

# Sequences and Limits

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**1 The sequence  $(x_n)$  is defined by the following formulas for the  $n$ th term. Write the first five terms in each case**

**1.1**  $x_n := 1 + (-1)^n$

$$x_1 = 0$$

$$x_2 = 1$$

$$x_3 = 0$$

$$x_4 = 1$$

$$x_5 = 0$$

**1.2**  $x_n := \frac{(-1)^n}{n}$

$$x_1 = -1$$

$$x_2 = \frac{1}{2}$$

$$x_3 = \frac{-1}{3}$$

$$x_4 = \frac{-1}{4}$$

$$x_5 = \frac{1}{5}$$

**1.3**  $x_n := \frac{1}{n(n+1)}$

$$\begin{aligned}x_1 &= \frac{1}{2} \\x_2 &= \frac{1}{6} \\x_3 &= \frac{1}{12} \\x_4 &= \frac{1}{20} \\x_5 &= \frac{1}{30}\end{aligned}$$

**1.4**  $x_n := \frac{1}{n^2+2}$

$$\begin{aligned}x_1 &= \frac{1}{3} \\x_2 &= \frac{1}{6} \\x_3 &= \frac{1}{11} \\x_4 &= \frac{1}{18} \\x_5 &= \frac{1}{27}\end{aligned}$$

**2** The first few terms of a sequence  $(x_n)$  are given below. Assuming that the "natural pattern" indicated by these term persists, give a formula for the  $n$ th term  $x_n$

**2.1** 5, 7, 8, 11, ...,

$$x_n = 3 + 2n$$

**2.2**  $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots,$

$$x_n = -\left(-\frac{1}{2}\right)^n$$

**2.3**  $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots,$

$$x_n = \frac{n}{n+1}$$

**2.4** 1, 4, 9, 16, ...,

$$x_n = n^2$$

**3** List the first five terms of the following inductively defined sequences

**3.1**  $x_1 := 1$ ,  $x_{n+1} := 3x_n + 1$

$$x_2 = 4$$

$$x_3 = 13$$

$$x_4 = 40$$

$$x_5 = 121$$

$$x_6 = 364$$

**4** For any  $b \in \mathbb{R}$ , prove that  $\lim(\frac{b}{n}) = 0$

If  $\epsilon > 0$  is given, then  $\frac{1}{\epsilon} > 0$ . By the Archimedean property, there exists a natural number  $K = K(\epsilon)$  such that  $\frac{b}{K} < \epsilon$ . Then, if  $n \geq K$ , we have that  $\frac{b}{n} \leq \frac{b}{K} < \epsilon$ . Consequently, if  $n \geq K$ , then

$$\left| \frac{b}{n} - 0 \right| = \frac{b}{n} < \epsilon$$

$$\therefore \lim(\frac{b}{n}) = 0$$

**5** Use the definition of the limit of a sequence to establish the following limits

**5.1**  $\lim(\frac{n}{n^2+1}) = 0$

Let  $\epsilon > 0$  be given. Let us first note that  $n \in \mathbb{N}$  and

$$\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

From the Archimedean Property there exists  $K \in \mathbb{N}$  such that,  $\frac{1}{K} < \epsilon$ . If  $n \geq K$ , then  $\frac{1}{n} \leq \frac{1}{K} < \epsilon$ , therefore

$$\left| \frac{n}{n^2+1} - 0 \right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \epsilon$$

$$\therefore \lim(\frac{n}{n^2+1}) = 0$$

### 5.2 $\lim(\frac{2n}{n+1}) = 2$

Let  $\epsilon > 0$  be given, we want to obtain the inequality

$$\left| \frac{2n}{n+1} - 2 \right| < \epsilon$$

when  $n \geq K$  for some  $K \in \mathbb{N}$ . We can simplify the expression on the left:

$$\begin{aligned} \left| \frac{2n}{n+1} - 2 \right| &= \left| \frac{2n - 2n - 2}{n+1} \right| \\ &= \left| \frac{-2}{n+1} \right| \\ &= \frac{2}{n+1} < \frac{2}{n} < \epsilon \end{aligned}$$

From section 4, it is clear that since  $\forall b \in R \Rightarrow \lim(\frac{b}{n}) = 0$ , then  $|\frac{2}{n} - 0| = \frac{2}{n} < \epsilon$

$$\therefore \lim(\frac{2n}{n+1}) = 2$$

### 5.3 $\lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$

Let  $\epsilon > 0$  be given, we want to obtain the inequality

$$\left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| < \epsilon$$

when  $n \geq K$  for some  $K \in \mathbb{N}$ . We can simplify the expression on the left:

$$\begin{aligned} \left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| &= \left| \frac{6n+2-6n-15}{4n+5} \right| \\ &= \left| \frac{-13}{4n+5} \right| \\ &= \frac{13}{4n+5} < \frac{13}{n} = \frac{13}{n} < \epsilon \end{aligned}$$

From section 4, it is clear that since  $\forall b \in R \Rightarrow \lim(\frac{b}{n}) = 0$ , then  $|\frac{13}{n} - 0| = \frac{13}{n} < \epsilon$   
 $\therefore \lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$

### 5.4 $\lim(\frac{n^2-1}{2n^2+3}) = \frac{1}{2}$

Let  $\epsilon > 0$  be given, we want to obtain the inequality

$$\left| \frac{n^2-1}{2n^2+3} - \frac{1}{2} \right| < \epsilon$$

when  $n \geq K$  for some  $K \in \mathbb{N}$ . We can simplify the expression on the left:

$$\begin{aligned} \left| \frac{n^2 - 1}{2n^2 + 3} - \frac{1}{2} \right| &= \left| \frac{2n^2 - 2 - 2n^2 - 3}{4n^2 + 6} \right| \\ &= \left| \frac{-5}{4n^2 + 6} \right| \\ &= \frac{5}{4n^2 + 6} < \frac{5}{4n^2} = \frac{5}{n^2} \leq \frac{5}{n} < \epsilon \end{aligned}$$

From section 4, it is clear that since  $\forall b \in R \Rightarrow \lim(\frac{b}{n}) = 0$ , then  $|\frac{5}{n} - 0| = \frac{5}{n} < \epsilon$   
 $\therefore \lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$