Problem Section 8

Learning Outcomes

The problems are designed to build conceptual understanding and problem-solving skills. The emphasis is on learning to find, evaluate and build confidence. The specific tasks include:

- Find probabilities using a PDF.

- Derive the PDF from a CDF

- Calculate probabilities in the uniform and exponential distributions

- Back up and support work with relevant explanations

Exercises

1. Semester grades (expressed as a fraction of points) in a Physics for Poets class are decribed fairly well by the PDF
$$f(x) = 6x \cdot (1 - x), \qquad 0 \leq x < 1.$$
Anyone earning less than 60% fails. Five housemates are among the students enrolled in the class. Suppose the random variable X denotes the number (out of the 5 housemates) who fail the class.

a. What is the distribution of X? State the name as well as the values for the parameters. (What assumption are you making? Do you think it is valid here?)

$X \sim Binom(n = 5, prob = \pi)$ where

$$\begin{aligned}
\pi &= P(X \leq 0.6) \\
&= \int_0^{0.6} f(x)dx, \\
&= \int_0^{0.6} 6x(1-x)dx \\
&= \frac{6x^2}{2} - \frac{6x^3}{3}\Big|_0^{0.6} \\
&= \frac{6 \times .6^2}{2} - \frac{6 \times .6^3}{3}
\end{aligned}$$

This is calculated and reported below.

```
prob_fail <- 6*0.6^2/2 - 6*0.6^3/3
prob_fail
```

```
## [1] 0.648
```

We need to assume that the housemates are independent of each other. This may be unreasonable to assume if they are friends and have similar study habits/work ethic.

b. The number of housemates who fail will be ___ give or take ___. Fill in the first blank with $E[X]$ and the second blank with the $SD[X]$.

The mean of a binomial random variable is

$$E[X] = n \times \pi = 3.24.$$

The standard deviation of a binomial random variable is

$$SD[X] = \sqrt{n \times \pi \times (1-\pi)} = 1.0679326.$$

2. Suppose X is a random variable with CDF

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^{3/2}} & x \geq 0 \end{cases}$$

Determine $f(x)$, the PDF of X.

By the Fundamental Theorem of Calculus, we have

$$f(x) = \frac{d}{dx} F(x).$$

For $x < 0$, clearly $f(x) = 0$. When $x > 0$, we can use the chain rule of differentiation to get

$$\begin{aligned} f(x) &= -e^{-x^{3/2}} \times \frac{d}{dx} - x^{3/2}, \\ &= e^{-x^{3/2}} \times \frac{3}{2} x^{3/2-1} \\ &= \frac{3}{2}\sqrt{x} \cdot e^{-x^{3/2}} \end{aligned}$$

When $x = 0$, we can see that the left hand and right hand derivatives are both 0, and therefore we can write

$$f(x) = \frac{3}{2}\sqrt{x} \cdot e^{-x^{3/2}}, \qquad x \geq 0.$$

3. Suppose X is uniformly distributed over $[0, 1)$. That is

$$f(x) = 1 \quad 0 \leq x < 1.$$

Show that

$$P(-\ln(X) \leq 5) = 1 - e^{-5}$$

where $\ln(X)$ denotes denotes the natural logarithm of X.

$$\begin{aligned} P(-\ln(X) \leq 5) &= P(X \geq e^{-5}), \\ &= 1 - P(X < e^{-5}), \\ &= 1 - \int_{-\infty}^{e^{-5}} f(x)\, dx, \\ &= 1 - \int_{0}^{e^{-5}} dx, \\ &= 1 - x\big|_0^{e^{-5}} \qquad\qquad = 1 - e^{-5} \end{aligned}$$

4. Among the most famous of all meteor showers are the Perseids, which occur each year in early August. In some areas the frequency of visible Perseids can be as high as forty per hour. Assume that sightings occur according to a Poisson process.

a. Let $X(t)$ denote the number of sightings in t hours. Then we are given $X(t) \sim Pois(40t)$. Calculate the probability using the Poisson model that an observer who has just seen a meteor will have to wait at least five minutes before seeing another one.

The PMF of $X(t) \sim Pois(40t)$ is given by

$$P(X(t) = x) = e^{-40t} \times \frac{(40t)^x}{x!}, \quad x = 0, 1, 2,$$

Since 5 minutes is equal to $\frac{1}{12}$ of an hour, we can say that

$$X(\frac{1}{12}) \sim Pois(40 \times \frac{1}{12})$$

and we are interested in calculating $P\left(X(\frac{1}{12}) = 0\right)$

Plugging into the PMF gives $e^{-40/12}$. This is calculated manually and also using the function `dpois`.

```
exp(-40/12)
```

```
## [1] 0.03567399
```

```
dpois(0, lambda = 40/12)
```

```
## [1] 0.03567399
```

b. Let the random variable Y denote the length of time (in hours) between consecutive sightings of the meteor.

i. What is the distribution of Y? State the name and also show its PDF.

$Y \sim Exp(40)$. The PDF of Y is

$$f(y) = 40e^{-40\,y}, \quad y \geq 0.$$

ii. Repeat the calculation from part a but using the distribution of Y.

Here we want the probability that $Y > \frac{1}{12}$. Since the CDF of Y is

$$F(y) = P(Y \leq y) = 1 - e^{-40y}$$

we can say that

$$P(Y > \frac{1}{12}) = 1 - F(1/12) = e^{-40/12}$$

This is calculated below.

```
1 - pexp(q = 1/12, rate = 40)
```

```
## [1] 0.03567399
```