Chapter 1

Set Theory

What is probability?

- **Probability** is a number between 0 and 1 (inclusive) that describes how likely it is that something will occur.
- As an area of mathematics, **probability** is the study of randomness, a particular form of uncertainty.
  - the possible outcomes are known; however
  - they are unpredictable, but
  - there is a well-defined rule for choosing among them.
- Probability theory uses the language of sets, hence we begin a brief review of concepts from set theory.

Sample space

Definition 1.1 The set S, of all possible outcomes of a random process is called the sample space. It is also known as the outcome set.

Random process	Sample space
Toss a coin once	$S = \{H; T\}$
Toss a coin twice	$S = \{(H, H); (H, T); (T, H); (T, T)\}$

Example 1.1

For each of the following "experiments", describe the sample space.

a. A local TV station advertises two newscasting positions. If two women (W_1, W_2) and two men (M_1, M_2) apply, list all the possible teams of two co-anchors that can be formed?

Example 1.1

For each of the following "experiments", describe the sample space.

b. A local TV station is seeking to hire a sports announcer and a weather forecaster. If two women (W_1, W_2) and two men (M_1, M_2) apply, list all the possible teams that can be formed?

Example 1.1

For each of the following “experiments”, describe the sample space.

c. Toss a coin repeatedly until the first head shows.

Example 1.1

For each of the following "experiments", describe the sample space.

d. Let S be the set of right triangles with a 5" hypotenuse and whose height and length are a and b, respectively. Characterize the outcomes in S.

Event

Definition 1.2 An event E is a collection of outcomes in S. That is, E is a subset of S, which is written in short hand as $E \subseteq S$.

Experiment	Sample space	Event
Toss two dice	$S = \left\{ \begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,6) \\ (2,1) & (2,2) & \cdots & (2,6) \\ \vdots & \vdots & \vdots & \vdots \\ (6,1) & (6,2) & \cdots & (6,6) \end{array} \right\}$	E : Sum of numbers is 3 $\{(1,2),(2,1)\}$

Example 1.2:

Consider the experiment of choosing coefficients for the quadratic equation

$$ax^2 + bx + c = 0.$$

Characterize the values of a, b, c associated with the event E: Equation has complex roots.

Set Operations

Given any two events A and B we have the following operations:

UNION: The union of A and B, written $A \cup B$ is the set of elements that belong to either A or B or both:

$$A \cup B \;=\; \{x : x \in A \text{ or } x \in B\}\,.$$

The symbol $\in$ means "in" or "a member of". Similarly $\notin$ means "not in" or "not a member of".

Set Operations

Given any two events A and B we have the following operations:

INTERSECTION: The intersection of A and B, written $A \cap B$ is the set of elements that belong to both A and B:

$$A \cap B \quad = \quad \{x : x \in A \text{ and } x \in B\} .$$

Set Operations

Definition 1.3 Two events A and B are disjoint (or mutually exclusive) if $A \cap B = \emptyset$ where $\emptyset$ denotes the void or empty set (consisting of no elements).

Example 1.3

A coin is tossed four times and the resulting sequence of heads and/or tails is recorded. Define the events:

E: exactly two heads appear
F: heads and tails alternate
G: first two tosses are heads

Which events, if any, are mutually exclusive?

Set Operations

Given any two events A and B we have the following operations:

COMPLEMENTATION: The complement of A, written A^c is the set of elements that are in S but not in A:

$$A^c = \{x \in S : x \notin A\}.$$

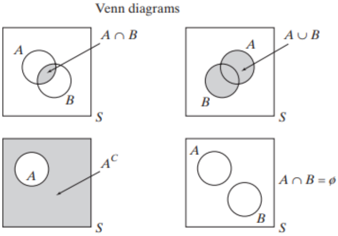
Set Operations

Given any two events A and B we have the following operations:

SET DIFFERENCE: The set difference of B and A, denoted by $B \setminus A$ (or $B - A$) is

$$\begin{aligned} B \setminus A &= (x \in S : x \in B \textit{ and } x \notin A\}\,, \\ &= B \cap A^c. \end{aligned}$$

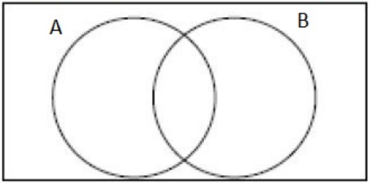
- The following figure shows the Venn diagrams for a set union, intersection, complement and set difference.



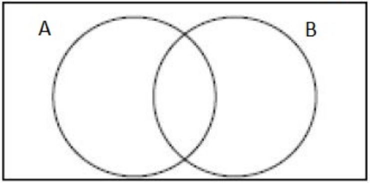
Example 1.4:

Shade the events corresponding to the following descriptions. Then write them in terms of set operations.

Precisely one of two events happen



At most one of two events happen



Example 1.5

Let events A, B and sample space S be defined as the following intervals:

$$S = \{x : 0 \leq x < 7\}$$
$$A = \{x : 0 < x < 5\}$$
$$B = \{x : 3 \leq x < 7\}$$

Characterize the following events:

a. A^c
b. $A \cap B$
c. $A \cup B$
d. $A \cap B^c$
e. $A^c \cup B$
f. $A^c \cap B^c$

- Unions and intersections can be defined for arbitrary collections of sets. For example, if $A_1, A_2, A_3, \ldots$ is a collection of sets, all defined on a sample space S, then

$$\begin{aligned}
\bigcup_{i=1}^{\infty} A_i &= \{x \in S : x \in A_i, \text{ some } i\}, \\
\bigcap_{i=1}^{\infty} A_i &= \{x \in S : x \in A_i, \text{ for all } i\}
\end{aligned}$$

Example 1.6

Let $S = (0,1]$ and $A_k = (1/k, 1]$. What is $\bigcup_{k=1}^{\infty} A_k$? (Note: A_k are nested and increasing in the sense that $A_1 \subset A_2 \subset A_3 \ldots$)

Properties of set operations

Theorem 1.1 For any three events E, F, and G defined on a sample space S, we have the following identities:

1. Commutative laws

- $E \cup F = F \cup E$

- $E \cap F = F \cap E$

2. Associative laws

- $E \cup (F \cup G) = (E \cup F) \cup G$

- $E \cap (F \cap G) = (E \cap F) \cap G$

3. Distributive laws

- $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$

- $E \cup (F \cap G) = (E \cup F) \cap (E \cup G)$

Properties of set operations

Theorem 1.1 For any three events E, F, and G defined on a sample space S, we have the following identities:

4. Identity laws

- $E \cap S = E$
- $E \cup \phi = E$

5. Complement laws

- $E \cup E^c = S$
- $E \cap E^c = \phi$

Properties of set operations

Theorem 1.2 (DeMorgan's Laws) For any two events E and F defined on a sample space S:

- $(E \cup F)^c = E^c \cap F^c$
- $(E \cap F)^c = E^c \cup F^c$

Proof of DeMorgan's first law

First we show $(E \cup F)^c \subseteq E^c \cap F^c$. Consider an element $x \in (E \cup F)^c$.

$$\begin{aligned} x \in (E \cup F)^c &\Rightarrow x \notin (E \cup F), \\ &\Rightarrow x \notin E \ AND \ x \notin F \\ &\Rightarrow x \in E^c \ AND \ x \in F^c \\ &\Rightarrow x \in (E^c \cap F^c). \end{aligned}$$

Since this holds for any arbitrary element x in $(E \cup F)^c$, we can say

$$(E \cup F)^c \subseteq E^c \cap F^c.$$

Proof of Demorgan's first law

Next we show $E^c \cap F^c \subseteq (E \cup F)^c$. Consider an element $y \in E^c \cap F^c$.

$$\begin{aligned} y \in E^c \cap F^c &\Rightarrow x \in E^c \ AND \ x \in F^c, \\ &\Rightarrow x \notin E \ AND \ x \notin F, \\ &\Rightarrow x \notin (E \cup F), \\ &\Rightarrow x \in (E \cup F)^c. \end{aligned}$$

Again, since y is any arbitrary element of $E^c \cap F^c$, we have the result

$$E^c \cap F^c \subseteq (E \cup F)^c$$

and hence

$$(E \cup F)^c = E^c \cap F^c$$

- De Morgan's laws commonly apply to text searching using Boolean operators AND, OR, and NOT.

  The queries

  - Search A: NOT (cars OR trucks)
  - Search B: (NOT cars) AND (NOT trucks)

  return the same results