1$ and $n \geq 1$).

Now we turn our attention to cycles. For $n \geq 3$, denote the cycle of length n by C_n. As $C_3 \cong K_3$, we find that $\chi(C_3) = 3$. Consider the next few values of n. Below, minimal proper vertex colorings are given. Observe that when n is even, two colors suffice. Also, two colors are necessary since the cycle contains at least one



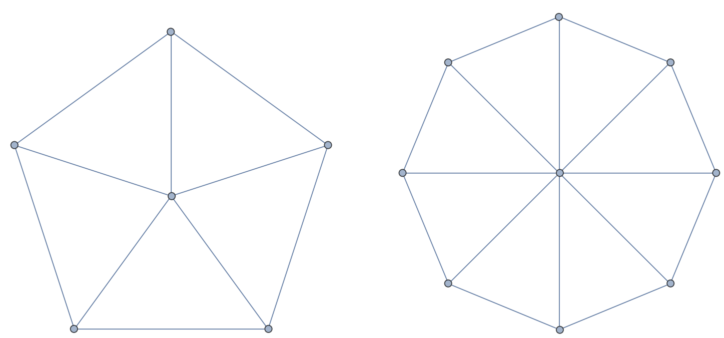
edge. In the case where n is odd, let $x_1 x_2 \cdots x_n x_1$ be such a cycle. Without loss of generality, suppose that x_1 is colored red and x_2 is colored blue. Since consecutive vertices must receive different colors, we can keep alternating between red and blue, coloring vertices with odd subscripts red and those with even subscripts blue. However, when we try to color vertex x_n, it cannot be colored red since x_1 is red and it cannot be colored blue since x_{n-1} is blue, so it requires a third color. Hence,



we have argued that for all $n \geq 3$,

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

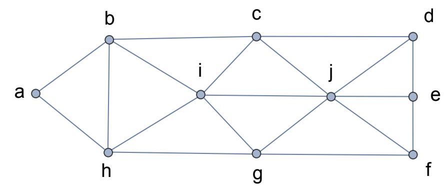
A **wheel graph** W_n consists of a cycle isomorphic to C_{n-1} and a single vertex x such that x is adjacent to every vertex in C_{n-1}. For example, the wheel graphs W_6 and W_9 are pictured below. In order to obtain a minimal proper vertex coloring



of W_n, we begin with a minimal proper vertex coloring of C_{n-1} (using two colors if $n-1$ is even and three colors if $n-1$ is odd). Since vertex x must be colored differently from all of the vertices in the cycle, we find that for $n \geq 4$,

$$\chi(W_n) = \begin{cases} 4 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

EXAMPLE 2.11. Recall the problem of scheduling committees at an organization such that committees with overlapping membership do not meet at the same time. One can represent this problem by letting the committees correspond with vertices and committees with overlapping membership are indicated with edges (as in the graph below). When we considered this problem previously, we began by finding a maximal independent set. Note that vertices in an independent set can receive the same color in a proper vertex coloring. We continued in this manner, partitioning the vertex set into independent sets, that corresponded to the hours that each collection of committees could meet. Hence, the minimum number of hours needed for all of the committees to meet corresponds to the chromatic number for the graph



being considered. Since this graph contains a subgraph isomorphic to K_3, at least three colors are needed to properly color the vertices. It remains to be seen if three colors are sufficient. In fact, there are many triangles in this graph and the three vertices in each triangle must receive different colors. Starting on the right side of the graph, without loss of generality, suppose that e, d, and j receive colors red, blue, and green, respectively. Then f can be colored blue, c and g can be colored red, i can be colored blue, and we have a problem when we get to b and h. They cannot be colored red (since c and g are red) and they cannot be colored blue (since i is blue). They also cannot receive the same color since they are adjacent. We can color b green, but must use a fourth color, say yellow, for h. a can then be colored red. Hence, the chromatic number for this graph is 4, meaning that four hours are needed for all of the committees to meet.

Returning to the problem of coloring maps, we must define what it means for a graph to be planar. A graph is **planar** if it can be drawn on a plane without any edges crossing. The graphs one obtains from identifying regions on a map with vertices and connecting regions with edges when they have a border in common are always planar. A **plane graph** refers to a planar depiction of a planar graph. For example, the graph on the left in Figure 15 is a planar graph, and the graph on the right is its plane graph depiction.

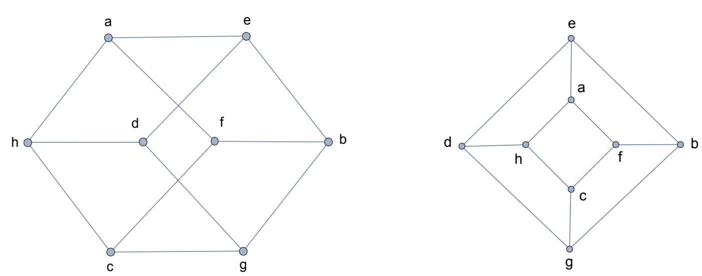


FIGURE 15. A planar graph and its plane graph depiction.

Given the connection planar graphs have with map-making, it is beneficial to have a classification of planar graphs. In 1930, Kuratowski offered the following theorem, which gives necessary and sufficient conditions for a graph to be planar. Before we state the theorem, a couple of definitions are in order. The process of adding vertices to the middle of edges in a graph is called **subdividing**. A subgraph of a graph G is called an ***H*-configuration** if it can be obtained from a graph H by subdividing. For example, Figure 16 contains a graph with a subgraph that is a K_4 configuration. The K_4 configuration is the subgraph induced by $\{b, c, e, f\}$

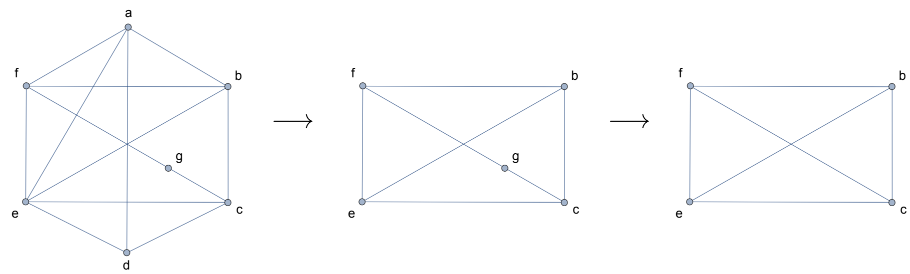


FIGURE 16. A graph containing a K_4-configuration.

since this subgraph can be formed by subdividing the edge cf using vertex g. Now we are in a position to state Kuratowski's Theorem. As the proof of this theorem is rather involved, we will refrain from providing a proof.

THEOREM 2.12 (Kuratowski, 1930). *A graph is planar if and only if it does not contain a subgraph that is a K_5-configuration or a $K_{3,3}$-configuration.*

Observe that it is implied by this theorem that K_5 and $K_{3,3}$ are not planar graphs. This is something that we will prove as applications of Euler's Formula, but first, we need a new definition. Given a plane graph G, a **region** of G is a maximal portion of the plane for which any two points may be joined by a curve C such that no point of C intersects a vertex or edge of G. In particular, note that every plane graph contains an exterior region that is unbounded. For example, the plane graph in Figure 17 contains three regions. At first it is not clear whether or not the number of regions of a plane graph depend upon the specific plane graph depiction that is drawn. This issue is resolved in the following theorem.

THEOREM 2.13 (Euler's Formula, 1752). *If G is a connected planar graph, then any plane graph depiction of G has $r = e - v + 2$ regions, where e and v are the size and order of G, respectively.*

PROOF. Since G is a connected planar graph, it is possible to draw a plane depiction of G edge-by-edge so that at each stage of the construction, one has a connected plane graph. Let G_n be the resulting graph after n edges have been drawn. Also, let r_n, e_n, and v_n denote the number of regions, size, and order or G_n, respectively. We proceed by (weak) induction on n. As a starting point, G_1 consists of a single edge, so $r_1 = 1$, $e_1 = 1$, and $v_1 = 2$, from which it is easily confirmed that $r_1 = e_1 - v_1 + 2$ holds. Now assume that G_{n-1} satisfies $r_{n-1} = e_{n-1} + v_{n-1} + 2$. In constructing G_n, we must consider two cases: either the new edge requires the addition of a vertex or it does not.

Case 1 Let xy be the edge added to form G_n from G_{n-1} and assume that x was not in G_{n-1}. Note that y must have been in G_{n-1} since G_n is assumed to be connected. In this case, no new regions are produced and the number of vertices and edges both increase by one:

$$r_n = r_{n-1}, \quad e_n = e_{n-1} + 1, \quad \text{and} \quad v_n = v_{n-1} + 1.$$

From the inductive hypothesis, it follows that

$$\begin{aligned} r_n &= r_{n-1} \\ &= e_{n-1} - v_{n-1} + 2 \\ &= (e_{n-1} + 1) - (v_{n-1} + 1) + 2 \\ &= e_n - v_n + 2. \end{aligned}$$

Case 2 Let xy be the edge added to form G_n from G_{n-1}, but this time assume that no new vertices are introduced. Then both x and$y$ are contained in G_{n-1} and the addition of this edge divides some region into two regions:

$$r_n = r_{n-1} + 1, \quad e_n = e_{n-1} + 1, \quad \text{and} \quad v_n = v_{n-1}.$$

From the inductive hypothesis, it follows that

$$\begin{aligned} r_n &= r_{n-1} + 1 \\ &= (e_{n-1} + 1) - v_{n-1} + 2 \\ &= e_n - v_n + 2. \end{aligned}$$

So, $r_n = e_n - v_n + 2$ holds for all n, and hence, the equation holds for the plane graph G. □

Analogous to the degree of a vertex, we can define the **degree of a region** in a plane graph to be the number of edges incident with that region (i.e., on the region's boundary). If a region is incident with both sides of an edge, then that edge gets counted twice in the region's degree. For example, consider the plane graph given in Figure 17. It contains three regions: A, B, and C. Here, A is the

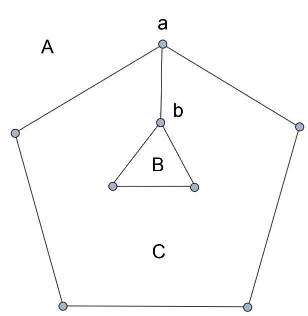


FIGURE 17. A plane graph with three regions.

exterior unbounded region. However, there are other plane graph depictions of this graph in which A would not be the exterior region. In this example, $deg(A) = 5$ and $deg(B) = 3$. When determining the degree of C, note that this region is incident with both sides of edge ab, so the edges gets counted twice, giving us $deg(C) = 10$.

With the concept of the degree of a region now in place, we obtain two useful corollaries to Euler's formula.

Corollary 2.14. *If G is a connected planar graph with size $e > 1$, then $e \leq 3v-6$, where v is the order of G.*

Proof. Consider a plane graph depiction of G. Since $e > 1$, every region must have degree ≥ 3. So, the sum of the degrees of all of the regions is at least $3r$. This sum must also equal $2e$, from which we obtain $2e \geq 3r$. Combining this inequality with Euler's formula results in

$$2e \geq 3(e - v + 2),$$

which is easily shown to be equivalent to the given inequality. □

Perhaps the most useful application of Corollary 2.14 comes from using its contrapositive. Consider the graph K_5, which has order 5 and size 10. Since $10 \not\leq 3 \cdot 5 - 6$, we can conclude that K_5 is not planar. Of course, if the inequality had been satisfied, then the corollary cannot be used to show that a connected graph is planar. Consider the bipartite graph $K_{3,3}$, which has order 6 and size 9, The inequality $9 \leq 3 \cdot 6 - 6$ is satisfied, but we will soon show that $K_{3,3}$ is not planar. Hence, the converse to Corollary 2.14 is not true.

Corollary 2.15. *If G is a connected bipartite planar graph with size $e > 1$, then $e \leq 2v - 4$, where v is the order of G.*

Proof. The proof is similar to that of the previous corollary except that every region must have degree ≥ 4 (since having degree 3 would imply the existence of a cycle of odd length). So, the sum of all of the degrees is at least $4r$, from which $2e \geq 4r$, along with Euler's formula, results in the given inequality. □

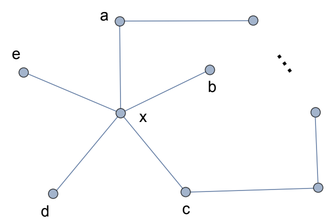
At this point, we check the inequality for $K_{3,3}$ and find that $9 \not\leq 2 \cdot 6 - 4$. Hence, $K_{3,3}$ is not planar.

Theorem 2.16 (The 5-Color Theorem). *Every planar graph can be properly 5-colored.*

Proof. Note that it is sufficient to prove this result for connected graphs. First, we claim that every connected planar graph G has a vertex of degree at most 5. Otherwise, every vertex has degree at least 6 and if we let v be the order of G, it follows that the sum of the degrees of the vertices in G is at least $6v$. Of course, the sum of the degrees of the vertices is $2e$, giving us $6v \leq 2e$. By Corollary 2.14, it follows that

$$6v \leq 2e \leq 2(3v - 6) = 6v - 12,$$

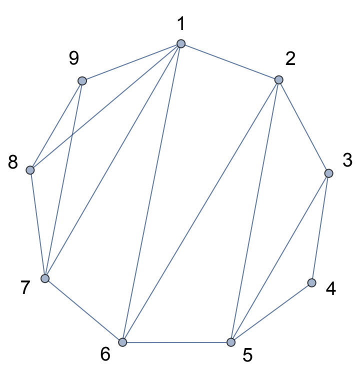
giving a contradiction. Thus, every connected planar graph contains some vertex x with degree at most 5. Now we proceed by induction on the number of vertices in G. A single vertex can be trivially 5-colored. Now suppose that every connected planar graph with n vertices can be 5-colored and let G be a connected planar graph of order $n + 1$. As we have seen, there exists some vertex x or degree at most 5. Removing this vertex from G, the subgraph induced by $V(G) - \{x\}$ is a connected planar graph of order n, and hence, it can be 5-colored. Fix a plane graph representation of G and color the vertices in $V(G) - \{x\}$ according to this

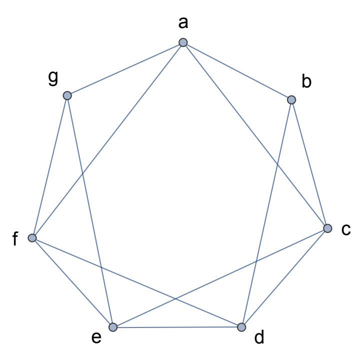


optimal 5-coloring. If $deg(x) \leq 4$, then there is a 5^{th} color that can be assigned to x, giving a 5-coloring of G. Otherwise, x has degree 5. Denote the vertices adjacent to x by a, b, c, d, e as in the picture. If any two vertices from $\{a, b, c, d, e\}$ have the same color, then there is at least one other color that can be assigned to x. So, suppose that a, b, c, d, e have different colors. Without loss of generality, suppose that a is red and c is blue. Now consider all red/blue paths emanating from a and the effect of switching the two colors of the vertices in these paths. If switching their colors does not affect c, then make this switch and color x red. If switching the colors does affect c, then there must be some red/blue path from a to c. In this case, consider the colors of b and d. Without loss of generality, suppose that b is green and d is yellow. Since we are considering a plane graph depiction of G, there can be no green/yellow path connecting b to d (as it would have to cross an edge in the red/blue path from a to c). So, we can switch the colors on all green/yellow paths emanating from b without affecting d and give x the color green. □

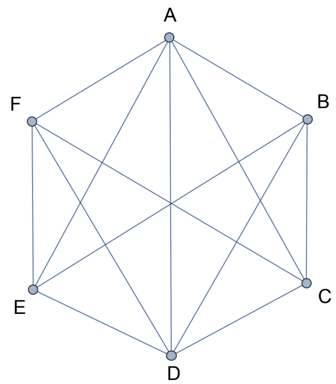
Exercises for Section 2.4

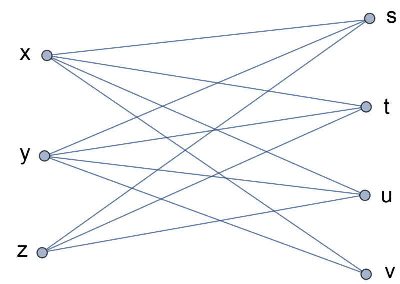
(1) Consider the following graphs.
   (a) Determine the chromatic numbers for each of the following graphs. In each case, provide a minimal proper coloring and explain how you know that no proper coloring with fewer colors exists.
   (b) Determine whether or not the graphs are planar. If so, provide a plane graph depiction. If not, justify your answer.





(2) For what values of n is the complete graph K_n planar? Justify your answer.





(3) For what values of m and n is the complete bipartite graph $K_{m,n}$ planar? Justify your answer.

(4) The *crossing number* $C(G)$ of a graph G is the minimum number of pairs of crossing edges in a depiction of G is a plane. So, if G is planar, then $C(G) = 0$. If G is not planar, then $C(G) > 0$. Determine the crossing numbers for the four graphs in Exercise 1.

(5) What are the crossing numbers for K_5 and K_6?

(6) If a connected plane graph has 14 edges and 9 regions, then what is its order?

(7) If a connected plane graph has 7 vertices, all of degree 4, then how many regions does it have?

(8) If a connected plane graph with n vertices, all of degree 4, has 10 regions, then what is n?

(9) Prove that a connected planar graph G with 8 vertices and 13 edges cannot be properly 2-colored.

(10) Let G be a graph of order n. Prove that

$$\chi(G) + \chi(\overline{G}) \leq n + 1.$$

Hint: use induction on a vertex-by-vertex construction of G.

(11) Without using the 4 or 5-Color Theorems, prove that every connected planar graph can be properly 6-colored. *Hint: you may use the fact that every connected planar graph has some vertex of degree at most 5.*

2.5. Chromatic Polynomials and Inclusion-Exclusion

The chromatic number $\chi(G)$ gives a measure of the connectivity of G. Another tool that we can use to study connectivity is the chromatic polynomial. For a graph G, define the *chromatic polynomial* $\mathcal{P}_G(k)$ to be the number of ways to properly color the vertices of G using k distinct colors. From this definition, we find that the least natural number m for which $\mathcal{P}_G(m) \neq 0$ is the chromatic number $\chi(G)$.

For example, consider the path P of length 5 (pictured below). If we wish to count the number of ways to properly color P using k colors, begin with vertex a, which can receive any of the k colors. After we select a color for a, then consider the number of possible colors for b. Regardless of which color a receives, there are $k-1$

possible colors that b can receive (note the independence of this selection). Next, there are $k-1$ colors that can be given to c (we just need to avoid the color that b receives. Continuing in this manner and applying the Multiplication Principle, we find that

$$\mathcal{P}_P(k) = k(k-1)^5.$$

In fact, the method employed above can be applied to a path P_ℓ of length ℓ to find that

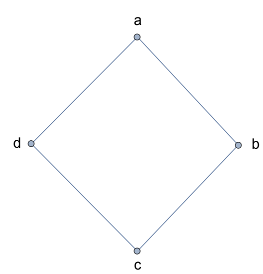
$$\mathcal{P}_{P_\ell}(k) = k(k-1)^\ell.$$

Observe that the first natural number that makes this polynomial nonzero is 2, the chromatic number for P_ℓ. For complete graphs, all of the vertices must receive different colors, from which we find

$$\mathcal{P}_{K_n}(k) = k(k-1)\cdots(k-n+1) = \frac{k!}{(k-n)!}.$$

If we consider cycles, we find that the determination of the chromatic polynomial becomes more challenging.

Let C_4 denote a cycle of length 4 (as pictured below). Beginning with vertex



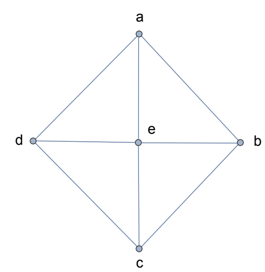
a, we can use any of the k colors. Then vertex b can receive any color other than the one used for a ($k-1$ possibilities). When we get to c, there are again $k-1$ possibilities, but arriving at d, we find that independence is not satisfied. Namely, if a and c happened to receive the same color, then there are $k-1$ possibilities for d. If a and c received different colors, then d has $k-2$ possibilities.

Thus, the determination of $\mathcal{P}_{C_4}(k)$ requires two separate (disjoint) cases. In the first case, we count the number of proper colorings in which a and c receive the same color. There are $k(k-1)^2$ such colorings. Second, we count the number of proper colorings in which a and c receive different colors. There are $k(k-1)(k-2)^2$ such colorings. By the Addition Principle, we find that

$$\begin{aligned}\mathcal{P}_{C_4}(k) &= k(k-1)^2 + k(k-1)(k-2)^2 \\ &= k(k-1)(k^2-3k+3)\end{aligned}$$

As with P_ℓ, the first natural number that makes this polynomial nonzero is 2, the chromatic number for C_4.

Now that we have $\mathcal{P}_{C_4}(k)$, consider the chromatic polynomial for the wheel graph W_5. Start by counting the number of possible colors for e, the central vertex.

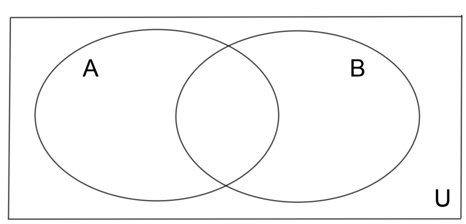


It can receive any of the k colors. Now, we color the remaining C_4 avoiding the color used for e. As before, we have two cases based on whether or not a and c receive the same color. If a and c receive the same color, then there are $(k-1)(k-2)^2$ ways to color the remaining vertices. If a and c do note receive the same color, then there are $(k-1)(k-2)(k-3)^2$ ways to color the remaining vertices. Hence, we find that

$$\begin{aligned}\mathcal{P}_{W_5}(k) &= k((k-1)(k-2)^2 + (k-1)(k-2)(k-3)^2) \\ &= k(k-1)(k-2)(k^2-5k+7).\end{aligned}$$

The first natural number that makes this polynomial nonzero is 3, the chromatic number for W_5.

As we continue working towards finding chromatic polynomials for cycles and wheels, we find that our arguements require many cases and subcases. To get around this issue, we will use the Inclusion-Exclusion Principle, which will provide a means of determining the cardinality of the intersection of a finite collection of sets. Before we state this principle in its most general form, consider the Venn diagram for two sets. Letting $|S|$ denote the cardinality of a set S, we wish to



determine $|A \cup B|$. When considering $|A| + |B|$, we find that the elements in $A \cap B$ are counted twice. Hence, it follows that

$$|A \cup B| = |A| + |B| - |A \cap B|. \tag{2.1}$$

To see how this equation can be applied, consider the following example.

EXAMPLE 2.17. How many natural numbers less than or equal to 300 are there that are not divisible by 3 and are not divisible by 5?

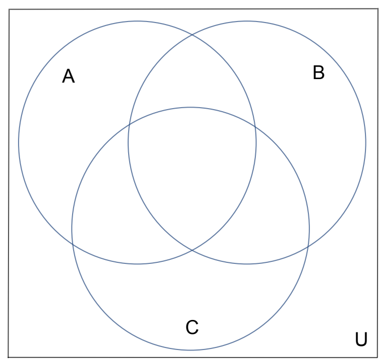
Solution: Let A denote the set of 3-digit natural numbers that are divisible by 3 and let B denote the set of 3-digit natural numbers that are divisible by 5. Then we are trying to find $|\overline{A} \cap \overline{B}|$. This set is equal to $|\overline{A \cup B}|$ by DeMorgan's Law. So, by Equation (2.1), we have

$$\begin{aligned} |\overline{A} \cap \overline{B}| &= |U| - |A \cup B| \\ &= |U| - |A| - |B| + |A \cap B|. \end{aligned}$$

The cardinality of U is 300. The number of such numbers divisible by 3 is given by $|A| = \frac{300}{3} = 100$. The number of such numbers divisible by 5 is $|B| = \frac{300}{5} = 60$. Elements in $A \cap B$ are precisely the numbers divisible by 15. Hence, $|A \cap B| = \frac{300}{15} = 20$. Thus, we find that

$$|\overline{A} \cap \overline{B}| = 300 - 100 - 60 + 20 = 160.$$

Now consider the case of three sets. Counting the number of elements in $|A| + |B| + |C|$ counts all intersections of two sets twice and the intersection of all three



sets three times. So, we must subtract off one copy of each intersection of two sets:

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|,$$

but this number does not count the intersection of all three sets. Thus,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

As with the previous example, this equation can be used to count $|\overline{A} \cap \overline{B} \cap \overline{C}|$. The Inclusion-Exclusion Principle provides a general form for enumerating such sets when an arbitrary number of sets are being considered.

Theorem 2.18 (Inclusion-Exclusion Principle). *Let $A_1, A_2, \dots A_n$ be n sets in some finite universal set U. If S_k denotes the sum of the cardinalities of all k-tuple intersections of the A_is, then*

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |U| - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n.$$

Observe that in this theorem,

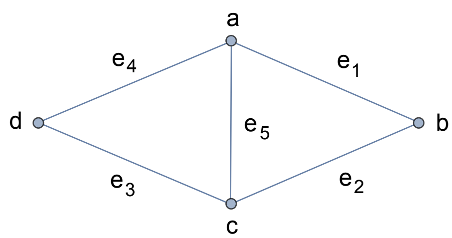
$$S_k = \sum_{\substack{i_j \in \{1,2,\dots,n\} \\ i_j \neq i_{j'}}} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|.$$

When $k = 1$, the interesections each consist of a single set:

$$S_1 = |A_1| + |A_2| + \cdots + |A_n|.$$

We will refrain from proving this theorem here, but will focus on its application. The following example demonstrates how Inclusion-Exclusion can be used to find chromatic polynomials.

EXAMPLE 2.19. Find the chromatic polynomial for the following graph. Solution:



Let the universal set be the number of ways of coloring the vertices in this graph (not necessarily properly) using k colors: $|U| = k^4$. Then let A_i be the number of ways of coloring the vertices (not necessarily properly) such that both endpoints of e_i receive the same color. Thus, $S_1 = 5k^3$ since $|A_i| = k^3$ for each i, corresponding to the endpoints of e_i having the same color and the other two vertices having any color. We also find that for every pair of distinct edges,

$$|A_i \cap A_j| = k^2 \quad \Longrightarrow \quad S_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} k^2 = 10k^2.$$

When counting intersections of three edges, we need to be more careful. Specifically, it is possible that three edges include only three vertices (a K_3) and it is possible that three edges include all 4 vertices. In the former case, we obtain k^2 and in the latter case, k. Since the graph contains exactly 2 triangles, it follows that

$$S_3 = 2k^2 + \left(\begin{pmatrix} 5 \\ 3 \end{pmatrix} - 2 \right) k = 2k^2 + 8k.$$

When counting intersections of four edges, all four vertices get included and there are k ways to choose a color. The same is true for intersections of five edges and we find that

$$S_4 = 5k \qquad \text{and} \qquad S_5 = k.$$

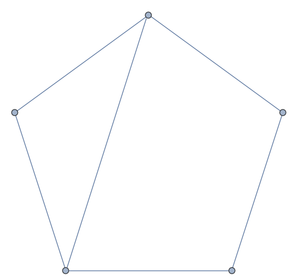
Putting together all of these cases, and using the Inclusion-Exclusion Principle, we find that if this graph is denoted by G, then

$$\begin{aligned} \mathcal{P}_G(k) &= |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| \\ &= k^4 - 5k^3 + 10k^2 - (2k^2 + 8k) + 5k - k \\ &= k^4 - 5k^3 + 8k^2 - 4k. \end{aligned}$$

Exercises for Section 2.5

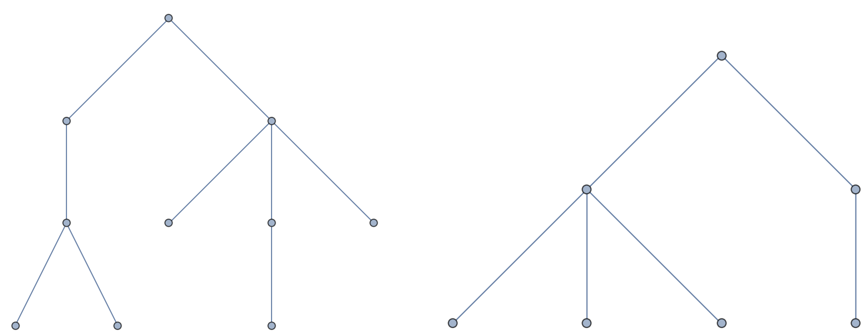
(1) Find $\mathcal{P}_{C_5}(k)$, giving a description of how you determined it.
(2) Find $\mathcal{P}_{W_6}(k)$, giving a description of how you determined it.
(3) Find $\mathcal{P}_{K_{2,n}}(k)$, giving a description of how you determined it.
(4) Find $\mathcal{P}_{K_{3,3}}(k)$, giving a description of how you determined it.
(5) How many natural numbers less than or equal to 2100 are not divisible by 3, are not divisible by 5, and are not divisible 7?

(6) How many ways are there to roll eight distinct (6-sided) dice such that all 6 faces appear?
(7) Use the Inclusion-Exclusion Principle to determine the chromatic polynomial for the following graph.



2.6. Trees

Due to their extensive use in applications as well as their extremal properties, trees serve many roles in graph theory. A **tree** is defined to be a connected graph that does not contain any cycles. In particular, trees do not contain cycles of even length, so by Theorem 2.9, they are bipartite. While paths are one type of tree, a couple of other examples are shown below. In any tree, the vertices that have



degree equal to one are called **leaves**. Vertices of degree greater than 1 are called **internal vertices**. We have characterized trees as being acyclic, but there are several equivalent definitions that are useful in various settings.

THEOREM 2.20. *Let T be a connected graph. Then the following are equivalent.*

(a) *T does not contain any cycles.*
(b) *There is a unique path between any distinct pair of vertices in T.*
(c) *T is minimally connected. That is, the removal of any edge of T (while keeping all vertices) disconnects T.*

PROOF. We prove a cyclic sequence of implications to obtain the given equivalence. ($(a) \implies (c)$) We prove this implication by proving the contrapositive statement. Suppose that T is not minimally connected. Then there exists some pair of vertices x and y such that the removal of edge xy from T does not disconnect T. If we

denote the resulting graph by $T - xy$, then we find that $T - xy$ is connected, and hence, it contains a path P between the vertices x and y. The path P along with edge xy form a cycle in T.
$((b) \implies (a))$ Again, we prove the contrapositive statement. Suppose that T contains a cycle C. Then the edges of C can be used to form two distinct paths between any pair of distinct vertices in the cycle. Hence, there does not exist a unique path between any pair of vertices in T.
$((c) \implies (b))$ Suppose that T is minimally connected, but there exists two different paths P_1 and P_2 between vertices x and y. Let $e = uv$ be the first edge on P_1, starting at x, that is not on P_2. By our assumption, $T - e$ must be disconnected with u and v in different connected components. If we let w be the next vertex in common with P_1 and P_2, then following P_2 from u to w, then coming back along P_1 from w to v creates a path from u to v, contradicting the assumption that $T - e$ was disconnected. Hence, any path between x and y must be unique. □

THEOREM 2.21. *A tree of order n has size $n - 1$.*

PROOF. As a connected graph, a tree can be formed edge-by-edge with the resulting graph connected each step of the way. If T is a tree, then let T_m be the resulting connected graph after m edges have been constructed. Then T_m must be minimally connected, and hence, is also a tree, for all m. We proceed by induction on m. When $m = 1$, T consists of a single edge, so it contains 2 vertices. Now suppose that T_{m-1} has order m and consider T_m. When constructing a new edge to form T_m, observe that exactly one new vertex must be added to the graph. If two vertices were added, then T_m would not be connected. If no vertices were added, then the new edge would connect some pair of vertices x and y that were already contained in T_{m-1}. Since T_{m-1} was assumed to be a tree, it contained a unique path between vertices x and y. This path, along with edge xy, form a cycle in T_m, contradicting the assumption that T_m is a tree. Hence, T_m must contain one more vertex than T_{m-1}, so T_m has order $m + 1$. □

Now we turn our attention to trees as subgraphs of connected graphs. For any graph G, let define the *minimum degree*

$$\delta(G) = \min\{\deg_G(x) \mid x \in V(G)\}.$$

THEOREM 2.22. *Let T be a tree of order $n \geq 1$. If G is any graph such that $\delta(G) \geq n - 1$, then G contains a subgraph that is isomorphic to T.*

PROOF. We proceed by induction on $n \geq 1$. When $n = 1$, any graph (which necessarily contains at least one vertex) will contain a subgraph isomorphic to a tree of order 1, even if G does not contain any edges. Now assume that the theorem is true for all trees of order n and let T be a tree of order $n+1$. Also assume that G is a graph such that $\delta(G) \geq n$. Let T' be the tree formed by removing a single leaf from T and let v be the vertex in T' that was adjacent to the removed leaf. By the inductive hypothesis, G must contain a subgraph isomorphic to T'. Since T' has order n, the vertex v is adjacent with at most $n-1$ vertices in T'. The assumption $\delta(G) \geq n$ then implies that v must also be adjacent to some vertex in G that is not in T'. This edge, along with T' form a subgraph of G that is isomorphic to T. □

A **spanning tree** of a graph G is a subgraph of G that is a tree and contains all vertices in G. Of course, one criterion that a graph must satisfy in order to have a spanning tree is that it must be connected. The following theorem shows that this condition is also sufficient.

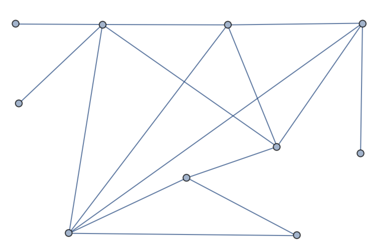
THEOREM 2.23. *Every connected graph G contains a spanning tree.*

PROOF. For this proof we use the fact that trees are minimally connected. Let G be a connected graph. If G is minimally connected, then it is a tree and we are done. Otherwise, there exists some edge e_1 whose removal does not disconnect G. Now let $G_1 = G - e_1$ be the graph formed by removing e_1 from G (while keeping all vertices). If G_1 is minimally connected, we are done. Otherwise, we continue in this manner, forming $G_i = G_{i-1} - e_i$ by removing an edge e_i that does not disconnect G_{i-1} whenever G_{i-1} is not minimally connected. As G contains a finite number of edges, this process must terminate with some G_k that is minimally connected and still spans the vertices in G. □

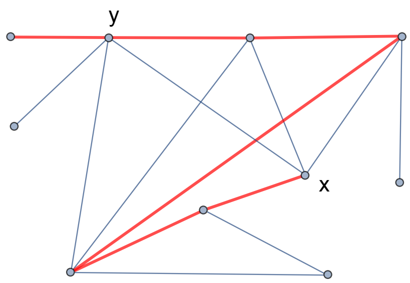
While the above proof shows a method for finding a spanning tree in a connected graph, there are two other methods worth noting as they use sequential choices that can be applied to computerized solutions to problems. In the first method, known as **depth-first**, we start with a given vertex x (called the **root**) in a connected graph G. Then we begin constructing a path from x until it is no longer possible to continue without repeating a vertex. The vertex where the path stops will be a leaf in the spanning tree. Next, we back up to the vertex in the path that was adjacent to this leaf and try to construct a path form this vertex to other vertices in G that were not included in the initial path. We continue this process, backtracking when necessary, until all vertices in G have been included. Since we have not allowed any vertices to be repeated, the result will be a spanning tree.

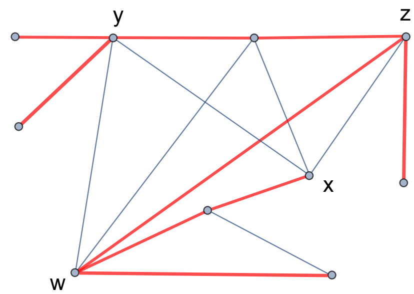
Another method for finding a spanning tree in a connected graph G is called **breadth-first**. As before, we start with a root x. However, this time we include all edges incident with x. We continue to successively add in edges leaving the vertices adjacent to x, provided they do not form a cycle with edges already included. This process is continued one level at a time, where each level corresponds to the distance from x. Since we avoid forming any cycles, the result will be a spanning tree.

In order the demonstrate these two methods, consider the following connected graph G. We start by identifying a vertex x to serve as the root. Using a depth-



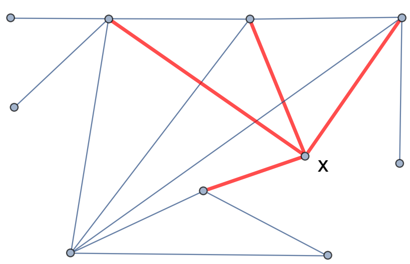
first construction, we begin by tracing out a path from x until we can no longer proceed without repeating a vertex. The terminal vertex for our initial path is a leaf. We back up to the previous vertex, y, and add try to produce a path from y to

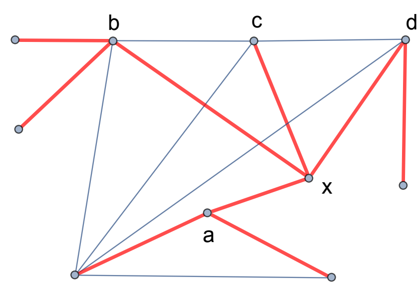




the remaining vertices. In this case, we are only allowed to include one additional vertex. So, we backtrack along the original path, including paths from z and w to form a spanning tree.

Using the same graph and the same choice of a root, we now consider a breadth-first construction. All edges incident with x are included. Label the vertices adjacent to x as a, b, c, and d. Next, we include the edges to vertices adjacent to a that

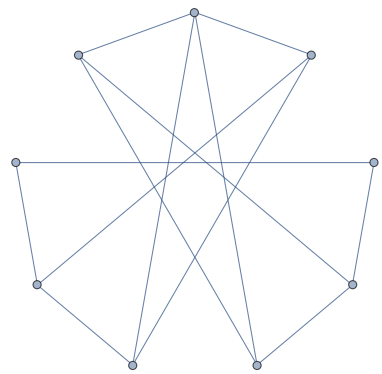




are not already included. Then we repeat this process with b, c, and d, including all vertices, and forming a spanning tree.

Exercises for Section 2.6

(1) List all nonisomorphic trees of order 4.
(2) List all nonisomorphic trees of order 5.
(3) Prove that every tree with order greater than one contains at least two leaves.
(4) Let T be a tree with size at least one. Then what is $\chi(T)$? Justify your answer.
(5) A **forest** is a disconnected graph that is the disjoint union of trees. If F is a forest of order n that consists of the disjoint union of $t > 1$ trees, then what is the size of F? Justify your answer.
(6) Prove that if T is a tree, then T is planar.
(7) Prove that a connected graph G is a tree if and only if it contains fewer edges than vertices.
(8) Find spanning trees in the following connected graph using both depth-first and breadth-first constructions.



CHAPTER 3

Extremal Problems

The underlying idea behind extremal problems in combinatorics and graph theory is that with quantity, comes structure. We seek to find precise bounds for which a large quantity implies the existence of certain properties. With regard to graphs, we will first focus on how many edges a graph must contain to guarantee the existence of certain subgraphs (Turán numbers). In Section **??**, we will turn our attention to edge-colorings, seeking to determine how many vertices an edge-colored complete graph must have to guarantee the existence of certain monochromatic subgraphs (Ramsey numbers). Finally, we will consider the combinatorial problem of coloring numbers in the set $\{1, 2, \ldots, n\}$ using k colors, while avoiding monochromatic triples $a + b = c$ (Schur's Theorem).

3.1. Turán Graphs

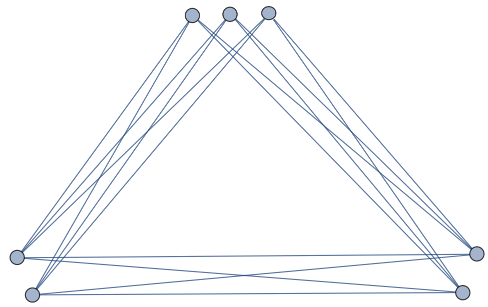
Recall that a clique is a complete subgraph of a graph. A *p-clique* of a graph G is a complete subgraph on p vertices. In 1941, Pál Turán [**15**] considered the problem of determining the largest size (number of edges) that a graph G on n vertices can have if it does not contain any p-cliques. In this note, we will investigate Turán's approach to this problem.

A graph G is called q-partite if $V(G)$ can be partitioned into q disjoint subsets $V_1, V_2, \ldots, V_q$ such that if two vertices are adjacent, then they must lie in distinct sets V_i and V_j $(i \neq j)$. The **Turán graph** $T_q(n)$ is then defined to be the complete q-partite graph on n vertices than is balanced (that is, the cardinalities of any two partite sets V_i and V_j differ by at most 1). Turán found that of all graphs on n vertices that lacked $(q+1)$-cliques, $T_q(n)$ had the largest size. Specifically, he proved that the size of any graph on n vertices that lacks $(q+1)$-cliques satisfies

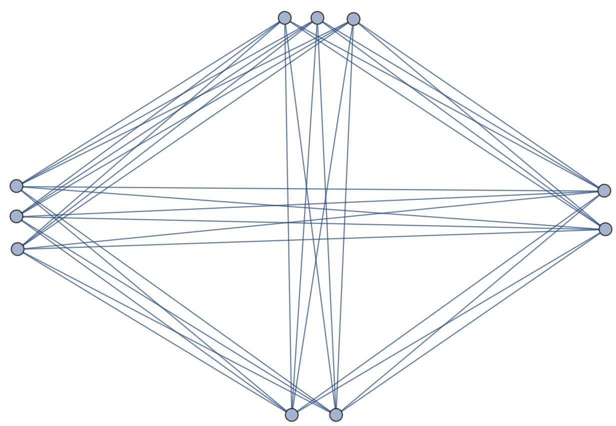
$$e \leq \left(1 - \frac{1}{q}\right) \frac{n^2}{2},$$

where e is the size of the graph. This is the content of Theorem 3.3 below, but we will first consider some examples of Turán graphs.

EXAMPLE 3.1. One can check that the graph $T_3(7)$ has size $16 = \lfloor \left(1 - \frac{1}{3}\right) \frac{7^2}{2} \rfloor$.



EXAMPLE 3.2. Verify that the graph $T_4(10)$ has size $37 = \lfloor \left(1 - \frac{1}{4}\right) \frac{10^2}{2} \rfloor$.



THEOREM 3.3 (Turán, 19??). *If a graph G on n vertices has size e and no p-cliques ($p \geq 2$), then*

$$e \leq \left(1 - \frac{1}{p-1}\right) \frac{n^2}{2}.$$

PROOF. We proceed by induction on n. If $n < p$, then it is not possible for such a graph to contain a p-clique. The maximum number of edges in a graph on n vertices is

$$\frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} \leq \frac{n^2}{2} - \left(\frac{n}{p-1}\right) \frac{n}{2}.$$

Suppose that the theorem is true for all graphs having fewer than n vertices and let G be a graph on n vertices that lacks p-cliques and has a maximal number of edges (where $n \geq p$). Notice that G must contain $(p-1)$-cliques; otherwise, more edges could be added without producing a p-clique. Let A be a subset of $V(G)$ whose induced subgraph of G is a $(p-1)$-clique. If e_A is the size of A, then it is easily seen that

$$e_A = \frac{(p-1)(p-2)}{2}.$$

Let $B = V(G) - A$ and define e_B to be the size of the subgraph of G induced by B. By the inductive hypothesis,

$$e_B \leq \left(1 - \frac{1}{p-1}\right)\frac{(n-p+1)^2}{2}.$$

The only edges that we have not counted in G are those that connect vertices in A to vertices in B. Let e_{AB} be the number of such edges. No vertex in B can be incident with more than $p-2$ vertices in A or G would contain a p-clique. Thus, we find that

$$e_{AB} \leq (p-2)(n-p+1).$$

Using the fact that $e = e_A + e_B + e_{AB}$, it follows that

$$\begin{aligned} e &\leq \frac{(p-1)(p-2)}{2} + \left(1 - \frac{1}{p-1}\right)\frac{(n-p+1)^2}{2} + (p-2)(n-p+1) \\ &\leq \frac{(p-1)(p-2)}{2} + \left(\frac{p-2}{p-1}\right)\frac{n^2 - 2(p-1)n + (p-1)^2}{2} + n(p-2) - (p-2)(p-1) \\ &\leq \left(\frac{p-2}{p-1}\right)\frac{n^2}{2} = \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}, \end{aligned}$$

completing the proof. □

Exercises for Section 3.1

(1) Determine the size of $T_3(14)$ and show that it has the maximal number of edges possible in any graph on 14 vertices that lacks 4-cliques.
(2) Determine the size of $T_5(22)$ and show that it has the maximal number of edges possible in any graph on 22 vertices that lacks 6-cliques.
(3) By the Division Algorithm, there exists unique $k, r \in \mathbb{Z}$ such that $n = qk + r$, where $0 \leq r < q$. Using the observation that the only possible cardinalities of partite sets in $T_q(n)$ are $\lfloor \frac{n}{q} \rfloor$ and $\lceil \frac{n}{q} \rceil$, write an expression that gives the size of $T_q(n)$ in terms of $\lfloor \frac{n}{q} \rfloor$ and $\lceil \frac{n}{q} \rceil$.
(4) Prove that whenever q divides n, $T_q(n)$ has exactly $\left(1 - \frac{1}{q}\right)\frac{n^2}{2}$ edges.

3.2. Ramsey Numbers

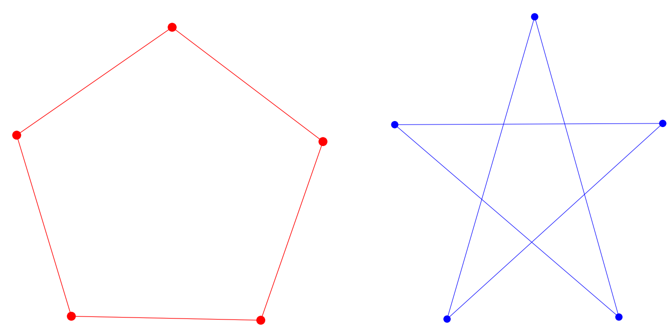
The work of Frank Ramsey [**13**] from 1930 leads to the following question:

How many people must be at a gathering to guarantee that there are at least three mutual acquaintances or at least three mutual strangers?

The solution to this question helped launch the area of Extremal Graph Theory known as Graph Ramsey Theory. Before we address this question, let us make precise the definition of a Ramsey number. For $s, t \in \mathbb{N}$, the **Ramsey number** $R(s, t)$ is the least $n \in \mathbb{N}$ such that every graph with n (or more) vertices contains a K_s-subgraph (a subgraph isomorphic to a complete graph on s vertices) or its complement contains a K_t-subgraph. Since the union of a graph and its complement produces a complete graph, $R(s, t)$ can also be defined in terms of the arbitrary coloring of the edges of a complete graph using two colors. Namely, if $s, t \in \mathbb{N}$, then $R(s, t)$ can be defined to be the least $n \in \mathbb{N}$ such that every arbitrary red/blue coloring of the edges of K_n contains a red K_s-subgraph or a blue K_t-subgraph.

3.2.1. Exact Values. From our first definition, we see that Ramsey's question is equivalent to determining the value of $R(3,3)$ (we identify the people at a gathering with vertices and two vertices are adjacent if and only if the corresponding individuals know each other). Before we tackle $R(3,3)$, consider a few simpler Ramsey numbers. Since every graph with at least one vertex contains a K_1-subgraph, it follows that $R(1,t) = 1$ for all $t \in \mathbb{N}$. We also see that $R(2,t) = t$ for all $t \in \mathbb{N}$ since a graph that lacks a K_2-subgraph does not have any edges, making its complement a complete graph. From our second definition, it is easily seen that $R(s,t) = R(t,s)$. Thus, the cases that remain are those for which $s, t \geq 3$.

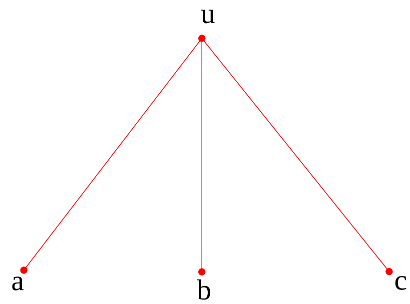
In general, the explicit evaluation of $R(s,t)$ is rather difficult, but lower bounds for such numbers are easily obtained by specific examples. For example, consider the following graph and its complement.



Neither of these graphs contain a triangle, from which we conclude that $R(3,3) > 5$. These graphs are examples of circulant graphs and the "randomness" of such graphs makes them less likely to contain *cliques* (complete subgraphs) of large order. Hence, they are often used to provide lower bounds of Ramsey numbers. The following theorem gives a complete answer to Ramsey's question.

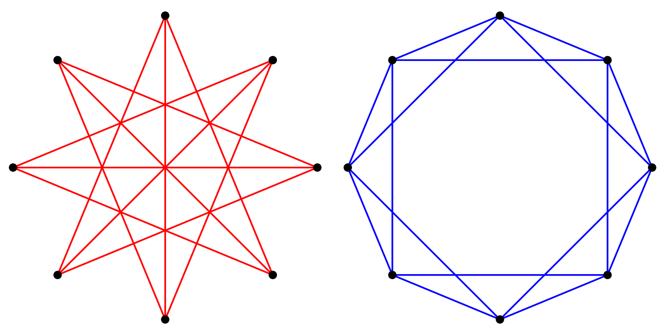
THEOREM 3.4 (Greenwood and Gleason, 1955). *$R(3,3) = 6$.*

PROOF. From the previous example, all that remains to be shown is that $R(3,3) \leq 6$. Of course, if every graph with 6 vertices contains a triangle or its complement contains a triangle, then the same would be true for any graph with more than 6 vertices (just remove the unnecessary vertices and the edges incident with those vertices). So, we will show that for any red/blue coloring of the edges of K_6, there must be a monochromatic triangle. Each vertex in K_6 has degree 5 and hence, at least 3 of the edges incident with a given vertex must share the same color. Without loss of generality, assume that vertex u is such that edges ua, ub, and uc are all red, resulting in the following subgraph.



If any of ab, bc, or ac is red, then we obtain a red triangle. Otherwise, they must all be blue and (a, b, c) forms a blue triangle. In either case, we have a monochromatic triangle, from which we conclude $R(3, 3) \leq 6$. □

As with $R(3, 3)$, we begin the determination of $R(3, 4)$ and $R(4, 4)$ by using circulant graphs to prove lower bounds. Of course, this will require us to be able to determine the order of a maximal complete subgraph (known as the *clique number*) for such graphs. Consider the following circulant graph on 8 vertices and its complement.



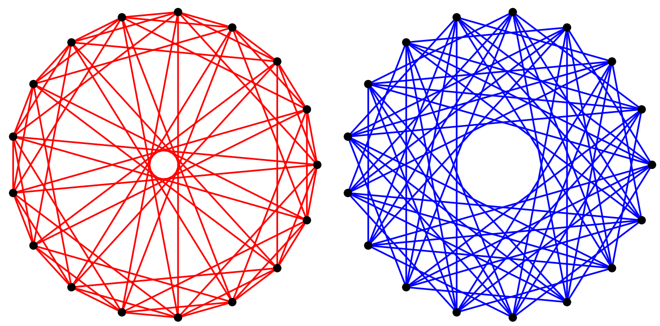
One can check by inspection that the red graph has clique number 2, while its complement has clique number 3. The lack of red triangles and blue K_4-subgraphs implies that $R(3, 4) > 8$. The following theorem will show this Ramsey number to be 9 (this was originally proved by Greenwood and Gleason [**10**] in 1955).

THEOREM 3.5 (Greenwood and Gleason, 1955). $R(3, 4) = 9$.

PROOF. It remains to be shown that $R(3, 4) \leq 9$. Consider an arbitrary red/blue coloring of the edges of K_9. We will need to consider 3 cases.
Case 1: Suppose that some vertex u is incident with at least 4 red edges: ua_1, ua_2, ua_3, and ua_4. If any two of $\{a_1, a_2, a_3, a_4\}$ are adjacent via a red edge, then including u produces a red triangle. Otherwise, the four vertices form a blue K_4.
Case 2: Suppose that some vertex v is incident with at least 6 blue edges: vb_1, vb_2, ..., vb_6. By Theorem 3.4, $\{b_1, b_2, \ldots, b_6\}$ contains a red triangle or a blue triangle. If it includes a blue triangle, then including v produces a blue K_4.
Case 3: The only graphs we have neglected are those in which every vertex is incident with 3 red edges and 5 blue edges. In this case, the graph spanned by the

blue edge contains 9 vertices of degree 5. Of course, it is not possible for a graph to have an odd number of vertices of odd degree, so this case does not occur. □

In general, the clique number of a graph is very difficult to determine. However, Mathematica is able to find clique numbers for graphs of relatively small order. Consider the following graph on 17 vertices and its complement.



Using Mathematica, one can check that the clique numbers for both of the above graphs are 3. Hence, we find that $R(4,4) > 17$. The following theorem is also due to Greenwood and Gleason [**10**].

THEOREM 3.6 (Greenwood and Gleason, 1955). *$R(4,4) = 18$.*

PROOF. It remains to be shown that $R(4,4) \leq 18$. Consider an arbitrary red/blue coloring of the edges of K_{18}. Each vertex has degree 17. Hence, a given vertex (call it u) must be incident with at least 9 edges of the same color. Without loss of generality, assume that ua_1, ua_2, ..., ua_9 are all red. By Theorem , the subgraph induced by $\{a_1, a_2, \ldots, a_9\}$ is a red/blue coloring of the edges of K_9 and must contain a red K_3 or a blue K_4. If it contains a red K_3, then including u results in a red K_4. Thus, we conclude that $R(4,4) \leq 18$. □

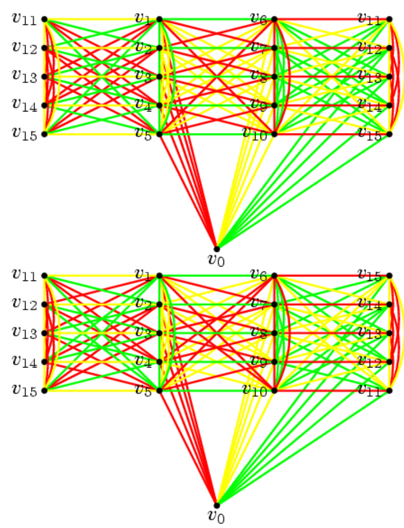
Besides the Ramsey numbers we have calculated so far, very few exact determinations of Ramsey numbers are currently known (although ranges of possible values are known for quite a few). In fact, the only other known Ramsey numbers (see [**12**]) are $R(3,6) = 18$, $R(3,7) = 23$, $R(3,8) = 28$, $R(3,9) = 36$, $R(4,5) = 25$. Although we are unable to give precise values of others, many ranges are known for "small" Ramsey numbers. For example,

$$\begin{array}{lll} 36 \leq R(4,6) \leq 41, & 49 \leq R(4,7) \leq 61, & 43 \leq R(5,5) \leq 49, \\ 58 \leq R(5,6) \leq 87, & 102 \leq R(6,6) \leq 165, & 113 \leq R(6,7) \leq 298, \\ 565 \leq R(9,9) \leq 6588. & & \end{array}$$

Besides his role in laying the foundation for Ramsey theory, Paul Erdös was well-known for his ability to judge the difficulty of various problems. He often remarked on the difference between determining the exact value of $R(5,5)$ and the exact value of $R(6,6)$. The following is a paraphrased version of his remarks.

"Suppose that evil aliens land on the earth and say that they are going to come back in five years and blow it up, unless humankind can tell them the value of $R(5,5)$ when they come back. Then all the mathematicians and computer scientists of the world should get together, and using all the computers in the world, we would probably be able to compute $R(5,5)$ and save the earth. But what if the aliens had instead said that they would blow up the earth unless we could calculate $R(6,6)$ in five years? In that case, the best strategy that humankind could follow would be to divert everyone's energy and resources into weapons research for the next five years."

3.2.2. Generalized Ramsey Numbers. There are three main ways that one typically generalizes Ramsey numbers: by adding more colors, by considering monochromatic subgraphs other than complete subgraphs, or by considering analogous definitions in the hypergraph setting. Of course, it isn't unreasonable to generalize in more than one way. When adding more colors, one can define the Ramsey number $R(s_1, s_2, \ldots, s_m)$ to be the smallest $n \in \mathbb{N}$ such that every arbitrary coloring of the edges of K_n using m colors results in a K_{s_i}-subgraph for some color i. The following result is also due to Greenwood and Gleason [**10**] and is the only known "nontrivial" multicolor diagonal Ramsey number. Recently, Codish, Frank, Itzhakov, and Miller [**7**] have shown that $R(3,3,4) = 30$.



THEOREM 3.7 (Greenwood and Gleason, 1955). *$R(3,3,3) = 17$.*

PROOF. From the previous 3-colorings of the edges of K_{16}, we see that $R(3,3,3) \geq 17$. Now consider an arbitrary red/blue/green coloring of the edges of K_{17}. Fix a vertex u and note that at least 6 of the edges incident with u must have the same color. With our loss of generality, assume ua_1, ua_2, $\ldots$, ua_6 are red. If any (a_i, a_j) is red, we obtain a red triangle. Otherwise, all such edges are blue and green and the subgraph induced by a_1, a_2, $\ldots$, a_6 is a blue/green coloring of K_6. Since $R(3,3) = 6$, it must contain either a blue triangle or a green triangle. $\square$

When generalizing Ramsey numbers to subgraphs that are not complete, assume that G and H are any graphs. Then $R(G, H)$ is defined to be the smallest $n \in \mathbb{N}$ such that every red/blue coloring of K_n results in a red subgraph isomorphic to G or a blue subgraph isomorphic to H. Next, we describe a result proved in 1972 by Chvátal and Harary [**6**]. We denote by $\chi(G)$ the chromatic number of G and by $c(H)$ the order of the largest connected component of H.

THEOREM 3.8 (Chvátal and Harary, 1972). $R(G, H) \geq (\chi(G) - 1)(c(H) - 1) + 1$.

PROOF. Let $m = (\chi(G) - 1)(c(H) - 1)$ and form a 2-coloring of the edges of K_m using $\chi(G) - 1$ copies of $K_{c(H)-1}$. Color the edges within each copy of $K_{c(H)-1}$ blue and all of the remaining edges red. Since red edges only connect vertices from different copies of $K_{c(H)-1}$, a vertex-coloring of any graph spanned by red edges can be obtained by assigning the vertices colors based on which copy of $K_{c(H)-1}$ they reside in. Thus, no red copy of G can exist or it would be possible to color it using $\chi(G) - 1$ colors. On the other hand, the subgraph spanned by the blue edges has a maximal connected component of order $c(H) - 1$, so no blue copy of H can exist. Hence, $R(G, H) > m$. □

THEOREM 3.9 (Chvátal, 1977). *If T_m is any tree of order m, then*

$$R(T_m, K_n) = (m - 1)(n - 1) + 1.$$

PROOF. Since a tree is assumed to be connected, we have that $c(T_m) = m$. From this and the fact that $\chi(K_n) = n$, Theorem 3.8 implies that

$$R(T_m, K_n) \geq (m - 1)(n - 1) + 1.$$

It remains to be shown that

$$R(T_m, K_n) \leq (m - 1)(n - 1) + 1.$$

Note that when $m = n = 2$, it is trivial that $R(T_2, K_2) \leq 2$. So, we proceed by strong induction and assume that the inequality

$$R(T_{m'}, K_{n'}) \leq (m' - 1)(n' - 1) + 1$$

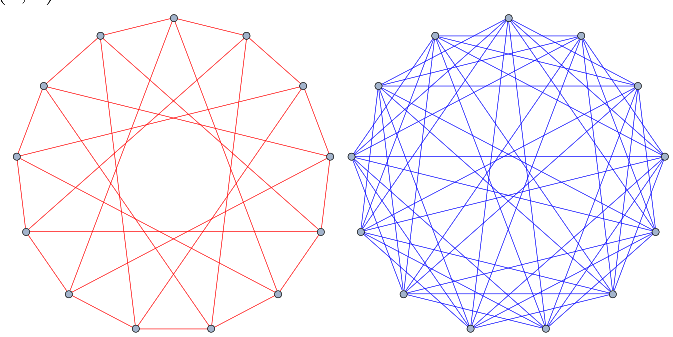
holds for all $m' + n' < m + n$. Fix a tree T_m of order m and consider an arbitrary red/blue coloring of the edges of $K_{(m-1)(n-1)+1}$. Let T' be a tree formed by removing a leaf x from T_m. Assume also that the removal of x resulted in the removal of the single edge xy. Since T' has order $m - 1$, it follows from the inductive hypothesis that $K_{(m-1)(n-1)+1}$ contains either a red T' or a blue K_n. Assume the former case and consider the red/blue coloring of the graph $K_{(m-1)(n-2)+1}$ formed by removing the vertices in the T'-subgraph from the original $K_{(m-1)(n-1)+1}$. Again, by the inductive hypothesis, this graph contains a red T_m or a blue K_{n-1}. Assume the latter. Thus, the original $K_{(m-1)(n-1)+1}$ contains a red T' and a blue K_{n-1} that are disjoint. Consider all of the edges in this graph connecting the vertex y to the vertices in the K_{n-1}. If any one of them are red, we obtain a red copy of T_m. Otherwise, they must all be blue and including the vertex y, we obtain a blue K_n. □

Exercises for Section 3.2

(1) Let G be a self-complementary graph (ie., it is isomorphic to its complement) on n vertices that has clique number m. What can be concluded about the Ramsey number $R(m+1, m+1)$?
(2) If $s, t, v \in \mathbb{N}$ satisfy $t < v$, what is the relationship between $R(s, t)$ and $R(s, v)$? Explain your answer.
(3) Determine the values of $R(1, t, v)$ and $R(2, 2, v)$.
(4) Prove that $R(2, t, v) = R(t, v)$. More generally, prove that

$$R(2, s_1, s_2, \ldots, s_m) = R(s_1, s_2, \ldots, s_m).$$

(5) Consider the following graph on 13 vertices and its complement. One can verify using Mathematica that the clique number of the red graph is 2, while its blue complement has a clique number of 4. It follows that $R(3, 5) > 13$.



Prove that $R(3, 5) = 14$ (this is also due to Greenwood and Gleason, 1955 [**10**]). *(Hint: Consider two cases: the case in which there exists a vertex u incident with at least 5 red edges and the case in which every vertex is incident with at most 4 red edges (and hence, at least 9 blue edges).)*
(6) Using induction on $m \geq 2$, prove that for all trees T_m on m vertices,

$$R(T_m, K_{1,n}) \leq m + n - 1.$$

This result is due to Burr [**4**], who was also able to prove equality when it is assumed that $m - 1$ divides $n - 1$.

Bonus Exercises (these are a bit harder and you will not need to know them for the test):

(7) Prove that whenever $s \geq 2$ an $t \geq 2$ are integers,

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

This is often referred to as Ramsey's Theorem.

(8) Using induction on $k = s+t$, prove that

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

(9) Using an optimal construction similar to the one used to prove Theorem 3.8, prove Abott's Theorem (Theorem 2.3.2 in [**1**]):

$$R(pq+1,pq+1) > (R(p+1,p+1)-1)(R(q+1,q+1)-1),$$

where $R_r(t)$ is the r-color Ramsey number $R(t,t,\ldots,t)$ having r copies of t.

(10) Prove that every 2-coloring of K_5 that lacks a monochromatic subgraph isomorphic to K_3 has a red subgraph isomorphic to C_5 and a blue subgraph isomorphic to C_5.

3.3. Schur's Theorem

In Exercise 7 of the Bonus Exercises of the previous section, it is stated that

$$R(s,t) \leq R(s-1,t) + R(s,t-1)$$

for all $2 \geq 2$ and $t \geq 2$. This is often referred to as Ramsey's Theorem, despite the fact that it bares little resemblance to Ramsey's original theorem [**13**]. In fact, this result was first proved by Greenwood and Gleason [**10**] in 1955 and there is no reason to assume that they were studying Ramsey's work as they did not cite him in their paper. Ramsey's original theorem is much more broad, but essentially guarantees the existence of Ramsey numbers (multicolor and r-uniform hypergraph included). The trivial Ramsey numbers along with the above inequality can be used to argue that all 2-color Ramsey numbers exist.

A key ingredient to proving Schur's Theorem [**14**] will be the assumption that the multicolor Ramsey number

$$R^k(3) := R(\underbrace{3,3,\ldots,3}_{k\ terms})$$

exists. We prove this fact in the next theorem.

Theorem 3.10. *The multicolor Ramsey number $R^k(3)$ exists for all $k \geq 2$.*

Proof. We proceed by induction on $k \geq 2$. In the first case, we have already shown that $R(3,3) = 6$ (Theorem 3.2.1). In fact, we know from Exercise 7 of Section 3.2 that all 2-color Ramsey numbers exist. Now suppose the theorem is true for the Ramsey number $R^{k-1}(3)$ and consider $R^k(3)$. Grouping the first $k-1$ colors together, we can consider the 2-colored Ramsey number

$$R(R^{k-1}(3),3) \geq R^k(3).$$

Thus, $R^k(3)$ is bounded above, and therefore, exists. $\square$

THEOREM 3.11 (Schur, 1916). *For any $k \in \mathbb{N}$ there exists a least positive integer $S(k)$ such that every k-coloring of the set $\{1, 2, \ldots, S(k)\}$ contains integers a, b, and c of the same color such that $a + b = c$.*

In the statement of Schur's Theorem, the triple a, b, and c is called a *monochromatic Schur solution* and $S(k)$ is called a *Schur number*. Before we prove the theorem, we consider a few examples. When $k = 1$, we find that $S(1) = 2$ since the equation $a + b = c$ cannot be satisfied using only $\{1\}$ and 1-coloring $\{1, 2\}$ gives the monochromatic Schur solution $1 + 1 = 2$.

Now consider the Schur number $S(2)$. First, we show that $S(2) \geq 5$ by 2-coloring the set $\{1, 2, 3, 4\}$ so that no monochromatic Schur solution exists:

$$\{1, 2, 3, 4\}.$$

Next, we show that every 2-coloring of $\{1, 2, 3, 4, 5\}$ contains a monochromatic Schur solution. The number 1 must receive a color, so without loss of generality, suppose that 1 is colored red. Then if we wish to avoid a monochromatic Schur solution, 2 must be colored blue since $1 + 1 = 2$. Also, 4 must be colored red since $2 + 2 = 4$. At this point, both 1 and 4 are red and $1 + 4 = 5$ forces 5 to be colored blue. Now 3 cannot be red since 1 and 4 are red. But 3 also cannot be blue since 2 and 5 are blue.

$$\{1, 2, \underline{3}, 4, 5\}$$

No matter how we 2-color the numbers, we always have a monochromatic Schur solution. Thus, $S(2) = 5$. The only other known Schur numbers are $S(3) = 14$, $S(4) = 45$, and $S(5) = 161$ [**11**].

PROOF OF SCHUR'S THEOREM. For any $k \geq 2$, we know that $R^k(3)$ exists by Theroem 3.10. Let $m = R^k(3)$ and number the vertices in K_m by 1, 2, $\ldots$, m and arbitrarily partition the set $\{1, 2, \ldots, m-1\}$ into k subsets corresponding to k colors. If i and j are two distinct vertices, then color the edge connecting them according to

$$|j - i| \in \{1, 2, \ldots, m-1\}.$$

Since $m = R^k(3)$, it follows that the k-coloring of K_m that we just constructed necessarily contains a monochromatic triangle. Suppose that $x < y < z$ are the vertices of such a triangle. Then $y - x$, $z - y$, and $z - x$ are all the same color. If we let $a = z - y$, $b = y - x$, and $c = z - x$, then it follows that

$$a + b = (z - y) + (y - x) = z - x = c$$

is a monochromatic Schur solution. Thus, we have shown that $S(k) \leq m - 1$. $\square$

Exercises for Section 3.3

(1) Give a 3-coloring of $\{1, 2, 3, 4, 5, 6\}$ that lacks a monochromatic Schur solution.
(2) Explain why $S(k) \geq k + 1$ for all $k \geq 1$.
(3) Prove that for any $n \geq 1$ and any $k \geq 3$, the diagonal k-color Ramsey number $R^k(n)$ exists. You may use the assumption that all 2-color Ramsey numbers exist.

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