Chapter 3

Equally Likely Rule and Counting Methods

References: Pruim 2.2, Larsen & Marx 2.6

A number of probabilities may be calculated simply using the axioms defined in the previous chapter. We have already seen the Rule of Complements, and the Addition Rule for the union of overlapping events. In this chapter, we will turn our attention to the task of calculating probabilities when all the possible outcomes are equally likely.

Theorem 3.1 (Equally Likely Rule)**.** *Suppose our sample space consists of n equally likely outcomes $s_1, s_2, \dots, s_n$. Then*

$$P(E) = \frac{|E|}{|S|} = \frac{|E|}{n} = \frac{\textit{number of elements in } E}{\textit{number of elements in } S}.$$

Proof. Let p be the probability of any outcome. Then

$$\begin{aligned} P(S) &= P(s_1 \cup s_2 \cup \cdots \cup s_n) \\ &= P(s_1) + P(s_2) + \dots P(s_n) \quad \text{axiom 3} \\ &= p + p + p + \cdots + p = 1. \quad \text{axiom 2} \end{aligned}$$

$$\Rightarrow np = 1 \Rightarrow p = \frac{1}{n}.$$

By axiom (3) again

$$
\begin{aligned}
P(E) &= \sum_{s_i \in E} P(s_i) = \sum_{s_i \in E} p, \\
&= p \cdot \sum_{s_i \in E} 1, \\
&= p \cdot |E| = \frac{1}{n}|E|.
\end{aligned}
$$

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Let's now use this rule to calculate probabilities in an example.

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Example 3.1. A local TV station advertises two news casting positions. If two women (W_1, W_2) and two men (M_1, M_2) apply, the "experiment" of hiring two coanchors generates a sample space with 6 outcomes:

$$S = \{(W_1, W_2), (W_1, M_1), (W_1, M_2), (W_2, M_1), (W_2, M_2), (M_1, M_2)\}.$$

Let E denote the event that at least one woman is hired. Calculate $P(E)$ assuming all six outcomes in S are equally likely.

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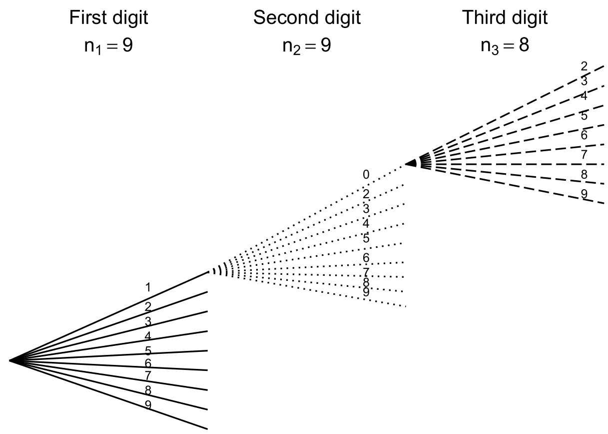
In complicated situations, or for larger sample spaces, counting the number of favorable outcomes itself may be challenging. In these cases, we will need a more systematic approach.

Combinatorics is an area of mathematics primarily concerned with counting. In combinatorics, the **rule of product** or **multiplication principle** is a basic counting principle and is stated below.

Lemma 3.1. ***(Multiplication Principle for Counting)*** *If a job consists of k separate tasks performed in series, the ith one of which can be done in n_i ways, then the entire job can be done in $n_1 \times n_2 \times \cdots \times n_k$ ways.*

For example, say we want to count the number of integers between 100 and 999 that have distinct digits. We can think of this task sequentially as involving three steps: choosing the first digit, then the second and finally the third. There are 9 choices for the first digit: 1 - 9. For each choice of first digit, there are still 9 choices for the second. Finally, for each choice of first and second digit, there are 8 choices for the third. Hence there are $9 \times 9 \times 8 = 648$ integers between 100 and 999. This is illustrated below.



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Example 3.2. In 1824, Louis Braille (1809-1852) invented the standard alphabet for the blind. It uses a six dot matrix where some of the dots are raised. For instance, the letter "e" has two raised dots and is written

The configuration with no raised dots is useless. How many letters are in the Braille alphabet?

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Example 3.3. A statistics student needs to take three STAT electives (403, 425, 435) in their final four quarters (Fall, Winter, Spring, Summer). In how many ways can they plan their schedule, assuming they don't want to take more than one statistics elective in a quarter? (All three electives are offered each quarter)

Example 3.3 illustrates that order is not always a meaningful characteristic of a collection of elements. For example, when counting the possible options for the three quarters, our only concern is with which quarters, not any particular rearrangement of them. We call this a <u>combination</u>. On the other hand, when choosing the electives we are forming an ordered arrangement of the courses. A choice of 425, 403, 435 is different from 403, 435, 425. Each arrangement is referred to as a <u>permutation</u> of the courses.

The multiplication rule can be used to calculate the number of ordered arrangements of a set of n items. For example, in order to count all the rearrangements of the three electives 425, 403, 435, we could have reasoned as follows:

$$Choices: \underset{\text{quarter 1}}{\underline{3}} \times \underset{\text{quarter 2}}{\underline{2}} \times \underset{\text{quarter 3}}{\underline{1}} = 3! = 6$$

In a similar manner, the number of ordered arrangements of k items from n objects - called a <u>partial permutation</u> - is equal to

$$\underset{1}{\underline{n}} \times \underset{2}{\underline{n-1}} \times \underset{3}{\underline{n-2}} \cdots \times \underset{\text{k}}{\underline{n-(k-1)}} = \frac{n!}{(n-k)!}. \tag{3.1}$$

Equation (3.1) provides the number of ways of selecting a group of k items from n distinct objects, when the order in which the item is selected is relevant. This means that for each group of k items, all of the permutations of the k items will be counted. That is, each group of k items will be counted $k!$ times.

As an example, consider the following collections of 3 items taken from 5: a, b, c, d, e.

A	partial permutations of the letters a,b,c,d, e taken 3 at a time **abc**, acb, bac, bca, cab, cba, **abd**, adb, bad, bda, dab, dba, **abe**, aeb, bae, bea, eab, eba, **acd**, adc, cda, cad, dac, dca, **ace**, aec, cae, cea, eac,eca, **ade**, aed, dae, dea, ead, eda, **bcd**, bdc, cbd, cdb, dcb, dbc, **bce**, bec, cbe, ceb, ebc, ecb, **bde**, bed, deb, dbe, edb, ebd, **cde**, ced, dce, dec, edc, ecd
B	groupings or *combinations* of the letters a, b, c, d,e taken 3 at a time **abc**, **abd**, **abe**, **acd**, **ace**, **ade**, **bcd**, **bce**, **bde**, **cde**

For each element in B there are six distinct elements in A corresponding to the number of re-arrangements of the three letters. The **division rule** provides us with a way to correct for this type of double counting.

Lemma 3.2. ***(Division Rule for Counting)*** *If there is a r to 1 correspondence between the elements in set A with the elements in set B, and there are n elements in A, then there are n/r elements in B.*

From the preceding discussion, it is clear that the number of combinations or (unordered) groupings of k items that can be formed from n objects is equal to the number of ordered groupings of k items divided by $k!$.

Definition 3.1. Let n and k be integers with $0 \leq k \leq n$. We define $\binom{n}{k}$ as the number of different groupings of size k that can be formed from n items when the order of selection is irrelevant. Then

$$\binom{n}{k} = \frac{n!}{k!\ (n-k)!},$$

The notation $\binom{n}{k}$ is called a binomial coefficient and is read as "n choose k". The R function to compute binomial coefficients is `choose(n, r)`.

By convention

$$\binom{n}{0} = 1, \quad \binom{n}{n} = 1, \quad \binom{n}{k} = 0, \;\; k < 0 \textit{ or } k > n$$

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Example 3.4. From a group of 5 adults and 7 children:

a. how many different carpools of 2 adults and 3 kids can be formed?

b. how many different carpools of 2 adults and 3 kids can be formed if 2 of the kids are fighting and refuse to carpool together?

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In a combinatorial setting, making the transition from an enumeration to a probability is easy. If there are n ways to perform a certain operation and a total of m of those satisfy some stated condition—call it E —then by the equally likely rule, $P(E)$ is defined to be the ratio m/n. This assumes, of course, that each of the n possible ways are equally likely.

Let's now calculate a combinatorial probability.

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Example 3.5. From a standard well-shuffled deck of 52 playing cards (consisting of 4 suits – $\clubsuit$, $\heartsuit$, $\spadesuit$, $\diamondsuit$), you are dealt five cards.

a. The sample space S associated with this "experiment" consists of all groupings of five cards from the deck. How many elements are in S?

b. Let E denote the event that the five cards are all of one suit. How many outcomes are in E?

c. Assume that each of the elements in S are equally likely. Calculate $P(E)$.

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3.1 Practice Problems

1. A tourist wants to visit six of America's ten largest cities. In how many ways can she do that if the order of her visits
   a. does not matter to her
   b. matters to her
2. Suppose we roll 5 standard dice and record the number that lands on each dice. For example, the outcome

   $$\underset{\text{die1}}{\underline{1}}\ ,\ \underset{\text{die2}}{\underline{1}}\ ,\ \underset{\text{die3}}{\underline{1}}\ ,\ \underset{\text{die4}}{\underline{1}}\ ,\ \underset{\text{die5}}{\underline{1}}$$

   corresponds to observing a 1 on all the dice.
   a. How many outcomes are in the sample space S?
   b. The event E consists of all the outcomes where the numbers on at least two dice match. How many outcomes are in E? (Hint: calculate the number of outcomes in the complement of E)
   c. Assume that every outcome in S is equally likely. What is the probability of E?
3. Calculate $\binom{12}{9}$. Do this by hand and verify in R.
4. A store manager receives a shipment of 30 microwave ovens, 5 of which are (unknown to the manager) defective. The store manager selects 4 ovens randomly from the shipment and tests them to see if they are defective.

a. The sample space S associated with this "experiment" consists of all groupings of 4 ovens selected from the lot. How many elements are in S?
b. Let E denote all the outcomes composed of 2 defective and 2 non-defective ovens? How many elements in E?
c. Assume that every outcome in S is equally likely. Calculate $P(E)$.