Homework 7

Autumn 2022

KEY

Instructions

- This homework is due in Gradescope on Wednesday Nov 23 by midnight PST.
- Please answer the following questions in the order in which they are posed.
- Don't forget to knit the document frequently to make sure there are no compilation errors.
- When you are done, download the PDF file as instructed in section and submit it in Gradescope.

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1. A tool-and-die company makes castings for steel stress-monitoring gauges. Their annual profit Q is a function of demand X, given by

$$t(X) = 1 - exp(-2X).$$

The demand X has the PDF

$$f(x) = 6e^{-6x} \quad x > 0.$$

a. Interpret the number "6" in the PDF of X. This means, describe in your own words what this tells you (in context) about the demand X.

We have that 6 is the rate parameter of the demand/year for castings on steel stress-monitoring gauges. Using the fact that

$$E\left[X\right] = \frac{1}{\lambda}$$

we can therefore say that they should expect a demand of 1/6 (unspecified units of demand) castings per year.

b. Find $E\left[t(X)\right]$. State any rules you use and show your work clearly.

$$
\begin{aligned}
E[t(X)] &= \int_0^\infty t(x)f(x)dx \text{ by the law of the Unconcious probabilist}\\
&= \int_0^\infty (1-exp(-2x))6e^{-6x}dx\\
&= \int_0^\infty (6e^{-6x} - 6e^{-2x-6x})dx\\
&= \int_0^\infty (6e^{-6x})dx - \int_0^\infty (6e^{-2x-6x})dx\\
&= 1 - \int_0^\infty (6e^{-8x})dx \text{ since the first term is the integration of the PDF f(x)}\\
&= 1 - 6(\frac{-1}{8}e^{-8x}|_0^\infty)\\
&= 1 - 6(0 - \frac{-1}{8})\\
&= 1 - \frac{6}{8}\\
&= \frac{1}{4}
\end{aligned}
$$

2. Suppose the duration of a phone call, X, can be described probabilistically by

$$P(X > x) = ae^{-\lambda_1 x} + (1-a)e^{-\lambda_2 x},\ 0 < x < \infty$$

where a, λ_1 and λ_2 are constants, $0 < a < 1,\ \lambda_1, \lambda_2 > 0$.

a. Write the CDF of X by filling in the question marks in align environment below. Justify your answers below.

$$
\begin{aligned}
F(x) &= P(X \leq x)\\
&= \begin{cases} 0 & x < 0 \\ 1 - [ae^{-\lambda_1 x} + (1-a)e^{-\lambda_2 x}] & x \geq 0. \end{cases}
\end{aligned}
$$

We have that our random variable X takes values of X=$x > 0$. Thus for x<0, P(X<x) = 0, since a phone call can never be less than 0 minutes.

For $x \geq 0$ we may find:

$$P(X < x) = 1 - P(X > x) = 1 - [ae^{-\lambda_1 x} + (1-a)e^{-\lambda_2 x}]$$

b. Find a PDF, $f(x)$, for x. Show your work. (Don't forget to mention the support)

We know the PDF, $f(x)$ will simply be the derivative of the CDF, $F(x) = P(X < x)$.

We have for $x < 0$, F(x) = 0, so $f(x) = 0$ for $x < 0$

For $x \geq 0$ we must find $\frac{d}{dx}(1 - [ae^{-\lambda_1 x} + (1-a)e^{-\lambda_2 x}])$.

We have that as:

$$
\begin{gathered}
\frac{d}{dx}(1 - [ae^{-\lambda_1 x} + (1-a)e^{-\lambda_2 x}]) =\\
a(\lambda_1 e^{-\lambda_1 x}) + (1-a)(\lambda_2 e^{-\lambda_2 x})
\end{gathered}
$$

Thus we have:

$$f(x) = \frac{d}{dx}F(x)$$
$$= \begin{cases} 0 & x < 0 \\ a(\lambda_1 e^{-\lambda_1 x}) + (1-a)(\lambda_2 e^{-\lambda_2 x}) & x \geq 0. \end{cases}$$

c. Calculate the expected duration of the call. State the answer in a sentence that shows you understand the meaning of expected value. In other words, we are looking for more than "the expected value is xxx".

We have that:

$$\begin{aligned} E[X] &= \int_0^\infty x \cdot f(x)dx \\ &= \int_0^\infty x \cdot [a(\lambda_1 e^{-\lambda_1 x}) + (1-a)(\lambda_2 e^{-\lambda_2 x})]dx \\ &= a\int_0^\infty \lambda_1 x \cdot e^{-\lambda_1 x}dx + (1-a)\int_0^\infty \lambda_2 x \cdot e^{-\lambda_2 x}dx \\ &= a\frac{1}{\lambda_1} + (1-a)\frac{1}{\lambda_2} \end{aligned}$$

where the last sentence follows by using Lemma 12.1 and the expectation of a Exponential RV. The expected value tells us the average duration of a single phone call over many separate replications of the experiment.

d. By how much will the duration of a call typically vary from expected? Calculate the standard deviation $SD(X)$. (*Hint:* begin by finding

$$Var\left[X\right] = E\left[X^2\right] - \mu^2$$

where μ is your answer to part c.)

Hint: for parts c and d, you may use the results from Lemma 12.1 without proof but you have to provide complete citations to results you use. This means you cannot just say "from the result we saw in class". This is not how proofs are written.

First we may find $E(X^2)$.

We know that for a general $Z \sim exponential(\lambda)$ RV, we have:

$Var(Z) = \frac{1}{\lambda^2}$. Thus $E(Z^2) = Var(Z) + E(Z)^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{2}{\lambda^2}$ by Lemma 12.1 .

We may use this lemma to find $E(X^2)$ and thus derive the variance for this RV X.

We have:

$$\begin{aligned} E[X^2] &= \int_0^\infty x^2 \cdot f(x)dx \\ &= \int_0^\infty x^2 \cdot [a(\lambda_1 e^{-\lambda_1 x}) + (1-a)(\lambda_2 e^{-\lambda_2 x})]dx \\ &= a\int_0^\infty x^2 \lambda_1 e^{-\lambda_1 x}dx + (1-a)\int_0^\infty x^2 \lambda_2 e^{-\lambda_2 x}dx \\ &= a(\frac{2}{\lambda_1^2}) + (1-a)\frac{2}{\lambda_2^2} \text{by realizing the inside of the integrals are exponential PDF's and using Lemma 12.1} \end{aligned}$$

From here we want to use the variance formula:

$Var(X) = E[X^2] - E[X]^2 = a(\frac{2}{\lambda_1^2}) + (1-a)\frac{2}{\lambda_2^2} - [a\frac{1}{\lambda_1} + (1-a)\frac{1}{\lambda_2}]^2.$

Thus we have that the typical deviation of a call length from expected is:

$\sqrt{a(\frac{2}{\lambda_1^2}) + (1-a)\frac{2}{\lambda_2^2} - [a\frac{1}{\lambda_1} + (1-a)\frac{1}{\lambda_2}]^2}$ minutes.