

Sequences and Limits

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1 The sequence (x_n) is defined by the following formulas for the n th term. Write the first five terms in each case

1.1 $x_n := 1 + (-1)^n$

$$x_1 = 0$$

$$x_2 = 1$$

$$x_3 = 0$$

$$x_4 = 1$$

$$x_5 = 0$$

1.2 $x_n := \frac{(-1)^n}{n}$

$$x_1 = -1$$

$$x_2 = \frac{1}{2}$$

$$x_3 = \frac{-1}{3}$$

$$x_4 = \frac{-1}{4}$$

$$x_5 = \frac{1}{5}$$

1.3 $x_n := \frac{1}{n(n+1)}$

$$\begin{aligned}x_1 &= \frac{1}{2} \\x_2 &= \frac{1}{6} \\x_3 &= \frac{1}{12} \\x_4 &= \frac{1}{20} \\x_5 &= \frac{1}{30}\end{aligned}$$

1.4 $x_n := \frac{1}{n^2+2}$

$$\begin{aligned}x_1 &= \frac{1}{3} \\x_2 &= \frac{1}{6} \\x_3 &= \frac{1}{11} \\x_4 &= \frac{1}{18} \\x_5 &= \frac{1}{27}\end{aligned}$$

2 The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these term persists, give a formula for the n th term x_n

2.1 $5, 7, 8, 11, \dots,$

$$x_n = 3 + 2n$$

2.2 $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots,$

$$x_n = -\left(-\frac{1}{2}\right)^n$$

2.3 $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots,$

$$x_n = \frac{n}{n+1}$$

2.4 1, 4, 9, 16, ...,

$$x_n = n^2$$

3 List the first five terms of the following inductively defined sequences

3.1 $x_1 := 1, x_{n+1} := 3x_n + 1$

$$x_2 = 4$$

$$x_3 = 13$$

$$x_4 = 40$$

$$x_5 = 121$$

$$x_6 = 364$$

4 For any $b \in \mathbb{R}$, prove that $\lim(\frac{b}{n}) = 0$

If $\epsilon > 0$ is given, then $\frac{1}{\epsilon} > 0$. By the Archimedean property, there exists a natural number $K = K(\epsilon)$ such that $\frac{b}{K} < \epsilon$. Then, if $n \geq K$, we have that $\frac{b}{n} \leq \frac{b}{K} < \epsilon$. Consequently, if $n \geq K$, then

$$\left| \frac{b}{n} - 0 \right| = \frac{b}{n} < \epsilon$$

$$\therefore \lim(\frac{b}{n}) = 0$$

5 Use the definition of the limit of a sequence to establish the following limits

5.1 $\lim(\frac{n}{n^2+1}) = 0$

Let $\epsilon > 0$ be given. Let us first note that $n \in \mathbb{N}$ and

$$\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

From the Archimedean Property there exists $K \in \mathbb{N}$ such that, $\frac{1}{K} < \epsilon$. If $n \geq K$, then $\frac{1}{n} \leq \frac{1}{K} < \epsilon$, therefore

$$\left| \frac{n}{n^2+1} - 0 \right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \epsilon$$

$$\therefore \lim(\frac{n}{n^2+1}) = 0$$

5.2 $\lim(\frac{2n}{n+1}) = 2$

Let $\epsilon > 0$ be given, we want to obtain the inequality

$$\left| \frac{2n}{n+1} - 2 \right| < \epsilon$$

when $n \geq K$ for some $K \in \mathbb{N}$. We can simplify the expression on the left:

$$\begin{aligned} \left| \frac{2n}{n+1} - 2 \right| &= \left| \frac{2n - 2n - 2}{n+1} \right| \\ &= \left| \frac{-2}{n+1} \right| \\ &= \frac{2}{n+1} < \frac{2}{n} < \epsilon \end{aligned}$$

From section 4, it is clear that since $\forall b \in R \Rightarrow \lim(\frac{b}{n}) = 0$, then $|\frac{2}{n} - 0| = \frac{2}{n} < \epsilon$

$$\therefore \lim(\frac{2n}{n+1}) = 2$$

5.3 $\lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$

Let $\epsilon > 0$ be given, we want to obtain the inequality

$$\left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| < \epsilon$$

when $n \geq K$ for some $K \in \mathbb{N}$. We can simplify the expression on the left:

$$\begin{aligned} \left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| &= \left| \frac{6n+2 - 6n-15}{4n+5} \right| \\ &= \left| \frac{-13}{4n+5} \right| \\ &= \frac{13}{4n+5} < \frac{13}{n} = \frac{13}{n} < \epsilon \end{aligned}$$

From section 4, it is clear that since $\forall b \in R \Rightarrow \lim(\frac{b}{n}) = 0$, then $|\frac{13}{n} - 0| = \frac{13}{n} < \epsilon$
 $\therefore \lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$

5.4 $\lim(\frac{n^2-1}{2n^2+3}) = \frac{1}{2}$

Let $\epsilon > 0$ be given, we want to obtain the inequality

$$\left| \frac{n^2-1}{2n^2+3} - \frac{1}{2} \right| < \epsilon$$

when $n \geq K$ for some $K \in \mathbb{N}$. We can simplify the expression on the left:

$$\begin{aligned} \left| \frac{n^2 - 1}{2n^2 + 3} - \frac{1}{2} \right| &= \left| \frac{2n^2 - 2 - 2n^2 - 3}{4n^2 + 6} \right| \\ &= \left| \frac{-5}{4n^2 + 6} \right| \\ &= \frac{5}{4n^2 + 6} < \frac{5}{4n^2} = \frac{5}{n^2} \leq \frac{5}{n} < \epsilon \end{aligned}$$

From section 4, it is clear that since $\forall b \in R \Rightarrow \lim(\frac{b}{n}) = 0$, then $|\frac{5}{n} - 0| = \frac{5}{n} < \epsilon$
 $\therefore \lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$

6 Show that

6.1 $\lim(\frac{1}{\sqrt{n+7}}) = 0$

We should first note that since $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}}$$

For a given $\epsilon > 0$, we obtain $1/\sqrt{n} < \epsilon$ iff $n > 1/\epsilon^2$. If we take $K > 1/\epsilon^2$ then it follows that for all $n \geq K$:

$$\left| \frac{1}{\sqrt{n}} - 0 \right| < \epsilon$$

6.2 $\lim(\frac{2n}{n+2}) = 2$

We should first note that since $n \in \mathbb{N}$,

$$\frac{2n}{\sqrt{n+2}} < \frac{2n}{n} = 2$$

It is trivial to show that $|2 - 2| < \epsilon$ for all $\epsilon > 0$

6.3 $\lim(\frac{\sqrt{n}}{n+1}) = 0$

We should first note that since $n \in \mathbb{N}$,

$$\frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$

It is sufficient from 6.1 to conclude that $\lim(\frac{\sqrt{n}}{n+1}) = 0$

$$\mathbf{6.4} \quad \lim\left(\frac{(-1)^n \cdot n}{n^2+1}\right) = 0$$

We should first note that since $n \in \mathbb{N}$,

$$\frac{(-1)^n \cdot n}{n^2+1} < \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

The result follows from section 4