Sequences and Limits

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1 The sequence (x_n) is defined by the following formulas for the nth term. Write the first five terms in each case

1.1 $x_n := 1 + (-1)^n$

$$x_1 = 0$$
$$x_2 = 1$$
$$x_3 = 0$$
$$x_4 = 1$$
$$x_5 = 0$$

1.2 $x_n := \frac{(-1)^n}{n}$

$$x_1 = -1$$
$$x_2 = \frac{1}{2}$$
$$x_3 = \frac{-1}{3}$$
$$x_4 = \frac{-1}{4}$$
$$x_5 = \frac{1}{5}$$

1.3 $x_n := \frac{1}{n(n+1)}$

$$x_1 = \frac{1}{2}$$
$$x_2 = \frac{1}{6}$$
$$x_3 = \frac{1}{12}$$
$$x_4 = \frac{1}{20}$$
$$x_5 = \frac{1}{30}$$

1.4 $x_n := \frac{1}{n^2+2}$

$$x_1 = \frac{1}{3}$$
$$x_2 = \frac{1}{6}$$
$$x_3 = \frac{1}{11}$$
$$x_4 = \frac{1}{18}$$
$$x_5 = \frac{1}{27}$$

2 The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these term persists, give a formula for the nth term x_n

2.1 5, 7, 8, 11, ...,

$$x_n = 3 + 2n$$

2.2 $\frac{1}{2}$, $-\frac{1}{4}$, $\frac{1}{8}$, $-\frac{1}{16}$, ...,

$$x_n = -(-\frac{1}{2})^n$$

2.3 $\frac{1}{2}$, $-\frac{2}{3}$, $\frac{3}{4}$, $-\frac{4}{5}$, ...,

$$x_n = \frac{n}{n+1}$$

2.4 1, 4, 9, 16, ...,

$$x_n = n^2$$

3 List the first five terms of the following inductively defined sequences

3.1 $x_1 := 1$, $x_{n+1} := 3x_n + 1$

$$\begin{aligned} x_2 &= 4 \\ x_3 &= 13 \\ x_4 &= 40 \\ x_5 &= 121 \\ x_6 &= 364 \end{aligned}$$

4 For any $b \in \mathbb{R}$, prove that $\lim(\frac{b}{n}) = 0$

If $\epsilon > 0$ is given, then $\frac{1}{\epsilon} > 0$. By the Archimedean property, there exists a natural number $K = K(\epsilon)$ such that $\frac{b}{K} < \epsilon$. Then, if $n \geq K$, we have that $\frac{b}{n} \leq \frac{b}{K} < \epsilon$. Consequently, if $n \geq K$, then

$$\left|\frac{b}{n} - 0\right| = \frac{b}{n} < \epsilon$$

$\therefore \lim(\frac{b}{n}) = 0$

5 Use the definition of the limit of a sequence to establish the following limits

5.1 $\lim(\frac{n}{n^2+1}) = 0$

Let $\epsilon > 0$ be given. Let us first note that $n \in \mathbb{N}$ and

$$\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

From the Archimedean Property there exists $K \in \mathbb{N}$ such that, $\frac{1}{K} < \epsilon$. If $n \geq K$, then $\frac{1}{n} \leq \frac{1}{K} < \epsilon$, therefore

$$\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \epsilon$$

$\therefore \lim(\frac{n}{n^2+1}) = 0$

5.2 $\lim(\frac{2n}{n+1}) = 2$

Let $\epsilon > 0$ be given, we want to obtain the inequality

$$\Big|\frac{2n}{n+1} - 2\Big| < \epsilon$$

when $n \geq K$ for some $K \in \mathbb{N}$. We can simplify the expression on the left:

$$\begin{aligned} \Big|\frac{2n}{n+1} - 2\Big| &= \Big|\frac{2n-2n-2}{n+1}\Big| \\ &= \Big|\frac{-2}{n+1}\Big| \\ &= \frac{2}{n+1} < \frac{2}{n} < \epsilon \end{aligned}$$

From section 4, it is clear that since $\forall b \in R \Rightarrow \lim(\frac{b}{n}) = 0$, then $|\frac{2}{n} - 0| = \frac{2}{n} < \epsilon$

$\therefore \lim(\frac{2n}{n^2+1}) = 2$

5.3 $\lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$

Let $\epsilon > 0$ be given, we want to obtain the inequality

$$\Big|\frac{3n+1}{2n+5} - \frac{3}{2}\Big| < \epsilon$$

when $n \geq K$ for some $K \in \mathbb{N}$. We can simplify the expression on the left:

$$\begin{aligned} \Big|\frac{3n+1}{2n+5} - \frac{3}{2}\Big| &= \Big|\frac{6n+2-6n-15}{4n+5}\Big| \\ &= \Big|\frac{-13}{4n+5}\Big| \\ &= \frac{13}{4n+5} < \frac{13}{n} = \frac{13}{n} < \epsilon \end{aligned}$$

From section 4, it is clear that since $\forall b \in R \Rightarrow \lim(\frac{b}{n}) = 0$, then $|\frac{13}{n} - 0| = \frac{13}{n} < \epsilon$
$\therefore \lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$

5.4 $\lim(\frac{n^2-1}{2n^2+3}) = \frac{1}{2}$

Let $\epsilon > 0$ be given, we want to obtain the inequality

$$\Big|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\Big| < \epsilon$$

when $n \geq K$ for some $K \in \mathbb{N}$. We can simplify the expression on the left:

$$\begin{aligned}\Big|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\Big| &= \Big|\frac{2n^2-2-2n^2-3}{4n^2+6}\Big| \\ &= \Big|\frac{-5}{4n^2+6}\Big| \\ &= \frac{5}{4n^2+6} < \frac{5}{4n^2} = \frac{5}{n^2} \leq \frac{5}{n} < \epsilon\end{aligned}$$

From section 4, it is clear that since $\forall b \in R \Rightarrow \lim(\frac{b}{n}) = 0$, then $|\frac{5}{n} - 0| = \frac{5}{n} < \epsilon$
$\therefore \lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$

6 Show that

6.1 $\lim(\frac{1}{\sqrt{n+7}}) = 0$

We should first note that since $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}}$$

For a given $\epsilon > 0$, we obtain $1/\sqrt{n} < \epsilon$ iff $n > 1/\epsilon^2$. If we take $K > 1/\epsilon^2$ then it follows that for all $n \geq K$:

$$\Big|\frac{1}{\sqrt{n}} - 0\Big| < \epsilon$$

6.2 $\lim(\frac{2n}{n+2}) = 2$

We should first note that since $n \in \mathbb{N}$,

$$\frac{2n}{\sqrt{n+2}} < \frac{2n}{n} = 2$$

It is trivial to show that $|2-2| < \epsilon$ for all $\epsilon > 0$

6.3 $\lim(\frac{\sqrt{n}}{n+1}) = 0$

We should first note that since $n \in \mathbb{N}$,

$$\frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$

It is sufficient from 6.1 to conclude that $\lim(\frac{\sqrt{n}}{n+1}) = 0$

6.4 $\lim(\frac{(-1)^n \cdot n}{n^2+1}) = 0$

We should first note that since $n \in \mathbb{N}$,

$$\frac{(-1)^n \cdot n}{n^2+1} < \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

The result follows from section 4

7 Let $x_n := 1/\ln(n+1)$ for $n \in \mathbb{N}$

7.1 Use the definition of limit to show that $\lim(x_n) = 0$

Since ln is a monotonically increasing function, we have that:

$$\frac{1}{\ln(n+1)} < \frac{1}{\ln(n)}$$

We can find the inequality

$$\begin{aligned} \epsilon &> \frac{1}{\ln(n)} \\ \frac{1}{\epsilon} &< \ln(n) \\ e^{1/\epsilon} &< n \end{aligned}$$

If we choose $K(\epsilon) = e^{1/\epsilon}$ and $n \geq K(\epsilon)$, it follows that

$$\left|\frac{1}{\ln(n+1)} - 0\right| < \left|\frac{1}{\ln(n)}\right| = \frac{1}{\ln(n)} < \epsilon$$

$\therefore \lim(x_n) = 0$

7.2 Find a specific value of $K(\epsilon)$ as required in the definition of limit for each of (i) $\epsilon = 1/2$, and (ii) $\epsilon = 1/10$

Using $K(\epsilon) = e^{1/\epsilon}$, we have for (i) $K(1/2) = e^2$, and (ii) $K(1/10) = e^{10}$

8 Prove that $\lim(x_n) = 0$ iff $\lim(|x_n|) = 0$. Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n)

($\Longrightarrow$) If $\lim(x_n) = 0$, we have that given $\epsilon > 0$

$$|x_n - 0| < \epsilon$$

For all $n \geq K(\epsilon) \in \mathbb{N}$.
We will define $|x|$ as $\max\{-x, x\}$, we will have that

$$\begin{aligned} ||x_n| - 0| &= \max\{\max\{x_n, -x_n\}, -\max\{x_n, -x_n\}\} \\ &= \max\{x_n, -x_n\} \\ &= |x_n - 0| \\ &< \epsilon \end{aligned}$$

($\Longleftarrow$) If $\lim(|x_n|) = 0$, we have that given $\epsilon > 0$

$$||x_n| - 0| < \epsilon$$

For all $n \geq K(\epsilon) \in \mathbb{N}$.
We will define $|x|$ as $\max\{-x, x\}$, we will have that

$$\begin{aligned} ||x_n| - 0| &= \max\{\max\{x_n, -x_n\}, -\max\{x_n, -x_n\}\} \\ &= \max\{x_n, -x_n\} \\ &= |x_n - 0| \\ &< \epsilon \end{aligned}$$

Thus concluding our proof. $x_n = (-1)^n$ is an example of a sequence with $\lim(|x_n|)$ converging but not $\lim(x_n)$

9 Show that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim(x_n) = 0$, then $\lim(\sqrt{x_n}) = 0$

If $\lim(x_n) = 0$ then we have for all $\epsilon > 0$:

$$|x_n - 0| < \epsilon$$

which is equivalent to saying that $x_n < \epsilon$ as x_n is positive (or zero). It follows that $\sqrt{x_n} < x_n < \epsilon$, and subsequently $|\sqrt{x_n}| < |x_n| < \epsilon$.

10 Prove that $\lim(x_n) = x$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$

If $\lim(x_n) = x$ and if $x > 0$ then for all $\epsilon > 0$

$$|x_n - x| < \epsilon$$

It follows that

$$-\epsilon < x_n - x < \epsilon$$

Since $\epsilon > 0$, we can choose $\epsilon = \frac{x}{2} > 0$

$$-\frac{x}{2} < x_n - x < \frac{x}{2}$$

We have that

$$\begin{aligned} -\frac{x}{2} &< x_n - x \\ -\frac{x}{2} + x &< x_n \\ \frac{x}{2} &< x_n \end{aligned}$$

Since $\frac{x}{2} > 0$, then by definition of limit, there exists $M = K(\epsilon) \in \mathbb{N}$ such that $x_n > 0$ for $n \geq M$