Homework 1

Autumn 2023

KEY

2023-10-21

Instructions

- This homework is due in Gradescope on Wednesday Oct 11 by midnight PST. There is a 15 minute grace period and submissions made during this time will not be marked as late. Any work submitted past this period is considered late.
- Please answer the following questions in the order in which they are posed.
- Don't forget to (i) make a local copy this document for your work and to (ii) knit the document frequently to make sure there are no compilation errors.
- When you are done, download the PDF file as instructed in section and submit it in Gradescope.

Exercises

1. a. Let's denote the three pilots by P_1, P_2, P_3 and the two geologists by G_1, G_2. Then (P_1, P_2, G_1) represents the outcome that pilots 1 and 2 were picked as was geologist 1. The sample space consisting of all the **six outcomes** for this "experiment" is shown below:

$$S = \{(P_1, P_2, G_1), (P_1, P_3, G_1), (P_2, P_3, G_1), (P_1, P_2, G_2), (P_1, P_3, G_2), (P_2, P_3, G_2)\}.$$

   b. Since the duties of the pilots are interchangeable in part a), we did not need to differentiate between the roles assigned to each pilot. We need to do this now. Suppose the first pilot in an outcome is the designated pilot, and the second is the co-pilot. Then the sample space consists of six outcomes from part a, but also the following additional six outcomes for a total of **twelve outcomes**:

$$(P_2, P_1, G_1), (P_3, P_1, G_1), (P_3, P_2, G_1), (P_2, P_1, G_2), (P_3, P_1, G_2), (P_3, P_2, G_2).$$

2. By **Bonferroni's inequality** which says that

$$P(A \cap B) \geq P(A) + P(B) - 1$$

we can say that the **smallest** value for the intersection probability is 0.3.

By the **subset inequality** which states that if $E \subseteq F$ then $P(E) \leq P(F)$, we can conclude that the **largest** value for the intersection probability is $P(A) = 0.5$ since

$$(A \cap B) \subseteq A \rightarrow P(A \cap B) \leq P(A) = 0.5$$

and also

$$(A \cap B) \subseteq B \rightarrow P(A \cap B) \leq P(B) = 0.8.$$

3. To answer this question, we want to examine where $P(A \cap B)$ can equal 0. The answer is no, because by Bonferroni's inequality we know that

$$\begin{aligned} P(A \cap B) &\geq P(A) + P(B) - 1, \\ &= P(A) - (1 - P(B)) \\ &= P(A) - P(B^c) \quad \text{law of complements} \\ &= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}. \end{aligned}$$

Since $\frac{1}{12} > 0$, we can say that A and B must have elements in common since the probability of their intersection is positive.

4. Let $E = (B \cup C)$. Then

$$\begin{aligned} P(A \cup B \cup C) &= P(A \cup E), \\ &\leq P(A) + P(E) \end{aligned}$$

where the second equation follows from the union bound for two events. Let's now apply the union bound to $P(E)$:

$$\begin{aligned} P(E) &= P(B \cup C), \\ &\leq P(B) + P(C). \end{aligned}$$

Hence, putting both sets of equations together, we have:

$$P(A \cup B \cup C) \leq P(A) + P(B) + P(C).$$