Chapter 7.1

Mean of a Discrete Random Variable

Chapter 7.1 1 / 17

Motivation

The Probability Mass Function provides a global overview of the behavior of a random variable.

Numerical summaries such as mean and variance can help us understand the *typical* behavior of the random variable.

Chapter 7.1 2 / 17

Suppose a high school student has taken 10 courses and received 5 A's, 4 B's and 1 C. Using the traditional numerical scale where an A is worth 4, B is worth 3 and a C is worth 2, what is this student's overall grade point average (GPA)?

Chapter 7.1 3 / 17

The grade point average in example 7.1 is a weighted sum of the numerical worth of a grade:

$$\textit{GPA} = \sum (\textit{grade})(\textit{proportion of times student gets that grade}).$$

The mean of a discrete random variable is a similar weighted sum with the probabilities serving as weights.

Chapter 7.1 4 / 17

Mean of a discrete random variable

Definition 7.1 Let X be a discrete random variable with PMF f. The mean (also called **expected value**) of X is denoted as μ or E[X] and is defined by

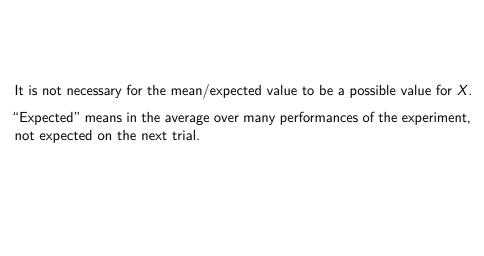
$$E[X] = \sum_{x} x \cdot P(X = x) = \sum_{x} x \cdot f(x).$$

The sum is taken over all the possible values of X. When the possible values are infinite, we require that the sum is well defined and is finite. If $E[X] = \pm \infty$, we simply say that E[X] does not exist.

5 / 17

Find E[X] where X is the outcome when we roll a fair die.

Chapter 7.1 6 / 17



Chapter 7.1 7 / 17

Mean of a Binomial Random variable

Theorem 7.1 Let $X \sim Binom(n, \pi)$. Then

$$E[X] = n\pi$$
.

Before diving into the proof, let's be mindful of a couple of useful facts:

• Fact 1: For any $x \ge 1$, we have the result:

$$x\binom{n}{x} = n\binom{n-1}{x-1}.$$

8 / 17

Mean of a Binomial Random variable

Theorem 7.1 Let $X \sim Binom(n, \pi)$. Then

$$E[X] = n\pi$$
.

Before diving into the proof, let's be mindful of a couple of useful facts:

• Fact 2: The binomial PMF sums to 1:

$$\sum_{x=0}^{n} \binom{n}{x} \pi^{x} (1-\pi)^{n-x} = 1.$$

Chapter 7.1 9 / 17

Mean of a Binomial Random variable: proof

$$E[X] = \sum_{x=0}^{n} x \cdot f(x) = \sum_{x=0}^{n} x \binom{n}{x} \pi^{x} (1 - \pi)^{n-x},$$

$$= \sum_{x=1}^{n} x \binom{n}{x} \pi^{x} (1 - \pi)^{n-x},$$

$$= \sum_{x=1}^{n} n \binom{n-1}{x-1} \pi^{x} (1 - \pi)^{n-x}, \text{ by fact } 1$$

$$= n \pi \sum_{x=1}^{n} \binom{n-1}{x-1} \pi^{x-1} (1 - \pi)^{n-x},$$

$$= n \pi \sum_{y=0}^{n-1} \binom{n-1}{y} \pi^{y} (1 - \pi)^{n-1-y}, \quad y = x - 1$$

$$= n \pi. \text{ by fact } 2$$

Chapter 7.1 10 / 17

Linearity of Expected Values

Suppose in the die roll example, we are interested in calculating E[X+3].

It seems plausible that the answer should be E[X] + 3, since we are just adding 3 to the number that results from rolling the die.

The definition below formalizes this idea and is stated for any arbitrary linear transformation of X.

Chapter 7.1 11 / 17

Linearity of Expected Values

Lemma 7.2 Let X be a discrete random variable, let a and b be constants and let Y = aX + b. Then Y is a discrete random variable and

$$E[Y] = aE[X] + b.$$

Chapter 7.1 12 / 17

Linearity of Expectation: proof

Suppose X takes values $x_1, x_2, x_3, ...$ and $f(x_1)$, $f(x_2)$ and so on, are the probabilities.

Then Y takes values $ax_1 + b$, $ax_2 + b$, $ax_3 + b$, ... and

$$P(Y = ax_i + b) = P(X = x_i) = f(x_i).$$

Therefore, the expected value of Y can now be found as follows:

$$E[Y] = \sum_{x} (ax + b) \cdot f(x),$$

= $a \sum_{x} x \cdot f(x) + b \sum_{x} f(x),$
= $aE[X] + b.$

Chapter 7.1 13 / 17

A typical day's production of a certain electronic component is twelve. The probability that one of these components needs rework is 0.11. Each component needing rework costs \$100. What is the average daily cost for defective components?

Chapter 7.1 14 / 17

Lemma 7.2 and example 7.4 illustrate that we do not need to find the PMF of Y in order to find it's expectation.

In fact, this idea applies to any transformation of X, not just a linear one, and is stated next.

Chapter 7.1 15 / 17

Law of Unconscious Probabilist

Lemma 7.3 Let X be a discrete random variable with PMF f and let t(X) be a transformation of X for some function t. Then Y = t(X) is a discrete random variable and

$$E[Y] = E[t(X)] = \sum_{x} t(x)f(x).$$

Chapter 7.1 16 / 17

Suppose \boldsymbol{X} is a discrete random variable with PMF as shown in the following table.

X	-2	-1	0	1	2
probability	0.2	0.1	0.4	0.1	0.2

Use Lemma 7.3 to calculate $E[X^2]$.

Chapter 7.1 17 / 17