Problem Section 4 KEY

Point Estimation

Learning Outcomes

The problems are designed to build conceptual understanding and problem-solving skills. The emphasis is on learning to find, evaluate and build confidence. The specific tasks include:

- Derive and calculate the method of moments estimator
- Be able to verify unbiasedness and consistency of an estimator
- Be able to "unbias" a biased estimator which is off by a multiplicative constant
- Compare unbiased estimators using their variance
- Compare biased estimators using the mean squared error
- Back up and support work with relevant explanations

Instructions

- Please work in groups of three to answer the problems below.
- Each member of the group must write up their work on a separate sheet of paper. Be sure to clearly write your full name on the top as it appears in the gradebook.

Exercises

1. Suppose $X_1, X_2, \ldots, X_n$ are independent discrete random variables drawn from the PMF

$$f(x) = \begin{cases} \frac{1}{3} - \theta_0 & x = 0 \\ \frac{1}{3} & x = 1 \\ \frac{1}{3} + \theta_0 & x = 2 \end{cases}$$

for some value θ_0. You can assume $-\frac{1}{3} \leq \theta_0 \leq \frac{1}{3}$ so all probabilities are non-zero.

a. Solve the equation $E\left[X\right] = \bar{x}$ for θ_0. This is your method of moments estimate $\hat{\theta}_0^{mom}$.

Since X is a discrete random variable, we have

$$\begin{aligned} E\left[X\right] &= \sum_x x \cdot f(x), \\ &= 0 \times \left(\frac{1}{3} - \theta_0\right) + 1 \times \left(\frac{1}{3}\right) + 2 \times \left(\frac{1}{3} + \theta_0\right) = \frac{1}{3} + \frac{2}{3} + 2\theta_0 = 1 + 2\theta_0 \end{aligned}$$

and the methods of moments estimate is the value of θ_0 which solves the equation

$$E\left[X\right] = \bar{x}.$$

That is, we are solving for θ_0 in the equation:

$$1 + 2\,\theta_0 = \bar{x}$$

.

This gives:

$$\hat{\theta_0}^{mom} = \frac{\bar{x}-1}{2}$$

b. Based on the sample 0, 0, 1, 0, 1, 2, 1, 0, 0, calculate the value of $\hat{\theta}_0^{mom}$ for this sample.

```
x <- c(0,0,1,0,1,2,1,0,0)

xbar <- mean(x)

thetahat <-(xbar-1)/2; thetahat
```

```
## [1] -0.2222222
```

Our estimate of θ_0 is -0.22.

2. Suppose $X_1, X_2, \ldots, X_n$ are independent continuous random variables from the beta distribution with PDF

$$f(x) = (\theta_0^2 + \theta_0)x^{\theta_0 - 1}(1-x) \quad 0 \le x < 1$$

for some value $\theta_0 > 0$.

a. Solve the equation $E[X] = \bar{x}$ for θ_0. This is your method of moments estimate $\hat{\theta}_0^{mom}$.

Since X is a continuous random variable, we have

$$\begin{aligned}
E[X] &= \int_0^1 x \cdot f(x) \quad dx \\
&= (\theta_0^2 + \theta_0) \int_0^1 x^{\theta_0}(1-x) \quad dx = (\theta_0^2 + \theta_0) \int_0^1 (x^{\theta_0} - x^{\theta_0+1}) \quad dx \\
&= (\theta_0^2 + \theta_0) \left. \frac{x^{\theta_0+1}}{\theta_0+1} - \frac{x^{\theta_0+1}}{\theta_0+1} \right|_0^1 \\
&= \frac{\theta_0}{\theta_0+2}
\end{aligned}$$

The methods of moments estimate is the value of θ_0 which solves the equation

$$E[X] = \bar{x}.$$

That is, we are solving for θ_0 in the equation:

$$\frac{\theta_0}{\theta_0+2} = \bar{x} \Rightarrow \hat{\theta}_0^{mom} = \frac{2}{(1-\bar{x})}.$$

b. Based on the sample shown below, calculate the value of $\hat{\theta}_0^{mom}$ for this sample.

```
#generate sample of n=10 from the PDF f(x) which is
#a beta distribution with shape parameters theta0 and 2.

set.seed(5361)
theta0 <- 3   #true value of theta0
```

```r
#simulate obs.

x <- rbeta(n = 10, shape1 = theta0, shape2 = 2)
x
```

```
##  [1] 0.5041897 0.6650960 0.7740926 0.3894557 0.6755842 0.8240522 0.3667064
##  [8] 0.7228622 0.6655768 0.2951356
```

```r
#write code to calculate your estimate below

xbar <- mean(x)
thetahat <- 2/(1-xbar)
print(thetahat)
```

```
## [1] 4.857613
```

c. Rerun just the code to generate a new sample of x values and re-calculate your estimate for each new sample 5 times. What was the largest value for your estimate? Smallest?

```r
for(i in 1:5){
  x <- rbeta(n = 10, shape1 = theta0, shape2 = 2)
  xbar <- mean(x)
  thetahat <- 2/(1-xbar);
  print(thetahat)
}
```

```
## [1] 5.279476
## [1] 7.129078
## [1] 4.768064
## [1] 4.560907
## [1] 3.902967
```

3. If $\hat{\theta}_1$ and $\hat{\theta}_2$ are two unbiased estimators of a parameter θ and $Var\left[\hat{\theta}_1\right] < Var\left[\hat{\theta}_2\right]$, then $\hat{\theta}_1$ is said to be *more efficient* than $\hat{\theta}_2$.

   Suppose X_1, X_2, X_3 are independently and identically sampled from a distribution with true mean μ and variance σ^2. Consider the following two estimators for μ:

$$\hat{\mu}_1 = \frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3$$
$$\hat{\mu}_2 = \frac{1}{3}X_1 + \frac{1}{3}X_2 + \frac{1}{3}X_3$$

a. Show that both estimators are unbiased.

   Hint: Use Theorem 2.1 (i) to find the expectations

   An estimator $\hat{\mu}$ is said to be unbiased for the true value of the parameter μ_0 if

$$E\left[\hat{\mu}\right] = \mu_0.$$

The following shows why $\hat{\mu}_1$ is an unbiased estimator of μ.

$$\begin{aligned}
E[\hat{\mu}_1] &= E\left[\frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3\right], \\
&= \frac{1}{4}E[X_1] + \frac{1}{2}E[X_2] + \frac{1}{4}E[X_3], && \text{(linearity of expectation)} \\
&= \frac{1}{4}\mu + \frac{1}{2}\mu + \frac{1}{4}\mu, \\
&= \mu.
\end{aligned}$$

A similar proof will show that $\hat{\mu}_2$ is also an unbiased estimator of μ. In fact, any linear combination of the X_i, $\sum_{i=1}^n a_i X_i$ will be an unbiased estimator of μ so long as $\sum\limits_{i=1}^n a_i = 1$.

b. Which is the more efficient estimator?

Hint: Use Theorem 2.1 (ii) to find the variances

In order to answer this question, we need to compare the variance of the two estimators.

$$\begin{aligned}
Var[\hat{\mu}_1] &= Var\left[\frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3\right], \\
&= \frac{1}{16}Var[X_1] + \frac{1}{4}Var[X_2] + \frac{1}{16}Var[X_3], && \text{(independence)} \\
&= \frac{3\sigma^2}{8}.
\end{aligned}$$

A similar derivation will show that $Var[\hat{\mu}_2] = \frac{3\sigma^2}{9}$. Hence $\hat{mu}_2$ is the more efficient estimator of μ. It makes better use of the information in the data and is more precise.

In fact, $\hat{\mu}_2$ is the sample mean estimator and it has the smallest variance in the class of linear estimators.

4. When sampling $X_1, X_2, \ldots, X_n$ independently from a $Unif(0, \theta_0)$ distribution, we saw that the $X_{max} = \max\{X_1, X_2, \ldots, X_n\}$ was a biased estimator of θ_0 because

$$E[X_{max}] = \frac{n}{n+1}\theta_0.$$

a. Create an unbiased estimator of θ_0 based on X_{max}. Call this estimator $\hat{\theta}_0$.

Hint: an estimator can be based on random variables and numbers like n. The only thing off limits are unknown parameters

For our estimator, $\hat{\theta}_0$ to be unbiased, we must have that $E\left[\hat{\theta}_0\right] = \theta_0$. From the info in the problem we know:

$$E[X_{max}] = \frac{n}{n+1}\theta_0 \Rightarrow E\left[\frac{n+1}{n}X_{max}\right] = \theta_0$$

Thus, the estimator $\frac{n+1}{n}X_{max}$ is unbiased.

b. Derive the variance of $\hat{\theta}_0$. Is its variance larger or smaller than the $Var[X_{max}]$?

Hint: Recall it was mentioned in class that $Var(X_{max}) = \frac{n\theta_0^2}{(n+2)(n+1)^2}$. You may use this fact without proof.

We wish to find $Var(\hat{\theta}_0) = Var(\frac{n+1}{n}X_{max})$.

By properties of Variance we know:

$$
\begin{aligned}
Var\left[\frac{n+1}{n}X_{max}\right] &= \left(\frac{n+1}{n}\right)^2 Var(X_{max}) \\
&= \left(\frac{n+1}{n}\right)^2 \frac{n\theta_0^2}{(n+2)(n+1)^2} \\
&= \frac{\theta_0^2}{n(n+2)}
\end{aligned}
$$

Since we know that $\frac{n+1}{n} > 1$ we have that $Var(\hat{\theta}_0) = \left(\frac{n+1}{n}\right)^2 Var(X_{max}) > Var(X_{max})$

c. A common way to compare estimators that are not both unbiased is via the Mean Square Error (MSE) which accounts for both the bias and variance of an estimator.

$$MSE(\hat{\theta}_0) = (bias)^2 + Var\left[\hat{\theta}_0\right]$$

where the bias term is defined as

$$bias = E\left[\hat{\theta}_0\right] - \theta_0.$$

An estimator with a smaller MSE is preferred. Find $MSE(X_{max})$, the MSE of X_{max} and also $MSE(\hat{\theta}_0)$.

We have the bias of $Bias(X_{max}) = \frac{n}{n+1}\theta_0 - \theta_0 = \frac{-\theta_0}{n+1}$

So putting this together with the Variance of X_{max} we derived above we have that:

$$MSE(X_{max}) = \frac{n\theta_0^2}{(n+2)(n+1)^2} + \frac{\theta_0^2}{(n+1)^2}$$

Since we know from part a that $\hat{\theta_0}$ is unbiased (bias = 0), this means that

$$MSE(\hat{\theta_0}) = Var(\hat{\theta_0}) = \frac{\theta_0^2}{n(n+2)}.$$

d. Which estimator has the smaller MSE? Compare $MSE(X_{max})$ with $MSE(\hat{\theta}_0)$.

To compare the MSE's we can simply analyze the ratio $\frac{MSE(X_{max})}{MSE(\hat{\theta}_0)}$.

Thus we are analyzing the ratio:

$$
\begin{aligned}
\frac{\frac{n\theta_0^2}{(n+2)(n+1)^2} + \frac{\theta_0^2}{(n+1)^2}}{\frac{\theta_0^2}{n(n+2)}} &= \frac{\frac{n}{(n+2)(n+1)^2} + \frac{1}{(n+1)^2}}{\frac{1}{n(n+2)}} \\
&= \frac{\frac{2n+2}{(n+2)(n+1)^2}}{\frac{1}{n(n+2)}} \\
&= \frac{n(2n+2)}{(n+1)^2} \\
&= \frac{2n(n+1)}{(n+1)^2} \\
&= \frac{2n}{n+1}
\end{aligned}
$$

This is larger than 1 whenever $n > 1$. So the MSE of X_{max} is larger than the MSE of $\hat{\theta}_0$.