Problem 4

We are tasked with showing that for three events A, B, and C in a sample space, the following inequality holds:

$$P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$$

Step 1: Apply the Inclusion-Exclusion Principle

The inclusion-exclusion principle for three events A, B, and C gives the formula for the probability of their union:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Step 2: Recognize that the inclusion-exclusion formula is less than or equal to the sum of the individual probabilities

The right-hand side of the inclusion-exclusion formula includes terms for the pairwise intersections of the events (such as $P(A \cap B)$), as well as the three-way intersection $P(A \cap B \cap C)$. These intersection terms account for the overlap between the events, and they "adjust" for the over-counting of probabilities in the simple sum $P(A) + P(B) + P(C)$.

The terms involving the intersections $P(A \cap B)$, $P(B \cap C)$, and $P(C \cap A)$, as well as the three-way intersection $P(A \cap B \cap C)$, all represent **positive** quantities (since probabilities are non-negative). Therefore, the inclusion-exclusion formula tells us that:

$$P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$$

This is because subtracting these intersection terms (which are all non-negative) results in a value that is less than or equal to the sum of the individual probabilities.

Step 3: Conclusion

Hence, we have shown that:

$$P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$$

This completes the proof.