Chapter 12

Mean, variance and higher moments

References: Pruim 3.2

12.1 Mean and variance of a continuous random variable

The mean and variance of a continuous random variable are computed much like they are for discrete random variables, except that we replace summations with integration.

Definition 12.1. Let X be a continuous random variable with PDF f. Then

- $E[X] = \mu = \int_{-\infty}^{\infty} x f(x) dx$.
- $Var[X] = \sigma^2 = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$.

As with discrete distributions, the following simplifies the calculation of the variance.

$$Var[X] = E\left[X^2\right] - \mu^2.$$

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Example 12.1. Let $X \sim Unif(a, b)$. Then

- $E[X] = \frac{(a+b)}{2}$.
- $Var[X] = \frac{(b-a)^2}{12}$.

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Example 12.2. Suppose $X \sim Unif(0,1)$. Calculate the probability that X is more than 1 standard deviation from the mean.

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Lemma 12.1. *Suppose $X \sim Exp(\lambda)$. Then*

- $E[X] = \frac{1}{\lambda}$.
- $Var[X] = \frac{1}{\lambda^2}$.

Proof. We will use the technique of integration by parts

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

to prove the first result. To use this formula, we need to identify u and dv, and then compute du and v. Note that v is simply the integral of dv.

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x} dx.$$

Defining $u = x$ and $dv = \lambda e^{-\lambda x} dx$, we have

$$du = dx$$

and

$$v = \int \lambda e^{-\lambda x} dx = \lambda \cdot \frac{-e^{-\lambda x}}{\lambda} = -e^{-\lambda x}.$$

Therefore

$$\begin{aligned} E[X] &= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx \\ &= 0 + \frac{1}{\lambda}. \end{aligned}$$

The variance is found similarly using integration by parts twice to evaluate $\int_0^\infty x^2 \lambda e^{-\lambda x} dx$. □

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Example 12.3. Suppose $X \sim Exp(\lambda)$. Find the probability that X is within

1 standard deviation of the mean.

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We have already proved the following claims for discrete random variables. They are also true for continuous random variables.

Lemma 12.2. *Let X be a continuous random variable with PDF f, and a and b are numbers. Then*

- $E\left[t(X)\right] = \int\limits_{-\infty}^{\infty} t(x) \cdot f(x)dx$
- $E\left[aX + b\right] = aE\left[X\right] + b$
- $Var\left[aX + b\right] = a^2 Var\left[X\right]$

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Example 12.4. A parking garage charges a flat fee of \$10 for the first hour (or fraction thereof) and any additional time at a rate of \$8 per hour. Suppose the time, in hours, that we park in this lot is an exponential random variable with $\lambda = 1$.

Suppose the time, X (in hours), that we park in this lot is an exponential random variable with $\lambda = 1$. Let the random variable Y denote the cost (in dollars) that we will pay to park in the garage.

a. How does Y relate to X?

b. What is our expected cost to park? That is, find $E[Y]$.

12.2 Practice Problems

1. The PDF of X is
$$f(x) = (a + bx^2), \quad 0 \leq x < 1.$$
If $E[X] = \frac{3}{5}$, find a and b.

2. Let $f(x) = \frac{3}{8}x^2 \quad 0 \leq x < 2$. Find $E[X]$. Also find $E\left[\frac{1}{X}\right]$.

3. Let X be a random variable with CDF
$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \begin{cases} 0 & x < 0 \\ \frac{x^{3/2}}{8} & 0 \leq x < 4 \\ 1 & 4 \leq x \end{cases} \end{aligned}$$

Find $E[X]$.

12.3 Higher Moments: skewness

The expected values $E[X]$ and $E[X^2]$ are examples of **moments** of a random variable and its distribution. They are called the first and second moment about the origin.

The variance $E\left[(X - \mu)^2\right]$ is an example of a **central moment** or moment about the mean.

Higher moments describe additional features of the shape of a distribution. For instance,
$$E\left[(X - \mu)^3\right]$$
is zero for symmetric distributions and non-zero for asymmetric/skewed distributions, and therefore is often used as a measure of **skewness**. It is positive

when the distribution is skewed to the right and negative when it is skewed to the left. Similarly,

$$E\left[(X-\mu)^4\right]$$

strongly emphasizes values that are far from the mean. It is therefore used to study the tail behavior of a distribution.