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RESEARCH

Performance Evaluations of Vehicle Sharing in Closed Queueing Networks System

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Abstract

This investigation deals with performance analysis of vehicle (taxi or motorbike or both) sharing in the structure of a closed queueing network system having M nodes and N number of vehicles. Vehicles provide service to the people who hire it at the junction (node) of the roads. Vehicles are assumed to be either in active mode or in idle mode or in vacation mode. Our main objective is to find normalizing constant by using convolution algorithm with the help of which the various performance measures-utilization of i^{th} vehicle consisting of N vehicles in the network, average number of vehicles at i^{th} node, expected total number of vehicles in the system, average time of a vehicle that it spends in waiting as well as in service in the i^{th} node, total waiting time of a vehicle in the queues in all the nodes, average throughput of a vehicle in i^{th} node in the system as well as system throughput and the probability that there are K or more vehicles in i^{th} node have been derived explicitly. To show real-life applications of the model the numerical results have been computed by using computer software. The model under study may attract the attention of researchers interested in the development of mathematical models of vehicle-sharing problems in the city.

Keywords Closed queueing system · Queueing networks · Vehicle sharing · Performance of networks

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1 Introduction

From the history of the vehicle-sharing system beginning in Zurich [1] to till-date researchers have been attracted to the mathematical study of the vehicle-sharing system by taking various assumptions into account so as to develop the model realistically. An intensive survey of literature by Hortelano et al. [2] on vehicle-sharing systems reveals that ride-sharing systems are becoming popular day by day in the world not due to economic only but due to the maintenance of a clean environment also. Ferrero et al. [3] gave insights into the recent development of vehicle technology, electric vehicles are tempting people to change their behavior to adapt the system of optimal use of vehicle-sharing technology. Some of the researchers developed the vehicle-sharing models taking route as the closed path and each station (stop or bay) as the node under the closed queueing network framework. Gordon et al. [4] studied M nodes, N customers closed queueing network and obtained the joint probability distribution, marginal probability distribution in product form. They also extended their results to open, closed, and mixed network systems under various job classes and service disciplines by using a mean value algorithm. Basket et al. [5] developed a mixed queueing network model of a computer system taking four classes of customers as the central processing unit, data channels, terminals, and routing delay with service discipline as first come first served, priority service, no queueing, and last come first served respectively wherein customer when gets to other station. They obtained product-form equilibrium probability distribution of each class of customer for each station. Tao et al. [1] claimed that tools for the study and analysis of the sharing system analytical, mixed logical analytical model, and discrete choice model are more effective. Smith [6] focused on material handling and optimal routing of transportation networks by using a mean value algorithm with the help of a closed queueing system. Buzen [7] derived the convolution algorithm for finding the normalization constant of the product-form solution in queueing network systems. Reiser and Lavenberg [8] computed performance measures in a closed multi-chain queueing network having product form solution without computing product term and normalization constant. Burke [9] worked on the investigation of a single channel queue network having the departure of the first node with Poisson arrival serving with identically distributed independent exponential service time as the input for the second node is Poisson. Recently Aliamadi et al. [10] developed some algorithms and used them to solve the problem in sensitivity analysis of the actual demand on pricing decisions for maximizing the net present value of the supply chain network. The active state is the situation in which each loaded vehicle circulates through a closed path network. Bhasker [11] introduced a closed queueing network of data flow in a multiprocessor computer system taking jobs circulating in the network as the active state. Kendall [12] systematized the study of the queueing system by classifying the arrival pattern as regular arrival and random arrival. Lagershasen [13] made a performance analysis of a closed queueing network while introducing the mathematical system in economics. Boss [14] summarized some of the classical formulae of open and close queueing networks. Tucci and Sauer [15] described three structural studies of the mean value analysis algorithm in detail. Kishor and Trivedi [16] contributed to probability, reliability, and queueing theory applications in computer systems. Wu [17] focused on the role of insurance in battery technology in the electric vehicle-sharing system.

An extensive survey of literature made on vehicle-sharing problems shows that the pioneering work of Li and Tio [18] opened the door for the study of such problems. During this nearly one and half decade rare research works have been made on vehicle-sharing closed queueing networks under the considerations of the limited number of cities (nodes) between which the vehicles ply with ideal parameters. The limitations of prior works in the field motivated us to study vehicle-sharing closed queueing networks under the assumptions in a more general framework that exhibits more practicability in real-life situations. This study is more general than the earlier study made on the problem which leads to the problem being more applicable in transportation systems in material handling work, road construction for driver-less vehicles, manufacturing systems, and assembly lines. In this investigation waiting for a vehicle until it gets a passenger is termed as a waiting customer and the departure of vehicles carrying passengers from a node is meant to get service. The vehicle acts as a server when it continues to carry passengers. There is no confusion that this is the phenomenon of vehicle shifting from customer to server and vice-versa during its operation. Each node containing n_i vehicles in three different modes, acting, idle, and vacation, has been provisioned. Vehicles arrive in each node randomly and this random arrival of vehicles follows the Poisson probability distribution and they also follow exponential inter-arrival time distribution. In each node, vehicles take heterogeneous departure rates μ_i which is defined to be the service rate. After completion of service from i^{th} node to j^{th} node the arrival rate to j^{th} node is λ_i. If the vehicle gets another request in j^{th} node, it moves to that node. Otherwise, it moves to either idle mode with rate γ or in vacation mode with rate ν within the node. If a vehicle leaves the vacation state, it moves to the active mode with the rate ξ or to the idle mode with the rate γ. The vehicle moves to acting mode from idle mode with rate η and it moves to vacation from idle mode with rate ν. The equilibrium equations have been set up by using a transition diagram and the joint probability distribution of vehicles is derived by using the generating function method. Average arrival rate Λ_i and normalized constant have also been obtained by using the probability of i^{th} node and convolution algorithm. Ample number of the authors paid their attention to the queueing network with the provision of either an identical service rate with two modes or a heterogeneous service rate with identical modes. The novelty of our model is that we have derived explicit formulae for the various performance measures with the assumption that vehicles serve customers with heterogeneous service rates in three modes. Results derived in this study can be found rarely in the existing literature on the vehicle-sharing system in closed queueing network.

2 Mathematical Model and Analysis

In our model, the following notations have been used.

- λ_i - Average arrival rate in each node, $i = 0, 1, \ldots, N-1$.
- μ_i - Average service rate in each node, $i = 1, 2, \ldots, N$.

- γ - Rate in which vehicles move to idle state from acting and vacation state.
- η - Rate in which vehicles move from idle state to acting state.
- ν - Rate in which vehicles move to vacation state from idle and acting state.
- ξ - Rate in which vehicles move from vacation state to acting state.
- N - Total number of vehicles in the networks.
- M - Total number of nodes in the networks.
- n_i - Total number of vehicles in the i^{th} node.
- $P(0, n)$ - Probability of n customer in the idle state.
- $P(1, n)$ - Probability of n customer in the acting state.
- $P(2, n)$ - Probability of n customer in the vacation state.
- $P(h, n)$ - Probability of n customer in the idle, acting and vacation state.
- λ_i' - Total average arrival rate in node i.
- μ_i' - Total average service rate in node i.
- $G(N)$ - Normalizing constant.
- Λ_j - Average arrival rate of the network from i^{th} to j^{th} node.
- ρ_i - Traffic intensity or offered load of the network in i^{th} node.
- α - Function.
- Λ_j^* - Average arrival rate of the network independent of population size N.
- $U_i(N)$ - The utilization of the i^{th} vehicle when there are N vehicles in the system.
- $P(N \geq m)$ - Probability that there are m or more vehicles at node i.
- $E[L_i(N)]$ - Average number of vehicles at node i when there are N vehicles in the system.
- L - The expected total number of vehicles in the system.
- $R_(sp_i)$ - The time spent by a vehicle in a queue and in the service in a node i.
- $E[W]$ - Total waiting time in the queue of all nodes.
- $E[T_i(N)]$ - Average throughput of the vehicles in i^{th} node of the system.
- $E[T(N)]$ - Total throughput of the system.

Under the description of our model, we have constructed the following transition diagram.

The network transition diagram consists of $i, (i = 1, 2, ..., M)$ nodes (stations) where in n_i is the number of servers (vehicles) in i^{th} node. Our paradigm is of $M/M/1/FIFO/N$ model having a request in i^{th} node, the server serves immediately when arrives in i^{th} station otherwise server stays in queue. A request in the i^{th} node server(customer) that provides service at the i^{th} stage proceeds directly to the jth with probability p_{ij}, which is independent of the state of the system and i^{th} node, each node number of vehicles n_i has three states (idle, working and vacation) of the server. $P(n_i)$ represents probability of n_i server in i^{th} node. First Come First Serve discipline is followed by the server to take the riders for traveling. Vehicles arrive j^{th} node from i^{th} node in the Poissonian process with rate λ_j which had been served in i^{th} node with service rate μ_i. From the transition diagram Fig. 1, under the equilibrium conditions, the system of equations for the three states is expressed (1) through (18).

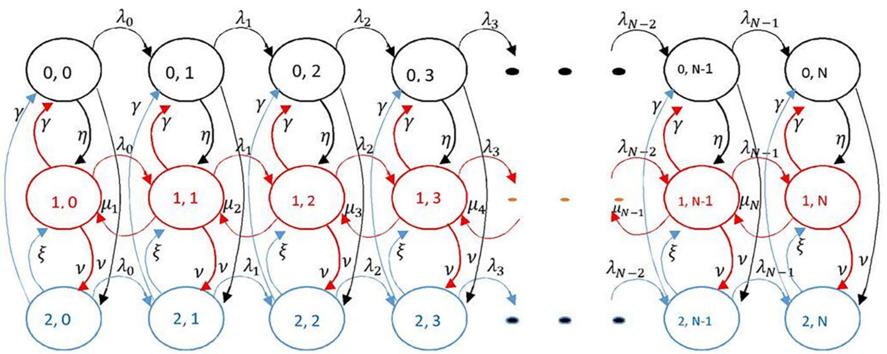


Fig. 1 Transition diagram of a server in i^{th} node in active, idle and vacation state

Vacation State of i^{th} Node

$$\gamma P(2,0) + \gamma P(1,0) - (\lambda_0 + \eta + \nu)P(0,0) = 0 \tag{1}$$

$$\gamma P_{(2,1)} + \gamma P(1,1) + \lambda_0 P(0,0) - (\lambda_1 + \eta + \nu)P(0,1) = 0 \tag{2}$$

$$\gamma P(2,2) + \gamma P(1,2) + \lambda_1 P(0,1) - (\lambda_2 + \eta + \nu)P(0,2) = 0 \tag{3}$$

$$\gamma P(2,n) + \gamma P(1,n) + \lambda_{n-1} P(0,n-1) - (\lambda_n + \eta + \nu)P(0,n) = 0 \tag{4}$$

$$\begin{aligned}&\gamma P(2,N-1) + \gamma P(1,N-1)\\ &\qquad + \lambda_{N-2} P(0,N-2) - (\lambda_{N-1} + \eta + \nu)P(0,N-1) = 0\end{aligned} \tag{5}$$

$$\gamma P(2,N) + \gamma P(1,N) + \lambda_{N-1} P(0,N-1) - (\eta + \nu)P(0,N) = 0. \tag{6}$$

Active State of i^{th} Node

$$\eta P(0,0) + \xi P(2,0) + \mu_1 P(1,1) - (\lambda_0 + \gamma + \nu)P(1,0) = 0 \tag{7}$$

$$\eta P(0,1) + \xi P(2,1) + \lambda_0 P(1,0) + \mu_2 P(1,2) - (\lambda_1 + \mu_1 + \gamma + \nu)P(1,1) = 0 \tag{8}$$

$$\eta P(0,2) + \xi P(2,2) + \lambda_1 P(1,1) + \mu_3 P(1,3) - (\lambda_2 + \mu_2 + \gamma + \nu)P(1,2) = 0 \tag{9}$$

$$\begin{aligned}&\eta P(0,n) + \xi P(2,n) + \lambda_{(n-1)} P(1,n-1)\\ &\qquad + \mu_{(n+1)} P(1,n+1) - (\lambda_n + \mu_n + \gamma + \nu)P(1,n) = 0\end{aligned} \tag{10}$$

$$\begin{aligned}&\eta P(0,N-1) + \xi P(2,N-1) + \lambda_{(N-2)} P(1,N-2)\\ &\qquad + \mu_N P(1,N) - (\lambda_{N-1} + \mu_{N-1} + \gamma + \nu)P(1,N-1) = 0\end{aligned} \tag{11}$$

$$\eta P(0,N) + \xi P(2,N) + \lambda_{(N-1)} P(1,N-1) - (\mu_N + \gamma + \nu)P(1,N) = 0. \tag{12}$$

Idle State of i^{th} Node

$$\nu P(0,0) + \nu P(1,0) - (\lambda_0 + \gamma + \xi)P(2,0) = 0 \tag{13}$$

$$\nu P(0,1) + \nu P(1,1) + \lambda_0 P(2,0) - (\lambda_1 + \gamma + \xi) P(2,1) = 0 \tag{14}$$

$$\nu P(0,2) + \nu P(1,2) + \lambda_1 P(2,1) - (\lambda_2 + \gamma + \xi) P(2,2) = 0 \tag{15}$$

$$\nu P(0,n) + \nu P(1,n) + \lambda_{n-1} P(2,n-1) - (\lambda_n + \gamma + \xi) P(2,n) = 0 \tag{16}$$

$$\begin{aligned} &\nu P(0,N-1) + \nu P(1,N-1) \\ &\qquad + \lambda_{N-2} P_{(2,N-2)} - (\lambda_{N-1} + \gamma + \xi) P(2,N-1) = 0 \end{aligned} \tag{17}$$

$$\nu P(0,N) + \nu P(1,N) + \lambda_{N-1} P(2,N-1) - (\gamma + \xi) P(2,N) = 0. \tag{18}$$

Equation (1) through Eq. (18), there are $3(N+1)$ independent equations that we have to solve. For higher accuracy of numerical solutions, solving a system of equations in explicit form always remains our first effort. Due to this fact, to determine joint probability distributions $P(0,i)$, $P(1,i)$, and $P(2,i)$, the probability generating function method has been used. For the system stability Eq. (29) is deployed. For the determination of normalized constants iteratively, the convolution algorithm has been used. Equations (4), (10), and (16) are general probability distribution functions of vacation, active, and idle states respectively in i^{th} node. From (4)

$$\begin{aligned} &\gamma \sum_{i=0}^{\infty} P(2,i) z^i + \gamma \sum_{i=0}^{\infty} P(1,i) z^i + z \sum_{i=0}^{\infty} \lambda_{i-1} P(0,i-1) z^{i-1} \\ &\qquad - \sum_{i=0}^{\infty} (\lambda_i + \eta + \nu) \sum_{i=0}^{\infty} P(0,i) z^i = 0 \end{aligned}$$

$$\text{with normalizing condition} \quad \sum_{i=0}^{N} P_{n_i}(0,i) + \sum_{i=0}^{N} P_{n_i}(1,i) + \sum_{i=0}^{N} P_{n_i}(2,i) = 1$$

$$i = 0, \ldots, N$$

$$\begin{aligned} \text{or} \quad \sum_{i=0}^{\infty} P(0,i) z^i &= \frac{\gamma z^i}{\left(\gamma - z \sum \lambda_i + \sum \lambda_i + \eta + \nu\right)} \\ &= \frac{\gamma z^i}{\left(\gamma + \sum \lambda_i + \eta + \nu\right)} \\ &\quad \left(1 + \frac{z \sum \lambda_i}{\gamma + \sum \lambda_i + \eta + \nu} + \ldots\right). \end{aligned}$$

By equating like the coefficient of z^i, we have

$$P(0,i) = \frac{\gamma}{\left(\gamma + \sum \lambda_i + \eta + \nu\right)}. \tag{19}$$

From (10), for our simplification simplicity, we choose $\xi = \eta$.

$$\begin{aligned} &\eta P(0,i) + \eta P(2,i) + \lambda_{i-1} P(1,i-1) + \mu_{i+1} P(1,i+1) \\ &- (\lambda_i + \mu_i + \gamma + \nu) P(1,i) = 0 \end{aligned}$$

$$\text{or}\quad \eta\sum_{i=0}^{\infty}P(0,i)z^i+\eta\sum_{i=0}^{\infty}P(2,i)z^i+\sum_{i=0}^{\infty}\lambda_{i-1}P(1,i-1)z^i+\sum_{i=0}^{\infty}\mu_{i+1}$$

$$P(1,i+1)z^i-\sum_{i=0}^{\infty}(\lambda_i+\mu_i+\gamma+\nu)P(1,i)z^i=0$$

$$\text{or,}\quad \sum_{i=0}^{\infty}P(1,i)z^i=\frac{\eta z^i}{\left(\eta-z\sum\lambda_i-z^{-1}\sum\mu_i+\sum\lambda_i+\sum\mu_i+\gamma+\nu\right)}$$

$$\text{or}\quad \sum_{i=0}^{\infty}P(1,i)z^i=\frac{\eta z^{i+1}}{z(\eta+\sum\lambda_i+\sum\mu_i+\gamma+\nu)-z^2\sum\lambda_i-\sum\mu_i}.$$

By equating like coefficient of z^{i+1}, we have

$$P(1,i+1)=\frac{\eta}{\left(\frac{-4\sum\lambda_i\sum\mu_i+(\eta+\sum\lambda_i+\sum\mu_i+\gamma+\nu)^2}{4\sum\lambda_i}\right)}. \tag{20}$$

Substituting j for $i+1$, we have

$$P(1,j)=\frac{4\eta\sum\lambda_{j-1}}{\left(-4\sum\lambda_{j-1}\sum\mu_{j-1}+(\eta+\sum\lambda_{j-1}+\sum\mu_{j-1}+\gamma+\nu)^2\right)}. \tag{21}$$

From (16)

$$\nu\sum_{i=0}^{\infty}P(0,i)z^i+\nu\sum_{i=0}^{\infty}P(1,i)z^i+z\sum_{i=0}^{\infty}\lambda_{i-1}P(2,i-1)z^{i-1}$$

$$-\sum_{i=0}^{\infty}(\lambda_i+\gamma+\eta)\sum_{i=0}^{\infty}P(2,i)z^i=0$$

$$\sum_{i=0}^{\infty}P(2,i)z^i=\frac{\nu z^i}{\left(\nu-z\sum\lambda_i+\sum\lambda_i+\gamma+\eta\right)}.$$

Equating like coefficient of z^i, we get

$$P(2,i)=\frac{\nu}{\nu+\sum\lambda_i+\gamma+\eta}. \tag{22}$$

ith node probabilities given by local balance Eqs. (19), (21), and (22) can also be expressed as:

$$P(0,i)=\frac{\gamma_i}{\left(\gamma_i+\lambda_i'+\eta_i+\nu_i\right)} \tag{23}$$

where $\lambda_i'=\sum_{i=1}^{ni}\lambda_i,\quad \sum_{i=1}^{M}ni=N.$

$$P(1,i) = \frac{4\eta_i \lambda_i'}{\left(-4\lambda_i'\mu_i' + (\eta_i + \lambda_i' + \mu_i' + \gamma_i + \nu_i)^2\right)} \quad \text{where} \quad \mu_i' = \sum_{i=1}^{ni} \mu_i. \tag{24}$$

$$P(2,i) = \frac{\nu_i}{\nu_i + \lambda_i' + \gamma_i + \eta_i}. \tag{25}$$

Probability of i^{th} node having n_i vehicles is

$$P(h,i) = P(0,i) + P(1,i) + P(2,i). \tag{26}$$

Require numerical solution of queueing networks is pragmatic only if joint probability distribution with M nodes given by

$$P(n_1, ..., n_M) = \frac{1}{G(N)} \prod_{i=1}^{M} (\rho_i)^{n_i} \tag{27}$$

exists with flow conservation law

$$\Lambda_j = \sum_{i=1}^{M} \lambda_i' P_{ij}, \tag{28}$$

where $G(N) = \sum_{n' \in D(N,M)} \prod_{i=0}^{M} (\rho_i)^{n_i}$ is normalizing constant chosen so that the sum of the steady state probability is unity, $n' = (n_1, n_2, ..., n_M)$, $D(N,M) = \{(n_1, n_2, ..., n_M) | \sum_{i=1}^{M} n_i = N \text{ and } n_i \geq 0, \quad \forall i$

$$\rho_i = \frac{\Lambda_i}{\mu_i} < 1 \tag{29}$$

is the stability condition. For the state space $\{n_1, ..., n_M / n_i \geq 0$ and $\sum_{i=1}^{M} n_i = N\}$, network flow rate of the model is expressed as

$$\Lambda_j = \sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i) \tag{30}$$

$$\text{where} \quad \sum_{j=1}^{M} P_{ij}(h,i) = 1.$$

Because of dependency, Eq. (30) does not give a unique solution; therefore, to make the solution unique, we perform

$$\Lambda_j(N) = \alpha(N)\Lambda_j^*, \qquad j = 1, ..., M \tag{31}$$

where Λ_j^* is independent of population size N. Having used Eq. (31), we are in the situation to find $G(N)$ by using convolution algorithm ([7]) as

$$G_m(i') = \rho_{i'} \times G_m(i'-1) + G_{m-1}(i') \qquad m = 1, ..., M \quad , \quad i' = 1, ..., N. \tag{32}$$

3 Performance Indices

For our model following various performance measures have been derived explicitly originally which have been used for the numerical computations:
The utilization of the i^{th} vehicle when there are N vehicles in the system

$$U_i(N) = \left\{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i) + \sum_{i=1}^M \lambda_i' P_{ij}(1,i) + \sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i}\right\} \times \left\{\frac{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i)+\sum_{i=1}^M \lambda_i' P_{ij}(1,i)+\sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-2) + G_{m-1}(N-1)}{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i)+\sum_{i=1}^M \lambda_i' P_{ij}(1,i)+\sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-1) + G_{m-1}(N)}\right\}. \tag{33}$$

The probability that there are m or more vehicles at node i is

$$P(N \geq m) = \left\{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i) + \sum_{i=1}^M \lambda_i' P_{ij}(1,i) + \sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i}\right\}^m \times \left\{\frac{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i)+\sum_{i=1}^M \lambda_i' P_{ij}(1,i)+\sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-m-1) + G_{m-1}(N-m)}{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i)+\sum_{i=1}^M \lambda_i' P_{ij}(1,i)+\sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-1) + G_{m-1}(N)}\right\}. \tag{34}$$

The average number of vehicles at node i when there are N vehicles in the system is

$$E[L_i(N)] = \sum_{l=1}^N \left\{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i) + \sum_{i=1}^M \lambda_i' P_{ij}(1,i) + \sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i}\right\}^l \times \left\{\frac{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i)+\sum_{i=1}^M \lambda_i' P_{ij}(1,i)+\sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-l-1)+G_{m-1}(N-l)}{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i)+\sum_{i=1}^M \lambda_i' P_{ij}(1,i)+\sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-1)+G_{m-1}(N)}\right\}. \tag{35}$$

The expected total number of vehicles in the system is

$$L = \sum_{i=1}^M \left\{\sum_{l=1}^N \left\{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i) + \sum_{i=1}^M \lambda_i' P_{ij}(1,i) + \sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i}\right\}^l\right\} \times \left\{\frac{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i)+\sum_{i=1}^M \lambda_i' P_{ij}(1,i)+\sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-l-1)+G_{m-1}(N-l)}{\frac{\sum_{i=1}^M \gamma_i P_{ij}(0,i)+\sum_{i=1}^M \lambda_i' P_{ij}(1,i)+\sum_{i=1}^M \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-1)+G_{m-1}(N)}\right\}. \tag{36}$$

The time spent by a vehicle in a queue and in the service in a node i is

$$R_{SPi} = \frac{1}{\lambda_i'} \sum_{l=1}^{N} \left\{ \frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} \right\}^{l} \times \left\{ \frac{\frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-l-1) + G_{m-1}(N-l)}{\frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-1) + G_{m-1}(N)} \right\}. \tag{37}$$

The total waiting time in the queue of all nodes is

$$E[W] = \sum_{i=1}^{M} \left[\frac{1}{\lambda_i'} \sum_{l=1}^{N} \left\{ \frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} \right\}^{l} \times \left\{ \frac{\frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-l-1) + G_{m-1}(N-l)}{\frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-1) + G_{m-1}(N)} \right\} - \frac{1}{\mu_i} \right]. \tag{38}$$

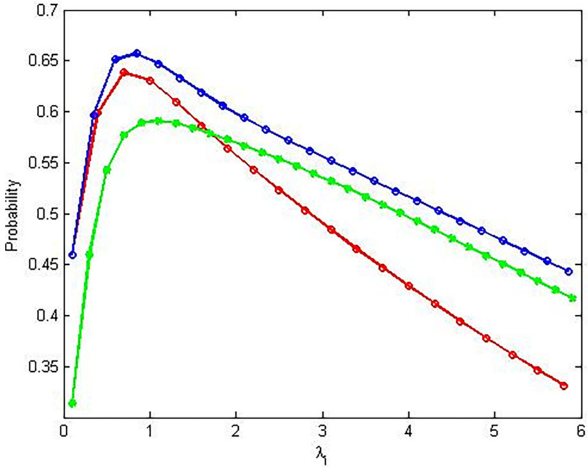
The average throughput of the vehicles in i^{th} node of the system is

$$\begin{aligned} E[Ti(N)] &= \mu_i \left\{ \frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} \right\} \\ &\quad \times \left\{ \frac{\frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-2) + G_{m-1}(N-1)}{\frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-1) + G_{m-1}(N)} \right\} \\ &= \left\{ \sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i) \right\} \\ &\quad \times \left\{ \frac{\frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-2) + G_{m-1}(N-1)}{\frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-1) + G_{m-1}(N)} \right\}. \end{aligned} \tag{39}$$

The total throughput of the system is

$$E[T(N)] = \sum_{i=1}^{M} \left\{ \sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i) \right\} \times \left\{ \frac{\frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-2) + G_{m-1}(N-1)}{\frac{\sum_{i=1}^{M} \gamma_i P_{ij}(0,i) + \sum_{i=1}^{M} \lambda_i' P_{ij}(1,i) + \sum_{i=1}^{M} \nu_i P_{ij}(2,i)}{\mu_i} G_m(N-1) + G_{m-1}(N)} \right\}. \tag{40}$$

Fig. 2 Probability of n_i vehicle versus arrival rate of customer in the i^{th} node



4 Numerical Results and Interpretations

To show practicability of our model, the performance of the vehicle-sharing network have been calculated by taking $M = 30$, $N = 50$ with the values of the parameters $\lambda_{i'} = 0.1 : 0.3 : 6$, $\lambda_{i''} = 0.1 : 0.25 : 5$, $\lambda_{i'''} = 0.1 : 0.2 : 4$, $\mu_{i'} = 0.1 : 0.4 : 8$, $\mu_{i''} = 0.1 : 0.3 : 7$, $\mu_{i'''} = 0.1 : 0.24 : 7.2$, $\gamma = 0.3$, $\eta = 1.2$ and $\nu = 0.4$ in (26) gives Fig. 2 which exhibits probability of $n_i = 20$ vehicles in i^{th} node.

The same set of parameters with the values has been used to obtain Figs. 3 to 11.

A subtle study can be made with the help of Fig. 2 that affects the change of the value of parameters on the probability of n_i vehicles in i^{th} node. Figure 2 reveals that the probability is in optimum value when average arrival rate $\lambda_{i'} = 0.7000$, $\lambda_{i''} = 0.8500$, $\lambda_{i'''} = 1.1000$ and slower service rates increase rapidly and decreases gradually with the faster service rates in node i. We observe that when service rates vary in ascending

Fig. 3 Average arrival rate in the network Λ_i versus arrival rate in i^{th} node λ_i

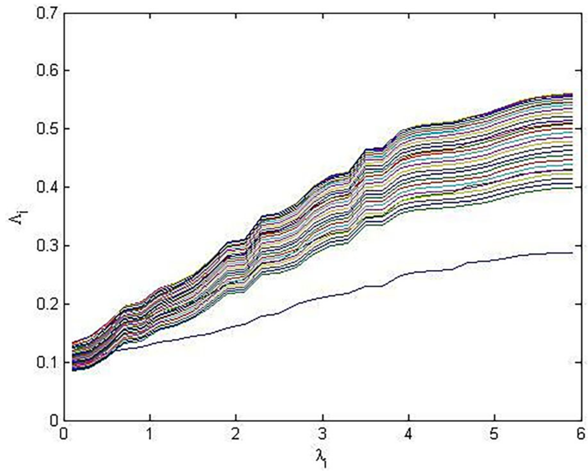


Fig. 4 Utilization of the i^{th} vehicle versus average arrival rate of the network Λ_i

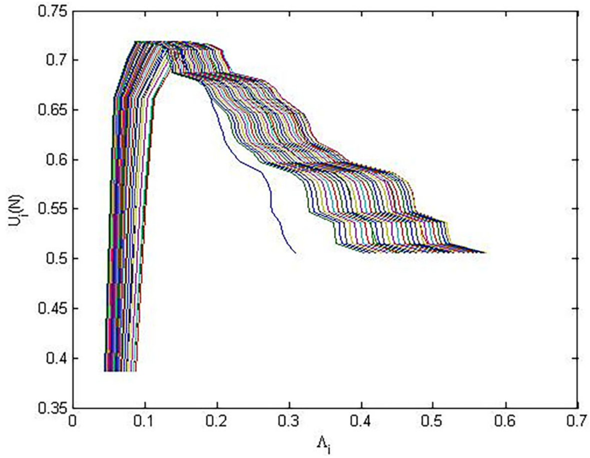


Fig. 5 Relative utilization of the i^{th} vehicle ρ versus average arrival rate of the network Λ_i

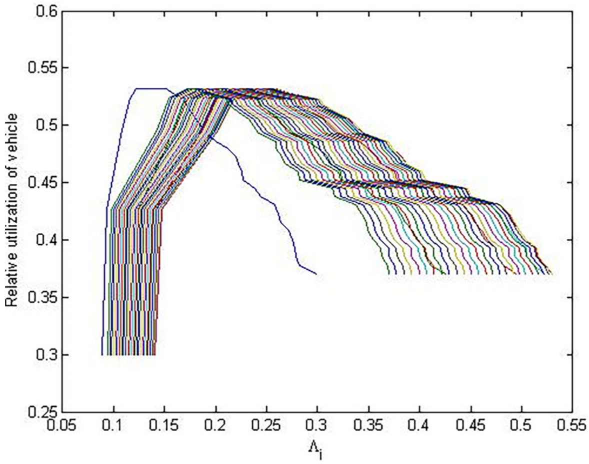
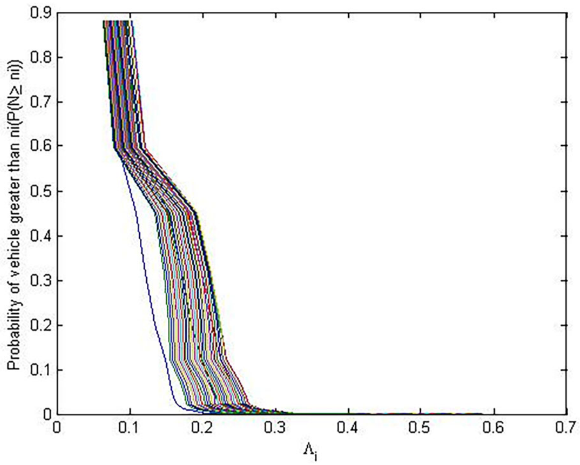


Fig. 6 The probability that there are n_i or more vehicle at node i versus average arrival rate of the network Λ_i



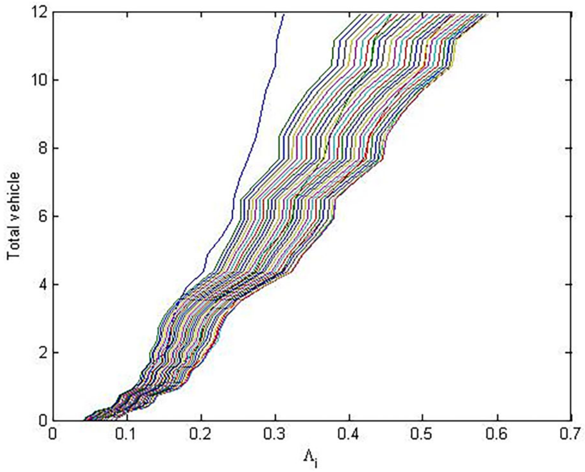


Fig. 7 Total vehicle in the system versus average arrival rate Λ_i

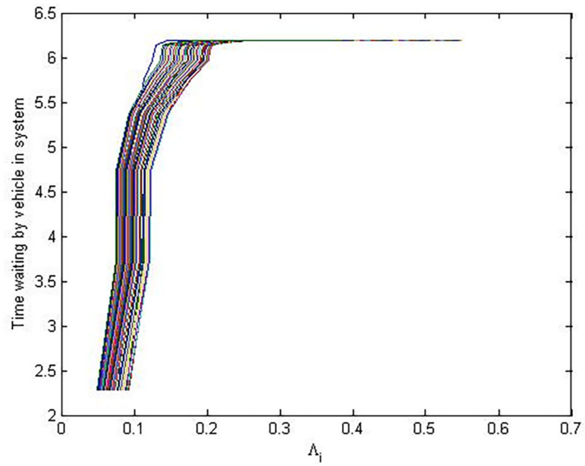


Fig. 8 Average sojourn time vs average arrival rate Λ_i

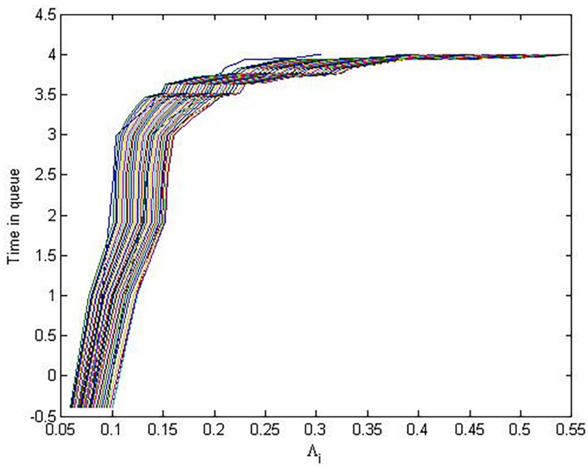
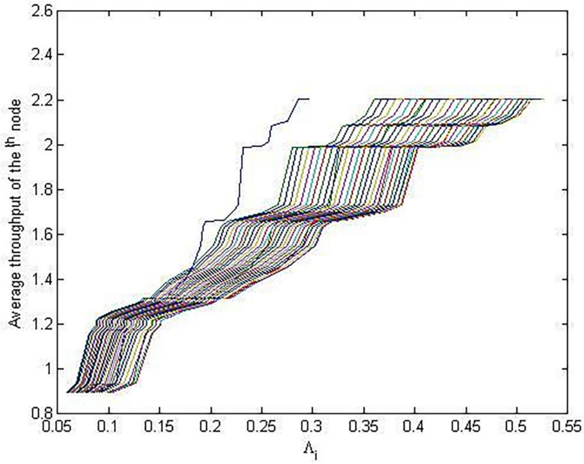


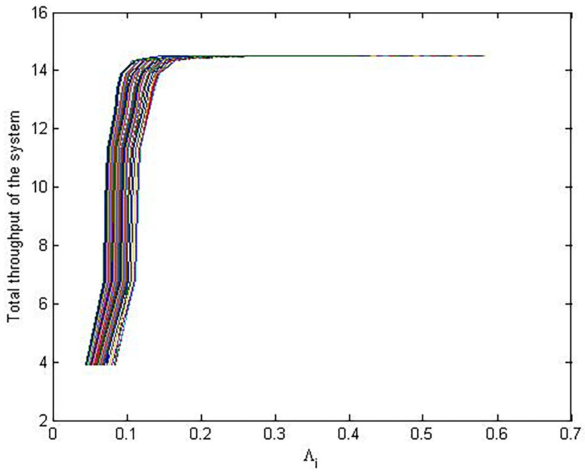
Fig. 9 Time spent by a job in queue vs average arrival rate Λ_i

Fig. 10 Throughput of i^{th} node versus Λ_i



order in the node that influences the system's average arrival rate and average throughput. Figure 3 shows the average arrival rate increases when the arrival rate of the i^{th} node increases. Utilization of the i^{th} vehicle in the network increases initially and happens to decrease because of an increase in average service rate which is demonstrated by Fig. 4. The same pattern of the graph in Fig. 5 is followed when the average arrival rate increases in the system. Probabilities of n_i or more vehicles in the networks decrease rapidly close to zero when the average arrival rate increases which has been displayed by Fig. 6. Figure 7 predicts that the total number of vehicles increases in a particular node with the increase in average arrival rate which agrees with the real situation. Initially total number of vehicles and total time waiting in nodes increases with slower average arrival rates and this number and time show asymptotic behavior with faster average arrival rates which is shown in Figs. 8 and 9. The average throughput illustrated in Fig. 10 shows upstream behavior getting wider with increasing average

Fig. 11 Total throughput of the system vs Λ_i



rates. Figure 11 expresses the fact that the throughput of the i^{th} node increases when the average arrival rate of the system increases. An increase in the average arrival rate increases the total throughput of the system and shows asymptotic behavior after some time.

5 Conclusion

In this study, a vehicle-sharing queueing model has been developed through a closed queueing network structure with M nodes and N vehicles under the assumption that the vehicle may remain either in active mode or in idle mode or in vacation mode. Having obtained joint probability distributions for all modes, the derivations of explicit formulae for utilization of i^{th} vehicle consisting of N vehicles in the network, the average number of vehicles at i^{th} node, expected total number of vehicles in the system, average time of a vehicle that it spends in waiting as well as in service in the i^{th} node, total waiting time of a vehicle in the queues in all the nodes, average throughput of a vehicle in i^{th} node in the system as well as system throughput and the probability that there are k or more vehicles in i^{th} node have been made for their numerical values predictions. This is the theoretical contribution of the work. This model has application in the transportation system in the material handling process in the optimal routing provision of closed queueing networks. Web-like networks of roads, the network of water supply in the urban, the network of electricity distribution in the country, and the plying of vehicles sharing system in the crowded areas in the city may be tackled with the help of results derived in this research work. This model may also be employed for further study of real-time traffic control systems in the transportation control policy formation in driverless car systems. The queueing model developed in this study may handle the grey data if we are confined to finding the short-term queue length of each node of the networks, which may be future research work and it is the limitation of the present work.

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Code Availability Code can be provided if needed.

Declarations

Ethical Approval Not applicable

Consent to Participate Not applicable

Consent for Publication Authors will give consent for publication.

Conflict of Interest The authors declare no competing interests.

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