NORTH-HOLLAND

On Using Linear Ordered Rank Statistics for Detecting Early Differences between Two Distributions

Sumedha Jayawardene

Department of Mathematical Sciences
The University of Memphis
Memphis, Tennessee 38152

and

Shie-Shien Yang

Department of Statistics
Kansas State University
Manhattan, Kansas 66506

ABSTRACT

A linear ordered rank statistic is considered for testing the equality of two distributions against the alternative that the two distributions are stochastically ordered. The statistic is proposed specifically for the purpose of detecting early stage stochastic ordering. Some asymptotic properties are derived. © Elsevier Science Inc., 1997

1. INTRODUCTION

Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent random samples from continuous distributions F, and G, respectively. Let $X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(m)}$ be the order statistics for the X-sample and $R_{(1)} \leqslant R_{(2)} \leqslant \cdots \leqslant R_{(m)}$, their ranks in the combined sample. The $R_{(i)}$'s are called the ordered ranks of the X_i's. Define the linear combination

$$T_N = a_1 R_{(1)} + \cdots + a_m R_{(m)}$$

where $a_1, \ldots, a_m$ are given constants and $N = m + n$. T_N is called a linear ordered rank statistic. Taking $a_i = 1,\ i = 1, \ldots, m$ yields the Wilcoxon

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rank sum statistic. Apart from this statistic, the class of linear ordered rank statistics is mutually exclusive with the class of linear rank statistics. However, many nonparametric statistics in the literature belong to the class of linear ordered rank statistics. For example, statistics that are functions of the number of exceeding observations [1, 2] belong to this class. Kamat [3], Haga [4], Sen [5], Govindarajulu [6], and Deshpandē [7] also considered statistics of the form T_N (or linear functions of T_N). In most of these papers, the main objective was in testing against the alternative hypothesis of a change in the location or scale parameter. In this paper, we are interested in testing the equality of two distributions against the alternative that the two distributions are stochastically ordered.

Two distributions F and G are said to have early or late stage stochastic ordering if the ordering is most prominent for values of x such that $F(x)$ and $G(x)$ are both less than 0.5, or greater than 0.5, respectively. The Logrank or Savage's test is known to be powerful at detecting late stage stochastic ordering between two distributions but weak at detecting early stage stochastic ordering. In this paper, weights a_i $i = 1, \dots, m$ that produce a test powerful at detecting early stage stochastic ordering, are proposed.

The linear ordered rank test with the proposed weights may be used either when the stochastic ordering is prominent only in the early stage or when the behavior of the two distributions in the late stage is not of interest and the objective is only to detect the ordering that exists in the early stage. An example of such a situation is when comparing two brands of pain relieving medications or two brands of sleep medications. In either case, interest is in knowing the brand that gives faster relief or sleep.

This paper is organized as follows. Section 2 describes how the weights of T_N can be chosen to detect early stage stochastic ordering. In Section 3, a Monte Carlo simulation study is conducted, to compare the linear ordered rank test with the proposed weights, with some known tests. An application to a real data set is also discussed in this section. Section 4 establishes asymptotic normality of T_N and also shows that asymptotically, the power of the test based on T_N is approximately an increasing function of a weighted difference of F and G.

2. THE PROPOSED TEST

Let F_m and G_n be the empirical distribution functions of the X- and Y-samples, respectively. Then $R_{(i)}$ can be expressed as

$$R_{(i)} = i + nG_n(X_{(i)})$$

Hence, the linear ordered rank statistic T_N can be written as

$$T_N = \sum_{i=1}^{m} ia_i + n \sum_{i=1}^{m} a_i G_n(X_{(i)}).$$

Note that $G_n(X_{(i)})$ $i = 1, \dots, m$ form the sample coverage considered by Fligner and Wolfe [8]. Using the expressions for the mean and covariances, given in that paper, it can be shown that,

$$E(T_N) = (N + 1) \sum_{i=1}^{m} \frac{i}{m+1} a_i$$

and

$$\operatorname{Var}(T_N) = \frac{n(N+1)}{m+2} \left(\sum_{i=1}^{m} \frac{i}{m+1} \left(1 - \frac{i}{m+1}\right) a_i^2 \right.$$
$$\left. + 2 \sum\sum_{i<j} \frac{i}{m+1} \left(1 - \frac{j}{m+1}\right) a_i a_j \right)$$

Consider

$$T_N - E(T_N) = n \sum_{i=1}^{m} a_i \left(G_n(X_{(i)}) - \frac{m}{m+1} F_m(X_{(i)}) \right). \tag{2.1}$$

Let

$$F_m^{-1}(p) = \inf_x \{ x\colon F_m(x) \geqslant p \}, \qquad F_m^*(x) = \frac{m}{m+1} F_m(x),$$

and μ_m a signed measure with mass a_i at (i/m), $i = 0, 1, \dots, m$. Then (2.1) can be expressed as

$$T_N - E(T_N) = n \int_0^1 \left[G_n \cdot F_m^{-1}(u) - F_m \cdot F_m^{-1}(u) \right] d\mu_m(u).$$

This integral representation provides another way of looking at the linear ordered rank statistic—as a weighted difference of G_n and F_m^*. Note that $G_n \cdot F_m^{-1}(u)$ is used to denote $G_n(F_m^{-1}(u))$. This notation for a composite function is used throughout the manuscript. Consider testing the hypothesis

$$H_0: F(x) = G(x) \text{ for all } x, \text{ against}$$

$$H_1: F(x) \leqslant G(x) \text{ for all } x \text{ and } F(x) < G(x) \text{ for some } x.$$

H_1 is equivalent to saying that F is stochastically greater than G. Under some conditions on a_i $i = 1, \ldots, m$ and F, and using a theorem of Govindarajulu [6], it can be shown that under H_0,

$$Z_N = [T_N - E(T_N)]/\sqrt{\operatorname{Var}(T_N)} \xrightarrow{D} N(0,1) \quad \text{as } m, n \to \infty.$$

A simpler derivation of the asymptotic normality of T_N under the alternative hypothesis is given in Section 4, assuming stronger conditions and using a centering constant that is data dependent.

If $\{a_1, a_2, \ldots, a_m\}$ is a nonnegative sequence of constants, then the integral representation of $T_N - E(T_N)$ suggests rejecting H_0 when $Z_N > z_\alpha$. It further suggests that a statistic of the form Z_N can be effective at detecting early stage stochastic ordering by taking a_i to be larger for $i < m/2$ than for $i > m/2$ $(i = 1, \ldots, m)$. We tried many decreasing sequences and recommend using

$$a_i = 1 - \Phi\left(\frac{i/m - 0.5}{0.05}\right) \qquad i = 1, \ldots, m \tag{2.2}$$

for detecting early stage stochastic ordering. An increasing sequence produces a test powerful at detecting late stage stochastic ordering. However, we have not been able to come up with a test that could outperform the Logrank test.

3. SIMULATION STUDY AND AN EXAMPLE

The linear ordered rank test with the recommended weights a_i $i = 1, \ldots, m$ given in (2.2) is considered in this simulation study. The results are based on 3000 generated samples.

TABLE 1

OBSERVED SIZE OF THE PROPOSED TEST

m	10	10	20	20	40	70	100
n	10	20	10	20	40	70	100
$\alpha = .10$	.1073	.1083	.1040	.1190	.1073	.0920	.0987
$\alpha = .05$	.0603	.0620	.0610	.0723	.0600	.0520	.0523

Table 1 compares the observed size of this proposed test with the exact size α. Both F and G were taken to be uniform $(0, 1)$ distributions as the proposed linear ordered rank statistic is distribution free under the null hypothesis.

In the power comparison study, the power of the proposed test was compared with that of the Wilcoxon rank sum and Logrank test. In the first case considered, F and G were taken to be piecewise exponential distributions with hazard functions λ_F and λ_G given below:

When $0 \leqslant x \leqslant 0.2$, $\lambda_F = 0.5$ and $\lambda_G = 3$;
When $0.2 < x \leqslant 0.4$, $\lambda_F = 3$ and $\lambda_G = 0.5$;
When $0.4 < x$, $\lambda_F = 1$ and $\lambda_G = 1$.

These values were selected to enhance the dominance G over F in the early stage and to make F and G identical in the late stage. The observed powers of the three tests are given in Table 2.

Table 3 gives the powers of the tests when F and G are taken to be Weibull distributions that cross at their 80th percentile. F is taken to be a Weibull ($\lambda = 0.5$, $\alpha = 0.5$) distribution and G a Weibull ($\lambda = 0.0149919$, $\alpha = 2.0$) distribution. This situation deviates slightly from the stated alternative hypothesis but is of interest as G clearly dominates F in the early stage.

TABLE 2

POWER COMPARISONS AT $\alpha = .05$

m	10	10	20	20	35	50
n	10	20	10	20	35	50
Proposed test	.3477	.4010	.4243	.5523	.7463	.8630
Wilcoxon test	.1623	.1683	.2137	.2710	.3760	.4733
Logrank test	.0837	.0873	.1070	.1250	.1383	.1627

TABLE 3
POWER COMPARISON AT $\alpha = .05$

m	10	10	20	20	35	50
n	10	20	10	20	35	50
Proposed test	.7997	.9183	.8667	.9683	.9993	1.000
Wilcoxon test	.4937	.6097	.5750	.7473	.9087	.9740
Logrank test	.1763	.1570	.1887	.1810	.2220	.2433

It is seen that the linear ordered rank test with the proposed weights fares very well at detecting early stage stochastic ordering. Note that the logrank test has very low power for the cases considered.

EXAMPLE. Wickersham et al. [9] discusses a study that was conducted to detect the effect of Verapamil on sheep, both before and after causing lung ischemia—reperfusing injury. The following data are the lung lymph thromboxane B_2 levels of two independent random samples of sheep, before causing the reperfusion injury. One sample was treated with Verapamil and the other was not.

Without Verapamil: 162, 188, 279, 418, 788
With Verapamil: 284, 298, 312, 330, 399, 452.

One empirical distribution clearly dominates the other in the early stage. The distributions cross around 0.5 and the ordering is reversed in the late stage. However, the differences in the late stage are not as significant as the differences in the early stage.

The proposed test gives a p-value of 0.0436. The Wilcoxon rank sum and Logrank tests give p-values of 0.2326 and 0.5334, respectively. Therefore, only the proposed test is successful in detecting the early stage stochastic ordering.

4. ASYMPTOTIC PROPERTIES OF T_N

The asymptotic distribution of T_N is derived under the following assumptions.

(1) $F' > 0$ and is bounded on (a, b)

(2) F'' is bounded on (a, b) where $a = \sup_x\{x | F(x) = 0\}$ and $b = \inf_x\{x | F(x) = 1\}$

(3) $a_i = h(i/m)$ where h is square Riemann integrable.
(4) When $m, n \to \infty$, $n/m \to \lambda$ where $0 < \lambda < \infty$.

In the following derivations, when $O(\cdot)$ is used, the mode of convergence is assumed to be "with probability one". Further, $I(\cdot)$ is used to denote the indicator function.

The following results are used in the derivation of the asymptotic distribution. From Theorem 5.3.1 of Csörgő and Rēvēsz [10, pg. 162],

$$\sup_{o<p<1} \left|\sqrt{m}\left(F_m^{-1}(p) - F^{-1}(p)\right)\right| = O\left[(\log\log m)^{1/2}\right]. \tag{4.1}$$

When $m, n \to \infty$, using (4.1) and Lemma 1 of Bahadur [11],

$$\sup_{o<p<1} \left|\left[G_n \cdot F_m^{-1}(p) - G \cdot F_m^{-1}(p)\right] - \left[G_n \cdot F^{-1}(p) - G \cdot F^{-1}(p)\right]\right|$$

$$= O\left(n^{-3/4}\log n\right) \tag{4.2}$$

and

$$\sup_{o<p<1} \left|\left[F_m \cdot F_m^{-1}(p) - F \cdot F_m^{-1}(p)\right] - \left[F_m \cdot F^{-1}(p) - F \cdot F^{-1}(p)\right]\right|$$

$$= O\left(m^{-3/4}\log m\right). \tag{4.3}$$

Since $\sup_x |F_m^*(x) - F_m(x)| = O((1/m))$, it is sufficient to prove the asymptotic normality of T_N with F_m^* replaced by F_m in the expression for $T_N - E(T_N)$. Consider

$$D_N = T_N - E(T_N) - n\int_0^1 \left[G \cdot F_m^{-1}(u) - F \cdot F_m^{-1}(u)\right] d\mu_N(u)$$

$$= n\int_0^1 \left\{\left[G_n \cdot F_m^{-1}(u) - F_m \cdot F_m^{-1}(u)\right]\right.$$

$$\left. -\left[G \cdot F_m^{-1}(u) - F \cdot F_m^{-1}(u)\right]\right\} d\mu_N(u)$$

$$= n\int_0^1 \{[G_n \cdot F_m^{-1}(u) - G \cdot F_m^{-1}(u)]$$

$$-[G_n \cdot F^{-1}(u) - G \cdot F^{-1}(u)]\} \, d\mu_N(u)$$

$$- n\int_0^1 \{[F_m \cdot F_m^{-1}(u) - F \cdot F_m^{-1}(u)]$$

$$-[F_m \cdot F^{-1}(u) - F \cdot F^{-1}(u)]\} \, d\mu_N(u)$$

$$+ n\int_0^1 [G_n \cdot F^{-1}(u) - G \cdot F^{-1}(u)] \, d\mu_N(u)$$

$$- n\int_0^1 [F_m \cdot F^{-1}(u) - F \cdot F^{-1}(u)] \, d\mu_N(u). \tag{4.4}$$

From (4.2), (4.3), and assumption 4, the first two integrals in (4.4) are of order $O(n^{-1/4}\log n)$ as $m, n \to \infty$. Writing the third integral in (4.4) as a sum, we get

$$\sqrt{n} \cdot \frac{1}{m}\int_0^1 [G_n \cdot F^{-1}(u) - G \cdot F^{-1}(u)] \, d\mu_N(u)$$

$$= \sqrt{n} \cdot \frac{1}{n} \sum_{j=1}^{n} \left[\sum_{i=1}^{m} \left(I\left[Y_j \leqslant F^{-1}\left(\frac{i}{m}\right)\right] - G \cdot F^{-1}\left(\frac{i}{m}\right)\right) \cdot \frac{1}{m} h\left(\frac{i}{m}\right)\right]$$

$$= \sqrt{n} \cdot \frac{1}{n} \sum_{j=1}^{n} V_{jm} \quad (\text{say}).$$

For large m and n, V_{jm} is essentially dependent on n.

Clearly, $E(V_{jm}) = 0$ and

$$\sigma_n^2 = \operatorname{Var}(V_{jm})$$

$$= \sum_{i=1}^{m} G \cdot F^{-1}\left(\frac{i}{m}\right)\left[1 - G \cdot F^{-1}\left(\frac{i}{m}\right)\right] h^2\left(\frac{i}{m}\right)\frac{1}{m^2}$$

$$+ 2\sum\sum_{i<k} G \cdot F^{-1}\left(\frac{i}{m}\right)\left[1 - G \cdot F^{-1}\left(\frac{k}{m}\right)\right] h\left(\frac{i}{m}\right) h\left(\frac{k}{m}\right)\frac{1}{m^2}.$$

Therefore, when $m, n \to \infty$,

$$\sigma_n^2 \to \sigma_G^2 = \int_0^1 \int_0^1 \left[G \cdot F^{-1}(p \wedge q) - G \cdot F^{-1}(p) G \cdot F^{-1}(q) \right] h(p) h(q)$$

It can be easily shown that the Lindeberg condition is satisfied for $V_{1m}, \ldots, V_{nm}$.

Hence,

$$\sqrt{n} \cdot \frac{1}{m} \int_0^1 \left[G_n \cdot F^{-1}(u) - G \cdot F^{-1}(u) \right] d\mu_N(u)$$

$$\to N\left(0, \sigma_G^2\right) \quad \text{as } m, n \to \infty \tag{4.5}$$

Similarly, we can show that

$$\sqrt{m} \cdot \frac{1}{m} \int_0^1 \left[F_m \cdot F^{-1}(u) - F \cdot F^{-1}(u) \right] d\mu_N(u)$$

$$\to N\left(0, \sigma_F^2\right) \quad \text{as } m, n \to \infty \tag{4.6}$$

Where σ_F^2 has the same form as σ_G^2 with G replaced by F.

Now it is clear from (4.4), (4.5), and (4.6) that

$$D_N / m\sqrt{n} \xrightarrow{D} N\left(0, \sigma_G^2 + \lambda \sigma_F^2\right) \quad \text{as } m, n \to \infty.$$

The following theorem summarizes the above result.

THEOREM. *Under assumptions* (1), (2), (3), *and* (4), *when* $m, n \to \infty$,

$$\left[T_N - E(T_N) - n \int_0^1 \left[G \cdot F_n^{-1}(u) - F \cdot F_n^{-1}(u) \right] d\mu_N(u) \right] \Big/ m\sqrt{n}$$

$$\to N\left(0, \sigma_G^2 + \lambda \sigma_F^2\right) \tag{4.7}$$

Note that when $F = G$, the integral term in (4.7) is identically zero and

$$\mathrm{Var}(T_N)/nm^2 \to (1+\lambda)\sigma_F^2 \quad \text{as } m, n \to \infty.$$

Therefore, under the null hypothesis,

$$Z_N = \left[T_N - E(T_N)\right] \Big/ \sqrt{\mathrm{Var}(T_N)} \xrightarrow{D} N(0,1) \quad \text{as } m, n \to \infty.$$

Further, the asymptotic results remain valid if the score function h_N depends on m and n, and converges uniformly to a square Riemann integrable function h on (0, 1).

Let

$$\hat{\Delta} = \frac{1}{m}\int_0^1 \left[G \cdot F_m^{-1}(u) - F \cdot F_m^{-1}(u)\right] d\mu_N(u)$$

and

$$\Delta = \int_0^1 \left[G \cdot F^{-1}(u) - F \cdot F^{-1}(u)\right] h(u)\, du.$$

It can be shown that $\hat{\Delta} \to \Delta$ with probability one as $m, n \to \infty$. This implies that the test based on Z_N is consistent and that the power is approximately an increasing function of Δ, for large m and n.

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