— Cubical Subtypes
— Initial Base Library
— Fast Type Check
— Strict Equality (HTS)
— Kan Operations

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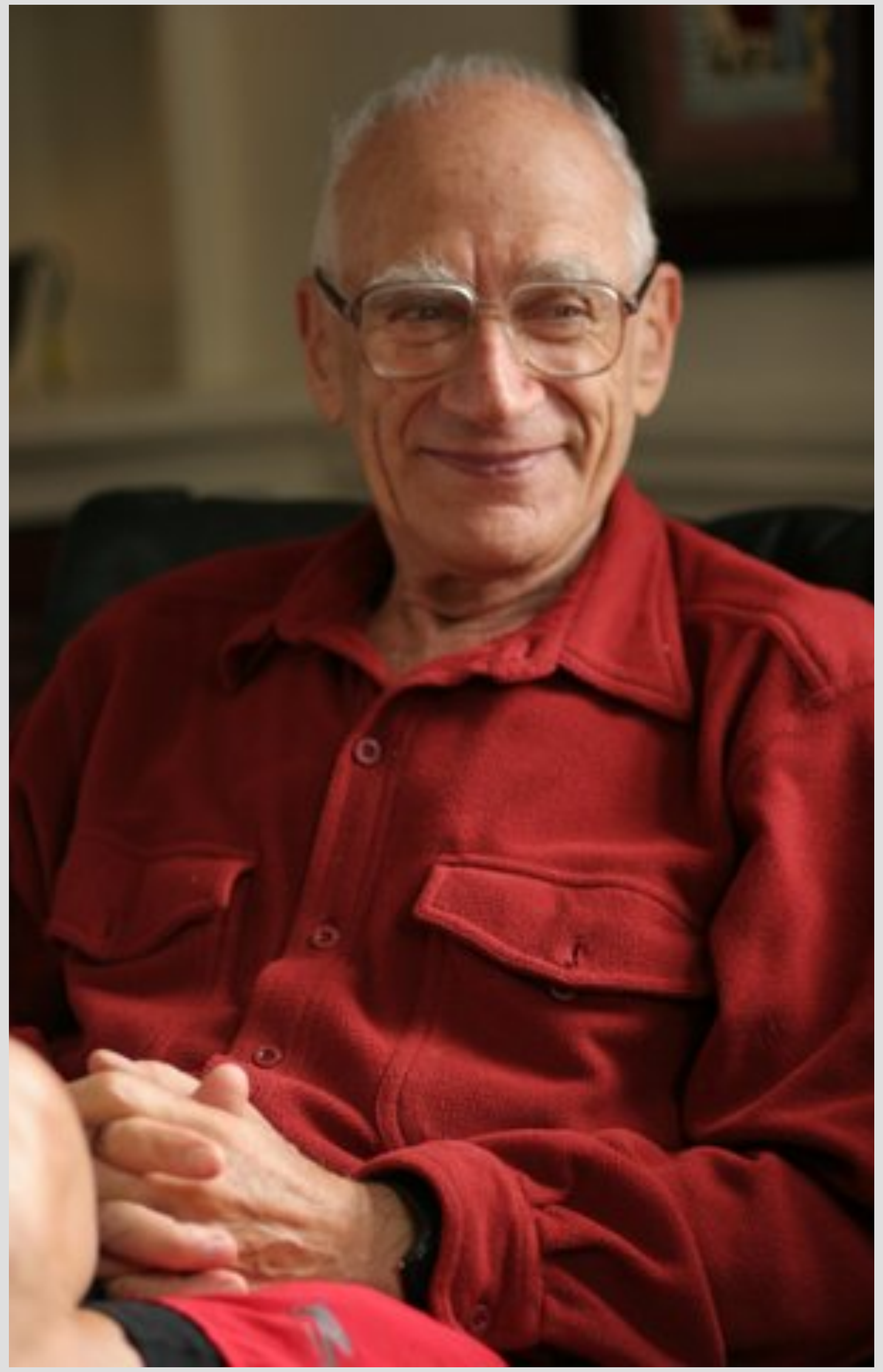
Groupoid Infinity Cafe

Groipoid Infinity, 2021, INFOTECH SE, Ukraine, Kyiv

@siegment
@5HT



Dan Kan

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Thierry Coquand

PTS	ML72	ML73	CCHM	HTS
UNI	PI	UNI	UNI	UNI
PI	SIGMA	PI	PI	PI
		SIGMA	SIGMA	SIGMA
		ID	PATH	ID
		NAT	GLUE	PATH
			HIT	GLUE
				HIT

Type Theory

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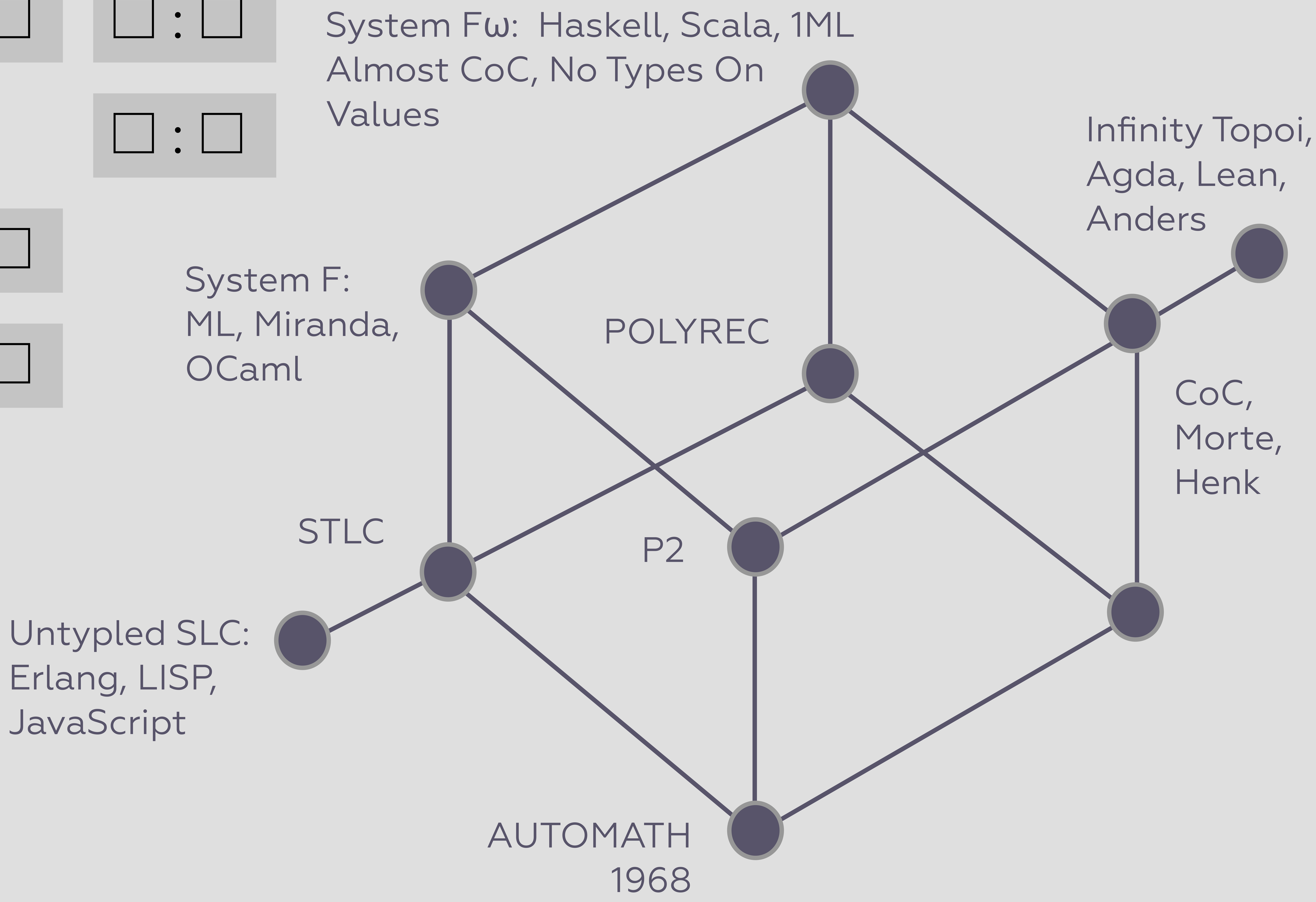
HW	VHDL, Verilog, Clash, Chisel, SystemC, Lava, BSV
ASM	PDP-11, VAX, S/360, M68K, PowerPC, MIPS, SPARC, Super-H, Intel, ARM, RISC-V
ALG	C, BCPL, ALGOL, SNOBOL, Simula, Pascal, Oberon, COBOL, PL/1
ML	SML, Alice ML, OCaml, UrWeb, Flow, F#
PURE	HOPE, Miranda, Clean, Charity, Joy, Mercury, Elm, PureScript, Fω Scala, Haskell, 1ML, Plutus
MACR	LISP, Scheme, Clojure, Racket, Dylan, LFE, CL, Nemerle, Nim, Haxe, Perl, Elixir
OOI	Simula, Smalltalk, Self, REBOL, Io, JS, Lua, Ruby, Python, PHP, TS, Java, Kotlin
CMP	C++, Rust, D, Swift, Fortran
SHELL	PowerShell, TCL, SH, CLIPS, BASIC, FORTH, SVC IDL, SOAP, ASN.1, GRPC
MARK	TeX, PS, XML, SVG, CSS, ROFF, OWL, SGML, RDF, SysML
LOGIC	AUT-68, ACL2, LEGO, ALF, Prolog, CPL, Mizar, Dedukti, HOL, Isabelle, Z
ΠΣ	Coq, F*, Lean, NuPRL, ATS, Epigram, Cayenne, Idris, Dhall, Cedile, Kind
HoTT	Menkar, Cubical, yacctt, redtt, RedPRL, Arend, Agda, Anders
CHKR	TLA+, Twelf, Promela, CSPM
PAR	Ling, Pony, Erlang, BPMN, Ada, E, Go, Occam, Oz
ARR	Julia, Wolfram, MATHLAB, Octave, Futhark, APL, SQL, cg, Clarion, Clipper, QCL, K, MUMPS, Q, R, S, J, O

CoC:	$\ast : \ast$	$\Box : \ast$	$\ast : \Box$	$\Box : \Box$
Fω:	$\ast : \ast$	$\Box : \ast$		$\Box : \Box$
P2:	$\ast : \ast$	$\Box : \ast$	$\ast : \Box$	
AUT:	$\ast : \ast$		$\ast : \Box$	
F:	$\ast : \ast$	$\Box : \ast$		



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OCaml Internal AST

```
type exp =
  | EPre of int | EKan of int
  | EVar of name | EHole
  | EPi of exp * (name * exp) | ELam of exp * (name * exp) | EApp of exp * exp
  | ESig of exp * (name * exp) | EPair  of exp * exp | EFst of exp | ESnd of exp
  | EId of exp | ERef of exp | EJ of exp
  | EPathP of exp | EPLam of exp | EAppFormula of exp * exp
  | EI | EDir of dir | EAnd of exp * exp | EOr of exp * exp | ENeg of exp
  | ETransp of exp * exp | EPartial of exp | ESystem of system
  | ESub of exp * exp * exp | EInc of exp | EOuc of exp
```

Informal BNF

cosmos := Uj | Vk
var := var name | hole
pi := Π name E E | λ name E E | E E
sigma := Σ name E E | (E,E) | E.1 | E.2
id:=Id E | ref E | idJ E
path := Path E | Ei | E @ E
I := I | 0 | 1 | E \/ E | E /\ E | ¬E
part := Partial E E | [(E=I)→E, ...]
sub := inc E | ouc E | E [I → E]
kan:=transp E E | hcomp E
glue:=Glue E | glue E | unglue E E

E := cosmos | var | MLTT | CCHM | HIT

HIT := inductive E E | ctor name E | match E E
CCHM := path | I | part | sub | kan | glue
MLTT := pi | sigma | id

Cosmos

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```
inductive cosmos : U
  | fibrant: nat → cosmos
  | pretypes: nat → cosmos
```

Agda 2.6.2

```
inductive cosmos : U
  | prop: nat → cosmos
  | fibrant: nat → cosmos
  | pretypes: nat → cosmos
  | strict: nat → cosmos
  | omega: cosmos
  | lock: cosmos
```

Pi


```
def Pi (A : U) (B : A → U) : U := Π (x : A), B x
def lambda (A: U) (B: A → U) (b: Pi A B) : Pi A B := λ (x : A), b x
def lam (A B: U) (f: A → B) : A → B := λ (x : A), f x
def apply (A: U) (B: A → U) (f: Pi A B) (a: A) : B a := f a
def app (A B: U) (f: A → B) (x: A): B := f x
def Π-β (A : U) (B : A → U) (a : A) (f : Pi A B)
    : Path (B a) (apply A B (lambda A B f) a) (f a) := idp (B a) (f a)
def Π-η (A : U) (B : A → U) (a : A) (f : Pi A B)
    : Path (Pi A B) f (λ (x : A), f x) := idp (Pi A B) f
```

Sigma

```
def Sigma (A: U) (B: A → U) : U := Σ (x: A), B x
def pair (A: U) (B: A → U) (a: A) (b: B a) : Sigma A B := (a, b)
def pr₁ (A: U) (B: A → U) (x: Sigma A B) : A := x.1
def pr₂ (A: U) (B: A → U) (x: Sigma A B) : B (pr₁ A B x) := x.2
def Sigma-β-1 (A : U) (B : A → U) (a : A) (b : B a)
   : Path A a (pr₁ A B (a ,b)) := idp A a
def Sigma-β-2 (A : U) (B : A → U) (a : A) (b : B a)
   : Path (B a) b (pr₂ A B (a, b)) := idp (B a) b
def Sigma-η (A : U) (B : A → U) (p : Sigma A B)
   : Path (Sigma A B) p (pr₁ A B p, pr₂ A B p) := idp (Sigma A B) p
```

Fibrations

Bundle: F → E → B

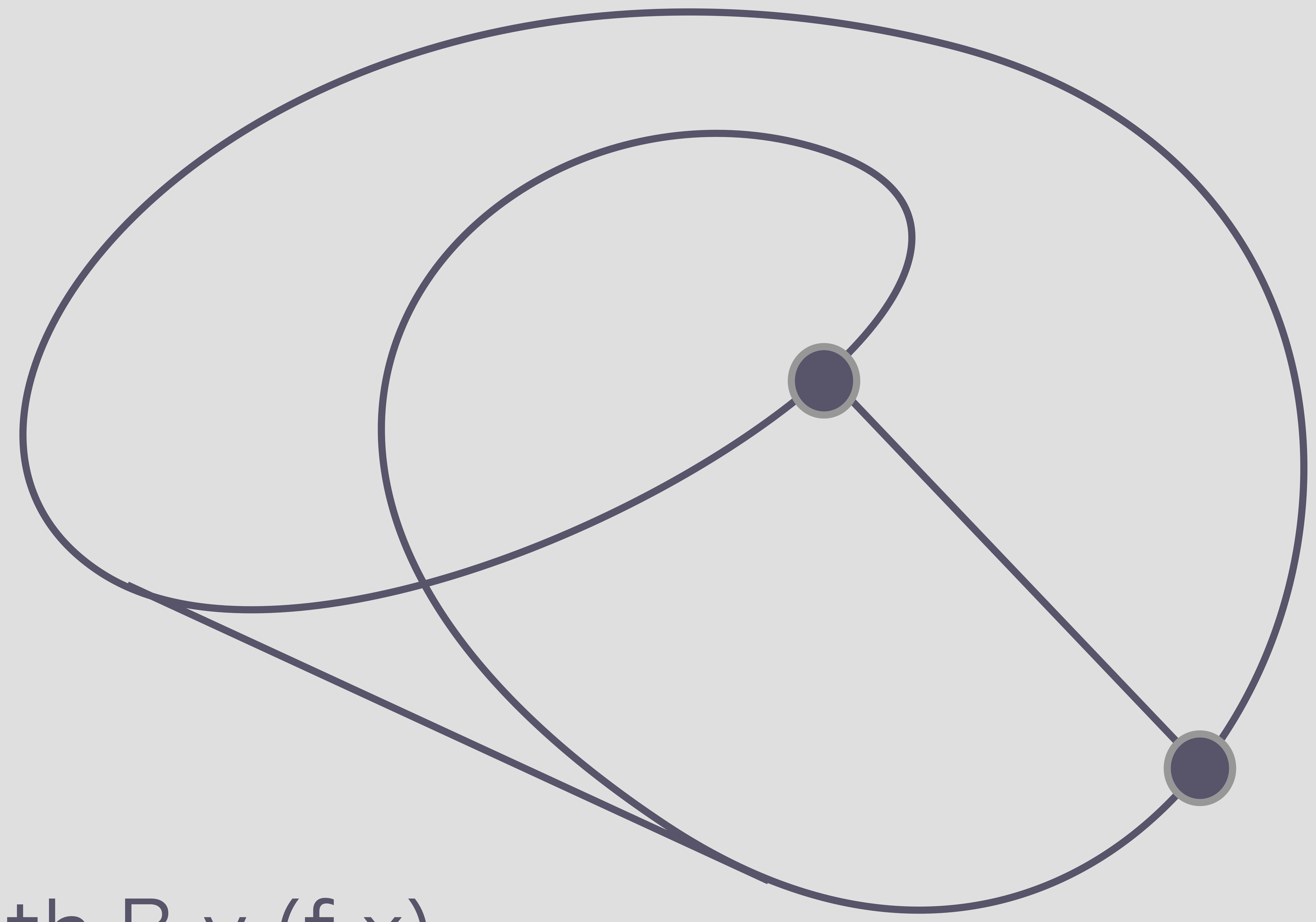
p : total -> B

F = fiber : B -> total

total = Σ (y: B), fiber(y)

Moebius E = S^1 ‘twisted *’ [0,1]

Trivial: E = B * F



```
def fiber (A B : U) (f: A → B) (y : B): U := Σ (x : A), Path B y (f x)
def isContr' (A: U) : U := Σ (x: A), Π (y: A), Path A x y
def isEquiv (A B : U) (f : A → B) : U := Π (y : B), isContr (fiber A B f y)
def equiv (A B : U) : U := Σ (f : A → B), isEquiv A B f
def idEquiv (A : U) : equiv A A := (id A, isContrSingl A)
```

Id (Strict Equality) in V

```
def 1= : I -> V := Id I 1
def 1=1 : 1= 1 := ref 1
def UIP (A : V) (a b : A) (p q : Id A a b) : Id (Id A a b) p q := ref p
def Jˢ (A : V) (B : Π (a b : A), Id A a b -> V) (a b : A)
        (d : B a a (ref a)) (p : Id A a b) : B a b p := idJ A B a d b p
def Jˢ-β (A : V) (B : Π (a b : A), Id A a b -> V) (a : A) (d : B a a (ref a))
    : Id (B a a (ref a)) (Jˢ A B a a d (ref a)) d := ref d
```

Path (Globular Equality) in U

```
def hmtpy (A : U) (x y : A) (p : Path A x y)
    : Path (Path A x x) (<_> x) (<i> p @ i /\ -i) := <j i> p @ j /\ i /\ -i

def isProp (A : U) : U := Π (a b : A), Path A a b
def isSet (A : U) : U := Π (a b : A) (a0 b0 : Path A a b), Path (Path A a b) a0 b0
def isGroupoid (A : U) : U := Π (a b : A) (x y : Path A a b)
            (i j : Path (Path A a b) x y), Path (Path (Path A a b) x y) i j
```

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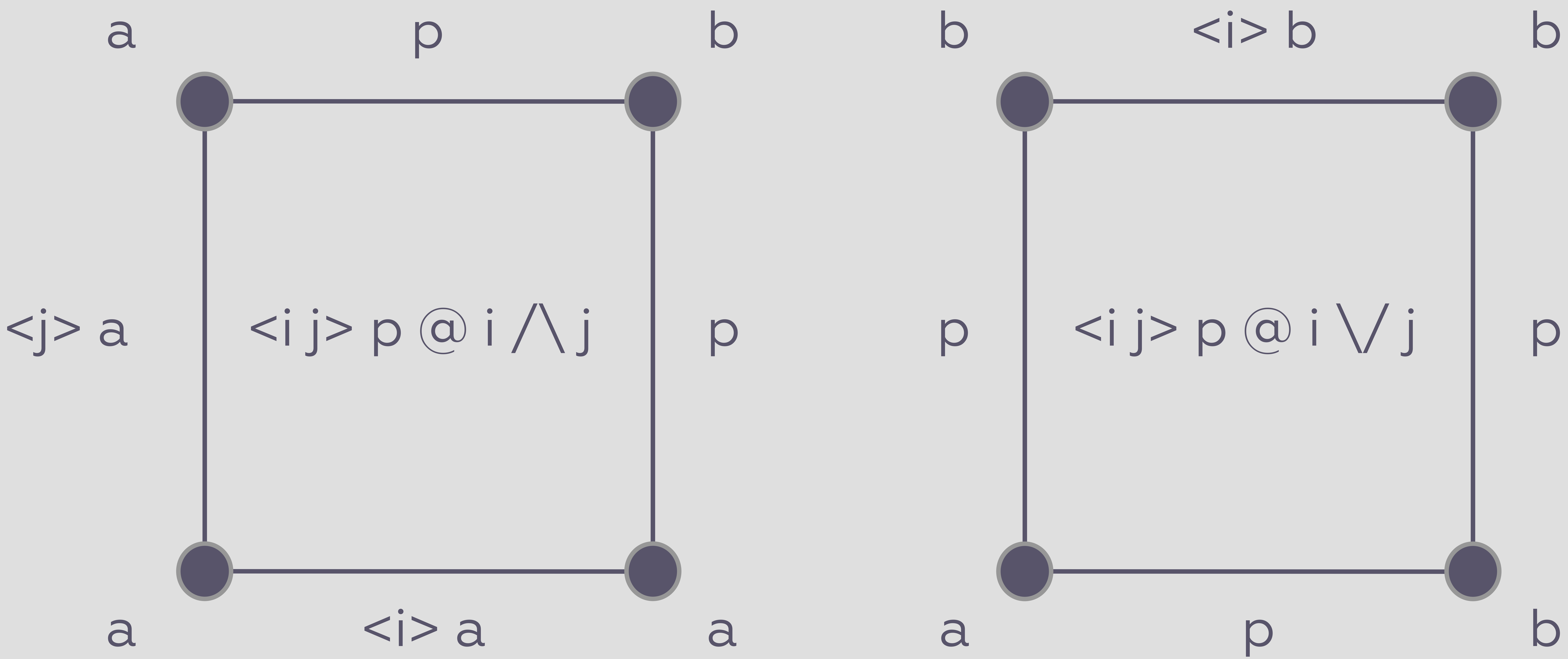
Path (Computational)

```
def transport (A B: U) (p: PathP (<_>U) A B) (a: A): B := transp p 0 a
def trans_comp (A:U)(a: A): Path A a (transport A A (<i> A) a) := <j> transp (<_> A) -j a
def subst (A: U) (P: A -> U) (a b: A) (p: Path A a b) (e: P a): P b := transp (<i> P (p @ i)) 0 e
def D (A : U) : U₁ := Π (x y : A), Path A x y → U
def J (A: U) (x: A) (C: D A) (d: C x x (idp A x)) (y: A) (p: Path A x y): C x y p
  := subst (singl A x) (\ (z: singl A x), C x (z.1) (z.2)) (eta A x) (y, p) (contr A x y p) d
def subst_comp (A: U) (P: A → U) (a: A) (e: P a)
    : Path (P a) e (subst A P a a (idp A a) e) := trans_comp (P a) e
def J-β (A : U) (a : A) (C : D A) (d: C a a (idp A a))
    : Path (C a a (idp A a)) d (J A a C d a (idp A a))
  := subst_comp (singl A a) (\ (z: singl A a), C a (z.1) (z.2)) (eta A a) d
```

Connections



DNF

```
def T (i j : I) : I := (i /\ -j) \/ (-i /\ j)
def T-comm (i j : I) : Id I (T i j) (T j i) := ref (T i j)
def /\-comm (i j : I) : Id I (i /\ j) (j /\ i) := ref (i /\ j)
def \/-comm (i j : I) : Id I (i \/ j) (j \/ i) := ref (i \/ j)
def ¬-of-/\ (i j : I) : Id I -(i /\ j) (-i \/ -j) := ref -(i /\ j)
def ¬-of-\/ (i j : I) : Id I -(i \/ j) (-i /\ -j) := ref -(i \/ j)
def /\-distrib-\/ (i j k : I) : Id I ((i \/ j) /\ k) ((i /\ k) \/ (j /\ k)) := ref ((i \/ j) /\ k)
def \/-distrib-/\ (i j k : I) : Id I ((i /\ j) \/ k) ((i \/ k) /\ (j \/ k)) := ref ((i /\ j) \/ k)
def /\-assoc (i j k : I) : Id I (i /\ (j /\ k)) ((i /\ j) /\ k) := ref (i /\ (j /\ k))
```

Generalized Transport

```
def subst' (A: U) (P: A → U) (a b: A) (p: Path A a b) (e: P a): P b
  := transp (<i> P (p @ i)) 0 e
def coerce (A B: U) (p: PathP (<_> U) A B): A → B := λ (x : A), trans A B p x
def pcomp (A: U) (a b c: A) (p: Path A a b) (q: Path A b c)
    : Path A a c := subst A (Path A a) b c q p
def transId (A : U) : A → A := transp (<_> A) 1
def transFill (A B : U) (p : PathP (<_> U) A B) (a : A)
    : PathP p a (transp p 0 a) := <j> transp (<i> p @ i /\j) -j a
```

transp (Huber)

$\mathrm{hcomp}^i\ N\ [\varphi \to 0]\ 0 = 0$
$\mathrm{hcomp}^i\ N\ [\varphi \to S\ u]\ (S\ u_0) = S\ (\mathrm{hcomp}^i\ N\ [\varphi \quad u]\ u_0)$
$\mathrm{hcomp}^i\ U\ [\varphi \to E]\ A = \mathrm{Glue}\ [\varphi\ (E(i/1), \mathrm{equiv}^i\ E(i/1{-}i))]\ A$
$\mathrm{hcomp}^i\ (\Pi\ (x : A), B)\ [\varphi \to u]\ u_0\ v = \mathrm{hcomp}^i\ B(x/v)\ [\varphi\ u\ v]\ (u_0\ v)$
$\mathrm{hcomp}^i\ (\Sigma\ (x : A), B)\ [\varphi \to u]\ u_0 = (v(i/1), \mathrm{comp}^i\ B(x/v)\ [\varphi\ u.2]\ u_0.2),\ v = \mathrm{hfill}^i\ A\ [\varphi \mapsto u.1]\ u_0.1$
$\mathrm{hcomp}^i\ (\mathrm{Path}^j\ A\ v\ w)\ [\varphi \to u]\ u_0 = \langle j \rangle\ \mathrm{hcomp}^i\ A\ [\ \varphi\ u\ j, (j = 0)\ v, (j = 1)\ w\ ]\ (u_0\ j)$
$\mathrm{hcomp}^i\ (\mathrm{Glue}\ [\varphi \to (T,w)]\ A)\ [\psi\ u]\ u_0 = \mathrm{glue}\ [\varphi \to t_1]\ a_1 = \mathrm{glue}\ [\varphi \to u(i/1)]\ (\mathrm{unglue}\ u(i/1))$
$= u(i/1) : \mathrm{Glue}\ [\varphi \to (T,w)]\ A,\ t_1 = u(i/1) : T,\ a_1 = \mathrm{unglue}\ u(i/1) : A,\ \mathrm{glue}\ [\varphi \to t_1]\ a1 = t_1 : T$

Homogeneous Composition

```
def kan (A : U) (a b c d : A) (p : Path A a c) (q : Path A b d) (r : Path A a b)
    : Path A c d := <i> hcomp A (∂ i) (λ (j : I), [(i = 0) → p @ j, (i = 1) → q @ j]) (inc (r @ i))

def comp (A : I → U) (r : I) (u : Π (i : I), Partial (A i) r) (u0 : (A 0)[r |→ u 0]) : A 1
  := hcomp (A 1) r (λ (i : I), [(φ : r = 1) → transp (<j> A (i ∨ j)) i (u i φ)])
                  (inc (transp (<i> A i) 0 (ouc u0)))

def ghcomp (A : U) (r : I) (u : I → Partial A r) (u0 : A[r |→ u 0]) : A
  := hcomp A (∂ r) (λ (j : I), [(φ : r = 1) → u j φ, (r = 0) → ouc u0]) (inc (ouc u0))
```

hcomp (Huber)

$\mathrm{transp}^i\ \mathrm{N}\ \varphi\ u_0 = u_0$

$\mathrm{transp}^i\ \mathrm{U}\ \varphi\ A = A$

$\mathrm{transp}^i\ (\Pi\ (x : A), B)\ \varphi\ u_0\ v = \mathrm{transp}^i\ B(x/w)\ \varphi\ (u_0\ w(i/0)),\ w = \mathrm{transpFill}^{-i}\ A\ \varphi\ v,\ v : A(i/1)$

$\mathrm{transp}^i\ (\Sigma\ (x : A), B)\ \varphi\ u_0 = (\mathrm{transp}^i\ A\ \varphi\ (u_0.1), \mathrm{transp}^i\ B(x/v)\ \varphi(u_0.2)),\ v = \mathrm{transpFill}^i\ A\ \varphi\ u_0.1$

$\mathrm{transp}^i\ (\mathrm{Path}^j\ A\ v\ w)\ \varphi\ u_0 = \ \langle j \rangle\ \mathrm{comp}^i\ A\ [\varphi \to u_0\ j,\ (j{=}0)\ v,\ (j{=}{>}1)\ w]\ (u_0\ j),\ u : A(j/0),\ v : A(j/1)$

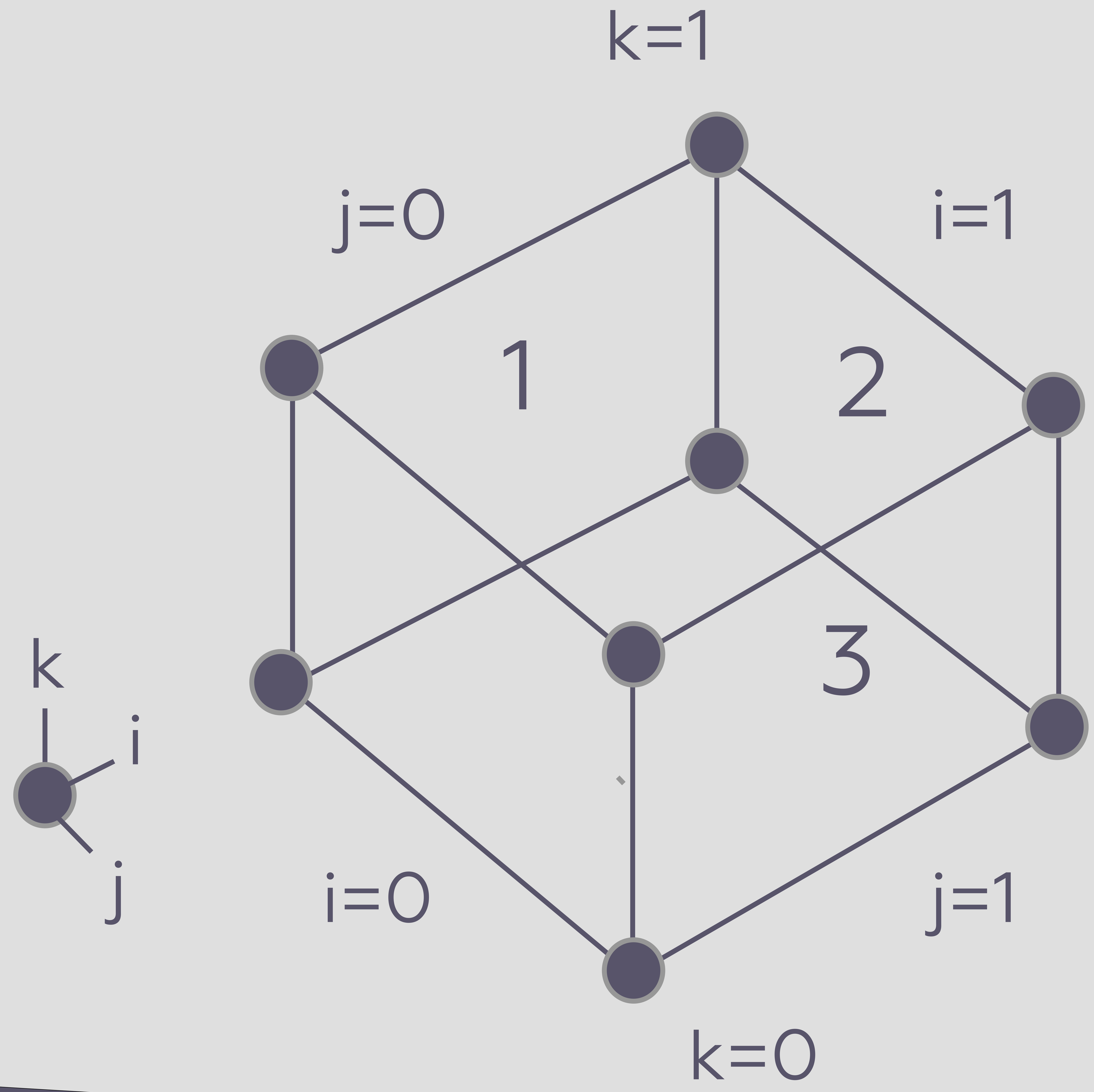
$\mathrm{transp}^i\ (\mathrm{Glue}\ [\varphi \to (T,w)]\ A)\ \psi\ u_0 = \mathrm{glue}\ [\varphi\ (i/1)\ t'_1]\ a'_1 : B(i/1)$

$\mathrm{transp}^{-i}\ A\ \varphi\ u = (\mathrm{transp}^i\ A(i/1{-}i)\ \varphi\ u)(i/1{-}i) : A(i/0)$

$\mathrm{transpFill}^i\ A\ \varphi\ u_0 = \mathrm{transp}^j\ A(i/i \wedge j)\ (\varphi\ (i{=}0))\ u_0 : A$

$\mathrm{hfill}^i\ A\ [\varphi \to u]\ u_0 = \mathrm{hcomp}^j\ A\ [\varphi \to u(i/i \wedge j),\ (i{=}0)\ u_0]\ u_0 : A$

hcomp



$$A : U, (k = 0) \to (M : A)$$

use hcomp to fill the lids:
(k = 0, k = 1):

$$\begin{aligned} \text{hcomp } A^k [&(j = 0) \to N1, \\ &(i = 1) \to N2, \\ &(j = 1) \to N3]\ M : A \end{aligned}$$

Base Library Assurance

1. MLTT Internalization ✓
2. Topos Theory ✓
3. Tesseract ← we are here
4. Category of Groupoids
5. Homological Algebra
6. Grothendieck Group

github.com/groupoid/anders

Thank You!