Cohomology and Spectra

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May 2025

Abstract

This article presents formal definitions and theorems for ordinary and generalized cohomology theories, unstable and stable spectra, and spectral sequences in Abelian categories, including the Serre, Atiyah-Hirzebruch, Leray, Eilenberg-Moore, Hochschild-Serre, Filtered Complex, Chromatic, Adams, and Bockstein spectral sequences. We define slopes, sheets, coordinates, quadrants, complex filtrations, and double complexes within this framework.

Contents

1 Cohomology and Spectra

1.1 Ordinary Cohomology

Definition 1. An *ordinary cohomology theory* on the category of topological spaces and pairs is a contravariant functor $H^*(-;G) : \mathrm{Top}^{\mathrm{op}} \to \mathrm{GrAb}$, assigning to each pair (X, A) a sequence of abelian groups $\{H^n(X, A; G)\}_{n\in\mathbb{Z}}$, with coefficient group G, satisfying:

1. *Homotopy*: If $f \simeq g : (X, A) \to (Y, B)$, then $f^* = g^* : H^n(Y, B; G) \to H^n(X, A; G)$.
2. *Exactness*: For (X, A), there is a long exact sequence:
$$\cdots \to H^n(X, A; G) \to H^n(X; G) \to H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \to \cdots$$
3. *Excision*: For $U \subset A$ with $\overline{U} \subset \mathrm{int}(A)$, the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces isomorphisms $H^n(X, A; G) \cong H^n(X \setminus U, A \setminus U; G)$.
4. *Additivity*: For $X = \coprod X_i$, $H^n(X; G) \cong \bigoplus H^n(X_i; G)$.
5. *Dimension*: For a point pt, $H^n(\mathrm{pt}; G) = \begin{cases} G & n = 0 \\ 0 & n \neq 0 \end{cases}$.

1.2 Generalized Cohomology Theories

Definition 2. A *generalized cohomology theory* is a contravariant functor $h^* : \mathrm{Top}^{\mathrm{op}} \to \mathrm{GrAb}$, assigning to each pair (X, A) a sequence $\{h^n(X, A)\}_{n\in\mathbb{Z}}$, satisfying:

1. *Homotopy*, *Exactness*, *Excision*, and *Additivity* as in Definition 1.
2. *Suspension*: There is a natural isomorphism $h^n(X, A) \cong h^{n+1}(\Sigma X, \Sigma A)$, where Σ is the reduced suspension.

The groups $h^n(\mathrm{pt})$ form a graded ring, the coefficients of h^*.

Theorem 1. *Every generalized cohomology theory h^* is representable by a spectrum $E = \{E_n, \sigma_n : \Sigma E_n \to E_{n+1}\}$, with $h^n(X) \cong [X, E_n]_*$, where $[-,-]_*$ denotes pointed homotopy classes.*

1.3 Unstable and Stable Spectra

Definition 3. A *spectrum* is a sequence of pointed spaces $\{E_n\}_{n\in I}$, where $I \subseteq \mathbb{Z}$, with structure maps $\sigma_n : \Sigma E_n \to E_{n+1}$. It is:

- *Unstable* if $I \subseteq \mathbb{Z}_{\geq 0}$.
- *Stable* if $I = \mathbb{Z}$ and each σ_n is a homotopy equivalence.

Theorem 2. *For an unstable spectrum E, the functor $X \mapsto [X, E_n]_*$ defines a cohomology theory on spaces of dimension $\leq n$. For a stable spectrum E, the functor $h^n(X) = [X, E_n]_*$ defines a generalized cohomology theory.*

1.4 Spectral Sequences

Definition 4. A *spectral sequence* in an Abelian category $\mathcal{A}$ is a collection of objects $\{E_r^{p,q}\}_{r\geq 1, p,q\in\mathbb{Z}}$, $E_r^{p,q} \in \mathcal{A}$, with differentials:

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+a_r, q+b_r},$$

such that:

1. $d_r \circ d_r = 0$.
2. $E_{r+1}^{p,q} = H^{p,q}(E_r, d_r) = \ker(d_r^{p,q})/\mathrm{im}(d_r^{p-a_r,q-b_r})$.
3. There exists a graded object $H^n \in \mathcal{A}$ with filtration $F_pH^{p+q} \subseteq H^{p+q}$, such that:
$$E_\infty^{p,q} \cong F_pH^{p+q}/F_{p-1}H^{p+q}.$$

The sequence is *first-quadrant* if $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$.

Definition 5. The *r-th sheet* of a spectral sequence is the collection $\{E_r^{p,q}\}_{p,q}$. The indices (p, q) are *coordinates*, with p the filtration degree and q the complementary degree, satisfying total degree $n = p + q$. The *slope* of $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is $\frac{-r+1}{r}$.

Definition 6. A *filtered complex* in $\mathcal{A} = \mathrm{Ab}$ is a chain complex (C_*, ∂) with a filtration $\cdots \subseteq F_{p-1}C_n \subseteq F_pC_n \subseteq F_{p+1}C_n \subseteq \cdots$, compatible with ∂. A *double complex* is a bigraded object $C_{p,q}$ with differentials $d^h : C_{p,q} \to C_{p-1,q}$, $d^v : C_{p,q} \to C_{p,q-1}$, satisfying $d^hd^h = d^vd^v = d^hd^v + d^vd^h = 0$. The total complex is $\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$.

Theorem 3. *A filtered complex (C_*, F_p) induces a spectral sequence with:*

$$E_0^{p,q} = F_pC_{p+q}/F_{p-1}C_{p+q}, \quad E_1^{p,q} = H_{p+q}(F_pC/F_{p-1}C) \implies H_{p+q}(C).$$

A double complex $C_{p,q}$ with filtration by p-index induces:

$$E_1^{p,q} = H_q^v(C_{p,*}), \quad d_1 = H(d^h) \implies H_{p+q}(Tot(C)).$$

1.5 Serre Spectral Sequence

Theorem 4. *For a fibration $F \to E \to B$ with B path-connected, there exists a first-quadrant spectral sequence:*

$$E_2^{p,q} = H^p(B; H^q(F; \mathbb{Z})) \implies H^{p+q}(E; \mathbb{Z}),$$

with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

1.6 Atiyah-Hirzebruch Spectral Sequence

Theorem 5. *For a generalized cohomology theory h^* and a CW-complex X, there exists a spectral sequence:*

$$E_2^{p,q} = H^p(X; h^q(pt)) \implies h^{p+q}(X),$$

with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

1.7 Leray Spectral Sequence

Theorem 6. *For a continuous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, there exists a spectral sequence:*

$$E_2^{p,q} = H^p(Y; R^q f_* \mathcal{F}) \implies H^{p+q}(X; \mathcal{F}),$$

with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

1.8 Eilenberg-Moore Spectral Sequence

Theorem 7. *For a pullback diagram with fibration $F \to E \to B$, there exists a spectral sequence:*

$$E_2^{p,q} = Tor_{H_*(B)}^{p,q}(H_*(F), R) \implies H_{p+q}(F; R),$$

with $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$.

1.9 Hochschild-Serre Spectral Sequence

Theorem 8. *For a group extension $1 \to N \to G \to Q \to 1$, there exists a spectral sequence:*

$$E_2^{p,q} = H^p(Q; H^q(N; R)) \implies H^{p+q}(G; R),$$

with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

1.10 Spectral Sequence of a Filtered Complex

Theorem 9. *For a filtered complex (C_*, F_p), there exists a spectral sequence:*

$$E_1^{p,q} = H_{p+q}(F_p C / F_{p-1} C) \implies H_{p+q}(C),$$

with $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$.

1.11 Chromatic Spectral Sequence

Theorem 10. *For a spectrum X, there exists a spectral sequence:*

$$E_1^{n,k} = \pi_{n-k}(L_{K(k)} X) \implies \pi_{n-k}(X),$$

where $L_{K(k)} X$ is the localization at the k-th Morava K-theory, with $d_r : E_r^{n,k} \to E_r^{n+1,k-r}$.

1.12 Adams Spectral Sequence

Theorem 11. *For a spectrum X and prime p, there exists a spectral sequence:*

$$E_2^{s,t} = Ext_A^{s,t}(Hom_*(X, \mathbb{Z}/p), \mathbb{Z}/p) \implies \pi_{t-s}(X_{(p)}),$$

where A is the Steenrod algebra, with $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$.

1.13 Bockstein Spectral Sequence

Theorem 12. *For a short exact sequence $0 \to R \to R' \to R'' \to 0$ of coefficient rings, there exists a spectral sequence:*

$$E_1^{p,q} = H^{p+q}(X; R'') \implies H^{p+q}(X; R),$$

with $d_r : E_r^{p,q} \to E_r^{p+1,q-r}$.

References

[1] Hatcher, A., *Algebraic Topology*, Cambridge University Press, 2002.

[2] Switzer, R., *Algebraic Topology - Homology and Homotopy*, Springer, 1975.

[3] Kato, G., *The Heart of Cohomology*, Springer, 2006