complex and cannot be written down as easily as for the two-link manipulator. The general procedure that we discuss in Chapter 3 establishes coordinate frames at each joint and allows one to transform systematically among these frames using matrix transformations. The procedure that we use is referred to as the **Denavit-Hartenberg** convention. We then use **homogeneous coordinates** and **homogeneous transformations** to simplify the transformation among coordinate frames.

Inverse Kinematics

Now, given the joint angles θ_1, θ_2 we can determine the end-effector coordinates x and y. In order to command the robot to move to location A we need the inverse; that is, we need the joint variables θ_1, θ_2 in terms of the x and y coordinates of A. This is the problem of **inverse kinematics**. In other words, given x and y in Equations (1.1) and (1.2), we wish to solve for the joint angles. Since the forward kinematic equations are nonlinear, a solution may not be easy to find, nor is there a unique solution in general. We can see in the case of a two-link planar mechanism that there may be no solution, for example if the given (x, y) coordinates are out of reach of the manipulator. If the given (x, y) coordinates are within the manipulator's reach there may be two solutions as shown in Figure 1.21, the so-called

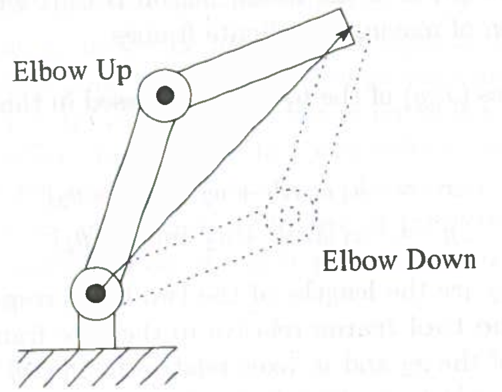

Figure 1.21: The two-link elbow robot has two solutions to the inverse kinematics except at singular configurations, the elbow up solution and the elbow down solution.

elbow up and **elbow down** configurations, or there may be exactly one solution if the manipulator must be fully extended to reach the point. There may even be an infinite number of solutions in some cases (Problem 1-20).

Consider the diagram of Figure 1.22. Using the **law of cosines**[1] we see

[1]See Appendix A

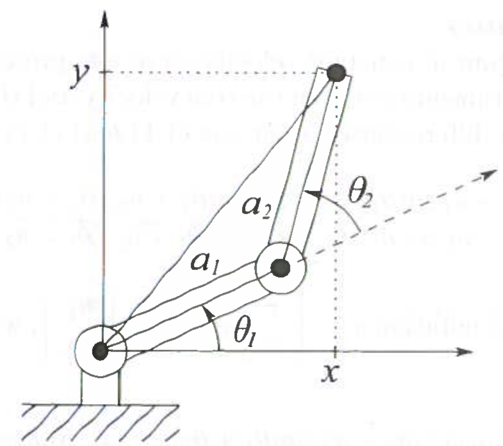

Figure 1.22: Solving for the joint angles of a two-link planar arm.

that the angle θ_2 is given by

$$\cos\theta_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2} := D \tag{1.5}$$

We could now determine θ_2 as $\theta_2 = \cos^{-1}(D)$. However, a better way to find θ_2 is to notice that if $\cos(\theta_2)$ is given by Equation (1.5), then $\sin(\theta_2)$ is given as

$$\sin(\theta_2) = \pm\sqrt{1 - D^2} \tag{1.6}$$

and, hence, θ_2 can be found by

$$\theta_2 = \tan^{-1}\frac{\pm\sqrt{1 - D^2}}{D} \tag{1.7}$$

The advantage of this latter approach is that both the elbow-up and elbow-down solutions are recovered by choosing the negative and positive signs in Equation (1.7), respectively.

It is left as an exercise (Problem 1-18) to show that θ_1 is now given as

$$\theta_1 = \tan^{-1}(y/x) - \tan^{-1}\left(\frac{a_2\sin\theta_2}{a_1 + a_2\cos\theta_2}\right) \tag{1.8}$$

Notice that the angle θ_1 depends on θ_2. This makes sense physically since we would expect to require a different value for θ_1, depending on which solution is chosen for θ_2.