Problem Section 6

Monday Nov 11 2024

Learning Outcomes

The problems are designed to build conceptual understanding and problem-solving skills. The emphasis is on learning to find, evaluate and build confidence. The specific tasks include:

- Implement a parametric and non-parametric bootstrap scheme

- Calculate bootstrap confidence intervals

- Back up and support work with relevant explanations

Instructions

- Please submit your completed file in CANVAS

Exercises

1. (Parametric bootstrap) Consider the following set of 25 numbers which are independently drawn from a normal distribution with mean 0 and variance $\sigma_0^2 = 1$.

```
set.seed(1225)

x <- rnorm(25, mean = 0, sd = 1)

x
```

```
##  [1]  1.373345216 -1.321717373 -1.130430171  0.174318722  1.572396621
##  [6] -1.033720544  0.528722173  0.433138549  0.359519501 -1.127509844
## [11]  0.572745921  1.164382652  1.058067005  0.195217444  0.156435394
## [16] -0.445555920 -0.930436079  1.185193882 -0.610439478 -0.068488257
## [21]  0.565900170 -0.322861038 -0.203976031  0.002425486  0.280508433
```

In this problem, we are interested in bootstrapping the sampling distribution of the method of moments estimator of σ_0^2, namely:

$$\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Since we know the data have been drawn from a normal distribution, we will use the parametric bootstrap method to accomplish this.

a. First, let's calculate the observed value of the estimator $\hat{\sigma}_0^2$ for this sample and save it in a variable called `obs_sigma2hat`. Report its value using inline code rounded to four digits.

```
nobs <- length(x)

obs_sigma2hat <- (nobs-1)*var(x)/nobs
```

The value of my estimate of σ_0^2 is 0.6549.

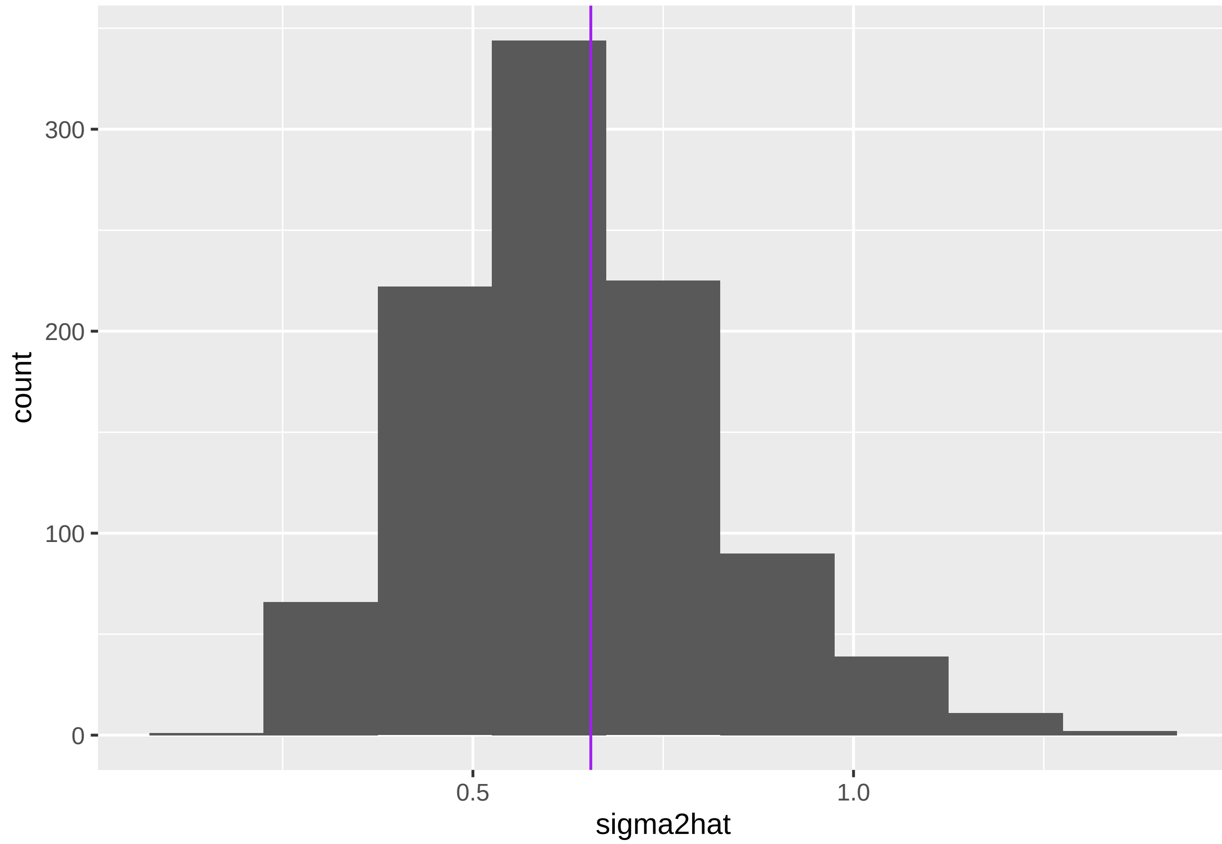
b. Now implement a parametric bootstrap scheme to approximate the sampling distribution of $\hat{\sigma}_0^2$. Generate $B = 1,000$ bootstrapped estimates and save them in a data frame called `boot_df`. For reproducibility, let's use 1234 as the random number seed. the first few rows to look at it. (Consult slides 9 - 13 in the slide deck 6.1 - 6.2.)

```
B=1000
set.seed(124)




boot_df <- tibble(
   sigma2hat = replicate( n = B,
                          expr = (nobs-1)*var(rnorm(n = nobs,
                                                    mean = 0,
                                                    sd = sqrt(obs_sigma2hat) ) )/nobs)
  )



boot_df %>% slice_head(n = 5)
```

```
## # A tibble: 5 x 1
##   sigma2hat
##       <dbl>
## 1     0.519
## 2     0.289
## 3     0.429
## 4     0.706
## 5     0.318
```

c. Create a histogram of the bootstrapped estimates. Comment on the shape.

```
ggplot( data = boot_df,
        mapping = aes(x = sigma2hat)) +
  geom_histogram(binwidth = 0.15) +
  geom_vline(xintercept = obs_sigma2hat, color = "purple")
```



The bootstrapped sampling distribution of $\hat{\sigma}_0^2$ is skewed to the right.

d. Calculate the average and standard deviation of the bootstrapped estimates. Calculate the bootstrap estimate of the bias in the estimator.

```
boot_summary <- boot_df %>%
                summarise( mean_boot_est = mean(sigma2hat),
                           sd_boot_est = sd(sigma2hat) )

boot_bias <- boot_summary$mean_boot_est - obs_sigma2hat
```

e. Revisit your answer from part a and bias correct your estimate in light of your answer in d. Call the bias corrected estimate as `unbias_obs_sigma2hat`

```
unbias_obs_sigma2hat <- obs_sigma2hat - boot_bias
```

The bias corrected estimate of σ_0^2 is 0.6758.

2. (Non-parametric bootstrap) Consider the following set of 20 numbers which were independently drawn from some (unknown) distribution with mean μ_0 and variance σ_0^2.

```
x  = c(3.56, 0.69, 0.10, 1.84, 3.93,
       1.25, 0.18, 1.13, 0.27, 1.21,
       0.50, 0.67, 0.01, 0.61, 0.82,
       1.70, 0.39, 0.11, 1.20, 0.72)
```

In this problem we are interested in constructing a confidence interval estimate for σ_0^2 using the method of moments estimator

$$\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

a. Calculate the observed value of the estimator $\hat{\sigma}_0^2$ for this sample and save it in a variable called `obs_sigma2hat`. Report its value using inline code rounded to four digits.

```
# calculate obs_sigma2hat for the observed sample
```

Now we wish to construct the sampling distribution of $\hat{\sigma}_0^2$. Since we do not know the actual distribution that these data were sampled from, we will use the non-parametric bootstrap method this time to help us create an approximate sampling distribution.

b. Follow the steps below to construct the bootstrapped sampling distribution of $\hat{\sigma}_0^2$. You can look at pages 16 - 19 of the slidedeck labeled "6.1 - 6.2")

- Take B random samples of size 20 each from the observed sample data with replacement and calculate the value of the bootstrapped estimate - call it sigma2hat - for each resample. Save the estimates in a data frame called `boot_df`.
- Make a histogram of the bootstrapped estimates. Examine the shape. Calculate the mean and standard deviation of the bootstrapped estimates.
- If the bootstrap results indicate evidence of bias in the estimator, adjust your point estimate from part a accordingly to calculate a bias-corrected version.

```
set.seed(1415151)

B = 1000
```

c. Calculate a 90% bootstrap percentile interval estimate for σ_0^2 and report it (rounded to 4 digits) in a sentence using inline code.