Chapter 12

Mean, variance and higher moments

Mean and variance

Definition 12.1 The mean and variance of a continuous random variable are computed much like they are for discrete random variables, except that we replace summations with integration. Let X be a continuous random variable with PDF f. Then

- $E[X] = \mu = \int\limits_{-\infty}^{\infty} xf(x)dx.$
- $Var[X] = \sigma^2 = E\left[(X-\mu)^2\right] = \int\limits_{-\infty}^{\infty} (x-\mu)^2 f(x)dx.$

As with discrete distributions, the following simplifies the calculation of the variance.

$$Var\left[X\right] = E\left[X^2\right] - \mu^2.$$

Example 12.1

Let $X \sim Unif(a, b)$. Then

- $E[X] = \frac{(a+b)}{2}$.
- $Var[X] = \frac{(b-a)^2}{12}$.

Example 12.2 contd.

Suppose $X \sim \mathit{Unif}(0, 1)$. Calculate the probability that X is more than 1 standard deviation from the mean.

Mean and variance of an exponential distribution

Lemma 12.1 Suppose $X \sim Exp(\lambda)$. Then

- $E[X] = \frac{1}{\lambda}$.
- $Var[X] = \frac{1}{\lambda^2}$.

Proof of Lemma 12.1

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x} dx.$$

We will use the technique of integration by parts

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

to prove the first result. To use this formula, we need to identify u and dv, and then compute du and v. Note that v is simply the integral of dv.

Proof of Lemma 12.1 contd.

Defining $u = x$ and $dv = \lambda e^{-\lambda x} dx$, we have

$$du = dx$$

and

$$v = \int \lambda e^{-\lambda x} dx = \lambda \cdot \frac{-e^{-\lambda x}}{\lambda} = -e^{-\lambda x}.$$

Therefore

$$\begin{aligned} E[X] &= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx \\ &= 0 + \frac{1}{\lambda}, \\ &= \frac{1}{\lambda}. \end{aligned}$$

Proof of Lemma 12.1 contd.

The variance is found similarly using integration by parts twice to evaluate $\int_0^\infty x^2 \lambda e^{-\lambda x} dx$.

Example 12.3

Suppose $X \sim Exp(\lambda)$. Find the probability that X is within 1 standard deviation of the mean.

Algebra of expected values

We have already proved the following claims for discrete random variables. They are also true for continuous random variables.

Lemma 12.2 Let X be a continuous random variable with PDF f, and a and b are numbers. Then

- $E\left[t(X)\right] = \int\limits_{-\infty}^{\infty} t(x) \cdot f(x) dx$
- $E\left[aX + b\right] = aE\left[X\right] + b$
- $Var\left[aX + b\right] = a^2 Var\left[X\right]$

Example 12.4

A parking garage charges a flat fee of \$10 for the first hour (or fraction thereof) and any additional time at a rate of \$8 per hour.

Suppose the time, X (in hours), that we park in this lot is an exponential random variable with $\lambda = 1$. Let the random variable Y denote the cost (in dollars) that we will pay to park in the garage.

a. How does Y relate to X?

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b. What is our expected cost to park? That is, find $E[Y]$.

Higher moments

The expected values $E[X]$ and $E\left[X^2\right]$ are examples of **moments** of a random variable and its distribution. They are called the first and second moment about the origin.

The variance $E\left[(X-\mu)^2\right]$ is an example of a **central moment** or moment about the mean.

Higher moments describe additional features of the shape of a distribution. For instance,

$$E\left[(X-\mu)^3\right]$$

is zero for symmetric distributions and non-zero for asymmetric/skewed distributions, and therefore is often used as a measure of **skewness**. It is positive when the distribution is skewed to the right and negative when it is skewed to the left.