Discrete Distributions

Expected value

Motivation

The Probability Mass Function provides a global overview of the behavior of a random variable.

Numerical summaries such as mean and variance can help us understand the *typical* behavior of the random variable.

Example 7.1

Suppose a high school student has taken 10 courses and received 5 A's, 4 B's and 1 C. Using the traditional numerical scale where an A is worth 4, B is worth 3 and a C is worth 2, what is this student's overall grade point average (GPA)?

The grade point average in example 7.1 is a weighted sum of the numerical worth of a grade:

$$GPA = \sum (\text{grade})(\text{proportion of times student gets that grade}).$$

The mean of a random variable is a similar weighted sum with the probabilities serving as weights.

Mean of a random variable

Definition 7.1 Let X be a discrete random variable with PMF f. The mean (also called **expected value**) of X is denoted as μ or $E[X]$ and is defined by

$$E[X] = \sum_x x \cdot P(X = x) = \sum_x x \cdot f(x).$$

The sum is taken over all the possible values of X. When the possible values are infinite, we require that the sum is well defined and is finite. If $E[X] = \pm\infty$, we simply say that $E[X]$ does not exist.

Example 7.2

Find $E[X]$ where X is the outcome when we roll a fair die.

It is not necessary for the mean/expected value to be a possible value for X.

"Expected" means in the average over many performances of the experiment, not expected on the next trial.

Mean of a Binomial Random variable

Theorem 7.1 Let $X \sim Binom(n, \pi)$. Then

$$E[X] = n\pi.$$

Before diving into the proof, let's be mindful of a couple of useful facts:

- Fact 1: For any $x \geq 1$, we have the result:

$$x\binom{n}{x} = n\binom{n-1}{x-1}.$$

Mean of a Binomial Random variable

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- Fact 2: The binomial PMF sums to 1:

$$\sum_{x=0}^{n} \binom{n}{x} \pi^x (1-\pi)^{n-x} = 1.$$

Mean of a Binomial Random variable: proof

$$
\begin{aligned}
E\left[X\right] &= \sum_{x=0}^{n} x \cdot f(x) = \sum_{x=0}^{n} x \binom{n}{x} \pi^{x} (1-\pi)^{n-x}, \\
&= \sum_{x=1}^{n} x \binom{n}{x} \pi^{x} (1-\pi)^{n-x}, \\
&= \sum_{x=1}^{n} n \binom{n-1}{x-1} \pi^{x} (1-\pi)^{n-x}, \quad \text{by fact 1} \\
&= n\,\pi \sum_{x=1}^{n} \binom{n-1}{x-1} \pi^{x-1} (1-\pi)^{n-x}, \\
&= n\,\pi \sum_{y=0}^{n-1} \underbrace{\binom{n-1}{y} \pi^{y} (1-\pi)^{n-1-y}}_{\text{PMF of a } Binom(n-1,\pi)}, \qquad y = x-1 \\
&= n\,\pi. \qquad \text{by fact 2}
\end{aligned}
$$

Linearity of Expected Values

Suppose in the die roll example, we are interested in calculating $E[X+3]$.

It seems plausible that the answer should be $E[X]+3$, since we are just adding 3 to the number that results from rolling the die.

The definition below formalizes this idea and is stated for any arbitrary linear transformation of X.

Linearity of Expected Values

Lemma 7.2 Let X be a discrete random variable, let a and b be constants and let $Y = aX + b$. Then Y is a discrete random variable and

$$E[Y] = aE[X] + b.$$

Linearity of Expectation: proof

Suppose X takes values $x_1, x_2, x_3, \ldots$ and $f(x_1)$, $f(x_2)$ and so on, are the probabilities.

Then Y takes values $ax_1 + b, ax_2 + b, ax_3 + b, \ldots$ and

$$P(Y = ax_i + b) = P(X = x_i) = f(x_i).$$

Therefore, the expected value of Y can now be found as follows:

$$\begin{aligned} E[Y] &= \sum_x (ax + b) \cdot f(x), \\ &= a \sum_x x \cdot f(x) + b \sum_x f(x), \\ &= aE[X] + b. \end{aligned}$$

Example 7.3

A typical day's production of a certain electronic component is twelve. The probability that one of these components needs rework is 0.11. Each component needing rework costs $100. What is the average daily cost for defective components?

Lemma 7.2 and example 7.4 illustrate that we do not need to find the PMF of Y in order to find it's expectation.

In fact, this idea applies to any transformation of X, not just a linear one, and is stated next.

Law of Unconscious Probabilist

Lemma 7.3 Let X be a discrete random variable with PMF f and let $t(X)$ be a transformation of X for some function t. Then $Y = t(X)$ is a discrete random variable and

$$E[Y] = E[t(X)] = \sum_{x} t(x)f(x).$$

Example 7.4

Suppose X is a discrete random variable with PMF as shown in the following table.

x	-2	-1	0	1	2
probability	0.05	0.10	0.35	0.3	0.20

Use Lemma 7.3 to calculate $E\left[(X-\frac{1}{2})^2\right]$.