

Intetics f(cafe) 25 July 2018 Freud House, Kyiv, Ukraine

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# HoTT: The Language of Space

Groupoid Infinity

# The HoTT Lineage

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- Nicolaas Govert de Bruijn [AUTOMATH]
- Henk Barendregt [ $\lambda$ -Cube]
- Robin Milner [ML, LCF,  $\pi$ -Calculus]
- Gérard Pierre Huet [CAML]
- Christine Paulin-Mohring [CiC]
- Peter Dybjer, Martin Hofmann, R.A.G. Seely [CwF, CwA, LCCC]
- Thierry Coquand [CoC, Coq, CCHM]
- Robert William "Bob" Harper, Jr. [HoTT, RedPRL, ML, Twelf]
- Steve Awodey, Vladimir Voevodsky [HoTT, C-Systems]
- Cyril Cohen, Simon Huber, Anders Mörtberg [CCHM, cubicaltt, yacctt]

# Abstract

## Cubical Base Library

Homotopy Type Theory (HoTT) is the most advanced programming language in the domain of intersection of several theories: algebraic topology, homological algebra, higher category theory, mathematical logic, and theoretical computer science. That is why it can be considered as a language of space, as it can encode any existent mathematics.

During this lecture on HoTT, we are trying to encode as much mathematics in the programming language as possible.

# Talk Structure

Slightly based on HoTT Chapters

## I. Foundations

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- MLTT
- Inductive Types, Induction
- IPL and Elements of Set Theory
- Control, Recursive Schemes
- Equiv, Iso, Univalence
- Higher Inductive Types
- Modalities

## II. Mathematics

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- Category Theory, Topos Theory
- Basic Algebra
- Ordinals, Mahlo Universe
- Differential Topology
- Fiber Bundles and Hopf Fibrations
- K-Theory
- Sequences, Chain Complexes

# I. Foundations

# $\lambda$ -Calculuses

## in Extended Lambda Cube

CoC:	$* \rightsquigarrow *$	$\square \rightsquigarrow *$	$* \rightsquigarrow \square$	$\square \rightsquigarrow \square$
F $\omega$ :	$* \rightsquigarrow *$	$\square \rightsquigarrow *$		$\square \rightsquigarrow \square$
P2:	$* \rightsquigarrow *$	$\square \rightsquigarrow *$	$* \rightsquigarrow \square$	
AUT:	$* \rightsquigarrow *$		$* \rightsquigarrow \square$	
F:	$* \rightsquigarrow *$	$\square \rightsquigarrow *$		

System F $\omega$ :  
 Haskell, Scala, 1ML  
 Almost CoC  
 No Types  
 On Values

System F:  
 ML, Miranda,  
 OCaml

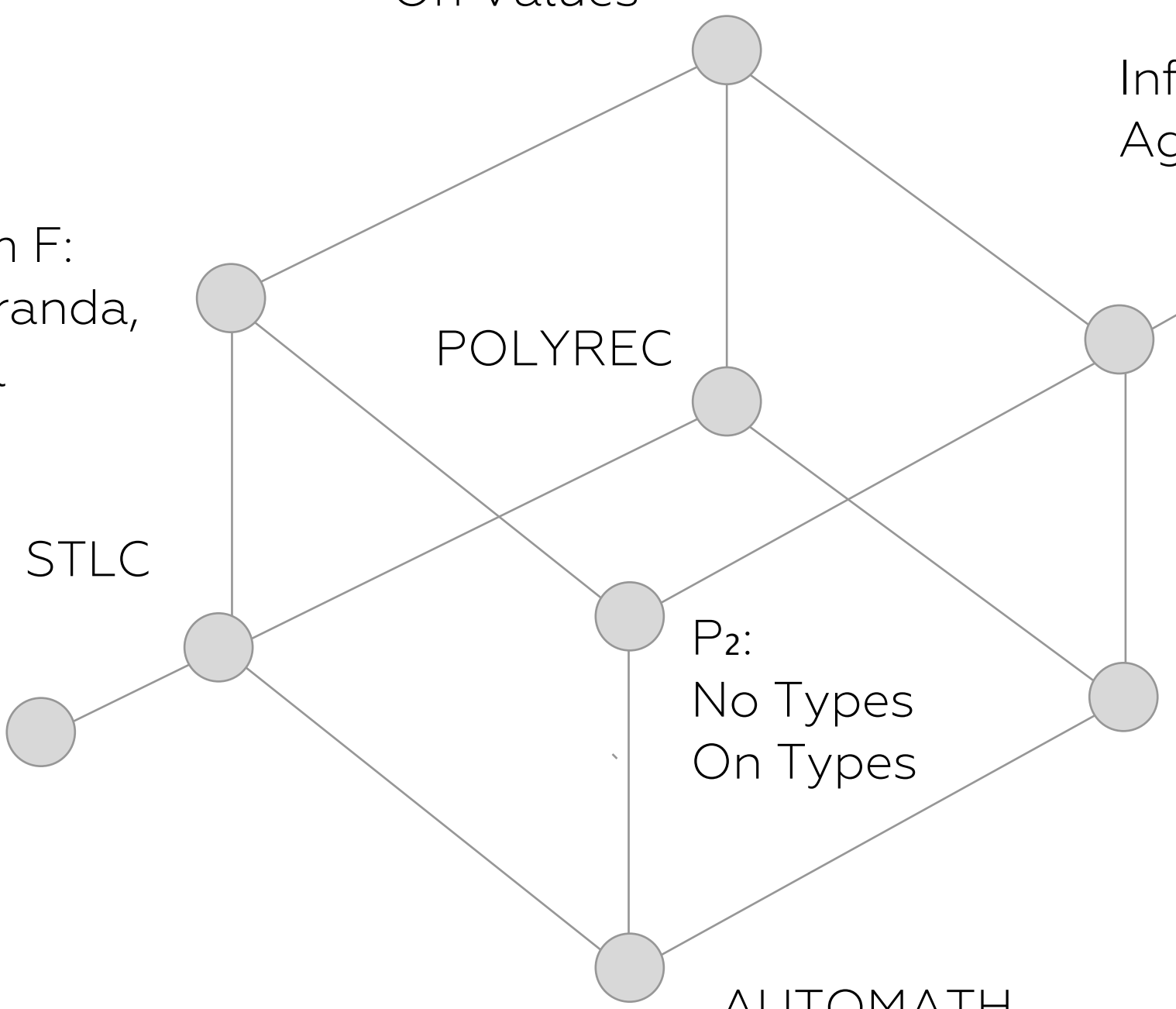
Infinity Topoi  
 Agda, Coq, Lean, Om

CoC: Morte, Henk

No Terms  
 On Types

P<sub>2</sub>:  
 No Types  
 On Types

AUTOMATH



Untyped SLC:  
 Erlang, LISP,  
 JavaScript

# MLTT 1972

Type Theory as new Foundations of Mathematics

$U : U$  — Single Universe Model — MLTT 1972, CoC 1988.

$x : A$  —  $x$  is a point (Star) in space  $A$  (Box)

$y = [ x : A ]$  —  $x$  and  $y$  are definitionally equal objects of type  $A$

$\Pi$

$\Sigma$

1. Formation Rules

$(x:A) \rightarrow B(x)$

$(x:A) * B(x)$

2. Introduction Rules

$\lambda (x:A) \rightarrow B(x)$

$(a,b)$

3. Elimination Rules

$f a = B(a)$

$.1, .2$

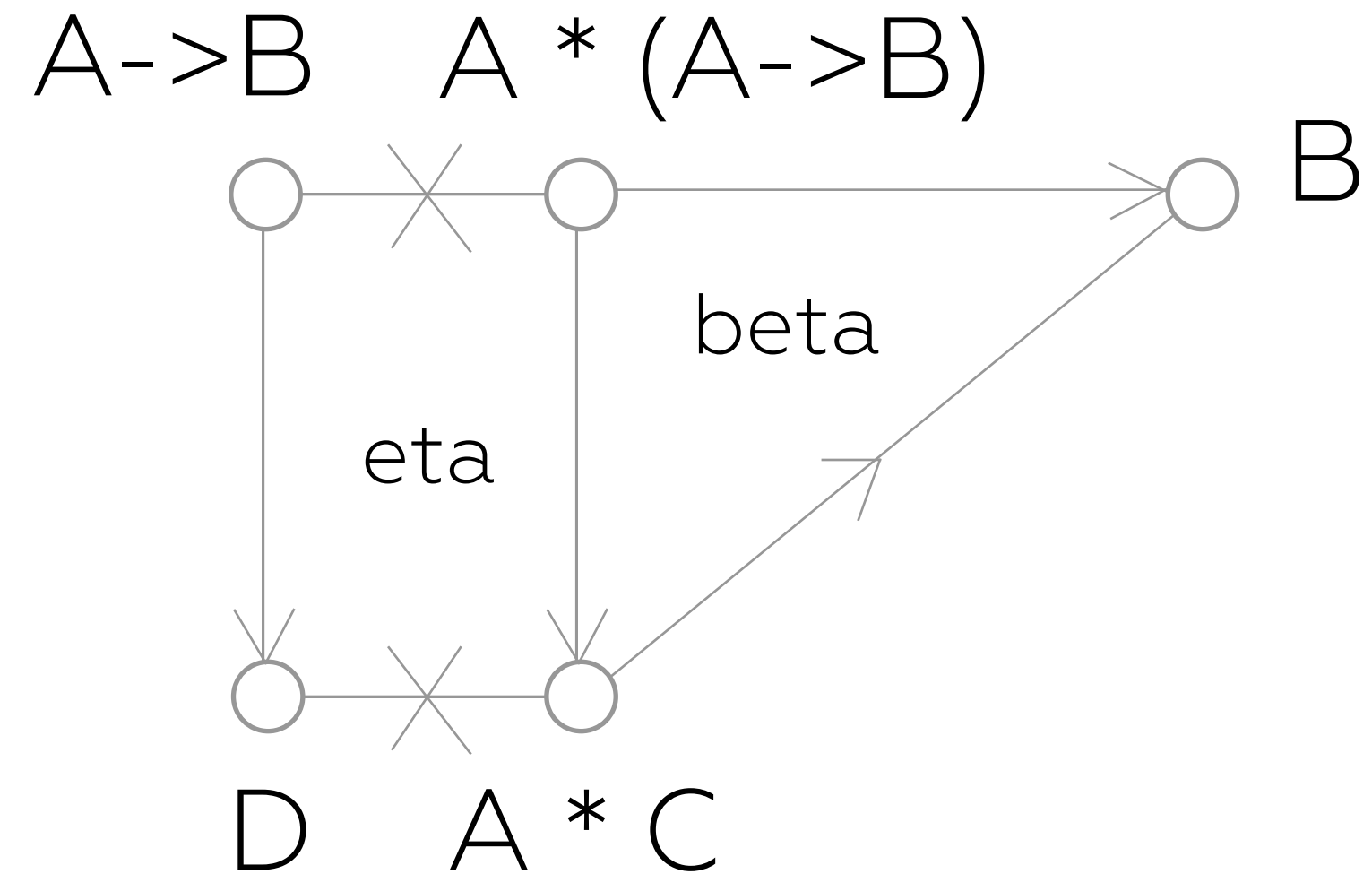
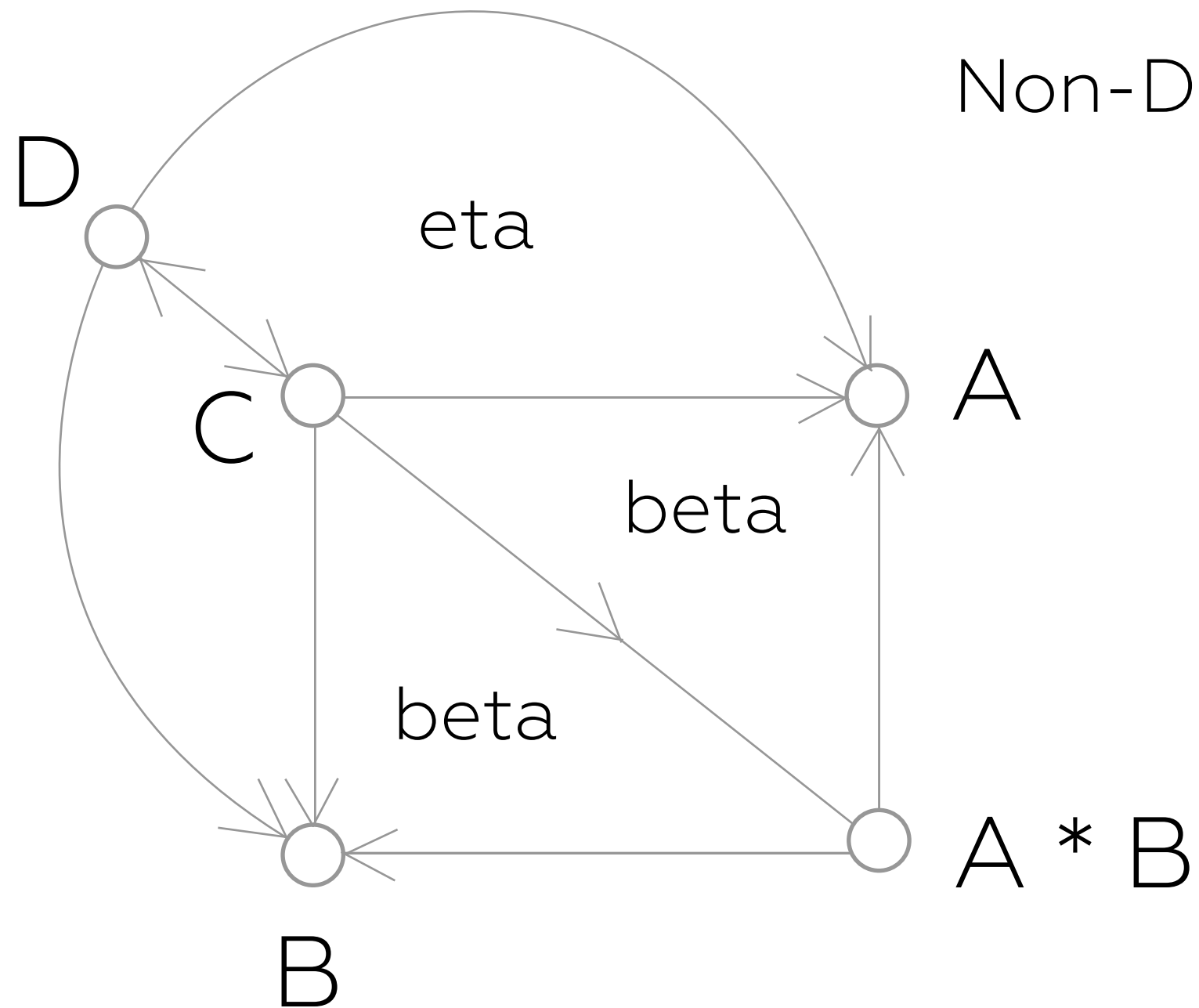
4. Computational Rules

two

three

# Beta and Eta

Duality of Intro and Elim and its Uniqueness  
Non-Dep Case (CCC). Homework: Proof LCCC case.





# MLTT 1975, 1984

Grothendieck Universe (containing all sets), Countable Universes

$U_0 : U_1 : U_2 : U_3 : \dots \infty$  — infinite hierarchy of universes

$S(n : \text{nat}) = U_n$

$A_1(n\ m : \text{nat}) = U_n : U_m$  where  $[m > n]$  — cumulative,  $[n+1=m]$  — non-cumulative

$R_1(m\ n : \text{nat}) = U_m \longrightarrow U_n : U_x$  where  $[x = \max(m, n)]$  — predicative,  
 $[x = n]$  — impredicative

1. Formation	data nat	data list	$x:A = y:A$	data W
2. Introduction	zero, succ	nil, cons	refl A x	sup
3. Elimination	natInd	listInd	J	wInd
4. Computational	Beta, Eta	Beta, Eta	Beta, Eta	Beta, Eta

# Intuitionistic Propositional Logic

According to Brouwer–Heyting–Kolmogorov interpretation

$\forall$	$\exists$	Path	0	1	+
$x:A \rightarrow B(x)$	$x:A * B(x)$	$x:A = y:A$	data empty	data unit	data either
$\lambda (x:A) \rightarrow B(x)$	$(x, B(x))$	refl A x		tt	inl, inr
$f a = B(a)$	pr1, pr2	J	elim0	elim1	eitherInd
Beta, Eta	Beta, Eta	Eta	Beta, Eta	Beta, Eta	Beta, Eta

# Proto (Prelude)

For run-time and I/O applications

maybe

$U \rightarrow U$

nothing, just

maybeInd

Beta, Eta

either

$U \rightarrow U \rightarrow U$

inl, inr

eitherInd

Beta, Eta

stream

$U \rightarrow U$

cons

streamInd

Beta, Eta

bool

$U$

true, false

boolInd

Beta, Eta

vector

$\text{nat} \rightarrow U$

vz, vs

vecInd

Beta, Eta

fin

$\text{nat} \rightarrow U$

fz, fs

finInd

Beta, Eta

Homework: Add I/O interface for finite and infinite loop.

# Induction Principle

## Natural Numbers Example

`natCase (C:U) (a b: C): nat -> C`  
`= split { zero -> a ; succ n -> b }`

`natRec (C:U) (z: C) (s: nat->C->C): (n:nat) -> C`  
`= split { zero -> z ; succ n -> s n (natRec C z s n) }`

`natInd (C:nat->U) (z: C zero) (s: (n:nat)->C(n)->C(succ n)): (n:nat) -> C(n)`  
`= split { zero -> z ; succ n -> s n (natInd C z s n) }`

Induction Principle could be the ultimate programming tool.

# Inductive AST

## Data Definition and Case Analysis

`data` tele (A: U) = emp | tel (n: name) (b: A) (t: tele A)

`data` branch (A: U) = br (n: name) (args: list name) (term: A)

`data` label (A: U) = lab (n: name) (t: tele A)

`data` ind = data\_ (n: name) (t: tele lang) (labels: list (label lang))  
          | case (n: name) (t: lang) (branches: list (branch lang))  
          | ctor (n: name) (args: list lang)

Homework: Add HITs elements as Path Equalities to data declarations.

Hint: use your imagination.

# Pi Type : Definition

Family of Types, Fibrations, Fiber Space  $A \rightarrow U$ , Fiber  $B(x)$ ,  
Section  $b(x)$ , Space of Sections  $\Pi(A, B)$

## Syntax

$\langle \rangle ::= \#option$   
 $T ::= \#identifier$   
 $U ::= * \langle \#number \rangle$   
 $O_1 ::= U \mid T \mid ( O ) \mid O O \mid O \rightarrow O$   
 $\mid \backslash (l: O) \rightarrow O \mid (l: O) \rightarrow O$

## Model

`data` pts = star (n: nat)  
| var (x: name) (l: nat)  
| pi (x: name) (l: nat) (d c: lang)  
| lambda (x: name) (l: nat) (d c: lang)  
| app (f a: lang)

Pure Type System (PTS), Single Axiom System, Calculus of Constructions (CoC)  
Henk, Morte, Om and many many others.

# Pi Type : Inference Rules

Formal Definition

$\text{Pi } (A: U) (P: A \rightarrow U) : U = (x:A) \rightarrow P(x)$

$\text{lambda } (A : U) (B: A \rightarrow U) (a : A) (b: B a): A \rightarrow B a = ?$

$\text{app } (A : U) (B: A \rightarrow U) (a : A) (f: A \rightarrow B a): B a = ?$

$\text{Beta } (A:U) (B:A \rightarrow U) (a:A) (f: A \rightarrow B a) : \text{Path } (B a) (\text{app } A B a (\text{lam } A B a (f a))) (f a)$

$\text{Eta } (A: U) (B: A \rightarrow U) (a: A) (f: A \rightarrow B a) : \text{Path } (A \rightarrow B a) f (\lambda(x:A) \rightarrow f x)$

One beta rule and one eta rule for Pi types.

# Sigma Type : Definition

Total Space  $\text{Sigma}(A,B)$ , Point in Base with Section  $(a,b)$

Syntax  $O_2 := (x: O) * O \mid (O,O) \mid O.1 \mid O.2$

Model `data` exists = sigma (n: name) (a b: lang)  
| pair (a b: lang)  
| fst (p: lang)  
| snd (p: lang)

Sigma is a part of the MLTT earliest core.

It models Type Refinement and Proofs by Existence (Construction).

Sigma is a chain link of telescopes (contexts), the curried notion of records.



# Sigma Type : Inference Rules

Existential Quantifier

$\text{Sigma } (A : U) (B : A \rightarrow U) : U = (x : A) * B x$

$\text{pair } (A : U) (B : A \rightarrow U) (a : A) (b : B a) : \text{Sigma } A B = ?$

$\text{pr1 } (A : U) (B : A \rightarrow U) (x : \text{Sigma } A B) : A = ?$

$\text{pr2 } (A : U) (B : A \rightarrow U) (x : \text{Sigma } A B) : B (\text{pr1 } A B x) = ?$

$\text{Beta1 } (B : A \rightarrow U) (a : A) (b : B a) \rightarrow \text{Path } A a (\text{pr1 } A B (\text{pair } A B a b))$

$\text{Beta2 } (B : A \rightarrow U) (a : A) (b : B a) \rightarrow \text{Path } (B a) b (\text{pr2 } A B (a, b))$

$\text{Eta } (B : A \rightarrow U) (p : \text{Sigma } A B) \rightarrow \text{Path } (\text{Sigma } A B) p (\text{pr1 } A B p, \text{pr2 } A B p)$

$\text{sigRec } (A : U) (B : A \rightarrow U) (C : U) (g : (x : A) \rightarrow B(x) \rightarrow C) (p : \text{Sigma } A B) : C = g p.1 p.2$

$\text{sigInd } (A : U) (B : A \rightarrow U) (C : \text{Sigma } A B \rightarrow U)$

$(p : \text{Sigma } A B) (g : (a : A) (b : B(a)) \rightarrow C(a, b)) : C p = g p.1 p.2$

# Sigma Type in Pi

Typing and Introduction Rules in Church-Bohm-Berarducci Encoding

```
-- Sigma/@  
  \ (A: *)  
-> \ (P: A -> *)  
-> \ (n: A)  
-> \ (Exists: *)  
-> \ (Intro: A -> P n -> Exists)  
-> Exists
```

```
-- Sigma/Intro  
  \ (A: *)  
-> \ (P: A -> *)  
-> \ (x: A)  
-> \ (y: P x)  
-> \ (Exists: *)  
-> \ (Intro: \ (x:A) -> P x -> Exists)  
-> Intro x y
```

# Sigma Type in Pi

Eliminators in Church-Bohm-Berarducci Encoding

-- Sigma/fst

\ (A: \*)

-> \ (B: A -> \*)

-> \ (n: A)

-> \ (S: #Sigma/@ A B n)

-> S A ( \ (x: A) -> \ (y: B n) -> x )

-- Sigma/snd

\ (A: \*)

-> \ (B: A -> \*)

-> \ (n: A)

-> \ (S: #Sigma/@ A B n)

-> S (B n) ( \ ( : A) -> \ (y: B n) -> y )

# Control (Haskell)

Port of Haskell-style erased 2-categorical structures for flow modeling

`pure_sig`       $(F:U \rightarrow U):U = (A:U) \rightarrow A \rightarrow F A$

`appl_sig`       $(F:U \rightarrow U):U = (A B:U) \rightarrow F (A \rightarrow B) \rightarrow F A \rightarrow F B$

`fmap_sig`       $(F:U \rightarrow U):U = (A B:U) \rightarrow (A \rightarrow B) \rightarrow F A \rightarrow F B$

`bind_sig`       $(F:U \rightarrow U):U = (A B:U) \rightarrow F A \rightarrow (A \rightarrow F B) \rightarrow F B$

`functor`:  $U = (F:U \rightarrow U) * \text{fmap\_sig } F$

`applicative`:  $U = (F:U \rightarrow U) * (\_ : \text{pure\_sig } F) * (\_ : \text{fmap\_sig } F) * \text{appl\_sig } F$

`monad`:  $U = (F:U \rightarrow U) * (\_ : \text{pure\_sig } F) * (\_ : \text{fmap\_sig } F) * (\_ : \text{appl\_sig } F) * \text{bind\_sig } F$

`FUNCTOR`:  $U = (f: \text{functor}) * \text{isFunctor } f$

`APPLICATIVE`:  $U = (f: \text{applicative}) * (\_ : \text{isFunctor } (f.1, f.2.2.1)) * \text{isApplicative } f$

`MONAD`:  $U = (f: \text{monad}) * (\_ : \text{isFunctor } (f.1, f.2.2.1))$

$* (\_ : \text{isApplicative } (f.1, f.2.1, f.2.2.1, f.2.2.2.1)) * \text{isMonad } f$

# F-Algebras

Inductive Type Modeling with Varmo Vene style Recursion Schemes

```
data fix (F:U->U) = Fix (point: F (fix F))
data nu (F:U->U) (A B:U) = CoBind (a: A) (f: F B)
data cofree (F:U->U) (A:U) = CoFree (_: fix (nu F A))
ind (F: U -> U) (A: U): U = (in_: F (fix F) -> fix F) * (in_rev: fix F -> F (fix F))
* ((F A -> A) -> fix F -> A) * (cofree_: (F (cofree F A) -> A) -> fix F -> A)
inductive (F: functor) (A: U): ind F.1 A = (in_ F.1,out_ F.1,cata A F,histo A F,tt)
```

Backported to cubicaltt.

# Bishop's Constructive Analysis

Reflexivity, Transitivity, Symmetry

Setoid  $(A: U): U$

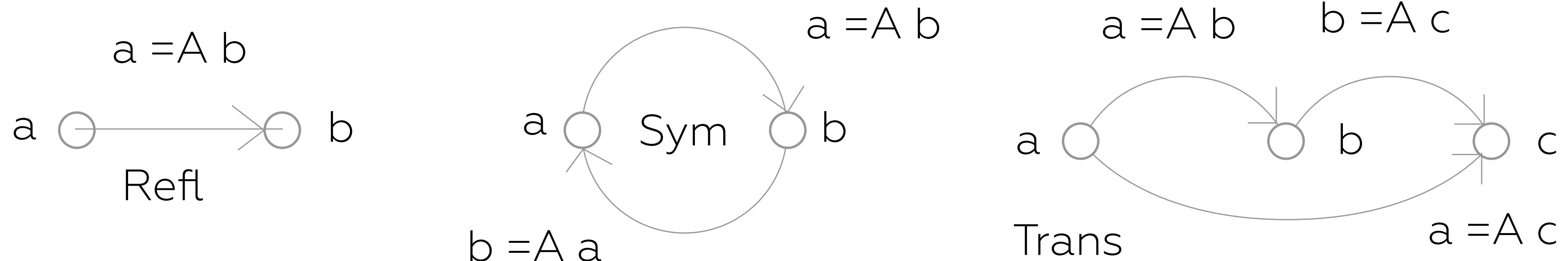
$= (\text{Carrier}: A)$

\*  $(\text{Equ}: (a\ b: A) \rightarrow \text{Path } A\ a\ b)$

\*  $(\text{Refl}: (x: A) \rightarrow \text{Equ } x\ x)$

\*  $(\text{Trans}: (x_1, x_2, x_3: A) \rightarrow \text{Equ } x_1\ x_2 \rightarrow \text{Equ } x_2\ x_3 \rightarrow \text{Equ } x_1\ x_3)$

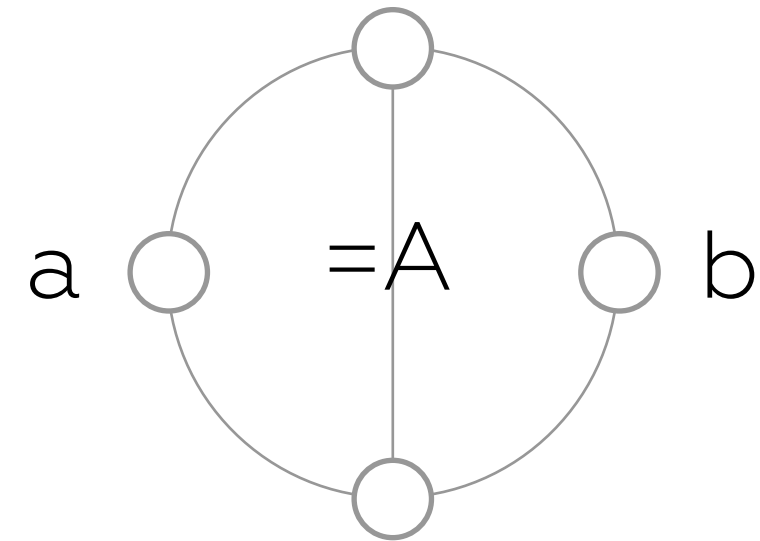
\*  $(\text{Sym}: (x_1, x_2: A) \rightarrow \text{Equ } x_1\ x_2 \rightarrow \text{Equ } x_2\ x_1)$



# Globular Theory

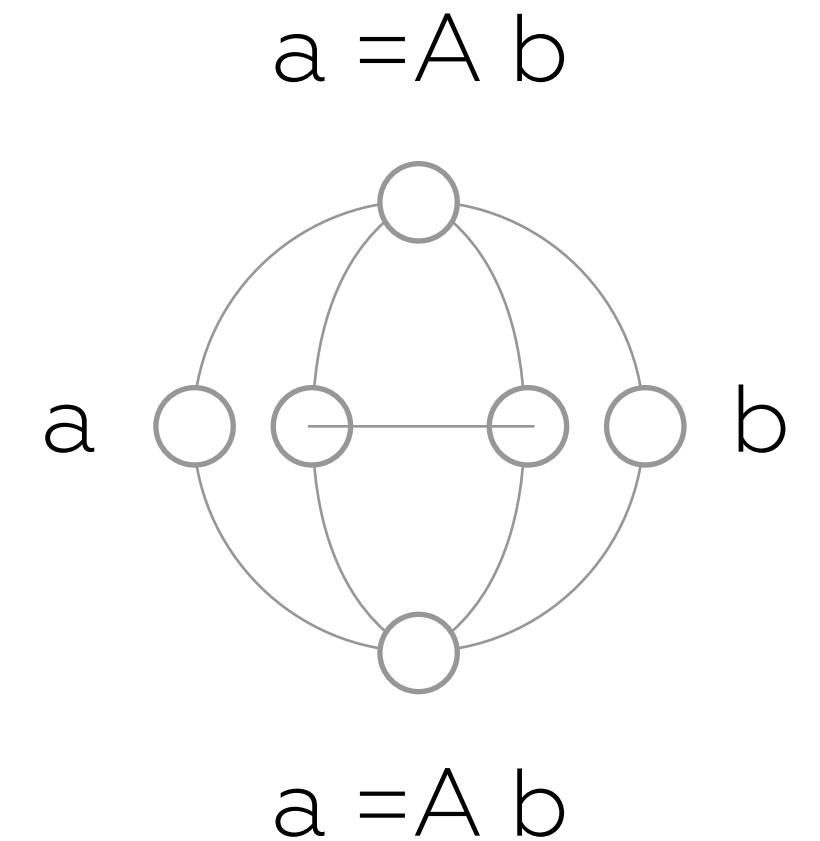
Multidimensional Equality

$a =_A b$



$((a =_A b) =_{(=A)} (a =_A b))$

$((a =_A b) =_{(=A)} (a =_A b)) =_{(=(=A))} ((a =_A b) =_{(=A)} (a =_A b))$



# Equ Type a la Martin-Löf

`Path` (A: U) (a b: A): U = `axiom` — `PathP` (<i>A) a b

`HeteroEqu` (A B: U) (a: A) (b: B) (P: Path U A B) : U = `axiom` — `PathP` P a b

`Equ` (A: U) (x y: A): U = `HeteroEqu` A A x y (<i>A)

`refl` (A: U) (a: A): `Equ` A a a = <i> a

`J` (A: U) (a: A) (C: (x : A) -> `Path` A a x -> U)

(d: C a (`refl` A a)) (x: A) (p: `Path` A a x): C x p

`Comp` (A : U) (a : A) (C : (x : A) -> `Path` A a x -> U)

(d : C a (`refl` A a)) : `Path` (C a (`refl` A a)) d (`J` A a C d a (`refl` A a))



# Path Types as Cubes

Syntax and Model

Syntax

$x : [ \text{PathP } p \ a \ b, p = (i: I) \rightarrow A ]$

$a : A \quad \bigcirc \longrightarrow \bigtriangleright \bigcirc \quad b : A$

de Morgan:  $1-i \mid i \mid i \wedge j \mid i \vee j$

Model

`data lang = hts | ...`

`data hts = path (a b: lang)`

`| path_lam (n: name) (a b: lang)`

`| path_app (f: name) (a b: lang)`

`| comp_ (a b: lang)`

`| fill_ (a b c: lang)`

`| glue_ (a b c: lang)`

`| glue_elem (a b: lang)`

`| unglue_elem (a b: lang)`

# Homotopy

## Syntax and Model

```
data I = i0
      | i1
      | seg <i> [(i=0) -> i0, (i=1) -> i1]
```

```
pathToHtpy (A: U) (x y: A) (p: Path A x y): I -> A
= split { i0 -> x; i1 -> y; seg @ i -> p @ i }
```

```
homotopy (X Y: U) (f g: X -> Y)
  (p: (x: X) -> Path Y (f x) (g x))
  (x: X): I -> Y = pathToHtpy Y (f x) (g x) (p x)
```

# n-Types

Path             $(A : U) : U = (a\ b : A) \rightarrow \text{PathP } (\lambda i \rightarrow A) a\ b$   
isContr         $(A : U) : U = (x : A) * ((y : A) \rightarrow \text{Path } A\ x\ y)$   
isProp         $(A : U) : U = (a\ b : A) \rightarrow \text{Path } A\ a\ b$   
isSet           $(A : U) : U = (a\ b : A) \rightarrow \text{isProp } (\text{Path } A\ a\ b)$   
isGroupoid     $(A : U) : U = (a\ b : A) \rightarrow \text{isSet } (\text{Path } A\ a\ b)$   
isGr\_2         $(A : U) : U = (a\ b : A) \rightarrow \text{isGroupoid } (\text{Path } A\ a\ b)$   
isGr\_3         $(A : U) : U = (a\ b : A) \rightarrow \text{isGr}_2\ (\text{Path } A\ a\ b)$

PROP :  $U = (X : U) * \text{isProp } X$

SET :  $U = (X : U) * \text{isSet } X$

GROUPOID :  $U = (X : U) * \text{isGroupoid } X$

INF\_GROUPOID :  $U = (X : U) * \text{isInfinityGroupoid } X$

# Subtyping in MLTT

## Subsets and Subtypes

hsubtypes (X: U): U = X -> PROP

subset (A: U) (\_: isSet A): U = A -> PROP

sethsubtypes (X : U) : isSet (hsubtypes X)

hsubtypespair (A B: U) (H0: hsubtypes A) (H1: hsubtypes B) (x: prod A B): PROP

subtypeEquality (A: U) (B: A -> U)

(pB: (x : A) -> isProp (B x))

(s t: Sigma A B) : Path A s.1 t.1 -> Path (Sigma A B) s t

iseqclass (X : U) (R : hrel X) (A : hsubtypes X) : U

propiseqclass (X : U) (R : hrel X) (A : hsubtypes X) : isProp (iseqclass X R A)

# Elements of Set Theory

## Set Theory Theorems

$$\begin{aligned} \text{ac} \quad & (A \ B: U) (R: A \rightarrow B \rightarrow U): (p: (x:A) \rightarrow (y:B) * (R \ x \ y)) \rightarrow (f:A \rightarrow B) * ((x:A) \rightarrow R(x)(f \ x)) \\ & = \backslash (g: (x:A) \rightarrow (y:B) * (R \ x \ y)) \rightarrow (\backslash (i:A) \rightarrow (g \ i).1, \backslash (j:A) \rightarrow (g \ j).2) \end{aligned}$$

$$\begin{aligned} \text{total} \quad & (A:U) (B \ C : A \rightarrow U) (f : (x:A) \rightarrow B \ x \rightarrow C \ x) (w:\text{Sigma } A \ B) : \text{Sigma } A \ C \\ & = (w.1, f \ (w.1) \ (w.2)) \end{aligned}$$

# Prop Logic

Set Theory Theorems

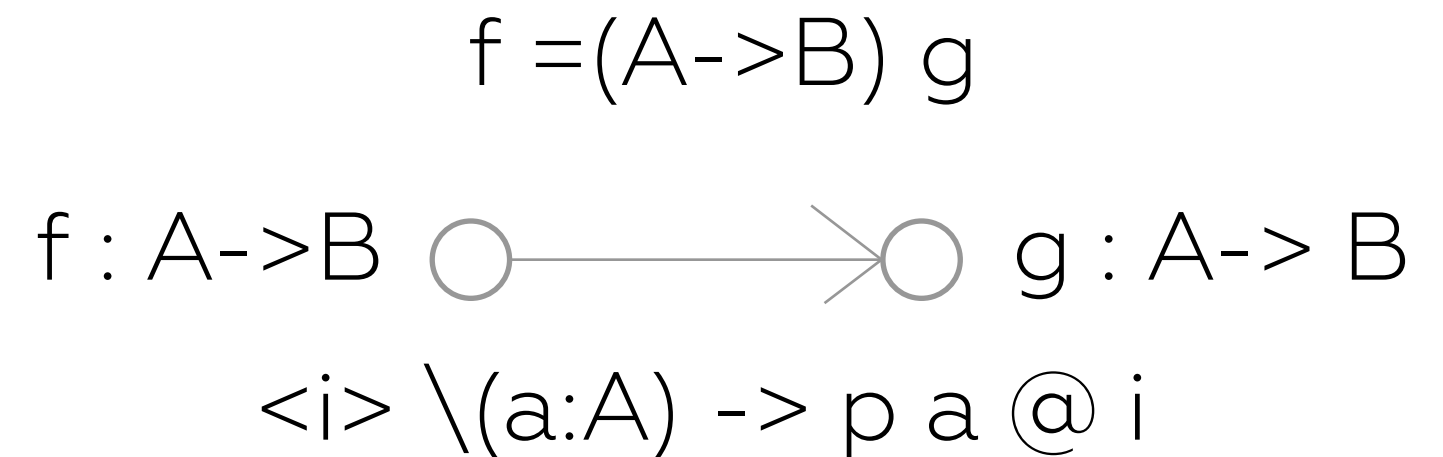
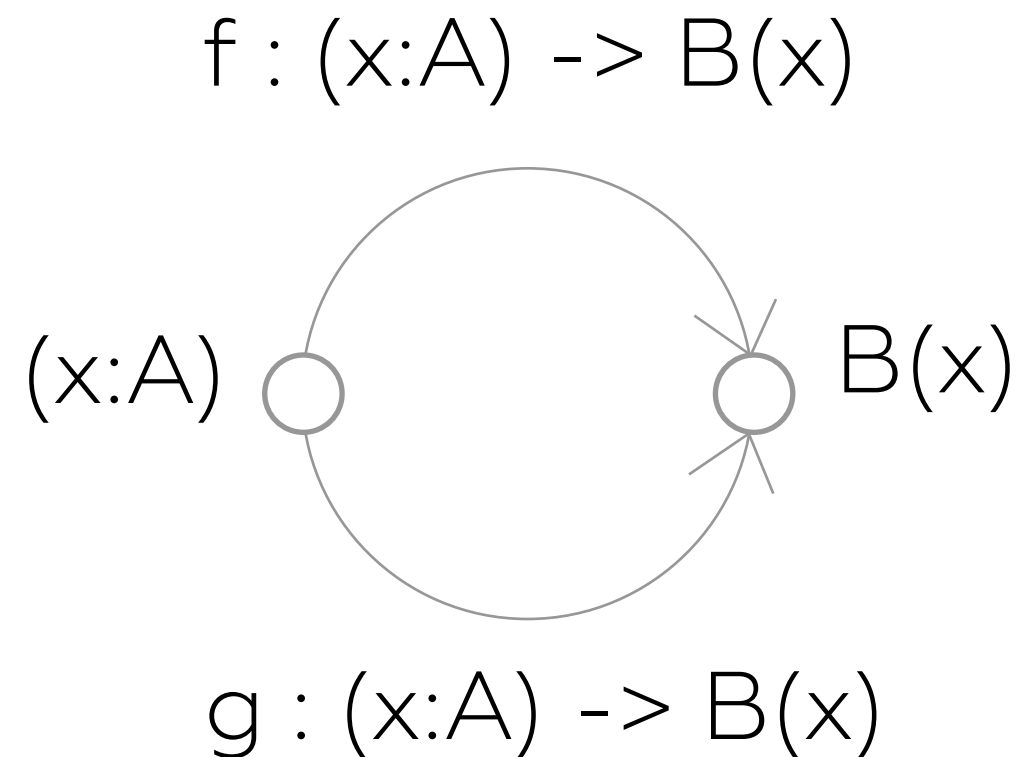
efq (A: U): **empty** -> A = emptyRec A  
neg (A: U): U = A -> **empty**  
dneg (A:U) (a:A): neg (neg A) = \ (h: neg A) -> h a  
neg (A: U): U = A -> **empty**  
dec (A: U): U = **either** A (neg A)  
stable (A: U): U = neg (neg A) -> A  
discrete (A: U): U = (a b: A) -> dec (Path A a b)

propDec (A : U) (a: isProp A) : isProp (dec A)  
propAnd (A B : U) (a: isProp A) (b: isProp B) : isProp (**prod** A B)  
propOr (A B : U) (a: isProp A) (b: isProp B) (x: A -> neg B) : isProp (**either** A B)  
propNeg (A : U) : isProp (neg A)  
propN0 : isProp **empty**

# FunExt

Syntax and Model

```
funext (A: U) (B: A -> U) (f g: (x:A) -> B x)
  (p: (x:A) -> Path (B x) (f x) (g x))
: Path ((y:A) -> B y) f g
= <i> \ (a: A) -> (p a) @ i
= <j> \ (x : A) -> homotopy A B f g p x (seg{I} @ j))
```



# FunExt

Formation, Intro, Elim, Beta, Eta

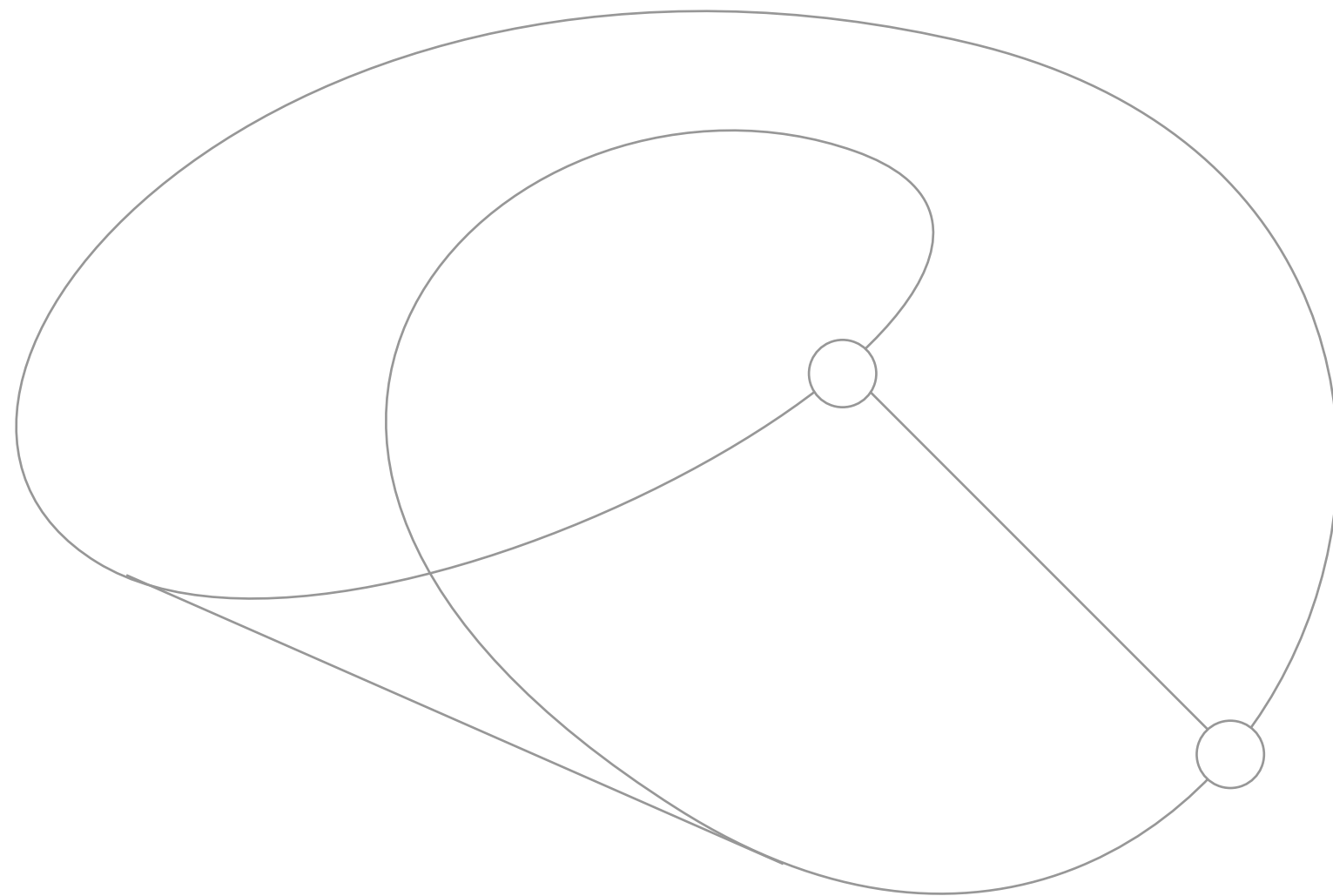
```
funext_form (A B: U) (f g: A -> B): U = Path (A -> B) f g
funext (A B: U) (f g: A -> B) (p: (x:A) -> Path B (f x) (g x)) : funext_form A B f g
happly (A B: U) (f g: A -> B) (p: funext_form A B f g) (x: A) : Path B (f x) (g x)
funext_Beta (A B: U) (f g: A -> B) (p: (x:A) -> Path B (f x) (g x))
    : (x:A) -> Path B (f x) (g x)
funext_Eta (A B: U) (f g: A -> B) (p: funext_form A B f g)
    : Path (funext_form A B f g) (funext A B f g (happly A B f g p)) p
```



# Weak Equivalence

Fibrational

```
fiber (A B: U) (f: A -> B) (y: B): U = (x: A) * Path B y (f x)
isEquiv (A B: U) (f: A -> B): U = (y: B) -> isContr (fiber A B f y)
equiv (A B: U): U = (f: A -> B) * isEquiv A B f
```



Fiber Bundle:  $F \rightarrow E \rightarrow B$

Moebius  $E = S^1$  'twisted \*'  $[0,1]$

Trivial:  $E = B * F$

$p : \text{total} \rightarrow B$

$F = \text{fiber} : B \rightarrow \text{total}$

$\text{total} = (y: B) * \text{fiber}(y)$

Fiber=Pi (B: U) (F: B -> U) (y: B)

: Path U (fiber (total B F) B (trivial B F) y) (F y)

# Isomorphism

isIso (A B: U): U      --- A = XML, B = JSON

= (f: A -> B)

\* (g: B -> A)

\* (s: section A B f g)

\* (t: retract A B f g)

\* unit

isoPath (A B: U) (f: A -> B) (g: B -> A)

(s: section A B f g) (t: retract A B f g): Path U A B

= <i> Glue B [ (i = 0) -> (A,f,isoToEquiv A B f g s t),  
(i = 1) -> (B,idfun B,idlsEquiv B) ]

iso: U

= (A: U)

\* (B: U)

\* isIso A B

isoToPath (i: iso): Path U i.1 i.2.1

= isoPath i.1 i.2.1 i.2.2.1 i.2.2.2.1 i.2.2.2.2.1 i.2.2.2.2.2.1

section (A B: U) (f: A -> B) (g: B -> A): U = (b: B) -> Path B (f (g b)) b

retract (A B: U) (f: A -> B) (g: B -> A): U = (a: A) -> Path A (g (f a)) a

# Univalence Axiom

All Equalities Should Be Equal

$\text{ua} (A B : U) : U = \text{equiv } A B \rightarrow \text{Path } U A B$

$\text{uaIntro} (A B : U) : \text{ua } A B$

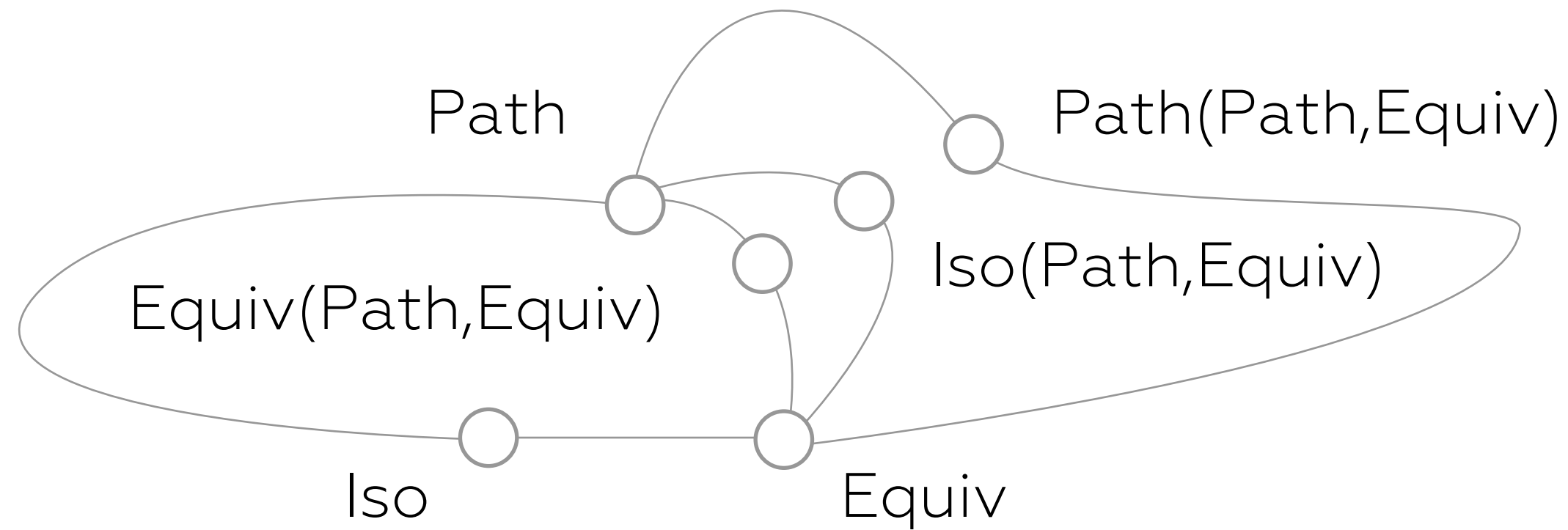
$\text{uaElim} (A B : U) (p : \text{Path } U A B) : \text{equiv } A B$

$\text{uaComp} (A B : U) (p : \text{Path } U A B)$   
 $: \text{Path} (\text{Path } U A B) (\text{uaIntro } A B (\text{uaElim } A B p)) p$

$\text{uaUniqueness} (A B : U) (w : \text{equiv } A B)$   
 $: \text{Path} (A \rightarrow B) w.1 (\text{uaElim } A B (\text{uaIntro } A B w)).1$

# Univalence Axiom

All Equalities Should Be Equal



```

lem2 (B: U) (F: B -> U) (y: B) (x: F y)
  : Path (F y) (comp (<i>F (refl B y @ i)) x []) x
  = <j> comp (<i>F ((refl B y) @ j /\ i)) x [(j=1) -> <k>x]

```

Trivial Fiber = Pi

```

lem3 (B: U) (F: B -> U) (y: B) (x: fiber (total B F) B (trivial B F) y)
  : Path (fiber (total B F) B (trivial B F) y) ((y,encode B F y x),refl B y) x
  = <i> ((x.2 @ -i,comp (<j> F (x.2 @ -i /\ j)) x.1.2 [(i=1) -> <_> x.1.2 ]),<j> x.2 @ -i \/ j)

```

```

FiberPi (B: U) (F: B -> U) (y: B) : Path U (fiber (total B F) B (trivial B F) y) (F y)
= isoPath T A f g s t where
  T: U = fiber (total B F) B (trivial B F) y
  A: U = F y
  f: T -> A = encode B F y
  g: A -> T = decode B F y
  s (x: A): Path A (f (g x)) x = lem2 B F y x
  t (x: T): Path T (g (f x)) x = lem3 B F y x

```

# I. Mathematics

# Category Theory

Categories

cat: U = (A: U) \* (A -> A -> U)

isPrecategory (C: cat): U

= (id: (x: C.1) -> C.2 x x)

\* (c: (x y z: C.1) -> C.2 x y -> C.2 y z -> C.2 x z)

\* (homSet: (x y: C.1) -> isSet (C.2 x y))

\* (left: (x y: C.1) -> (f: C.2 x y) -> Path (C.2 x y) (c x x y (id x) f) f)

\* (right: (x y: C.1) -> (f: C.2 x y) -> Path (C.2 x y) (c x y y f (id y)) f)

\* ((x y z w: C.1) -> (f: C.2 x y) -> (g: C.2 y z) -> (h: C.2 z w) ->  
Path (C.2 x w) (c x z w (c x y z f g) h) (c x y w f (c y z w g h)))

precategory: U = (C: cat) \* isPrecategory C

Instances:

Set, Functions, Category, Functors, Commutative Monoids, Abelian Groups

# Category Theory

## Functors

```
catfunctor (A B: precategory): U
= (ob: carrier A -> carrier B)
* (mor: (x y: carrier A) -> hom A x y -> hom B (ob x) (ob y))
* (id: (x: carrier A) -> Path (hom B (ob x) (ob x)) (mor x x (path A x)) (path B (ob x)))
* ((x y z: carrier A) -> (f: hom A x y) -> (g: hom A y z) ->
  Path (hom B (ob x) (ob z)) (mor x z (compose A x y z f g))
  (compose B (ob x) (ob y) (ob z) (mor x y f) (mor y z g)))
```

Category Equivalence, Id and Composition Functors, Slice and Coslice



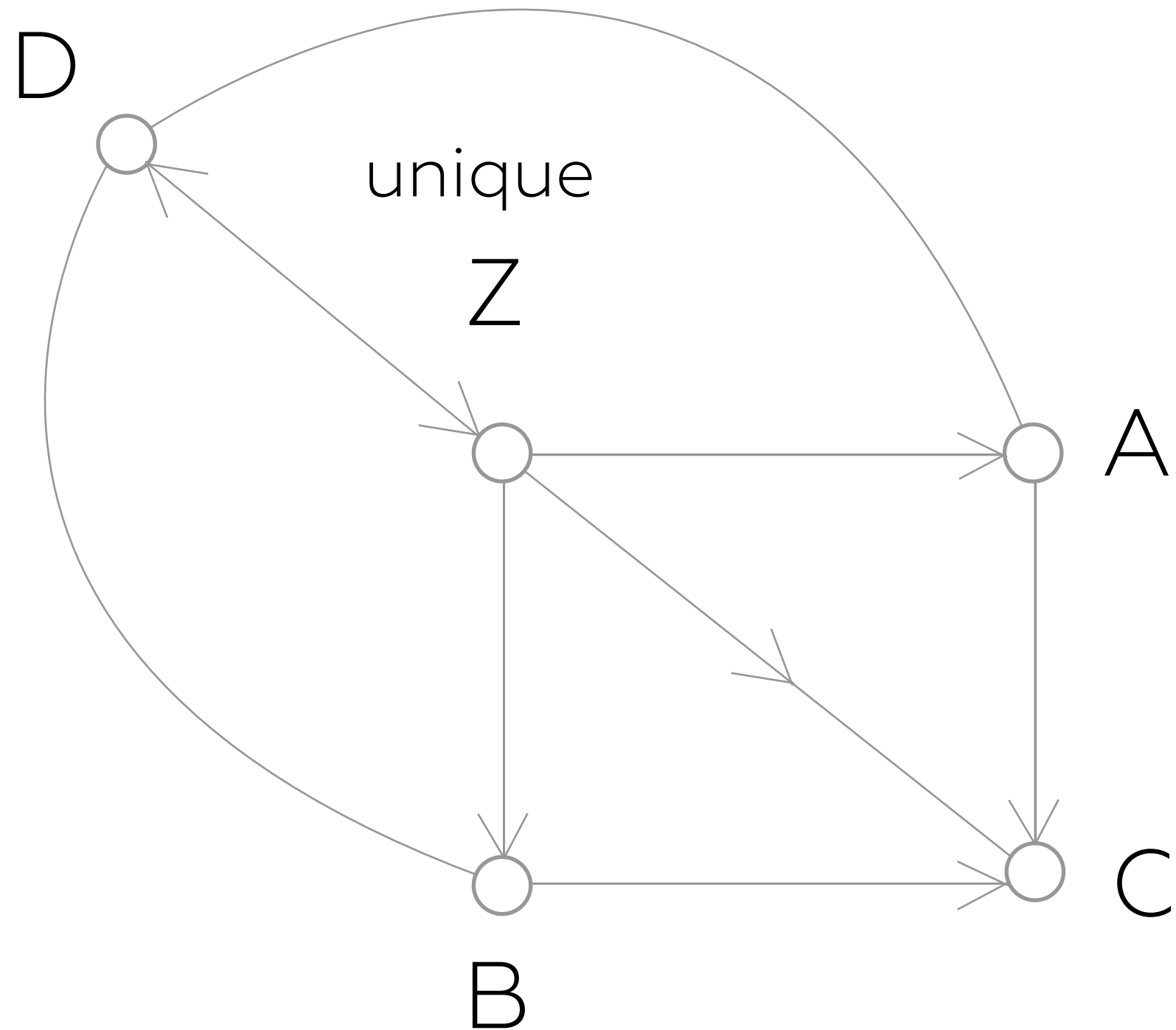
# Category of Sets

Formal Model of Set Theory

```
Set: precategory = ((Ob,Hom),id,c,HomSet,L,R,Q) where
  Ob: U = SET
  Hom (A B: Ob): U = A.1 -> B.1
  id (A: Ob): Hom A A = idfun A.1
  c (A B C: Ob) (f: Hom A B) (g: Hom B C): Hom A C = o A.1 B.1 C.1 g f
  HomSet (A B: Ob): isSet (Hom A B) = setFun A.1 B.1 B.2
  L (A B: Ob) (f: Hom A B): Path (Hom A B) (c A A B (id A) f) f = refl (Hom A B) f
  R (A B: Ob) (f: Hom A B): Path (Hom A B) (c A B B f (id B)) f = refl (Hom A B) f
  Q (A B C D: Ob) (f: Hom A B) (g: Hom B C) (h: Hom C D)
    : Path (Hom A D) (c A C D (c A B C f g) h) (c A B D f (c B C D g h))
    = refl (Hom A D) (c A B D f (c B C D g h))
```

# Pullback Completeness

Pullbacks and Fibers as edge case



Examples:  
Products, Fibers

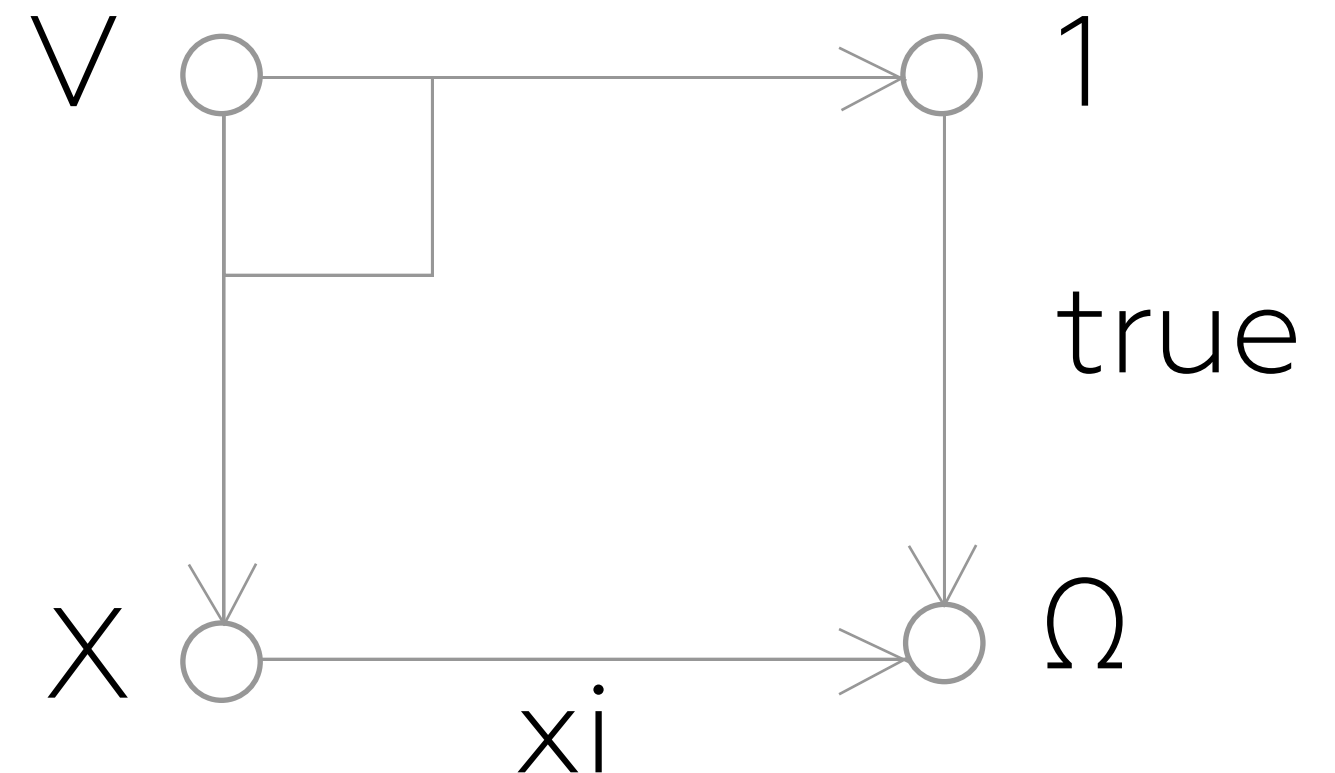
Dual Examples (Pushout):  
Coproducts, Cofibers

# Topos Theory

Categories

```
subobjectClassifier (C: precategory): U
= (omega: carrier C)
* (end: terminal C)
* (trueHom: hom C end.1 omega)
* (xi: (V X: carrier C) (j: hom C V X) -> hom C X omega)
* (square: (V X: carrier C) (j: hom C V X) -> mono C V X j
  -> hasPullback C (omega,(end.1,trueHom),(X,xi V X j)))
* ((V X: carrier C) (j: hom C V X) (k: hom C X omega)
  -> mono C V X j
  -> hasPullback C (omega,(end.1,trueHom),(X,k))
  -> Path (hom C X omega) (xi V X j) k)
```

```
Topos (cat: precategory): U
= (rezk: isCategory cat)
* (cartesianClosed: isCCC cat)
* subobjectClassifier cat
```



# Basic Abstract Algebra

## Structures

isMonoid (M: SET): U

= (op: M.1 -> M.1 -> M.1)

\* (\_: isAssociative M.1 op)

\* (id: M.1)

\* (hasIdentity M.1 op id)

isCMonoid (M: SET): U

= (m: isMonoid M)

\* (isCommutative M.1 m.1)

isGroup (G: SET): U

= (m: isMonoid G)

\* (inv: G.1 -> G.1)

\* (hasInverse G.1 m.1 m.2.2.1 inv)

isAbGroup (G: SET): U

= (g: isGroup G)

\* (isCommutative G.1 g.1.1)

isRing (R: SET): U

= (mul: isMonoid R)

\* (add: isAbGroup R)

\* (isDistributive R.1 add.1.1.1 mul.1)

isAbRing (R: SET): U

= (mul: isCMonoid R)

\* (add: isAbGroup R)

\* (isDistributive R.1 add.1.1.1 mul.1.1)

# Basic Abstract Algebra

Objects and Morphisms for Categorical Setup

monoidhom (a b: monoid): U  
= (f: a.1.1 -> b.1.1)  
\* (ismonoidhom a b f)

monoid: U = (X: SET) \* isMonoid X  
cmonoid: U = (X: SET) \* isCMonoid X  
group: U = (X: SET) \* isGroup X  
abgroup: U = (X: SET) \* isAbGroup X  
ring: U = (X: SET) \* isRing X  
abring: U = (X: SET) \* isAbRing X

cmonoidhom (a b: cmonoid): U = monoidhom (a.1, a.2.1) (b.1, b.2.1)  
grouphom (a b: group): U = monoidhom (a.1, a.2.1) (b.1, b.2.1)  
abgrouphom (a b: abgroup): U = monoidhom (a.1, a.2.1.1) (b.1, b.2.1.1)  
cmonabgrouphom (a: cmonoid) (b: abgroup): U = monoidhom (a.1, a.2.1) (b.1, b.2.1.1)

# Ordinals

## Structures

```
data ord = zero
         | succ (n: ord)
         | lim (f: nat -> ord)
```

```
data ord2 = zero
          | succ (n: ord2)
          | lim (f: nat -> ord2)
          | lim2 (f: ord -> ord2)
```

```
data ord3 = zero
          | succ (n: ord3)
          | lim (f: nat -> ord3)
          | lim2 (f: ord -> ord3)
          | lim3 (f: ord2 -> ord3)
```

data V

= pi\_ (x: V) (y: Elv x -> V)

| uni\_ (f: (x: V) -> (Elv x -> V) -> V)

(g: (x: V) -> (y: Elv x -> V) -> (Elv (f x y) -> V) -> V)

Elv: V -> U = split

pi\_ a b -> (x: Elv a) -> Elv (b x)

uni\_ f g -> Universe f g

[http://www.cs.swan.ac.uk/  
~csetzer/articles/uppermahlo.ps](http://www.cs.swan.ac.uk/~csetzer/articles/uppermahlo.ps)

cubical: Resolver.hs:(293,26)-  
(316,29): Non-exhaustive patterns  
in case

# Mahlo Universe

Structures

data Universe

(f: (x: V) -> (Elv x -> V) -> V)

(g: (x: V) -> (y: Elv x -> V) -> (Elv (f x y) -> V) -> V)

= fun\_ (x: Universe f g) (\_: Elt f g x -> Universe f g)

| f\_ (x: Universe f g) (\_: Elt f g x -> Universe f g)

| g\_ (x: Universe f g)

(y: Elt f g x -> Universe f g)

(z: Elv (f (Elt f g x) (\(a: Elt f g x) -> y a)))

Elt: (f: (x: V) -> (Elv x -> V) -> V) ->

(g: (x: V) -> (y: Elv x -> V) -> (Elv (f x y) -> V) -> V) ->

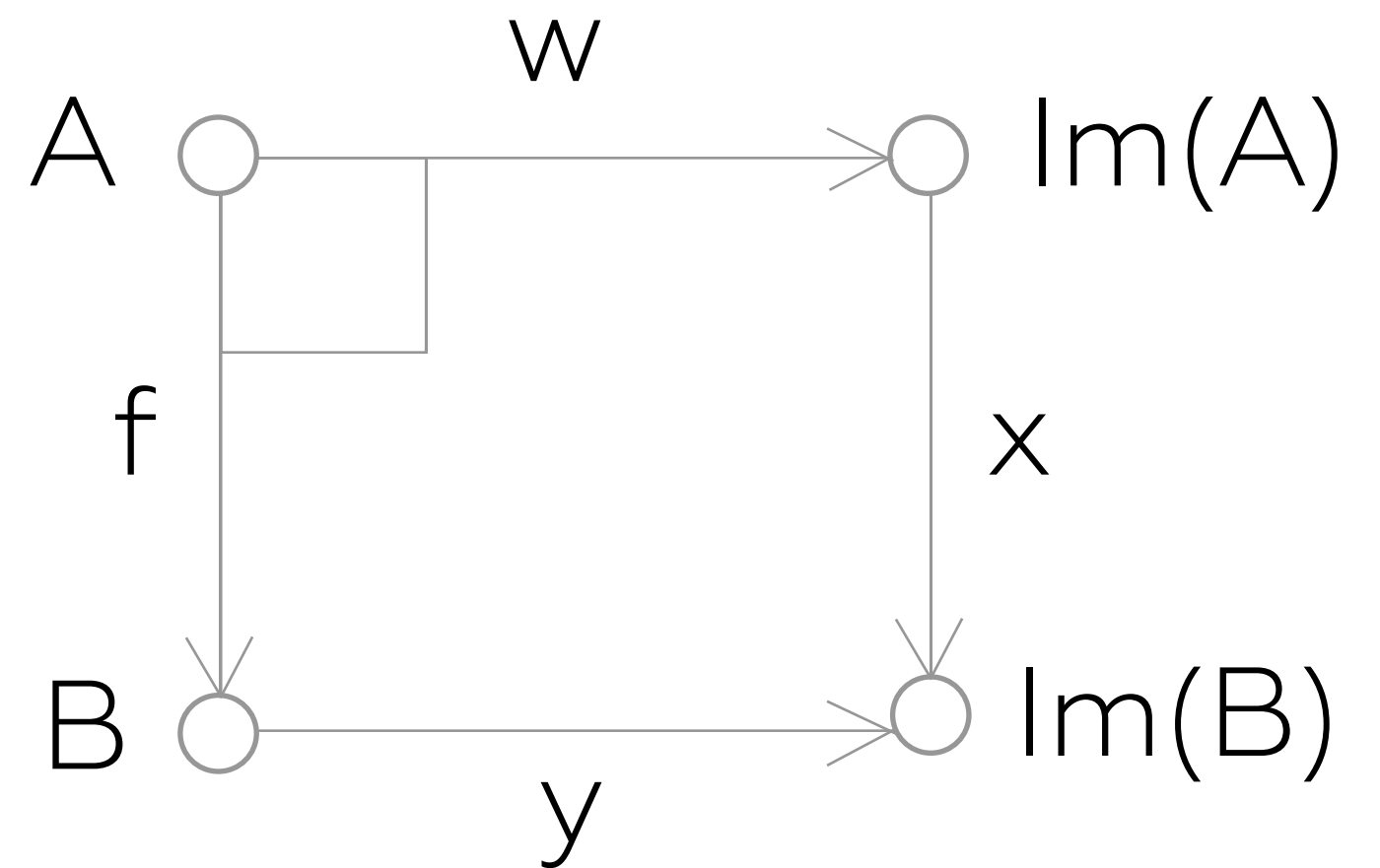
Universe f g -> V = undefined

# Differential Topology

Etale Maps

```
EtaleMap (A B: U): U
= (f: A -> B)
* isÉtaleMap A B f
```

```
isÉtaleMap (A B: U) (f: A -> B): U
= isPullbackSq A iA B (Im B) x y w f h where
iA : U = Im A
iB : U = Im B
x: iA -> iB = ImApp A B f
y: B -> iB = ImUnit B
w: A -> iA = ImUnit A
c1: A -> iB = o A iA iB x w
c2: A -> iB = o A B iB y f
T2: U = (a:A) -> Path iB (c1 a) (c2 a)
h: T2 = \ (a : A) -> <i> ImNaturality A B f a @ -i
```





# Differential Topology

Manifolds

HomogeneousStructure (V: U): U  
et (A B: U): EtaleMap A B -> (A -> B)  
isSurjective (A B: U) (f: A -> B): U

manifold (V': U) (V: HomogeneousStructure V'): U  
= (M: U)  
\* (W: U)  
\* (w: EtaleMap W M)  
\* (covers: isSurjective W M (et W M w))  
\* ( EtaleMap W V')

<https://ncatlab.org/schreiber/show/thesis+Wellen>

# Infinitesimal Modality

in Cohesive Topos

$\text{Im} : U \rightarrow U = \text{undefined}$

$\text{ImUnit} (A: U) : A \rightarrow \text{Im } A = \text{undefined}$

$\text{isCoreduced} (A:U): U = \text{isEquiv } A (\text{Im } A) (\text{ImUnit } A)$

$\text{ImCoreduced} (A:U): \text{isCoreduced} (\text{Im } A)$

$\text{ImApp} (A \ B: U) (f: A \rightarrow B): \text{Im } A \rightarrow \text{Im } B$

$= \text{ImRecursion } A (\text{Im } B) (\text{ImCoreduced } B) (\circ A \ B (\text{Im } B) (\text{ImUnit } B) f)$

$\text{ImNaturality} (A \ B:U) (f:A \rightarrow B): (a:A) \rightarrow \text{Path} (\text{Im } B)((\text{ImUnit } B)(f \ a))((\text{ImApp } A \ B \ f)(\text{ImUnit } A \ a))$

$\text{ImInduction} (A:U)(B:\text{Im } A \rightarrow U)(x: (a: \text{Im } A) \rightarrow \text{isCoreduced}(B \ a))$

$(y:(a: A) \rightarrow B(\text{ImUnit } A \ a)):(a:\text{Im } A) \rightarrow B \ a$

$\text{ImComputeInduction} (A:U)(B:\text{Im } A \rightarrow U) (c:(a:\text{Im } A) \rightarrow \text{isCoreduced}(B \ a))$

$(f:(a:A) \rightarrow B(\text{ImUnit } A \ a))(a:A)$

$: \text{Path} (B (\text{ImUnit } A \ a)) (f \ a) ((\text{ImInduction } A \ B \ c \ f) (\text{ImUnit } A \ a))$

# Higher Spheres

Fiber Bundle of Spheres

```
data S1 = base
      | loop <i> [ (i=0) -> base, (i=1) -> base]
```

```
data susp (A : U) = north
                  | south
                  | merid (a : A) <i> [ (i=0) -> north, (i=1) -> south ]
```

```
S2 : U = susp S1
```

```
S3 : U = susp S2
```

```
S4 : U = susp S3
```

```
S: nat -> U = split
  zero -> susp bool
  succ x -> susp (S x)
```

# Hopf Fibrations

Fiber Bundle of Spheres

ua (A B : U) (e : equiv A B) : Path U A B = <i> Glue B [ (i = 0) -> (A,e), (i = 1) -> (B,idEquiv B) ]  
rot: (x : S1) -> Path S1 x x = split { base -> loop1 ; loop @ i -> constSquare S1 base loop1 @ i }

mu : S1 -> equiv S1 S1 = split  
base -> idEquiv S1  
loop @ i -> equivPath S1 S1 (idEquiv S1) (idEquiv S1) (<j> \ (x : S1) -> rot x @ j) @ i

H : S2 -> U = split { north -> S1 ; south -> S1 ; merid x @ i -> ua S1 S1 (mu x) @ i }  
TH : U = (c : S2) \* H c

# Sequences

```
data Seq (A: U) (B: A -> A -> U) (X Y: A)
  = seqNil (_: A)
  | seqCons (X Y Z: A) (_: B X Y) (_: Seq A B Y Z)
```

fiberSeq: pointed -> pointed -> U = Seq pointed pmap

fiberNil (X: pointed): fiberSeq X X = seqNil X

fiberCons (X Y Z: pointed) (h: pmap X Y) (t: fiberSeq Y Z): fiberSeq X Z = seqCons X Y Z h t

homSeq: group -> group -> U = Seq group grouphom

homNil (X: group): homSeq X X = seqNil X

homCons (X Y Z: group) (h: grouphom X Y) (t: homSeq Y Z): homSeq X Z = seqCons X Y Z h t

abSeq: abgroup -> abgroup -> U = Seq abgroup abgrouphom

abNil (X: abgroup): abSeq X X = seqNil X

abCons (X Y Z: abgroup) (h: abgrouphom X Y) (t: abSeq Y Z): abSeq X Z = seqCons X Y Z h t

# Chain Complexes

ChainComplex: U

= (head: abgroup)

\* (chain: **nat** -> abgroup)

\* (augment: abgrouphom (chain **zero**) head)

\* ((n: **nat**) -> abgrouphom (chain (**succ** n)) (chain n))

CochainComplex: U

= (head: abgroup)

\* (cochain: **nat** -> abgroup)

\* (augment: abgrouphom head (cochain **zero**))

\* ((n: **nat**) -> abgrouphom (cochain n) (cochain (**succ** n)))

# Impredicative Encoding

As in version of Steve Awodey, HITs encoding

$\text{Nat\_Church} = (X: U) \rightarrow (X \rightarrow X) \rightarrow X \rightarrow X$

$\text{Nat} = (X: U) \rightarrow \text{isSet } X \rightarrow (X \rightarrow X) \rightarrow X \rightarrow X$

$\text{Unit} = (X: U) \rightarrow \text{isSet } X \rightarrow X \rightarrow X$

$1 = (\text{one}: \text{Unit}) * ((X \ Y: U) (x: \text{isSet } X) (y: \text{isSet } Y) (f: X \rightarrow Y) \rightarrow \text{naturalness } X \ Y \ f \ (\text{one } X \ x) (\text{one } Y \ y))$

$N = (\text{one}: \text{Nat}) * ((X \ Y: U) (x: \text{isSet } X) (y: \text{isSet } Y) (f: X \rightarrow Y) \rightarrow \text{naturalness } X \ Y \ f \ (\text{one } X \ x) (\text{one } Y \ y))$

Truncation  $\|A\|$  parametrized by  $(A:U)$  type  $= (X: U) \rightarrow \text{isProp } X \rightarrow (A \rightarrow X) \rightarrow X$

$S^1 = (X:U) \rightarrow \text{isGroupoid } X \rightarrow (x:X) \rightarrow \text{Path } X \ x \ x \rightarrow X$

Arbitrary  $(A:U)$  type  $= (X: U) \rightarrow \text{isSet } X \rightarrow (A \rightarrow X) \rightarrow X$

# Impredicative Encoding

Encode Unit. Homework: Bool, Circle, Sphere

$\text{unitEnc}' : U = (X : U) \rightarrow \text{isSet } X \rightarrow X \rightarrow X$

$\text{naturality } (X \ Y : U) (f : X \rightarrow Y) (a : X \rightarrow X) (b : Y \rightarrow Y) : U = \text{Path } (X \rightarrow Y) (\text{o } X \ X \ Y \ f \ a) (\text{o } X \ Y \ Y \ b \ f)$

$\text{isUnitEnc } (\text{one} : \text{unitEnc}') : U$

$= (X \ Y : U) (x : \text{isSet } X) (y : \text{isSet } Y) (f : X \rightarrow Y) \rightarrow \text{naturality } X \ Y \ f \ (\text{one } X \ x) (\text{one } Y \ y)$

$\text{unitEnc} : U = (x : \text{unitEnc}') * \text{isUnitEnc } x$

$\text{unitEncStar} : \text{unitEnc} = \backslash (X : U) (\_ : \text{isSet } X) \rightarrow \text{idfun } X, \backslash (X \ Y : U) (\_ : \text{isSet } X) (\_ : \text{isSet } Y) \rightarrow \text{refl } (X \rightarrow Y)$

$\text{unitEncRec } (C : U) (s : \text{isSet } C) (c : C) : \text{unitEnc} \rightarrow C = \backslash (z : \text{unitEnc}) \rightarrow z.1 \ C \ s \ c$

$\text{unitEncBeta } (C : U) (s : \text{isSet } C) (c : C) : \text{Path } C (\text{unitEncRec } C \ s \ c \ \text{unitEncStar}) \ c$

$\text{unitEncEta } (z : \text{unitEnc}) : \text{Path } \text{unitEnc} \ \text{unitEncStar } z$

$\text{unitEncInd } (P : \text{unitEnc} \rightarrow U) (a : \text{unitEnc}) : P \ \text{unitEncStar} \rightarrow P \ a$

$\text{unitEncCondition } (n : \text{unitEnc}') : \text{isProp } (\text{isUnitEnc } n)$



<https://github.com/groupoid/cafe>

Thank You!

<https://groupoid.space>