ERTH655/CEE623
Groundwater Modeling

Finite Element Technique

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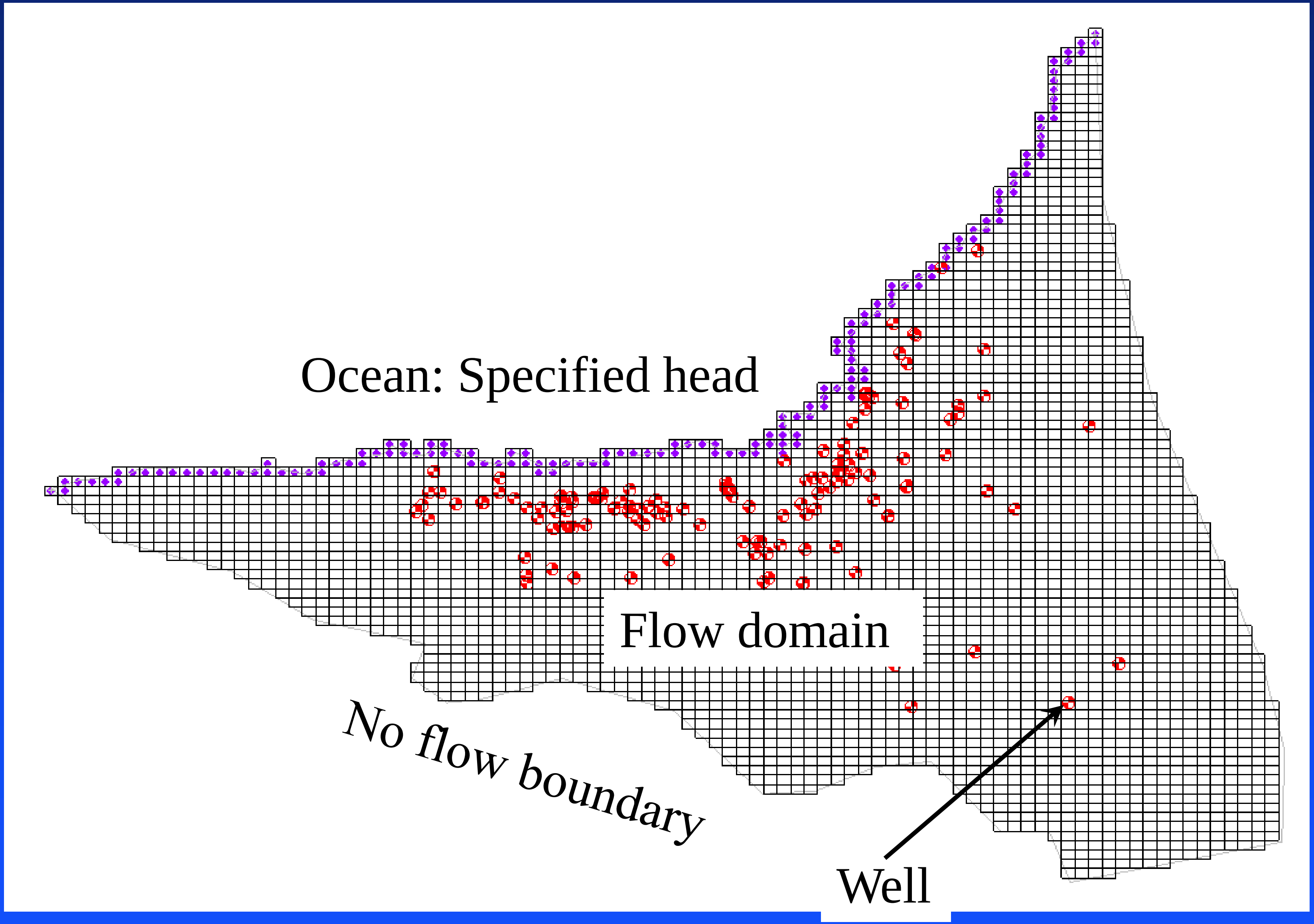
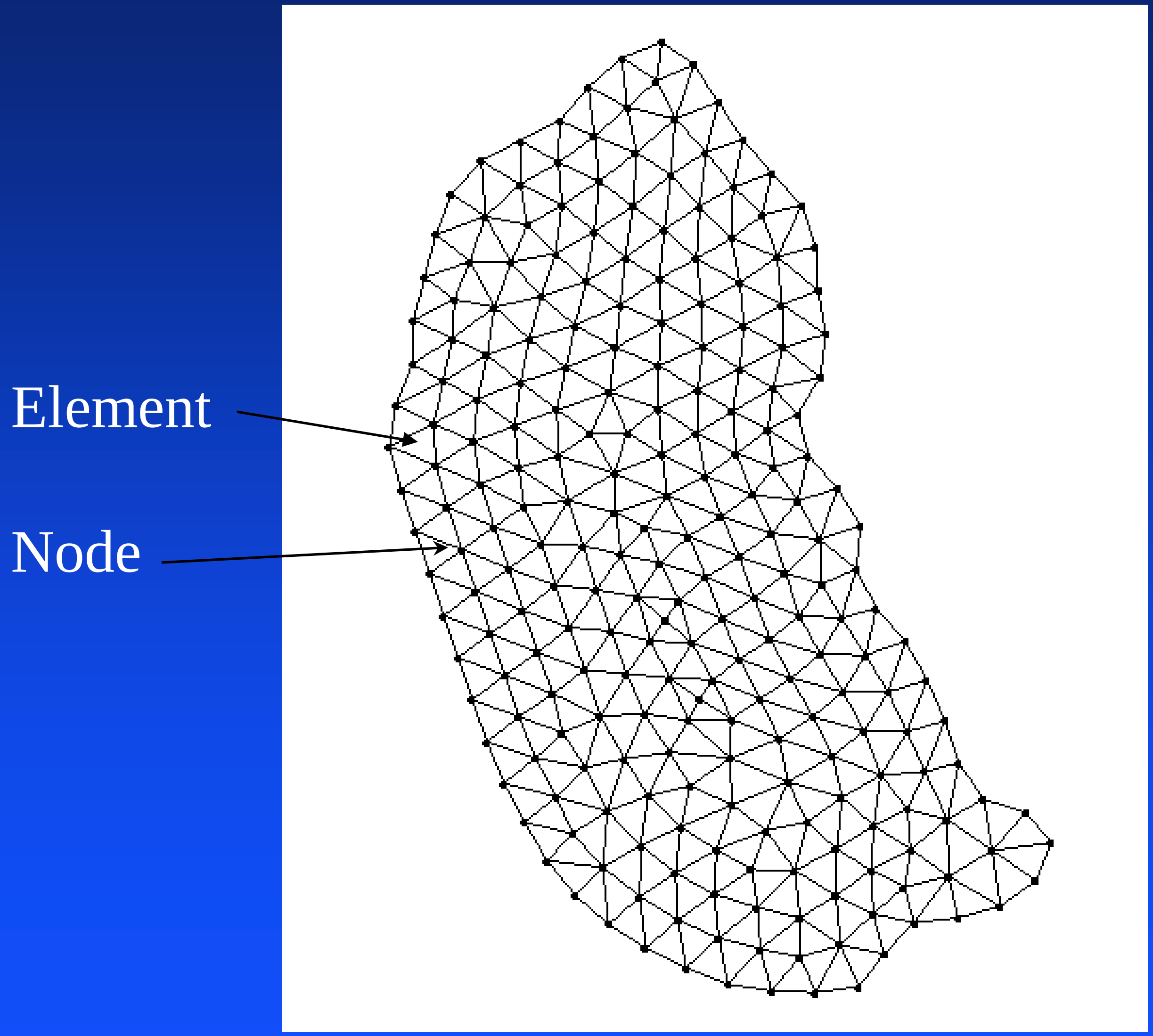
Reminder: Finite difference grid (MODFLOW)



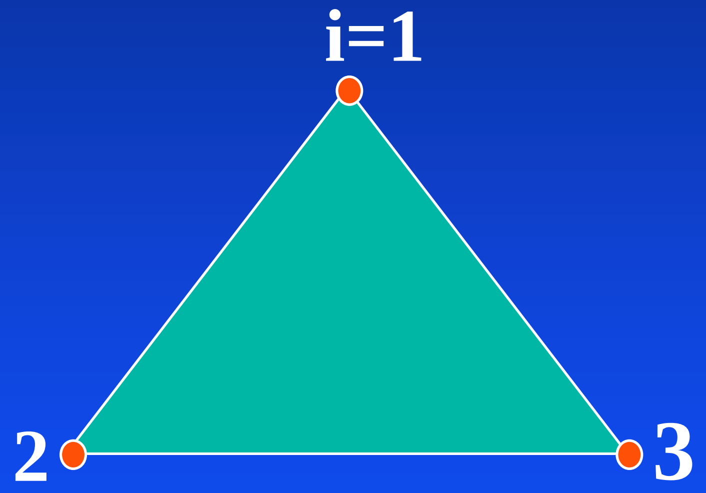
Finite element



2-dimensional, steady state, saturated flow

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) = 0 \qquad (1)$$

$$H^{(e)} = \sum_{i=1}^{n} N_i^{(e)} h_i \qquad (2)$$



H is approximate solution for h, N are shape functions, n number of nodes in element e

Inserting (1) into (2), the residual error will be:

$$R_i^{(e)} = -\iint_{A^{(e)}} W_i^{(e)}(x,y)\left[\frac{\partial}{\partial x}\left(K_x^{(e)}\frac{\partial H^{(e)}}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y^{(e)}\frac{\partial H^{(e)}}{\partial y}\right)\right]dxdy \qquad (3)$$

$$R_i^{(e)} = -\iint_{A^{(e)}} N_i^{(e)}\left[K_x^{(e)}\frac{\partial^2 H^{(e)}}{\partial x^2} + K_y^{(e)}\frac{\partial^2 H^{(e)}}{\partial y^2}\right]dxdy \qquad (4)$$

$$R_i^{(e)} = -\iint_{A^{(e)}} N_i^{(e)}\left[K_x^{(e)}\frac{\partial N_i^{(e)}}{\partial x}\frac{\partial H^{(e)}}{\partial x} + K_y^{(e)}\frac{\partial N_i^{(e)}}{\partial y}\frac{\partial H^{(e)}}{\partial y}\right]dxdy \qquad (5)$$

W = N; Green’s theory is applied

The conductance matrix:

$$\left[K^{(e)}\right] = \iint\limits_{A^{(e)}} \begin{bmatrix} \frac{\partial N_i^{(e)}}{\partial x} & \frac{\partial N_i^{(e)}}{\partial y} \\ \frac{\partial N_j^{(e)}}{\partial x} & \frac{\partial N_j^{(e)}}{\partial y} \\ \frac{\partial N_k^{(e)}}{\partial x} & \frac{\partial N_k^{(e)}}{\partial y} \end{bmatrix} \begin{bmatrix} K_x^{(e)} & 0 \\ 0 & K_y^{(e)} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i^{(e)}}{\partial x} & \frac{\partial N_j^{(e)}}{\partial x} & \frac{\partial N_k^{(e)}}{\partial x} \\ \frac{\partial N_i^{(e)}}{\partial y} & \frac{\partial N_j^{(e)}}{\partial y} & \frac{\partial N_k^{(e)}}{\partial y} \end{bmatrix} dxdy \qquad (6)$$

The expression for residuals:

$$\begin{Bmatrix} R_1^{(e)} \\ R_2^{(e)} \\ R_3^{(e)} \end{Bmatrix} = \left[K^{(e)} \right] \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix} \qquad (7)$$

The global system of equations for p elements:

$$\begin{Bmatrix} R_1 \\ . \\ . \\ . \\ R_p \end{Bmatrix} = [K] \begin{Bmatrix} h_1 \\ . \\ . \\ . \\ h_p \end{Bmatrix} \qquad (8)$$

Setting the residuals to zero:

$$[K]\{h\} = \{0\} \tag{9}$$

Generalization

1. Three dimensional

$$\left[K^{(e)}\right] = \iiint\limits_{V^{(e)}} \begin{bmatrix} \frac{\partial N_1^{(e)}}{\partial x} & \frac{\partial N_1^{(e)}}{\partial y} & \frac{\partial N_1^{(e)}}{\partial z} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{\partial N_n^{(e)}}{\partial x} & \frac{\partial N_n^{(e)}}{\partial y} & \frac{\partial N_n^{(e)}}{\partial z} \end{bmatrix} \begin{bmatrix} K_x^{(e)} & 0 & 0 \\ 0 & K_y^{(e)} & 0 \\ 0 & 0 & K_z^{(e)} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1^{(e)}}{\partial x} & \cdot & \cdot & \frac{\partial N_n^{(e)}}{\partial x} \\ \frac{\partial N_1^{(e)}}{\partial y} & \cdot & \cdot & \frac{\partial N_n^{(e)}}{\partial y} \\ \frac{\partial N_1^{(e)}}{\partial z} & \cdot & \cdot & \frac{\partial N_n^{(e)}}{\partial z} \end{bmatrix} dxdydz \tag{10}$$

2. Source/sink term

$$F_i^{(e)} = \iint_{A^{(e)}} \left(N_i^{(e)} q^{(e)} \right) dx\,dy \qquad (11)$$

$$[K]\{h\} = \{F\} \qquad (12)$$

3. Unsteady state

$$[C]\begin{Bmatrix} \dfrac{\partial h_1}{\partial t} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial h_p}{\partial t} \end{Bmatrix} + [K]\begin{Bmatrix} h_1 \\ \cdot \\ \cdot \\ \cdot \\ h_p \end{Bmatrix} = \{F\} \tag{13}$$

$$\left[C^{(e)}\right] = \iint \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \end{bmatrix} \left[S_s^{(e)}\right]\left[N_1^{(e)} \quad N_2^{(e)} \quad N_3^{(e)}\right] dxdy \tag{14}$$

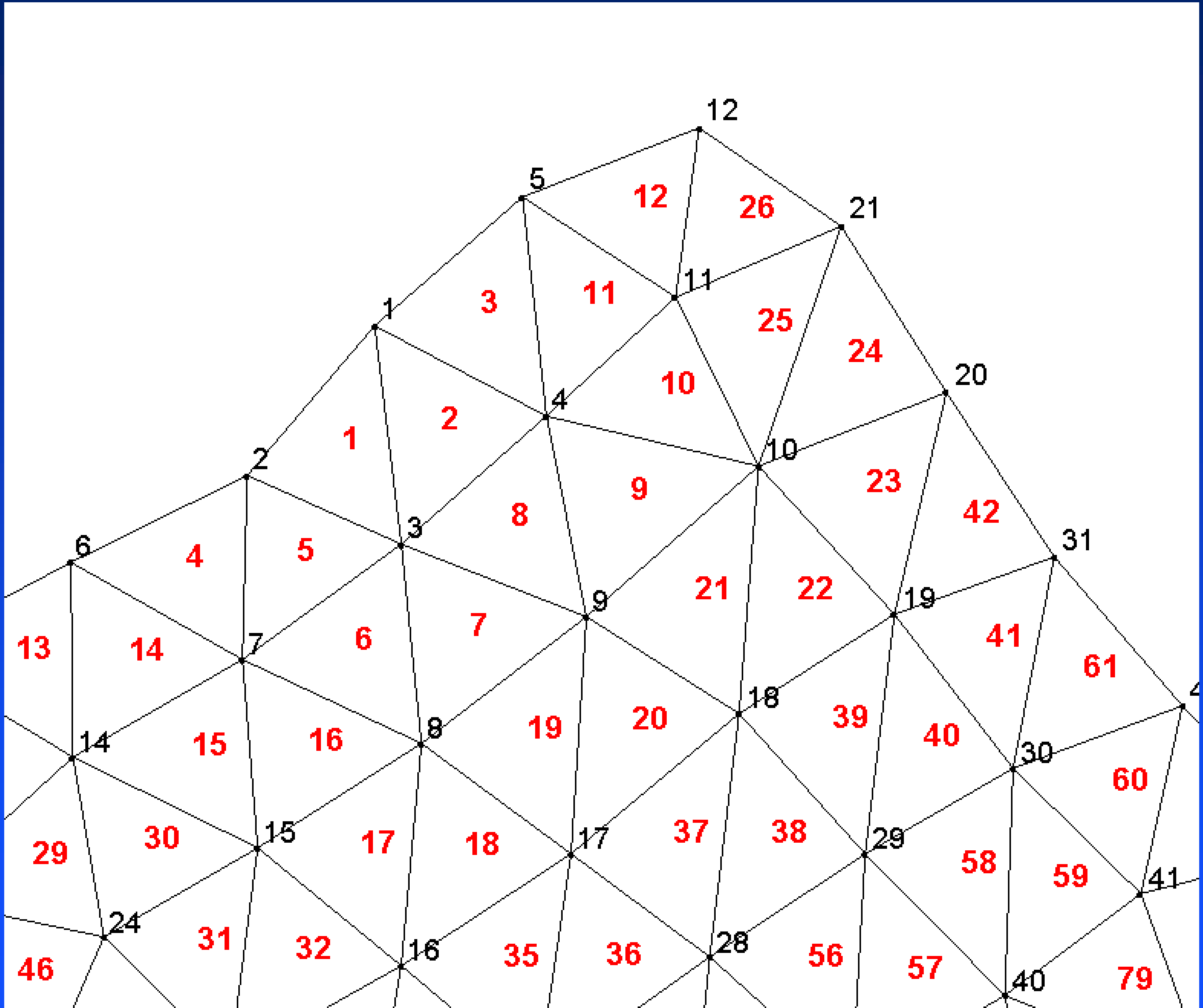
$$\left[C^{(e)}\right] = S_s^{(e)} \frac{A^{(e)}}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (15)$$

$$\{h\} = (1-\omega)\{h\}_t + \omega\{h\}_{t+\Delta t} \qquad (16)$$

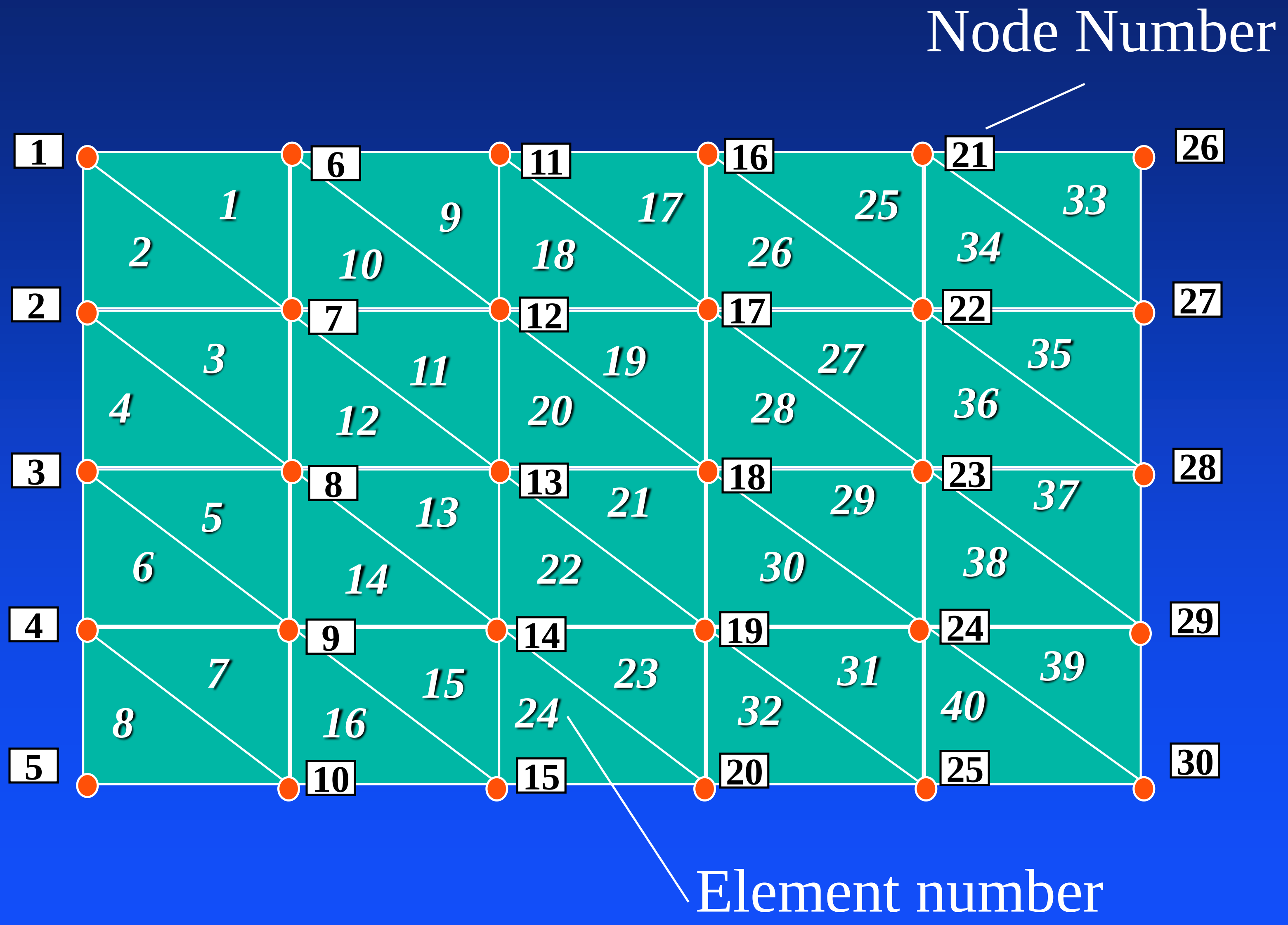
$$\{F\} = (1-\omega)\{F\}_t + \omega\{F\}_{t+\Delta t} \qquad (17)$$

$$\left([C]+\omega\Delta t[K]\right)\{h\}_{t+\Delta t} = \left([C]-(1-\omega)\Delta t[K]\right)\{h\}_t + \Delta t(1-\omega)\{F\}_t + \omega\{F\}_{t+\Delta t} \qquad (18)$$

Node and element numbering



Example mesh



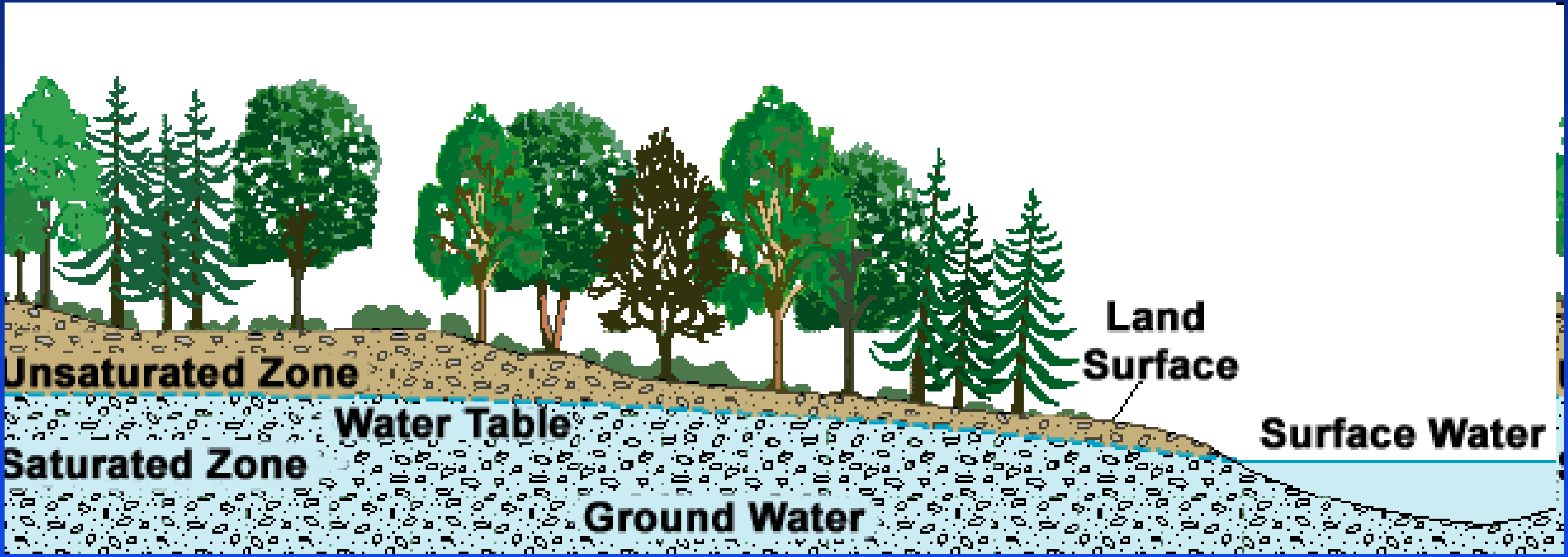
Node coordinates

Node No.	X	Y
1	0	10
2	0	7.5
3	0	5
4	0	2.5
.	.	.
.	.	.
.	.	.
28	20	5
29	20	2.5
30	20	0

Element/node numbers

Element	i	j	k
1	6	1	7
2	1	2	7
3	7	2	8
.	.	.	.
.	.	.	.
.	.	.	.
38	23	24	29
39	29	24	30
40	24	25	30

Saturated/Unsaturated system


Land
Surface
Unsaturated Zone
Water Table
Surface Water
Saturated Zone
Ground Water

Two dimensional, vertical cross section:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial H}{\partial z}\right) = \frac{\partial \theta}{\partial t}$$

$$H = h + z$$

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z} + K_z\right) = \frac{\partial \theta}{\partial t}$$

θ = water content = volume of water/total sample volume
t = time
H = hydraulic head
h = pressure head
x = vertical distance
K = unsaturated hydraulic conductivity = K(h) = K(θ)

$$\theta(h) = \begin{cases} \theta_r + \dfrac{\theta_s - \theta_r}{[1+|\alpha h|^n]^m} & h < 0 \\ \theta_s & h \geq 0 \end{cases}$$

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r}$$

$$K(h) = K_s S_e^l [1 - (1 - S_e^{1/m})^m]^2$$

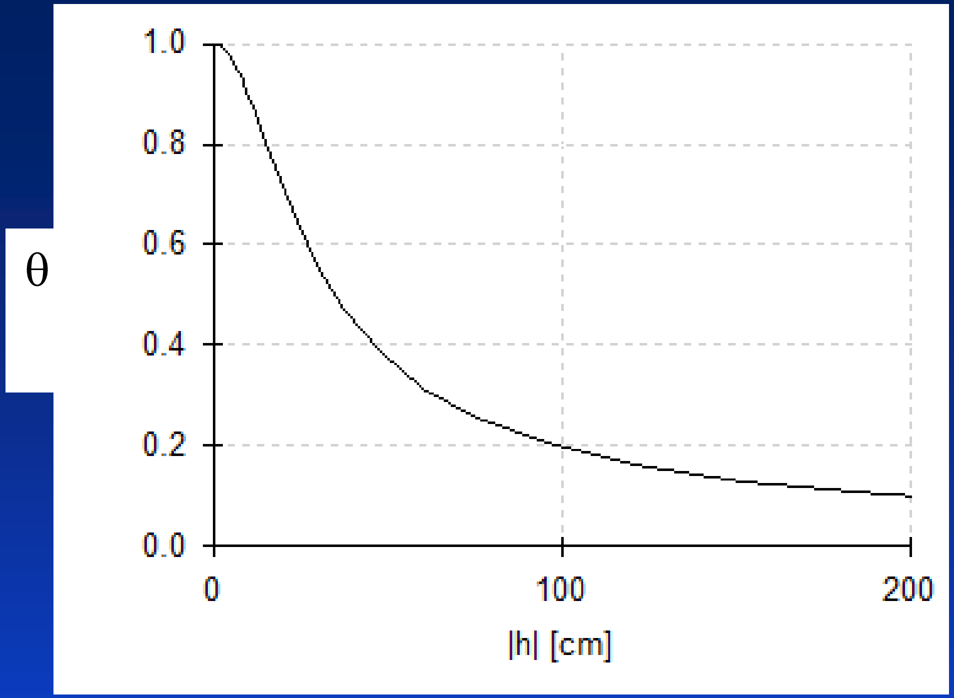
$$m = 1 - 1/n \quad , \quad n > 1$$

θ_s and θ_r = saturated and residual water content, respectively

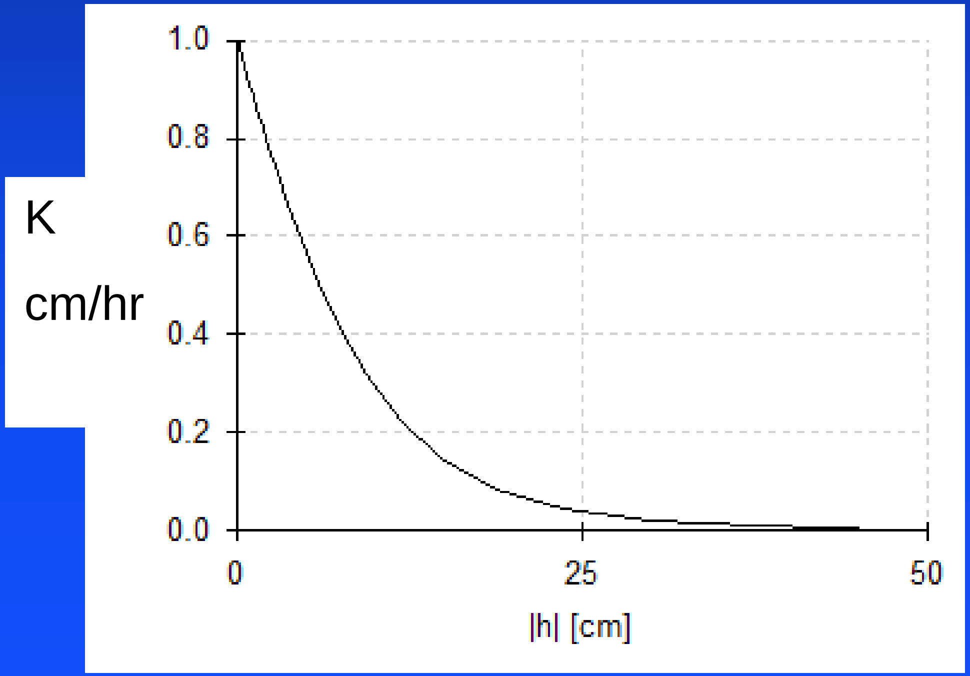
l = parameter ~ 0.5

S_e = effective saturation (between zero and 1)

m and n = parameters



Soil-water characteristic curve



Hydraulic conductivity function

Additional complications: Hysteresis