Chapter 1

Set Theory

References: Pruim Appendix B.1, Larsen & Marx 2.2

The purpose of probability theory is to build mathematical models for experiments whose outcome is not predictable with certainty. We call such an experiment a **random experiment**.

The mathematics of probability can be expressed most naturally using the language of sets, and so we will begin with a review of basic definitions from set theory.

Definition 1.1. The set S, of all possible outcomes of a random experiment is called the **sample space**. It is also known as the outcome set.

Experiment	Sample space
Toss a coin once	$S = \{H; T\}$
Toss a coin twice	$S = \{(H,H); (H,T); (T,H); (T,T)\}$

....

Example 1.1. For each of the following "experiments", describe the sample space.

a. A local TV station advertises two news casting positions. If two women (W_1, W_2) and two men (M_1, M_2) apply, list all the possible teams of two co-anchors that can be formed?

b. A local TV station is seeking to hire a sports announcer and a weather forecaster. If two women (W_1, W_2) and two men (M_1, M_2) apply, list all the possible teams that can be formed?

c. Toss a coin repeatedly until the first head shows.

d. Let S be the set of right triangles with a 5" hypotenuse and whose height and length are a and b, respectively. Characterize the outcomes in S.

Once the sample space is defined, we can then start to consider subsets of possible outcomes, which are called events.

Definition 1.2. An **event** E is a collection of outcomes in S. That is, E is a subset of S, which is written in short hand as $E \subseteq S$.

Experiment	Sample space	Event
Toss two dice	$S = \left\{ \begin{matrix} (1,1) & (1,2) & \cdots & (1,6) \\ (2,1) & (2,2) & \cdots & (2,6) \\ \vdots & \vdots & \vdots & \vdots \\ (6,1) & (6,2) & \cdots & (6,6) \end{matrix} \right\}$	$E:$ Sum of numbers is 3 $\{(1,2),(2,1)\}$

Example 1.2. Consider the experiment of choosing coefficients for the quadratic equation

$$ax^2 + bx + c = 0.$$

Characterize the values of a, b, c associated with the event E: Equation has complex roots.

Associated with events on a sample space are several operations. They represent the ways in which one event can be combined with another to form new events.

UNION: The union of A and B, written $A \cup B$ is the set of elements that belong to either A or B or both:

$$A \cup B = \{x : x \in A \text{ or } x \in B\} .$$

The symbol $\in$ means “in” or “a member of”. Similarly $\notin$ means “not in” or “not a member of”.

INTERSECTION: The intersection of A and B, written $A \cap B$ is the set of elements that belong to both A and B:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Definition 1.3. Two events A and B are disjoint (or mutually exclusive) if $A \cap B = \emptyset$ where $\emptyset$ denotes the void or empty set (consisting of no elements).

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Example 1.3. A coin is tossed four times and the resulting sequence of heads and/or tails is recorded. Define the events:

E: exactly two heads appear

F: heads and tails alternate

G: first two tosses are heads

Which events, if any, are mutually exclusive?

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COMPLEMENTATION: The complement of A, written A^c is the set of elements that are in S but not in A:

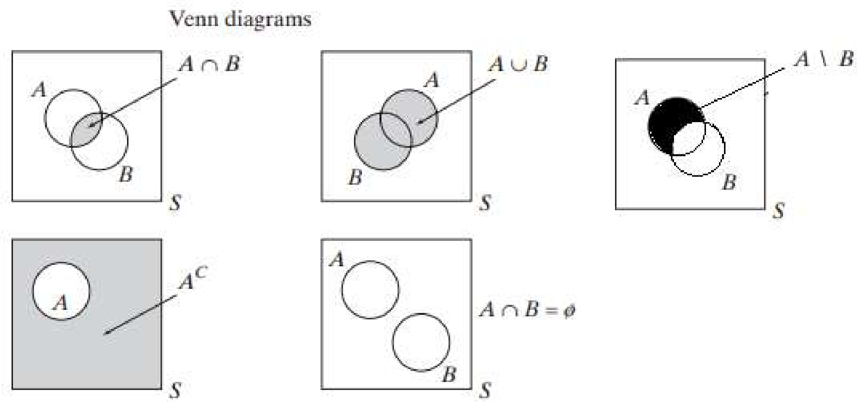
$$A^c = \{x \in S : x \notin A\}.$$

SET DIFFERENCE: The set difference of A and B, denoted by $A\ \ B$ is

$$\begin{aligned} A\ \ B &= (x \in S : x \in A \textit{ and } x \notin B\}, \\ &= A \cap B^c. \end{aligned}$$

The set difference $B\ \ A$ is like the complement of A but with respect to the set B.

Relationships based on two or more events can sometimes be hard to describe using only equations or verbal descriptions. An alternative approach that can be effective is to represent the events graphically in a format known as Venn diagrams. The following figure shows the Venn diagrams for a set union, intersection, complement and set difference. In each case the shaded interior represents the desired region.



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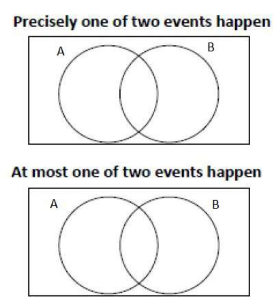
Example 1.4. Shade the events corresponding to the following descriptions. Then write them in terms of set operations.



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Example 1.5. Let events A and B and sample space S be defined as the following intervals:

$$\begin{aligned} S &= \{x : 0 \leq x < 7\} \\ A &= \{x : 0 < x < 5\} \\ B &= \{x : 3 \leq x < 7\} \end{aligned}$$

Characterize the following events:

a. A^c

b. $A \cap B$

c. $A \cup B$

d. $A \cap B^c$

e. $A^c \cup B$

f. $A^c \cap B^c$

. . . .

The definitions have been stated in terms of unions and intersections of pairs of events for simplicity of exposition, but they can be extended to arbitrary collections of sets.

For example, if $A_1, A_2, A_3, ...$ is a collection of sets, all defined on a sample space S, then

$$\begin{aligned} \bigcup_{k=1}^{\infty} A_k &= \{x \in S : x \in A_k, \text{ some } k\}, \\ \bigcap_{k=1}^{\infty} A_k &= \{x \in S : x \in A_k, \text{ for all } k\}. \end{aligned}$$

. . . .

Example 1.6. Let $S = (0, 1]$ and $A_k = (1/k, 1]$. What is $\bigcup_{k=1}^{\infty} A_k$? (Note: A_k are nested and increasing in the sense that $A_1 \subset A_2 \subset A_3 ...$)

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We will now state some useful properties of set operations.

Theorem 1.1. *For any three events E, F, and G defined on a sample space S, we have the following identities:*

1. *Commutative laws*
   - $E \cup F = F \cup E$
   - $E \cap F = F \cap E$
2. *Associative laws*
   - $E \cup (F \cup G) = (E \cup F) \cup G$
   - $E \cap (F \cap G) = (E \cap F) \cap G$
3. *Distributive laws*
   - $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$
   - $E \cup (F \cap G) = (E \cup F) \cap (E \cup G)$
4. *Identity laws*
   - $E \cap S = E$
   - $E \cup \phi = E$
5. *Complement laws*
   - $E \cup E^c = S$
   - $E \cap E^c = \phi$

The above laws display the following interesting pattern. Each of the identities stated above is one of a pair of identities, such that, each can be transformed into the other by interchanging $\cup$ and $\cap$, and also ϕ and S. Every valid proposition about the algebra of sets can be derived from the five fundamental laws stated above.

The following states an important one involving all three operators: unions, intersections and complements.

Theorem 1.2 (DeMorgan's Laws)**.** *For any two events E and F defined on a sample space S:*

- $(E \cup F)^c = E^c \cap F^c$
- $(E \cap F)^c = E^c \cup F^c$

Proof. We will walk through the proof of DeMorgan's first law. The second is left to you as an exercise. The proof is completed in two steps, by proving that each element in $(E \cup F)^c$ is also in $E^c \cap F^c$ and vice versa. Hence, the two sets must be identical.

Consider an element $x \in (E \cup F)^c$.

$$\begin{aligned} x \in (E \cup F)^c &\Rightarrow x \notin (E \cup F), \\ &\Rightarrow x \notin E \ AND \ x \notin F \\ &\Rightarrow x \in E^c \ AND \ x \in F^c \\ &\Rightarrow x \in (E^c \cap F^c). \end{aligned}$$

Since this holds for any arbitrary element x in $(E \cup F)^c$, we can say

$$(E \cup F)^c \subseteq E^c \cap F^c.$$

Let's now consider an element $y \in E^c \cap F^c$.

$$\begin{aligned} y \in E^c \cap F^c &\Rightarrow y \in E^c \ AND \ y \in F^c, \\ &\Rightarrow y \notin E \ AND \ y \notin F, \\ &\Rightarrow y \notin (E \cup F), \\ &\Rightarrow y \in (E \cup F)^c. \end{aligned}$$

Again, since y is any arbitrary element of $E^c \cap F^c$, we have the result

$$E^c \cap F^c \subseteq (E \cup F)^c$$

and hence

$$(E \cup F)^c = E^c \cap F^c$$

□

De Morgan's laws commonly apply to text searching using Boolean operators AND, OR, and NOT. Consider a set of documents containing the words "cars" and "trucks". De Morgan's laws hold that these two searches will return the same set of documents:

- Search A: NOT (cars OR trucks)
- Search B: (NOT cars) AND (NOT trucks)

1.1 Practice Problems

1. Suppose an archer shoots a dart randomly at a point in the half open unit interval $[0, 1)$.

   a. List the sample space for this experiment.

   b. The event "the number chosen is less than or equal to 1/3' ' is the subset