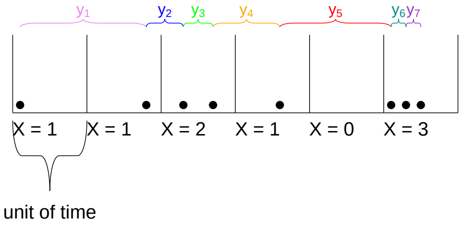
Chapter 11

The Exponential Distribution

Exponential Distribution

Situations sometimes arise, like a Poisson process, where the time interval between consecutively occurrences of a random event is important. Pictured below are two random variables X and Y, where X denotes the number of occurrences in a unit of time and Y denotes the interval between consecutive occurrences. The variable Y - also called the waiting time - is important in its own right.



Exponential Distribution

Theorem 11.1 Suppose a series of events satisfying the Poisson process are occurring at the rate of λ per unit time. Let the random variable Y denote the interval between consecutive events. Then Y has an **exponential** distribution and its PDF is

$$f(y) = \lambda e^{-\lambda y}, \quad y \geq 0.$$

We will denote this by $Y \sim Exp(\lambda)$.

Proof of Theorem 1.1

Suppose the event has occurred at time t. Consider the interval that extends from $[t, t+y)$ where $y > 0$. Let X denote the number of occurrences in this time period. Then

$$X \sim Pois(\lambda y)$$

since a rate of λ occurrences per unit time implies a rate of λy over $[t, t+y)$.

Notice that there will be no occurrences of the event during the time period $[t, t+y)$ only if $X = 0$. Therefore

$$P(Y > y) = P(X = 0) = e^{-\lambda y}.$$

Proof of Theorem 11.1

Hence:

$$F(y) = P(Y \leq y)$$
$$\begin{cases} 0 & y < 0, \\ 1 - e^{-\lambda y} & y \geq 0 \end{cases}$$

To find the PDF of Y, we differentiate $F(y)$ with respect to y

$$\begin{aligned} f(y) &= \frac{d}{dy}F(y) = \frac{d}{dy}(1 - e^{-\lambda y}) \\ &= -\frac{d}{dy}e^{-\lambda y} \\ &= \lambda \cdot e^{-\lambda y}. \quad y \geq 0 \end{aligned}$$

Example 11.1

A certain piece of equipment is subject to breakdowns, which occurs according to a Poisson process with rate $\lambda = 6$ per year. (You may assume that the time needed to repair a breakdown is negligible)

a. Find the probability that the next two breakdowns will occur within 1 month of each other.

Example 11.1

A certain piece of equipment is subject to breakdowns, which occurs according to a Poisson process with rate $\lambda = 6$ per year. (You may assume that the time needed to repair a breakdown is negligible)

b. Find the probability that there will be at least 3 months between the next two breakdowns.

Example 11.1

A certain piece of equipment is subject to breakdowns, which occurs according to a Poisson process with rate $\lambda = 6$ per year. (You may assume that the time needed to repair a breakdown is negligible)

c. Given that the equipment has functioned properly for 2 months since the last breakdown, calculate the probability that it will function properly for at least another 3 months.

- We say that a non-negative random variable, X is *memoryless* if for $x, k \geq 0$

$$P(X \geq x + k | X \geq k) = P(X \geq x).$$

- If we think of X as the lifetime (in hours say) of an instrument, the above equation says that if an instrument is alive at time k hours, then the probability that it survives an additional x hours is the same as the original lifetime distribution. (That is, the instrument does not remember that it has already been in use for k hours)
- Example 11.1 (b) illustrates that the exponential random variable has the memoryless property.

Finally, here are the essential facts about the probabilities for a Poisson process.

1. The number of arrivals for some specific length of time t can be modeled by a $Pois(\lambda t)$ distribution where λ is the rate per unit time
2. Arrivals in non-overlapping intervals are independent
3. The time between two consecutive arrivals has an exponential distribution
4. From any fixed time point a, the waiting time until the next arrival also has the same exponential distribution as in 3.

Calculations in R

```r
dexp(x = 5, rate = 6)    #P(X = x)
```

```
## [1] 5.614574e-13
```

```r
pexp(q = 1/6, rate = 6)  #P(X <= q)
```

```
## [1] 0.6321206
```

```r
qexp(p = 0.5, rate = 6)  #50th percentile or median
```

```
## [1] 0.1155245
```

Probability Density Function