# Problem Section 6

Mon Nov 6, 2023

---

**Learning Outcomes**

The problems are designed to build conceptual understanding and problem-solving skills. The emphasis is on learning to find, evaluate and build confidence. The specific tasks include:

- Use the Poisson models to calculate probabilities
- Compare binomial probabilities with their Poisson approximation
- Use the 'uniroot' function to solve a non-linear equation
- Back up and support work with relevant explanations

---

**Exercises**

1. Suppose telephone calls arrive at a help line at the rate of two per minute. A Poisson process is assumed to provide a good model for the arrivals. For each question below, write the probability you wish to calculate mathematically, then use built-in functions in R to do the calculation.

a. Calculate the probability that exactly five calls will arrive in the next 2 minutes.

   Let $X$ denote the number of arrivals in the next 2 minutes. Then $X \sim Poisson(\lambda = 4)$.

   Recall the PMF of $X \sim Pois(\lambda)$ is given by

$$f(x) = \frac{e^{-\lambda}\,\lambda^x}{x!} \quad x = 0, 1, 2, \ldots \tag{1}$$

   We want to find $P(X = 5)$ which involves plugging in $\lambda = 4$ and $x = 5$ in equation (1).

   This is calculated using R.

```
dpois(x = 5, lambda = 4)
```

```
## [1] 0.1562935
```

b. Calculate the probability that exactly five calls will arrive in the next 2 minutes and then five more calls will arrive in the following two minutes.

   Let $X$ denote the number of arrivals in the next 2 minutes and $Y$ denote the number of arrivals in the following 2 minutes. Then $X \sim Pois(\lambda = 4)$ and $Y \sim Pois(\lambda = 4)$ independently of each other since the intervals are non-overlapping. We want to calculate $P(X = 5 \cap Y = 5)$.

   By independence,

$$P(X = 5 \cap Y = 5) = P(X = 5) \times P(Y = 5)$$

   and each probability is the same as the one from part a.

```r
dpois(x=5, lambda = 4)*dpois(x = 5, lambda = 4)
```

```
## [1] 0.02442764
```

c. Calculate the probability that the next twenty five calls will occur within 10 minutes of each other.

   Let $X$ denote the number of arrivals in a 10 minute interval. Then $X \sim Pois(\lambda = 20)$ and we want to calculate $P(X \geq 24)$ which is the infinite sum:

$$P(X \geq 24) = \sum_{x=24}^{\infty} \frac{e^{-20} 20^x}{x!}$$

   If we were calculating this by hand, we would need to calculate its complement. However, it can be done in R as shown below.

```r
ppois(q = 23, lambda = 20, lower.tail = FALSE)
```

```
## [1] 0.2125072
```

2. Suppose $X \sim Pois(\lambda)$. Find $P(X = x | X \geq 1)$ for $x = 0, 1, 2, \ldots$. Write your steps clearly, beginning with the PMF of a Poisson.

   *Hint*: Try calculating this probability for specific values of $x$, such as $x = 0$, $x = 1$, $x = 2$. You will understand the pattern better that way.

   First we know that since $X \sim Pois(\lambda)$, therefore

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, 3 \ldots$$

   Using the definition of conditional probabilities, we can write

$$\begin{aligned} P(X = x | X \geq 1) &= \frac{P(X = x \cap X \geq 1)}{P(X \geq 1)}, \\ &= \frac{P\left(X = x \cap (X = 1 \cup X = 2 \cup X = 3 \cup \ldots)\right)}{P(X \geq 1)} \end{aligned} \tag{2}$$

   Let's calculate the denominator of equation (2) first since that does not change with the $x$.

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0), \\ &= 1 - e^{-\lambda}. \end{aligned}$$

   Now for the numerator of equation (2). If $x = 0$, then clearly this probability is 0 since the events $X = 0$ and $X \geq 1$ are disjoint. For any $x \geq 1$, the event $(X = x) \subset (X \geq 1)$ and therefore

$$\{(X = x) \cap (X \geq 1)\} = \{X = x\}$$

   Hence

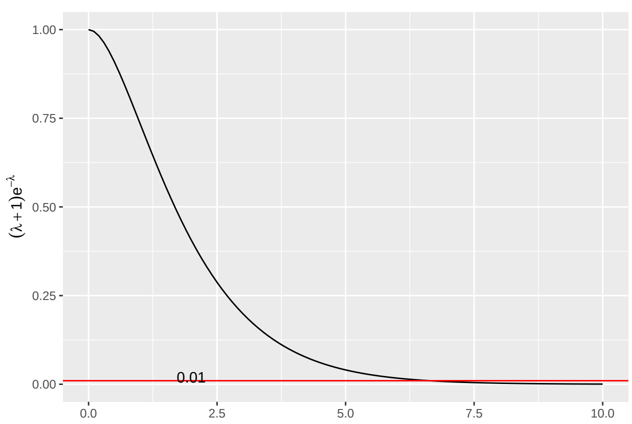
$$\begin{aligned} P(X = x | X \geq 1) &= \frac{P(X = x)}{P(X \geq 1)}, \\ &= \frac{e^{-\lambda} \lambda^x}{x!(1 - e^{-\lambda})}, \quad x = 1, 2, 3, \ldots \end{aligned}$$

3. Suppose $X$, the number of chocolate chips in a certain type of cookie has a Poisson distribution. We want the probability that a randomly chosen cookie has at least 2 chocolate chips to be greater than or equal to 0.99. Find the smallest value of the mean of this distribution that ensures this probability.

$$\begin{aligned} P(X \geq 2) &\geq 0.99 \\ \iff 1 - P(X = 0) - P(X = 1) &\geq 0.99 \\ \iff 1 - \frac{\lambda^0 e^{-\lambda}}{0!} - \frac{\lambda^1 e^{-\lambda}}{1!} &\geq 0.99 \\ \iff 1 - e^{-\lambda} - \lambda e^{-\lambda} &\geq 0.99 \\ \iff e^{-\lambda}(\lambda + 1) &\leq 0.01 \end{aligned}$$

We will need to use numerical methods to find the smallest value of $\lambda$ which satisfies this equation. Let's first make a plot of the function $e^{-\lambda}(\lambda + 1)$ to see what it looks like.

```
ggplot() +
  geom_function( fun = function(x){exp(-x)*(x+1)},
                 xlim=c(0,10) ) +
  geom_hline(yintercept = 0.01,
             color = "red" )+
  annotate(geom = "text",
           x = 2,
           y = 0.02,
           label = "0.01" ) +
  labs(x = expression(lambda),
       y = expression((lambda+1)*e^{-lambda} ) )
```




From the picture, we can see that the function $e^{-\lambda}(\lambda + 1)$ is a decreasing function of $\lambda$. Therefore, in order to find the smallest value of $\lambda$ that ensures

$$e^{-\lambda}(\lambda + 1) \leq 0.01$$

we need to find the $\lambda$ that satisfies the equation exactly. In other words, we need to find root of the equation

$$e^{-\lambda}(\lambda + 1) - 0.01 = 0.$$

The R function `uniroot` is a root finding method which uses a bisection algorithm.

```r
# Looks like we need a value between 6 and 7
# Get lambdaMin using uniroot function

lambdaMin <- uniroot(
                f = function(x){exp(-x)*(x + 1) - 0.01} ,
                lower = 0,
                upper = 10)$root

paste("The minimum value for the mean is lambda",
       round(lambdaMin, 3) )
```

```
## [1] "The minimum value for the mean is lambda 6.638"
```

```r
# Check for probability of at least 2 chocolate chips - should be 0.99
prob2plus <- ppois(1, lambdaMin, lower.tail = FALSE)

paste("The probability of at least 2 chocolate chips is",
       round(prob2plus, 3) )
```

```
## [1] "The probability of at least 2 chocolate chips is 0.99"
```