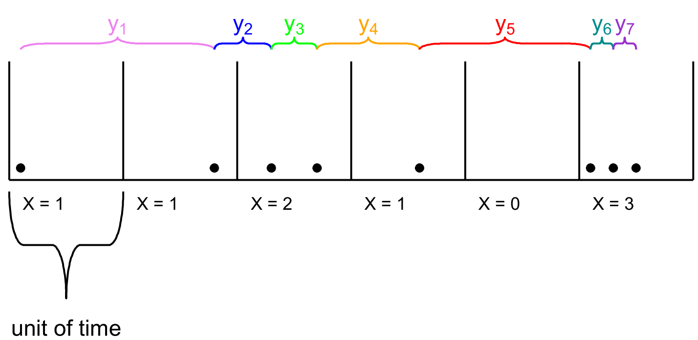
Chapter 11

The Exponential Distribution

References: Pruim 3.1.1, Larsen & Marx 4.2

Situations sometimes arise where the time interval between consecutively occurring events is an important random variable. Imagine being responsible for the maintenance on a network of computers. Clearly, the number of technicians you would need to employ in order to be capable of responding to service calls in a timely fashion would be a function of the "waiting time" from one breakdown to another.

The following figure shows the relationship between the random variables X and Y, where X denotes the number of occurrences in a unit of time and Y denotes the interval between consecutive occurrences.



Pictured are six intervals: $X = 0$ on one occasion, $X = 1$ on three occasions, $X = 2$ once, and $X = 3$ once. Resulting from those eight occurrences are seven measurements on the random variable Y. Obviously, the pdf for Y will depend on the pdf for X. One particularly important special case of that dependence is the Poisson/exponential relationship outlined in the theorem below.

Theorem 11.1. *Suppose a series of events satisfying the Poisson model are occurring at the rate of λ per unit time. Let the random variable Y denote the interval between consecutive events. Then Y has an* ***exponential*** *distribution and its PDF is*

$$f(y) = \lambda e^{-\lambda y}, \quad y \geq 0.$$

We will denote this by $Y \sim Exp(\lambda)$.

Proof. Suppose the event has occurred at time t. Consider the interval that extends from $[t, t+y)$ where $y > 0$. Let X denote the number of occurrences in this time period. Then

$$X \sim Pois(\lambda y)$$

since a rate of λ occurrences per unit time implies a rate of λy over $[t, t+y)$.

Define the random variable Y to denote the interval between consecutive events. We begin by deriving the CDF of Y. Once we have the CDF, we can differentiate it to get the PDF.

Notice that there will be no occurrences of the event during the time period $[t, t+y)$ only if $Y > y$.

$$P(Y > y) = P(X = 0) = e^{-\lambda y}.$$

Therefore:

$$\begin{aligned} F(y) &= P(Y \leq y) = 1 - P(Y > y) \\ &= 1 - e^{-\lambda y}. \quad y > 0 \end{aligned}$$

Taking derivatives with respect to y

$$\begin{aligned} f(y) &= \frac{d}{dy} F(y) = \frac{d}{dy}(1 - e^{-\lambda y}) \\ &= -\frac{d}{dy} e^{-\lambda y} \\ &= -\frac{d}{d(\lambda y)} e^{-\lambda y} \cdot \frac{d(\lambda y)}{dy} \\ &= \lambda \cdot e^{-\lambda y}. \quad y > 0 \end{aligned}$$

□

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Example 11.1. A certain piece of equipment is subject to breakdowns, which occurs according to a Poisson process with rate $\lambda = 6$ per year. (You may assume that the time needed to repair a breakdown is negligible)

a. Find the probability that the next two breakdowns will occur within 1 month of each other.

b. Find the probability that there will be at least 3 months between the next two breakdowns.

c. Given that the equipment has functioned properly for 2 months since the last breakdown, calculate the probability that it will function properly for at least another 3 months.

We say that a non-negative random variable, X is memoryless if for $x > k \geq 0$

$$P(X \geq x | X \geq k) = P(X \geq (x - k)).$$

If we think of X as the lifetime (in hours say) of an instrument, the previous equation says that if an instrument is alive at time k hours, then the probability that it survives an additional $x-k$ hours is the same as the original lifetime distribution. (That is, the instrument does not remember that it has already been in use for k hours)

As we see in example 11.1, the exponential distribution is a memoryless distribution. This has important consequences for deciding if an exponential is a good model for a given situation. If for example, we use the exponential distribution to model the time until failure, the model implies that the distribution of future time until failure does not depend on how old the part or system is. Presumably they fail because of things that happen to them, like being dropped or struck by lightining.

11.1 Practice Problems

1. Let $X \sim Exp(\lambda = 4)$ where λ is the rate parameter. That is:

$$f(x) = \lambda e^{-\lambda\, x}.$$

a. $P(X \geq 5)$
b. $P(2 < X < 5)$

c. $P(|X - 3.5| \geq 1.5)$

2. Records show that deaths occur according to a Poisson process at the rate of 0.1 per day among patients residing in a large nursing home. If someone dies today, what are the chances that a week or more will elapse before another death occurs?