

Twisted K-Theory and Fredholm Operators

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Abstract

This article explores the mathematical foundations of topological quantum programming, focusing on the role of twisted complex K-theory ($KU_G^\tau(X)$) and the space of self-adjoint, odd-graded Fredholm operators (Fred_C^0) as presented in the framework of Twisted Equivariant Differential K-theory (TED-K). We provide a detailed explanation of the type-theoretic definition of $KU_G^\tau(X)$ in cohesive homotopy type theory, its categorical interpretation, and its significance in encoding anyonic quantum states. Additionally, we analyze the properties of Fred_C^0 , its connection to L^2 spaces, and its role in K-theory via index theory and integration. The article is motivated by the TED-K framework for hardware-aware quantum programming, as developed by Sati and Schreiber.

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1 Twisted K-Theory and Fredholm Operators

Topological quantum computations (TQC) leverages topological properties of quantum materials to achieve fault-tolerant quantum computing, primarily through the braiding of anyons in topologically ordered states [1]. The TED-K framework, introduced by Sati and Schreiber [2], proposes a novel approach to topological quantum programming that is hardware-aware, reflecting the physics of anyonic ground states via twisted equivariant differential K-theory (TED-K). Central to this

framework are the twisted K-theory group $KU_G^\tau(X)$ and the space of Fredholm operators Fred_C^0 , which encode quantum states and their braiding operations.

This article synthesizes detailed explanations of $KU_G^\tau(X)$ and Fred_C^0 , drawing from the TED-K framework. We elucidate their definitions in cohesive homotopy type theory (HoTT), their categorical interpretations in ∞ -topoi, and their physical significance in TQC. The exposition is aimed at readers with a background in algebraic topology, functional analysis, or quantum computing, seeking to understand the mathematical underpinnings of topological quantum programming.

1.1 Definition in Cohesive HoTT

In the TED-K framework, the twisted complex K-theory group $KU_G^\tau(X)$ is defined in cohesive homotopy type theory as a dependent type:

$$X : \text{Type}, \tau : X \rightarrow \text{BPU} \vdash KU_G^\tau(X) \equiv \left| \int_{\text{BPU}} (X \rightarrow \text{Fred}_C^0 \beta \text{PU}) \right|_0 : \text{Type} \quad (1)$$

Here, X is a type representing a topological space or orbifold (e.g., the configuration space of anyonic defects), τ is a twist map to the classifying space BPU , Fred_C^0 is the space of self-adjoint, odd-graded Fredholm operators, βPU denotes the homotopy fiber adjusting for the twist, \int_{BPU} is the shape modality over BPU , and $|\cdot|_0$ is the 0-truncation modality.

1.2 Components of the Definition

- **PU**: The projective unitary group, defined as $\text{PU} = \text{U}(\mathcal{H})/\text{U}(1)$, where $\text{U}(\mathcal{H})$ is the unitary group of a separable complex Hilbert space \mathcal{H} . It acts on Fred_C^0 by conjugation and has $\pi_3(\text{PU}) \cong \mathbb{Z}$, making it suitable for classifying twists.
- **BPU**: The classifying space of PU , with $\pi_4(\text{BPU}) \cong \mathbb{Z}$. A map $\tau : X \rightarrow \text{BPU}$ classifies a gerbe or projective bundle, encoding topological obstructions in twisted K-theory.
- **Fred_C^0** : The space of self-adjoint, odd-graded Fredholm operators with index 0, discussed in detail in Section 1.7.
- **$\text{Fred}_C^0 \beta \text{PU}$** : The homotopy fiber of the PU -action map $\text{Fred}_C^0 \rightarrow \text{BPU}$, ensuring compatibility with the twist τ .
- **$X \rightarrow \text{Fred}_C^0 \beta \text{PU}$** : The type of sections of the twisted bundle of Fredholm operators over X .
- **\int_{BPU}** : The shape modality, extracting the homotopy type, combined with the dependent product over BPU , computing all sections compatible with τ .
- **$|\cdot|_0$** : The 0-truncation, yielding a set (the K-theory group $KU^0(X, \tau)$).

1.3 PU: Projective Unitary Group

The projective unitary group PU is defined as the quotient:

$$\text{PU} = \text{U}(\mathcal{H})/\text{U}(1) \quad (2)$$

where $U(\mathcal{H})$ is the group of unitary operators on a separable complex Hilbert space \mathcal{H} (e.g., $L^2(\mathbb{R}^n; \mathbb{C})$), equipped with the strong operator topology, and $U(1) \cong S^1$ is the circle group acting by scalar multiplication. An element $u \in U(\mathcal{H})$ is unitary if $u^*u = uu^* = I$, preserving the inner product, and PU identifies unitaries differing by a phase $e^{i\theta} \in U(1)$.

The topological structure of PU is significant for K-theory, as it has non-trivial homotopy groups, notably:

$$\pi_3(\text{PU}) \cong \mathbb{Z} \quad (3)$$

This makes PU suitable for classifying twists in twisted K-theory, as twists are often associated with elements in the third cohomology group $H^3(X; \mathbb{Z})$. In the TED-K framework, PU acts on Fred_C^0 by conjugation:

$$T \mapsto uTu^{-1}, \quad u \in \text{PU}, T \in \text{Fred}_C^0 \quad (4)$$

This action preserves the self-adjointness, odd-graded property, and Fredholm nature of the operators, enabling the encoding of the twist τ . Physically, the PU-action reflects symmetries in the quantum system, such as those arising from gauge fields or anyonic statistics in topological quantum materials.

In the categorical setting of the cohesive ∞ -topos of smooth ∞ -groupoids, PU is a group object (a 0-type with a group structure). Its role is to mediate the twisting of K-theory classes, ensuring that the quantum states encoded by Fred_C^0 respect the topological phase specified by τ .

1.4 BPU: Classifying Space of PU

The classifying space BPU is the delooping of PU, a 1-type in the ∞ -topos characterized by:

$$\pi_1(\text{BPU}) \cong \text{PU}, \quad \pi_i(\text{BPU}) = 0 \text{ for } i \neq 1 \quad (5)$$

However, since PU itself has higher homotopy groups, BPU inherits non-trivial homotopy in higher degrees:

$$\pi_4(\text{BPU}) \cong \pi_3(\text{PU}) \cong \mathbb{Z} \quad (6)$$

Maps $\tau : X \rightarrow \text{BPU}$ classify principal PU-bundles or gerbes over X , corresponding to elements in $H^1(X; \text{PU})$, which are related to $H^3(X; \mathbb{Z})$ via the long exact sequence of the fibration:

$$U(1) \rightarrow U(\mathcal{H}) \rightarrow \text{PU} \quad (7)$$

In the TED-K framework, $\tau : X \rightarrow \text{BPU}$ encodes a topological obstruction, such as a gauge field or a gerbe, that affects the braiding statistics of anyons. The classifying space BPU serves as the base for the twist, and the homotopy fiber $\text{Fred}_C^0 \beta \text{PU}$ ensures that the Fredholm operators are compatible with the PU-bundle defined by τ .

Categorically, BPU is a pointed 1-type in the cohesive ∞ -topos, and the map τ is a morphism in \mathcal{H} . The dependent product \int_{BPU} in the definition of $\text{KU}^\tau(X)$ integrates over all possible twists, ensuring that the K-theory classes reflect the topological structure imposed by τ . Physically, BPU mediates the connection between the topological phase of the quantum material and the quantum states encoded in Fred_C^0 .

1.5 Categorical Interpretation

The definition is interpreted in the ∞ -topos of smooth ∞ -groupoids, a cohesive ∞ -topos \mathcal{H} equipped with modalities like the shape functor $\int : \mathcal{H} \rightarrow \text{HoTop}$. The type $\text{KU}_G^\tau(X)$ corresponds to the homotopy classes of sections of a twisted bundle in the slice ∞ -topos $\mathcal{H}_{/\text{BPU}}$. Categorically:

- The mapping space $X \rightarrow \text{Fred}_C^0 \text{BPU}$ is the object $\text{Map}(X, \text{Fred}_C^0 \text{BPU})$ in \mathcal{H} .
- The dependent product \int_{BPU} is the right adjoint to the base change along τ , and \int extracts the homotopy type.
- The 0-truncation $|\cdot|_0$ maps to the set of connected components, yielding the abelian group $\text{KU}_G^\tau(X)$.

1.6 Physical Significance

In TQC, $\text{KU}_G^\tau(X)$ encodes the ****ground states of $\text{su}(2)$ -anyons**** (e.g., Majorana or Fibonacci anyons) in topological quantum materials [3]. The configuration space X represents defect positions, and the twist τ accounts for gauge fields or gerbes affecting anyonic statistics. The ****braid group****, arising from the cohesive shape $\int X$, acts on $\text{KU}_G^\tau(X)$ via transport in HoTT , implementing quantum gates through adiabatic braiding. This makes $\text{KU}_G^\tau(X)$ a hardware-aware construct, enabling formal verification and classical simulation of quantum computations [2].

1.7 Fredholm Operators: Fred_C^0

1.7.1 Definition of Fredholm Operators

A ****Fredholm operator**** $T : \mathcal{H} \rightarrow \mathcal{H}$ on a complex Hilbert space \mathcal{H} is a bounded linear operator with:

1. Finite-dimensional kernel: $\dim(\ker T) < \infty$.
2. Finite-dimensional cokernel: $\dim(\text{coker } T) = \dim(\mathcal{H}/\text{im } T) < \infty$.
3. Closed image: $\text{im } T$ is closed in \mathcal{H} .

The index is:

$$\text{index}(T) = \dim(\ker T) - \dim(\text{coker } T) \quad (8)$$

In Fred_C^0 , operators are ****self-adjoint****, ****odd-graded****, and typically have ****index 0****.

1.7.2 Properties of Fred_C^0

- **Self-Adjointness:** $T = T^*$, where T^* is the adjoint satisfying $\langle Tx, y \rangle = \langle x, T^*y \rangle$. This ensures a real spectrum and is common in spectral K-theory.
- **Odd-Graded:** $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ is $\mathbb{Z}/2$ -graded, with a grading operator $\gamma : \mathcal{H} \rightarrow \mathcal{H}$, $\gamma^2 = I$, and T satisfies $\gamma T = -T\gamma$. Thus, $T = \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix}$.

- **Index 0:** $\dim(\ker T) = \dim(\operatorname{coker} T)$, corresponding to the degree-0 component of K-theory.
- **PU-Action:** The projective unitary group PU acts by conjugation, $T \mapsto uTu^{-1}$, preserving the Fredholm properties and enabling twisted K-theory.

1.7.3 Connection to L^2 Spaces

The Hilbert space \mathcal{H} is often an L^2 space, such as $L^2(\mathbb{R}^n; \mathbb{C})$, the space of square-integrable functions:

$$L^2(\mathbb{R}^n; \mathbb{C}) = \left\{ f : \mathbb{R}^n \rightarrow \mathbb{C} \mid \int_{\mathbb{R}^n} |f(x)|^2 dx < \infty \right\}$$

Fredholm operators on L^2 spaces include elliptic differential operators, e.g., $T = -\Delta + V$, where Δ is the Laplacian and V is a potential. For grading, consider $\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}) \oplus L^2(\mathbb{R}^n; \mathbb{C})$, with $\gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. Operators in Fred_C^0 act on this graded space, encoding anyonic wavefunctions in TQC.

1.7.4 Integration and Index Theory

The analytical properties of Fred_C^0 involve integration, particularly in ****index theory****. The index of a Fredholm operator can be computed via the Atiyah-Singer index theorem:

$$\operatorname{index}(T) = \int_M \operatorname{ch}(E) \wedge \operatorname{Td}(TM) \quad (9)$$

where $\operatorname{ch}(E)$ is the Chern character and $\operatorname{Td}(TM)$ is the Todd class. In differential K-theory, the Chern character of a K-theory class involves integrals of curvature forms:

$$\operatorname{ch}([T]) = \int_X \operatorname{tr}(e^{F/2\pi i}) \quad (10)$$

In cohesive HoTT, the shape modality \int acts as a homotopical integration, mapping $\operatorname{KU}_G^\tau(X)$ to topological invariants, crucial for computing braiding statistics in TQC.

1.7.5 Physical Role

Fred_C^0 operators act on the Hilbert space of anyonic wavefunctions, with self-adjointness and grading reflecting their symmetries and statistics. The PU-action enables twisting, and the index 0 condition aligns with the degree-0 K-theory group, classifying stable quantum states. Integration via the Chern character connects these operators to physical observables, facilitating simulation and verification in the TED-K framework.

1.8 Conclusion

The twisted K-theory group $\operatorname{KU}_G^\tau(X)$ and the space of Fredholm operators Fred_C^0 are foundational to the TED-K framework for topological quantum programming. $\operatorname{KU}_G^\tau(X)$, defined in cohesive HoTT, encodes anyonic ground states and supports braid quantum gates via the braid group's action. Fred_C^0 , consisting of self-adjoint, odd-graded Fredholm operators on L^2 spaces, provides the

analytical backbone, with integration playing a key role in index theory and differential K-theory. Together, these constructs enable a hardware-aware, formally verifiable approach to TQC, bridging topology, functional analysis, and quantum computing. For further details, see [4].

References

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