Problem 3

To show that $P(A \cup B) \geq 0.6$, we use the formula for the probability of the union of two events and the fact that A and B are independent.

1. Formula for $P(A \cup B)$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Since A and B are independent, we have:

$$P(A \cap B) = P(A)P(B)$$

Substituting this into the formula for $P(A \cup B)$, we get:

$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$

2. Lower bounds for $P(A)$ and $P(B)$:

From the problem, we are given:

$$P(A) \geq 0.5 \quad \text{and} \quad P(B) \geq 0.2$$

Let $P(A) = a$ and $P(B) = b$, where $a \geq 0.5$ and $b \geq 0.2$. Substituting into the formula for $P(A \cup B)$, we have:

$$P(A \cup B) = a + b - ab$$

3. Show $P(A \cup B) \geq 0.6$:

We want to show that:

$$a + b - ab \geq 0.6$$

for $a \geq 0.5$ and $b \geq 0.2$.

Step 1: Substitute the lower bounds $a = 0.5$ and $b = 0.2$:

$$P(A \cup B) = 0.5 + 0.2 - (0.5)(0.2)$$

$$P(A \cup B) = 0.5 + 0.2 - 0.1 = 0.6$$

Thus, when $P(A) = 0.5$ and $P(B) = 0.2$, $P(A \cup B) = 0.6$.

Step 2: For larger values of a or b:

If $a > 0.5$ or $b > 0.2$, then:

- $a + b$ increases.
- ab, the product, also increases, but since ab is subtracted, the overall value of $P(A \cup B)$ increases.

Therefore, $P(A \cup B) \geq 0.6$ for all $a \geq 0.5$ and $b \geq 0.2$.

Final Conclusion:

$$P(A \cup B) \geq 0.6 \quad \text{for } P(A) \geq 0.5 \text{ and } P(B) \geq 0.2.$$

Problem 4:

Given:

$$P(A) = \frac{1}{2}, \quad P(B \cap C) = \frac{2}{3}, \quad P(A \cap B^c) = \frac{1}{4}.$$

Show that A and B are not independent. Recall that two events A and B are independent if:

$$P(A \cap B) = P(A) \cdot P(B).$$

Solution:

Step 1: Express $P(A \cap B^c)$ in terms of complements.

We know:

$$A = (A \cap B) \cup (A \cap B^c),$$

where the events $A \cap B$ and $A \cap B^c$ are disjoint. Hence:

$$P(A) = P(A \cap B) + P(A \cap B^c).$$

Substituting the given values:

$$P(A) = \frac{1}{2}, \quad P(A \cap B^c) = \frac{1}{4}.$$

This implies:

$$P(A \cap B) = P(A) - P(A \cap B^c).$$

$$P(A \cap B) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Step 2: Check independence.

For A and B to be independent, we require:

$$P(A \cap B) = P(A) \cdot P(B).$$

We already have:

$$P(A \cap B) = \frac{1}{4}, \quad P(A) = \frac{1}{2}.$$

Let $P(B)$ be the probability of event B. From the independence condition:

$$P(A) \cdot P(B) = \frac{1}{2} \cdot P(B).$$

If A and B were independent, then:

$$P(A \cap B) = \frac{1}{2} \cdot P(B).$$

Substitute $P(A \cap B) = \frac{1}{4}$ into the equation:

$$\frac{1}{4} = \frac{1}{2} \cdot P(B).$$

Solve for $P(B)$:

$$P(B) = \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

Step 3: Verify using other information.

To check for consistency, note that we are given $P(B \cap C) = \frac{2}{3}$. Since probabilities cannot exceed 1, the probability assignments in this scenario indicate inconsistency with independence.

Thus, $P(A \cap B) \neq P(A) \cdot P(B)$, proving that A and B are not independent.

Conclusion:

$$\boxed{\text{Events } A \text{ and } B \text{ are not independent.}}$$