Solution of the equation $z_1^{-1} c_1 z_1 c_1^{-1} =_F 1$ in a Free Group

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Contents

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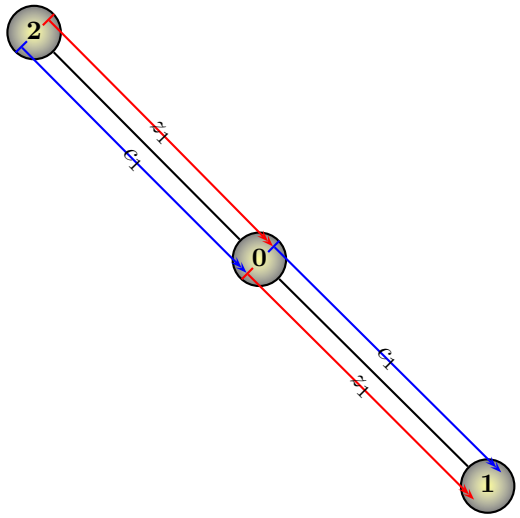
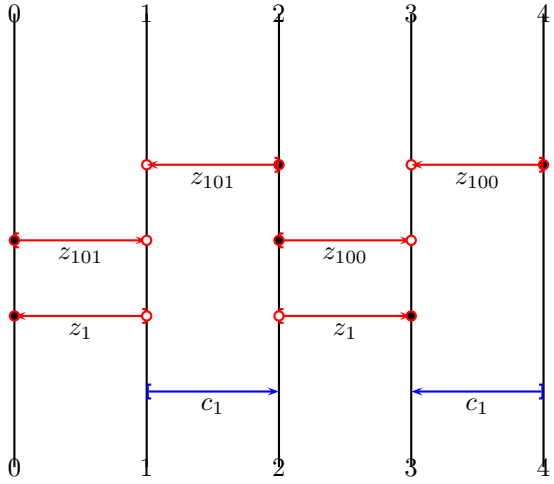
1 Cancellation scheme #1



z_1^{-1}	$1 \leftarrow 0$
c_1	$0 \leftarrow 1$
z_1	$2 \leftarrow 0$
c_1^{-1}	$0 \leftarrow 2$

Generalized Equation root-1

Below is the root GE obtained from the cancellation diagram above.



GE Information: Carrier: [0-1:z1-.] ; Carrier Dual: [2-3:z1+.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=3*<1=2*
```

We proceed.

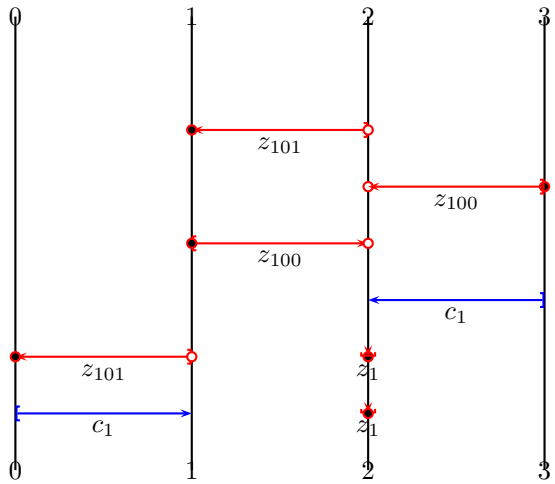
Generalized Equation root-1.1

We begin from the GE root-1 (see pp. 2). We consider its print

```
Print 1: =0=3*<1=2*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z1-.] to (new) boundaries 3 - 2.
Step 2: Moved (old) base [0-1:z101+.] to (new) boundaries 3 - 2.
Step 3: Collapsed (new) base [2-3:z1+.] to the empty base (3,3).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-1.1—is illustrated below:



GE Information: Carrier: [0-1:z101-.] ; Carrier Dual: [1-2:z101-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*
```

This completes the consideration of root-1.1, as derived from the application of a print to root-1.

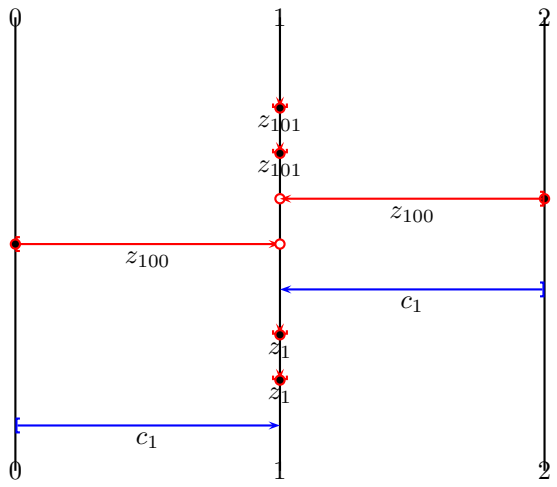
Generalized Equation root-1.1.1

We begin from the GE root-1.1 (see pp. 3). We consider its print

```
Print 1: =0=1*<1=2*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z101-.] to (new) boundaries 1 - 2.
Step 2: Moved (old) base [0-1:c1+.] to (new) boundaries 1 - 2.
Step 3: Collapsed (new) base [1-2:z101-.] to the empty base (2,2).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-1.1.1—is illustrated below:



GE Information: Carrier: [0-1:z100+.] ; Carrier Dual: [1-2:z100-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=2*<1=1*
```

This completes the consideration of root-1.1.1, as derived from the application of a print to root-1.1.

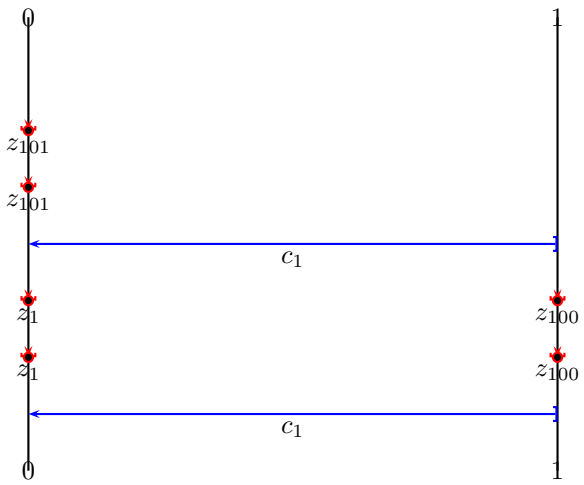
Generalized Equation root-1.1.1.1

We begin from the GE root-1.1.1 (see pp. 4). We consider its print

```
Print 1: =0=2*<1=1*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z100+.] to (new) boundaries 2 - 1.
Step 2: Moved (old) base [0-1:c1+.] to (new) boundaries 2 - 1.
Step 3: Collapsed (new) base [1-2:z100-.] to the empty base (2,2).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

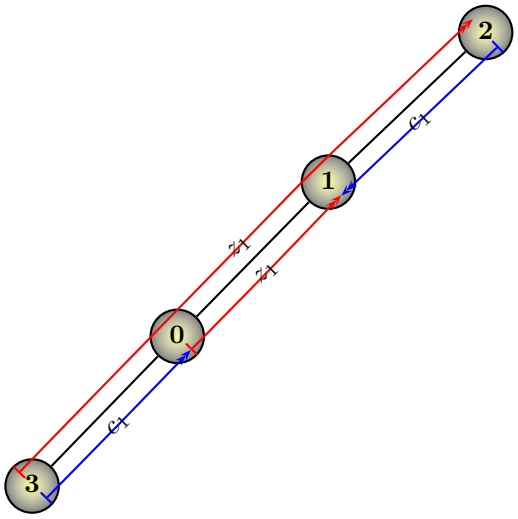
Upon applying the print, the GE we obtain—which we refer to as root-1.1.1.1—is illustrated below:



GE Information: Carrier: [0-0:z1+.] ; Carrier Dual: [0-0:z1+.] ; Critical Boundary: 0; The GE above is non-degenerate. This GE is a leaf in the GE tree. We have effectively found a solution!

This completes the consideration of root-1.1.1.1, as derived from the application of a print to root-1.1.1.

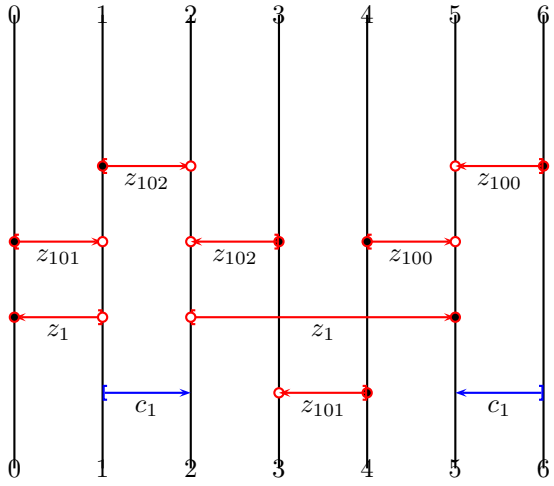
2 Cancellation scheme #2



z_1^{-1}	$1 \leftarrow 0$
c_1	$2 \leftarrow 1$
z_1	$3 \leftarrow 0 \leftarrow 1 \leftarrow 2$
c_1^{-1}	$0 \leftarrow 3$

Generalized Equation root-2

Below is the root GE obtained from the cancellation diagram above.



GE Information: Carrier: [0-1:z1-.] ; Carrier Dual: [2-5:z1+.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=5*<4*<3*<1=2*
```

We proceed.

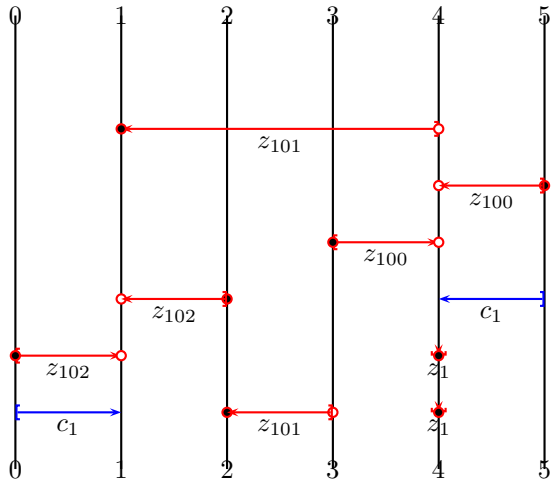
Generalized Equation root-2.1

We begin from the GE root-2 (see pp. 6). We consider its print

```
Print 1: =0=5*<4*<3*<1=2*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z1-.] to (new) boundaries 5 - 2.
Step 2: Moved (old) base [0-1:z101+.] to (new) boundaries 5 - 2.
Step 3: Collapsed (new) base [2-5:z1+.] to the empty base (5,5).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-2.1—is illustrated below:

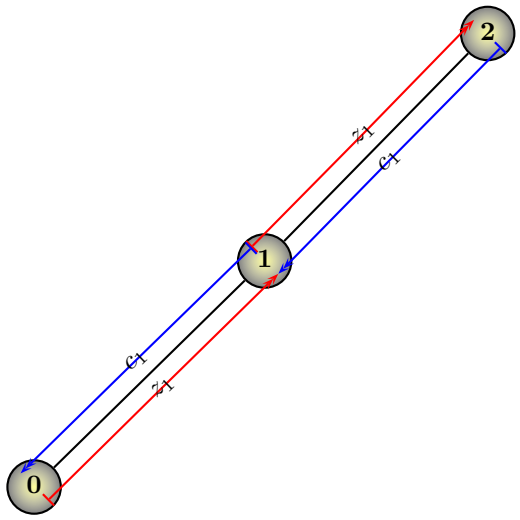


GE Information: Carrier: [0-1:z102+.] ; Carrier Dual: [1-2:z102-.] ; Critical Boundary: 1; Observe the following facts about this GE: The base [1-4:z101-.] and its dual are of the same polarity, yet one properly contains the other. The

base [2-3:z101-.] and its dual are of the same polarity, yet one properly contains the other. These observations show that the GE above is degenerate. This GE is a leaf in the GE tree. This branch of the tree has led us to a dead end.

This completes the consideration of root-2.1, as derived from the application of a print to root-2.

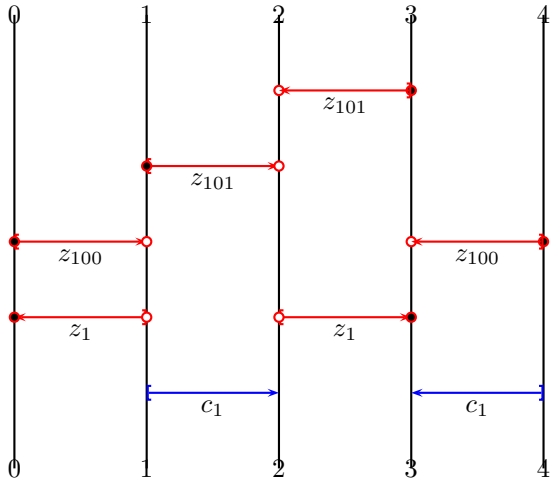
3 Cancellation scheme #3



z_1^{-1}	$1 \leftarrow 0$
c_1	$2 \leftarrow 1$
z_1	$1 \leftarrow 2$
c_1^{-1}	$0 \leftarrow 1$

Generalized Equation root-3

Below is the root GE obtained from the cancellation diagram above.



GE Information: Carrier: [0-1:z1-.] ; Carrier Dual: [2-3:z1+.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=3*<1=2*
```

We proceed.

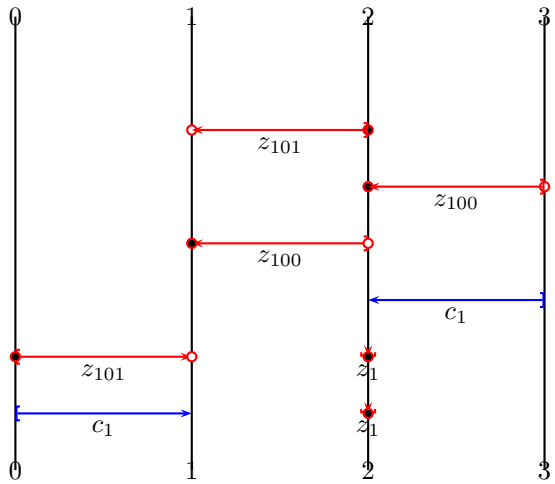
Generalized Equation root-3.1

We begin from the GE root-3 (see pp. 9). We consider its print

```
Print 1: =0=3*<1=2*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z1-.] to (new) boundaries 3 - 2.
Step 2: Moved (old) base [0-1:z100+.] to (new) boundaries 3 - 2.
Step 3: Collapsed (new) base [2-3:z1+.] to the empty base (3,3).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-3.1—is illustrated below:



GE Information: Carrier: [0-1:z101+.] ; Carrier Dual: [1-2:z101-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=2*<1=1*
```

This completes the consideration of root-3.1, as derived from the application of a print to root-3.

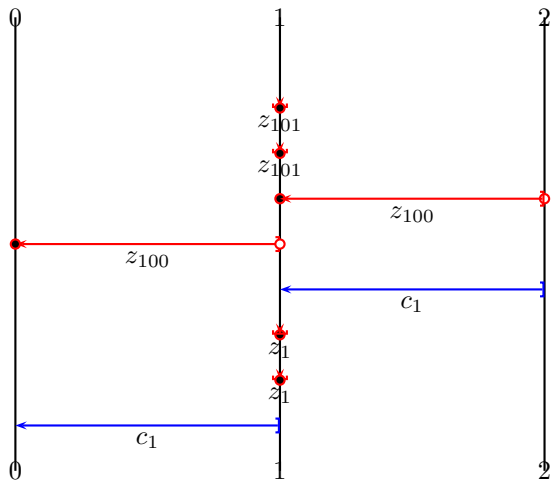
Generalized Equation root-3.1.1

We begin from the GE root-3.1 (see pp. 10). We consider its print

```
Print 1: =0=2*<1=1*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z101+.] to (new) boundaries 2 - 1.
Step 2: Moved (old) base [0-1:c1+.] to (new) boundaries 2 - 1.
Step 3: Collapsed (new) base [1-2:z101-.] to the empty base (2,2).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-3.1.1—is illustrated below:



GE Information: Carrier: [0-1:z100-.] ; Carrier Dual: [1-2:z100-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*
```

This completes the consideration of root-3.1.1, as derived from the application of a print to root-3.1.

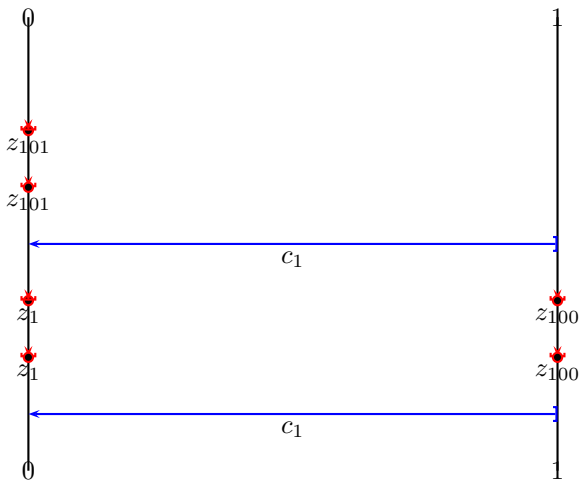
Generalized Equation root-3.1.1.1

We begin from the GE root-3.1.1 (see pp. 11). We consider its print

```
Print 1: =0=1*<1=2*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z100-.] to (new) boundaries 1 - 2.
Step 2: Moved (old) base [0-1:c1-.] to (new) boundaries 1 - 2.
Step 3: Collapsed (new) base [1-2:z100-.] to the empty base (2,2).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

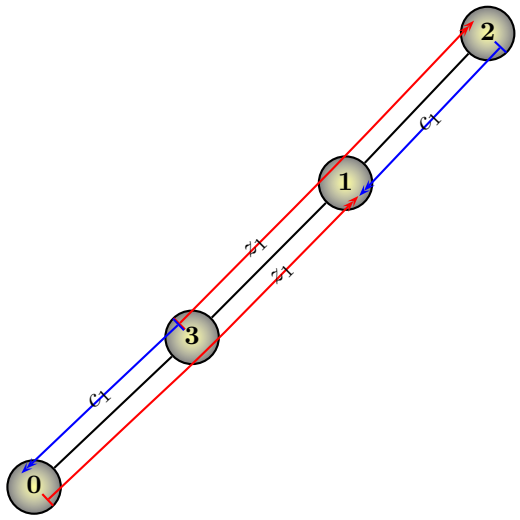
Upon applying the print, the GE we obtain—which we refer to as root-3.1.1.1—is illustrated below:



GE Information: Carrier: [0-0:z1+.] ; Carrier Dual: [0-0:z1+.] ; Critical Boundary: 0; The GE above is non-degenerate. This GE is a leaf in the GE tree. We have effectively found a solution!

This completes the consideration of root-3.1.1.1, as derived from the application of a print to root-3.1.1.

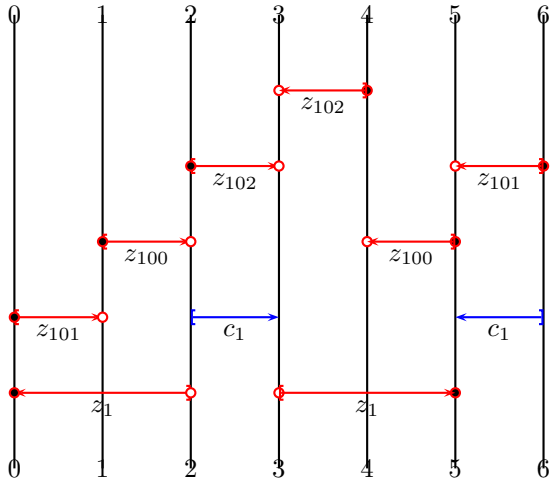
4 Cancellation scheme #4



z_1^{-1}	$1 \leftarrow 3 \leftarrow 0$
c_1	$2 \leftarrow 1$
z_1	$3 \leftarrow 1 \leftarrow 2$
c_1^{-1}	$0 \leftarrow 3$

Generalized Equation root-4

Below is the root GE obtained from the cancellation diagram above.



GE Information: Carrier: [0-2:z1-.] ; Carrier Dual: [3-5:z1+.] ; Critical Boundary: 2; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 3 valid prints (descendents).

It has 3 legal carrier-to-dual prints, as follows:

```
Print 1: =0=5*<1=4*<2=3*
Print 2: =0=5*<1<4*<2=3*
Print 3: =0=5*<4*<1<2=3*
```

We proceed.

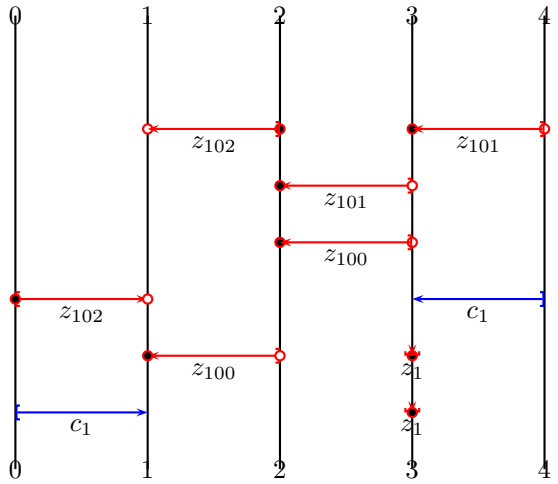
Generalized Equation root-4.1

We begin from the GE root-4 (see pp. 13). We consider its print

```
Print 1: =0=5*<1=4*<2=3*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-2:z1-.] to (new) boundaries 5 - 3.
Step 2: Moved (old) base [1-2:z100+.] to (new) boundaries 4 - 3.
Step 3: Moved (old) base [0-1:z101+.] to (new) boundaries 5 - 4.
Step 4: Collapsed (new) base [3-5:z1+.] to the empty base (5,5).
Step 5: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.
Step 6: Deleted (new) boundary 1 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-4.1—is illustrated below:



GE Information: Carrier: [0-1:z102+.] ; Carrier Dual: [1-2:z102-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=2*<1=1*
```

This completes the consideration of root-4.1, as derived from the application of a print to root-4.

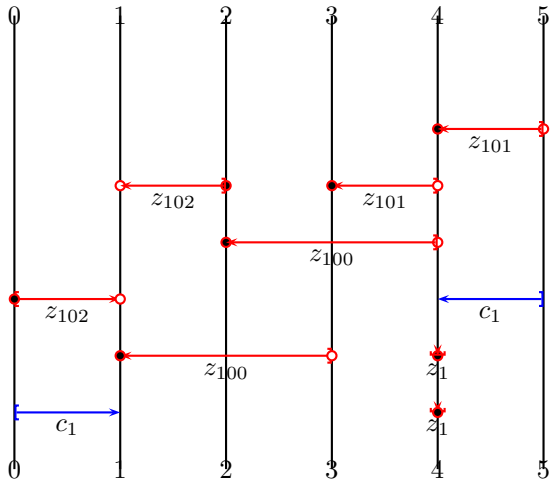
Generalized Equation root-4.2

We begin from the GE root-4 (see pp. 13). We consider its print

```
Print 2: =0=5*<1<4*<2=3*
```

Sequence of actions in performing the Print 2:
Step 1: Added (new) boundary 5.
Step 2: Moved (old) base [0-2:z1-.] to (new) boundaries 6 - 3.
Step 3: Moved (old) base [1-2:z100+.] to (new) boundaries 5 - 3.
Step 4: Moved (old) base [0-1:z101+.] to (new) boundaries 6 - 5.
Step 5: Collapsed (new) base [3-6:z1+.] to the empty base (6,6).
Step 6: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.
Step 7: Deleted (new) boundary 1 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-4.2—is illustrated below:



GE Information: Carrier: [0-1:z102+.] ; Carrier Dual: [1-2:z102-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=2*<1=1*
```

This completes the consideration of root-4.2, as derived from the application of a print to root-4.

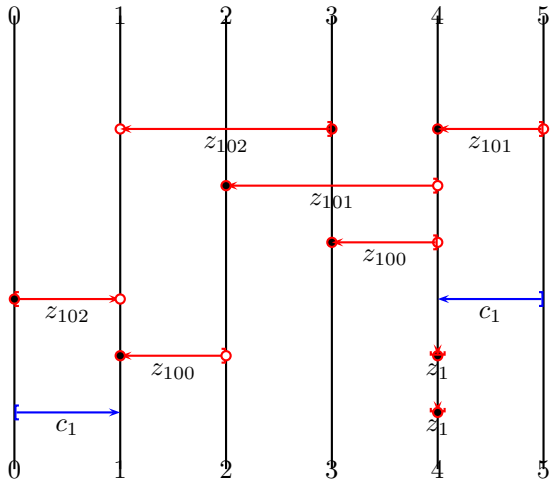
Generalized Equation root-4.3

We begin from the GE root-4 (see pp. 13). We consider its print

```
Print 3: =0=5*<4*<1<2=3*
```

Sequence of actions in performing the Print 3:
Step 1: Added (new) boundary 4.
Step 2: Moved (old) base [0-2:z1-.] to (new) boundaries 6 - 3.
Step 3: Moved (old) base [1-2:z100+.] to (new) boundaries 4 - 3.
Step 4: Moved (old) base [0-1:z101+.] to (new) boundaries 6 - 4.
Step 5: Collapsed (new) base [3-6:z1+.] to the empty base (6,6).
Step 6: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.
Step 7: Deleted (new) boundary 1 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-4.3—is illustrated below:



GE Information: Carrier: [0-1:z102+.] ; Carrier Dual: [1-3:z102-.] ; Critical Boundary: 1; Observe the following facts about this GE: The base [0-1:z102+.] has constraints with its dual that stretch the constant segment 0 - 1 to length different from 1. The base [4-5:z101-.] has constraints with its dual that stretch the constant segment 4 - 5 to length different from 1. These observations show that the GE above is degenerate. This GE is a leaf in the GE tree. This branch of the tree has led us to a dead end.

This completes the consideration of root-4.3, as derived from the application of a print to root-4.

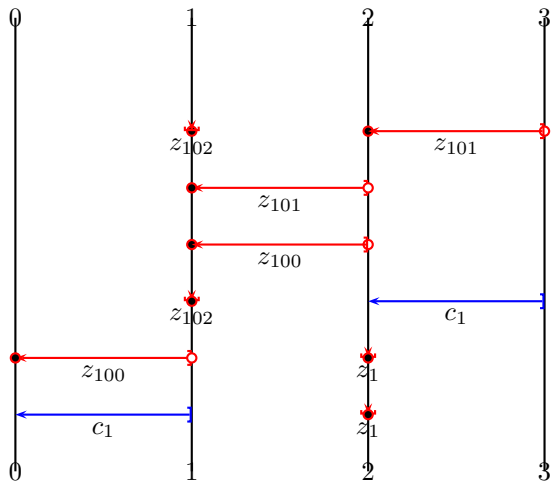
Generalized Equation root-4.1.1

We begin from the GE root-4.1 (see pp. 14). We consider its print

```
Print 1: =0=2*<1=1*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z102+.] to (new) boundaries 2 - 1.
Step 2: Moved (old) base [0-1:c1+.] to (new) boundaries 2 - 1.
Step 3: Collapsed (new) base [1-2:z102-.] to the empty base (2,2).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-4.1.1—is illustrated below:



GE Information: Carrier: [0-1:z100-.] ; Carrier Dual: [1-2:z100-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE

tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*
```

This completes the consideration of root-4.1.1, as derived from the application of a print to root-4.1.

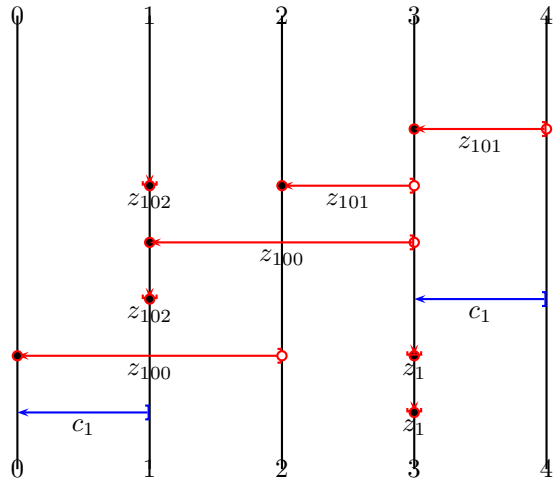
Generalized Equation root-4.2.1

We begin from the GE root-4.2 (see pp. 15). We consider its print

```
Print 1: =0=2*<1=1*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z102+.] to (new) boundaries 2 - 1.
Step 2: Moved (old) base [0-1:c1+.] to (new) boundaries 2 - 1.
Step 3: Collapsed (new) base [1-2:z102-.] to the empty base (2,2).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-4.2.1—is illustrated below:



GE Information: Carrier: [0-2:z100-.] ; Carrier Dual: [1-3:z100-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 2 valid prints (descendents).

It has 2 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*<2=3*
Print 2: =0=1*<1<2*<2=3*
```

This completes the consideration of root-4.2.1, as derived from the application of a print to root-4.2.

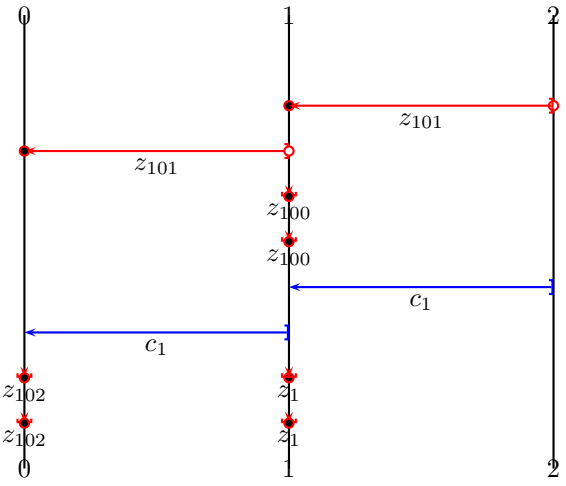
Generalized Equation root-4.1.1.1

We begin from the GE root-4.1.1 (see pp. 17). We consider its print

```
Print 1: =0=1*<1=2*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z100-.] to (new) boundaries 1 - 2.
Step 2: Moved (old) base [0-1:c1-.] to (new) boundaries 1 - 2.
Step 3: Collapsed (new) base [1-2:z100-.] to the empty base (2,2).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-4.1.1.1—is illustrated below:



GE Information: Carrier: [0-1:z101-.] ; Carrier Dual: [1-2:z101-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*
```

This completes the consideration of root-4.1.1.1, as derived from the application of a print to root-4.1.1.

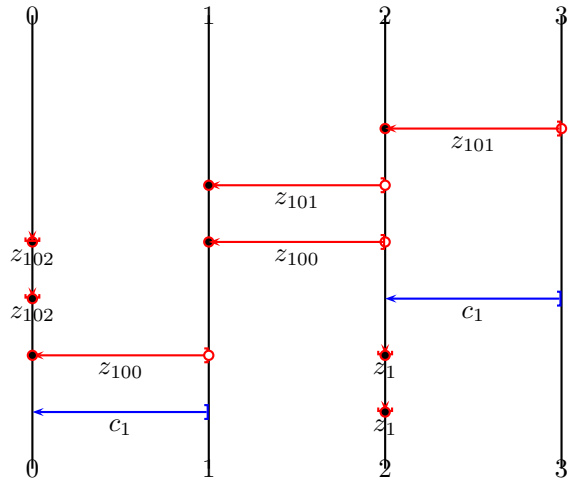
Generalized Equation root-4.2.1.1

We begin from the GE root-4.2.1 (see pp. 18). We consider its print

```
Print 1: =0=1*<1=2*<2=3*
```

Sequence of actions in performing the Print 1:
Step 1: Deleted constraint between boundary 0 in (old) base [0-2:z100-.] and boundary 1 in its dual.
Step 2: Moved (old) base [0-1:c1-.] to (new) boundaries 1 - 2.
Step 3: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-4.2.1.1—is illustrated below:



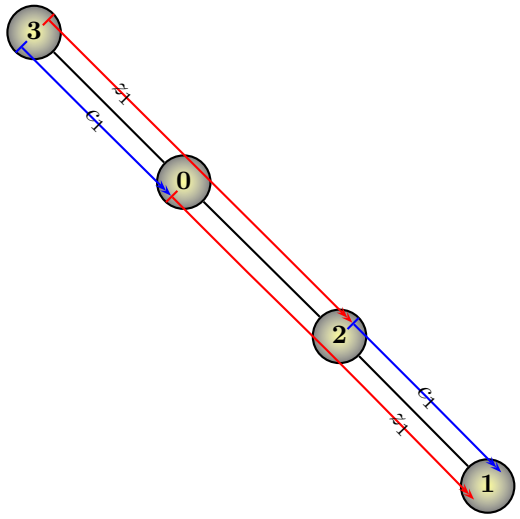
GE Information: Carrier: [0-1:z100-.] ; Carrier Dual: [1-2:z100-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*
```

This completes the consideration of root-4.2.1.1, as derived from the application of a print to root-4.2.1.

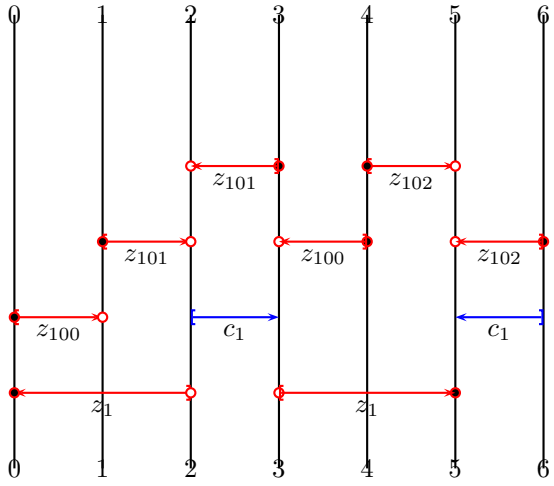
5 Cancellation scheme #5



z_1^{-1}	$1 \leftarrow 2 \leftarrow 0$
c_1	$2 \leftarrow 1$
z_1	$3 \leftarrow 0 \leftarrow 2$
c_1^{-1}	$0 \leftarrow 3$

Generalized Equation root-5

Below is the root GE obtained from the cancellation diagram above.



GE Information: Carrier: [0-2:z1-.] ; Carrier Dual: [3-5:z1+.] ; Critical Boundary: 2; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 3 valid prints (descendents).

It has 3 legal carrier-to-dual prints, as follows:

```
Print 1: =0=5*<1=4*<2=3*
Print 2: =0=5*<1<4*<2=3*
Print 3: =0=5*<4*<1<2=3*
```

We proceed.

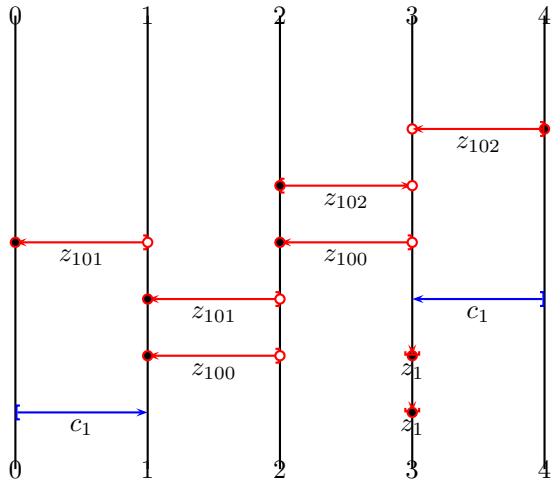
Generalized Equation root-5.1

We begin from the GE root-5 (see pp. 21). We consider its print

```
Print 1: =0=5*<1=4*<2=3*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-2:z1-.] to (new) boundaries 5 - 3.
Step 2: Moved (old) base [0-1:z100+.] to (new) boundaries 5 - 4.
Step 3: Moved (old) base [1-2:z101+.] to (new) boundaries 4 - 3.
Step 4: Collapsed (new) base [3-5:z1+.] to the empty base (5,5).
Step 5: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.
Step 6: Deleted (new) boundary 1 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-5.1—is illustrated below:



GE Information: Carrier: [0-1:z101-.] ; Carrier Dual: [1-2:z101-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*
```

This completes the consideration of root-5.1, as derived from the application of a print to root-5.

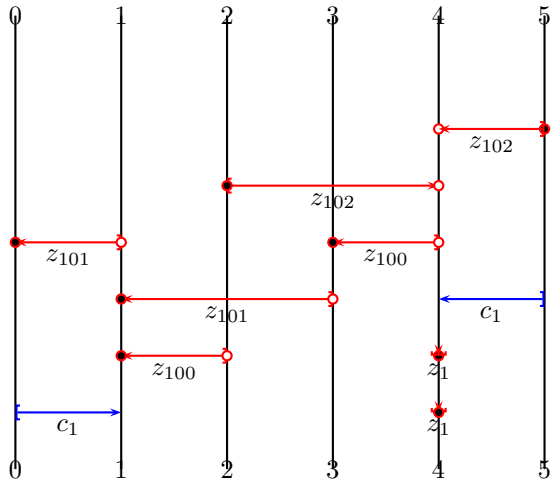
Generalized Equation root-5.2

We begin from the GE root-5 (see pp. 21). We consider its print

```
Print 2: =0=5*<1<4*<2=3*
```

Sequence of actions in performing the Print 2:
Step 1: Added (new) boundary 5.
Step 2: Moved (old) base [0-2:z1-.] to (new) boundaries 6 - 3.
Step 3: Moved (old) base [0-1:z100+.] to (new) boundaries 6 - 5.
Step 4: Moved (old) base [1-2:z101+.] to (new) boundaries 5 - 3.
Step 5: Collapsed (new) base [3-6:z1+.] to the empty base (6,6).
Step 6: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.
Step 7: Deleted (new) boundary 1 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-5.2—is illustrated below:



GE Information: Carrier: [0-1:z101-.] ; Carrier Dual: [1-3:z101-.] ; Critical Boundary: 1; Observe the following facts about this GE: The base [0-1:z101-.] has constraints with its dual that stretch the constant segment 0 - 1 to length different from 1. The base [4-5:z102-.] has constraints with its dual that stretch the constant segment 4 - 5 to length different from 1. These observations show that the GE above is degenerate. This GE is a leaf in the GE tree. This branch of the tree has led us to a dead end.

This completes the consideration of root-5.2, as derived from the application of a print to root-5.

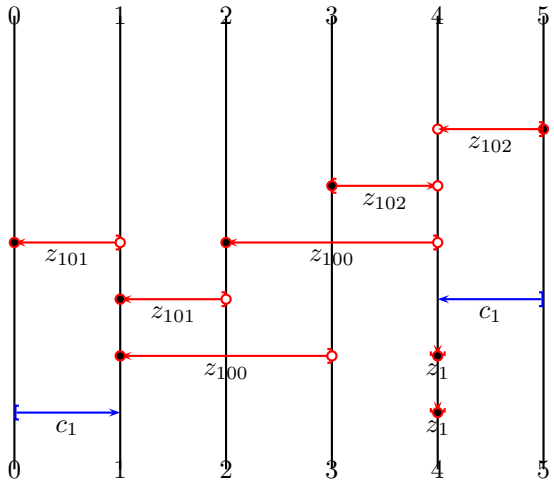
Generalized Equation root-5.3

We begin from the GE root-5 (see pp. 21). We consider its print

```
Print 3: =0=5*<4*<1<2=3*
```

Sequence of actions in performing the Print 3:
Step 1: Added (new) boundary 4.
Step 2: Moved (old) base [0-2:z1-.] to (new) boundaries 6 - 3.
Step 3: Moved (old) base [0-1:z100+.] to (new) boundaries 6 - 4.
Step 4: Moved (old) base [1-2:z101+.] to (new) boundaries 4 - 3.
Step 5: Collapsed (new) base [3-6:z1+.] to the empty base (6,6).
Step 6: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.
Step 7: Deleted (new) boundary 1 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-5.3—is illustrated below:



GE Information: Carrier: [0-1:z101-.] ; Carrier Dual: [1-2:z101-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*
```

This completes the consideration of root-5.3, as derived from the application of a print to root-5.

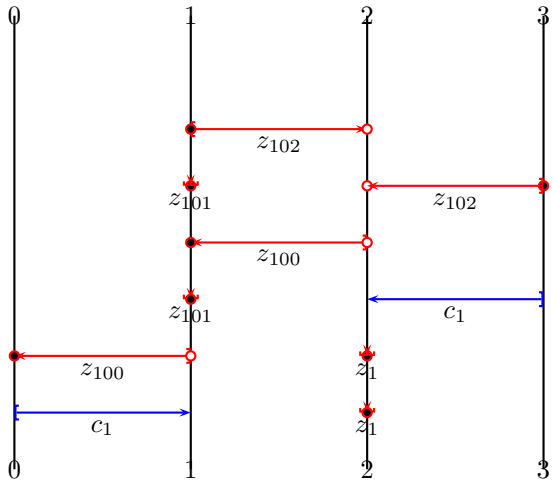
Generalized Equation root-5.1.1

We begin from the GE root-5.1 (see pp. 22). We consider its print

```
Print 1: =0=1*<1=2*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z101-.] to (new) boundaries 1 - 2.
Step 2: Moved (old) base [0-1:c1+.] to (new) boundaries 1 - 2.
Step 3: Collapsed (new) base [1-2:z101-.] to the empty base (2,2).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-5.1.1—is illustrated below:



GE Information: Carrier: [0-1:z100-.] ; Carrier Dual: [1-2:z100-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*
```

This completes the consideration of root-5.1.1, as derived from the application of a print to root-5.1.

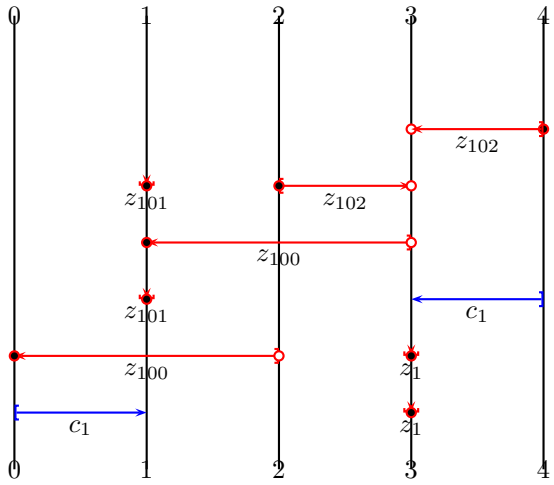
Generalized Equation root-5.3.1

We begin from the GE root-5.3 (see pp. 24). We consider its print

```
Print 1: =0=1*<1=2*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z101-.] to (new) boundaries 1 - 2.
Step 2: Moved (old) base [0-1:c1+.] to (new) boundaries 1 - 2.
Step 3: Collapsed (new) base [1-2:z101-.] to the empty base (2,2).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-5.3.1—is illustrated below:



GE Information: Carrier: [0-2:z100-.] ; Carrier Dual: [1-3:z100-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 2 valid prints (descendents).

It has 2 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*<2=3*
Print 2: =0=1*<1<2*<2=3*
```

This completes the consideration of root-5.3.1, as derived from the application of a print to root-5.3.

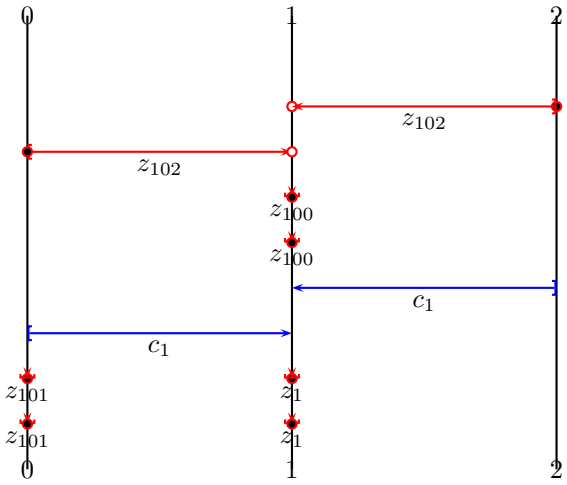
Generalized Equation root-5.1.1.1

We begin from the GE root-5.1.1 (see pp. 25). We consider its print

```
Print 1: =0=1*<1=2*
```

Sequence of actions in performing the Print 1:
Step 1: Moved (old) base [0-1:z100-.] to (new) boundaries 1 - 2.
Step 2: Moved (old) base [0-1:c1+.] to (new) boundaries 1 - 2.
Step 3: Collapsed (new) base [1-2:z100-.] to the empty base (2,2).
Step 4: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-5.1.1.1—is illustrated below:



GE Information: Carrier: [0-1:z102+.] ; Carrier Dual: [1-2:z102-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=2*<1=1*
```

This completes the consideration of root-5.1.1.1, as derived from the application of a print to root-5.1.1.

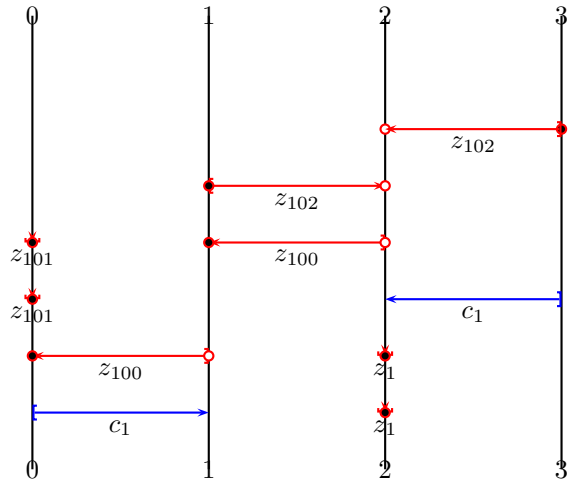
Generalized Equation root-5.3.1.1

We begin from the GE root-5.3.1 (see pp. 26). We consider its print

```
Print 1: =0=1*<1=2*<2=3*
```

Sequence of actions in performing the Print 1:
Step 1: Deleted constraint between boundary 0 in (old) base [0-2:z100-.] and boundary 1 in its dual.
Step 2: Moved (old) base [0-1:c1+.] to (new) boundaries 1 - 2.
Step 3: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-5.3.1.1—is illustrated below:



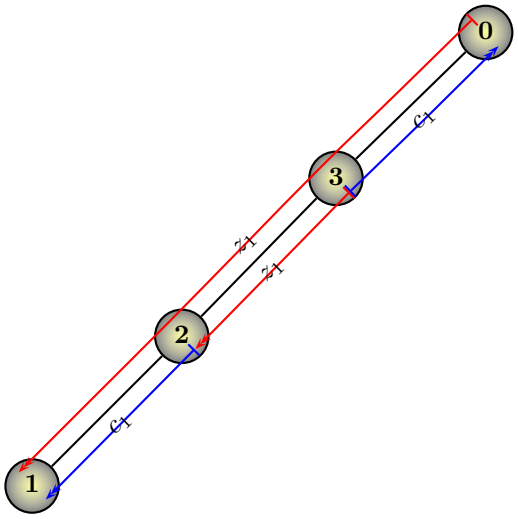
GE Information: Carrier: [0-1:z100-.] ; Carrier Dual: [1-2:z100-.] ; Critical Boundary: 1; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=1*<1=2*
```

This completes the consideration of root-5.3.1.1, as derived from the application of a print to root-5.3.1.

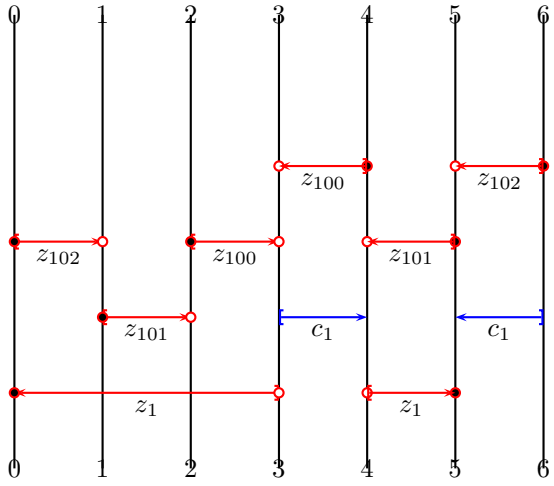
6 Cancellation scheme #6



z_1^{-1}	$1 \leftarrow 2 \leftarrow 3 \leftarrow 0$
c_1	$2 \leftarrow 1$
z_1	$3 \leftarrow 2$
c_1^{-1}	$0 \leftarrow 3$

Generalized Equation root-6

Below is the root GE obtained from the cancellation diagram above.



GE Information: Carrier: [0-3:z1-.] ; Carrier Dual: [4-5:z1+.] ; Critical Boundary: 3; The GE above is non-degenerate. This GE is *not* a leaf in the GE tree. It has 1 valid prints (descendents).

It has 1 legal carrier-to-dual prints, as follows:

```
Print 1: =0=5*<1<2<3=4*
```

We proceed.

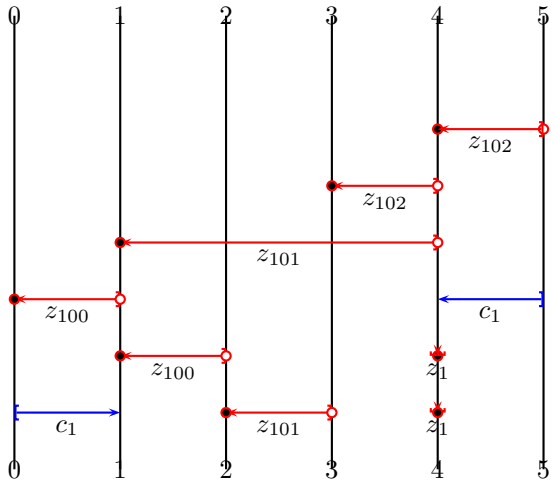
Generalized Equation root-6.1

We begin from the GE root-6 (see pp. 30). We consider its print

```
Print 1: =0=5*<1<2<3=4*
```

Sequence of actions in performing the Print 1:
Step 1: Added (new) boundary 5.
Step 2: Added (new) boundary 6.
Step 3: Moved (old) base [0-3:z1-.] to (new) boundaries 7 - 4.
Step 4: Moved (old) base [2-3:z100+.] to (new) boundaries 5 - 4.
Step 5: Moved (old) base [1-2:z101+.] to (new) boundaries 6 - 5.
Step 6: Moved (old) base [0-1:z102+.] to (new) boundaries 7 - 6.
Step 7: Collapsed (new) base [4-7:z1+.] to the empty base (7,7).
Step 8: Deleted (new) boundary 0 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.
Step 9: Deleted (new) boundary 1 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.
Step 10: Deleted (new) boundary 2 because it is not used inside any base. This will cause renumbering of higher numbered boundaries.

Upon applying the print, the GE we obtain—which we refer to as root-6.1—is illustrated below:



GE Information: Carrier: [0-1:z100-.] ; Carrier Dual: [1-2:z100-.] ; Critical Boundary: 1; Observe the following facts about this GE: The base [2-3:z101-.] and its dual are of the same polarity, yet one properly contains the other. The base [1-4:z101-.] and its dual are of the same polarity, yet one properly contains the other. These observations show that the GE above is degenerate. This GE is a leaf in the GE tree. This branch of the tree has led us to a dead end.

This completes the consideration of root-6.1, as derived from the application of a print to root-6.

7 Acknowledgements

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