Homework 3 Key

Independence

Instructions

Please answer the following questions in the order in which they are posed. Add a few empty lines below each and write your answers there. **Focus on answering in complete sentences and show work whether we ask for it or not**. You will also need scratch paper/pen to work out the answers before typing it.

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Exercises

1. The random variables X and Y are independent, each taking the values 1, 2 or 3. Complete the following table of the joint PMF. Show your work for each entry below the table.

$y \backslash x$	1	2	3
1	0.03	0.04	0.03
2	0.15	(b) =.2	(c) =.15
3	(a) = .12	(d) =.16	(e) = .12

We wish to find f(y,x). Since these are independent RV's we know f(x,y) = f(x)f(y).

From our table we can extract that f(y=1) = .03+.04+.03 = .1.

This implies that f(x=1) = f(x=1,y=1)/f(y=1) = .03/.1 = .3.

Similarly we have, f(x=2) = .4, and f(x=3) = .3.

Using this information we know that for (a), we must have that .03 + .15 + (a) = f(x=1) = .3.

This gives us that (a) = .12.

We also see that f(x=1,y=2) = 0.15 = f(x=1)f(y=2) giving us that f(y=2) = .5.

Thus all in all so far we have that:

f(y=1) = 0.1

f(y=2) = .5

f(y=3) = .4 (since .1+.5+.4 = 1)

f(x=1) = .3

f(x=2) = .4

f(x=3) = .3

From here we see that:

(b) = f(x=2)f(y=2) = .5*.4 = .2

(c) = f(y=2)f(x=3) = .5*.3 = .15

(d) = f(y=3)f(x=2) = .4*.4 = .16

(e) = f(y=3)f(x=3) = .4*.3 = .12

2. Suppose the probabilistic behavior of $\langle X, Y \rangle$ is governed by the joint PDF

$$f(x, y) = cx^2\, y^4\, e^{-y}\, e^{-\frac{x}{2}} \quad 0 < x,\ 0 < y.$$

a. Determine c. Show work.

Hint: you will need to remember properties of the Gamma function $\Gamma(k)$ which is defined for $k > 0$ as:

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$$

We must use the property here that

$$\int_x \int_y f(x, y) dy dx = 1$$

So we have:

$$\int_0^\infty \int_0^\infty cx^2\, y^4\, e^{-y}\, e^{-\frac{x}{2}} dx dy = c \int_0^\infty y^4 e^{-y} dy \int_0^\infty x^2 e^{-x/2} dx$$

Where we use properties of multiple integrals, since the equation can be separated into a function of x times a function of y.

Let us consider the dy integral first:

$$\int_0^\infty y^4 e^{-y} dy$$

Using our hint, we have that this resembles the form $\Gamma(k = 5)$.

Now we can move onto our dx integral.

$$\int_0^\infty x^2 e^{-x/2} dx$$

Letting $u = x/2$, we get the integral:

$$2 \int_0^\infty (2u)^2 e^{-u} du = 8 \int_0^\infty u^2 e^{-u} du = 8\Gamma(k = 3).$$

Thus we have that:

$$\int_0^\infty \int_0^\infty cx^2\, y^4\, e^{-y}\, e^{-\frac{x}{2}} dx dy = \Gamma(5) \times 8 \times \Gamma(3).$$

Thus we have that $c = \frac{1}{8\Gamma(5)\Gamma(3)} = 0.0026042$

b. Are X and Y are independent? Explain.

By Def 15.1, X and Y are independent if and only if

$$f(x,y) = f_1(x)\, f_2(y) \quad \forall\, x, y$$

where f_1 and f_2 are the marginal distributions of X and Y respectively. Let's find them.

By definition, for $x > 0$:

$$\begin{aligned}
f_1(x) &= \int_{-\infty}^{\infty} f(x,y)\, dy \\
&= \frac{1}{8\Gamma(5)\Gamma(3)} \int_0^{\infty} x^2\, y^4\, e^{-y}\, e^{-\frac{x}{2}}\, dy \\
&= \frac{1}{8\Gamma(5)\Gamma(3)} x^2 e^{-\frac{x}{2}} \underbrace{\int\limits_0^{\infty} y^4 e^{-y} dy}_{\Gamma(5)} \\
&= \frac{1}{8\Gamma(3)} x^2 e^{-\frac{x}{2}} \qquad x > 0
\end{aligned}$$

Similarly for $y > 0$:

$$\begin{aligned}
f_2(y) &= \int_{-\infty}^{\infty} f(x,y)\, dx \\
&= \frac{1}{8\Gamma(5)\Gamma(3)} \int_0^{\infty} x^2\, y^4\, e^{-y}\, e^{-\frac{x}{2}}\, dx \\
&= \frac{1}{8\Gamma(5)\Gamma(3)} y^4 e^{-y} \int\limits_0^{\infty} x^2 e^{-\frac{x}{2}} dx \\
&= \frac{1}{8\Gamma(5)\Gamma(3)} y^4 e^{-y}\, 2 \int\limits_0^{\infty} (2u)^2\, e^{-u} du \qquad u = \frac{x}{2} \\
&= \frac{1}{8\Gamma(5)\Gamma(3)} y^4 e^{-y}\, 2^3 \underbrace{\int\limits_0^{\infty} u^2\, e^{-u} du}_{\Gamma(3)} \\
&= \frac{1}{\Gamma(5)} y^4 e^{-y} \qquad y > 0
\end{aligned}$$

Therefore, X and Y are independent.

You could also "prove" this by noting that the joint PDF is a product of two functions, one of which depends on x and the other depends on y. In addition, the support of the distribution is *rectangular*. These conditions ensure independence of X and Y.

3. Two friends - let's call them Henry and Vincent - agree to meet at Tully's for coffee. Suppose that the random variables

$$X = \text{the time that Henry arrives at Tully's and}$$

$$Y = \text{the time that Vincent arrives at Tully's}$$

are independent uniform random variables on the interval $[5, 6]$. (The units are hours after noon)

a. Calculate the probability that both of them arrive before 5:30 PM.

We know that the probability that both people arrive before 5:30, means that $X < 5.5 \cap Y < 5.5$. Since these are independent RV's we have that:

$$P(X < 5.5 \cap Y < 5.5) = P(X < 5.5) \times P(Y < 5.5) = (\frac{6-5.5}{6-5})^2 = .25$$

So the probability that both of them arrive before 5:30 is .25.

We can also consider this problem using the maximum. We know that both arriving before 5:30 is equivalent to $Max(X, Y) < 5.5$. Using our Theorems we know that:

$$F_{max}(x) = [F(x)]^n = [\frac{5-x}{6-5}]^2$$

This will yield the equivalent answer by plugging in x=5.5.

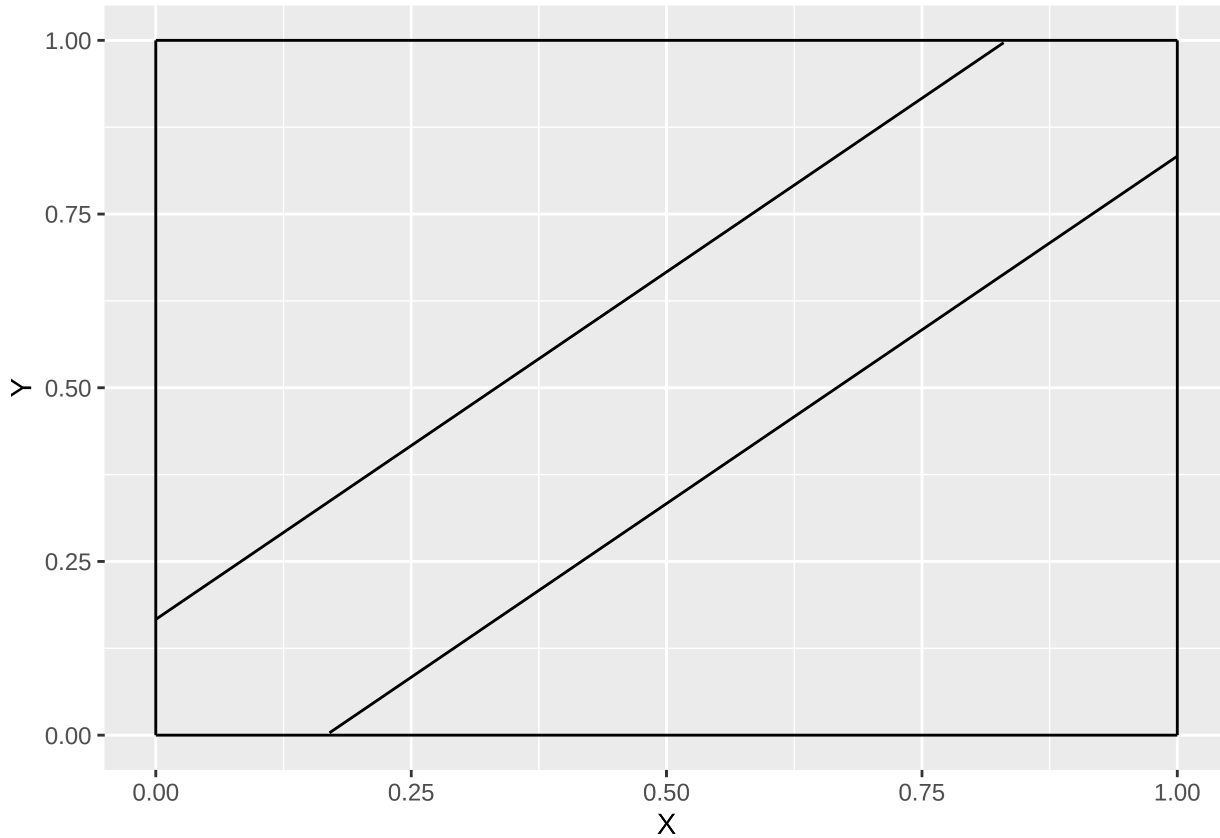
b. If neither of them is willing to wait more than 10 minutes for the other to show up, what is the probability they have a coffee together? That is, what is the probability they arrive within 10 minutes of each other.

Hint: Here you want to find the probability that $|X - Y|$ is less than some number.

We wish to find $P(|X - Y| < 1/6)$. We see that in this case we have $f(x, y) = 1 \times 1 = 1$

Drawing this out we have:

```
xline <- seq(0,1,.01)
y_0 <- rep(0,length(xline))
x_0 <- y_0
y_1 <- rep(1,length(xline))
x_1 <- y_1
yline <- seq(0,1,.01)
ggplot(data=NULL)+geom_line(aes(x=xline,y=y_0))+
  geom_line(aes(x=yline,y=y_1))+
  geom_line(aes(x=x_0,y=yline))+
  geom_line(aes(x=x_1,y=yline))+
  geom_line(aes(x=xline,y=xline-1/6))+
  geom_line(aes(x=xline,y=xline+1/6))+
  xlim(c(0,1))+
  ylim(c(0,1))+
  labs(x = "X",
       y="Y")
```



In this picture, we wish to integrate $f(x, y) = 1$ between those two lines (the area where the difference is less than 1/6). We see that this is equivalent to solving the integrals:

$$P(|X - Y| < 1/6) = \int_0^{1/6} \int_0^{x+1/6} 1dydx + \int_{1/6}^{5/6} \int_{x-1/6}^{x+1/6} 1dydx + \int_{5/6}^{1} \int_{x-1/6}^{1} 1dydx$$

Solving these integrals, we yield that $P(|X - Y| < 1/6) = \frac{11}{36}$.

Thus the probability that the two meet is 11/36.

4. An individual makes 100 check transactions between receiving his December and his January bank statements. Rather than subtracting the amounts exactly, he rounds off each checkbook entry to the nearest dollar. Let X_i denote the round off error on the *ith* check. A reasonable assumption is that

$$X_i \sim Unif\left(-\frac{1}{2}, \frac{1}{2}\right)$$

independently of each other.

Use the Bienaymé-Chebyshev inequality to get an upper bound for the probability that the accumulated error after his 100 transactions is $5 or more. Show work.

Let T denote the total round off errors for the transactions.

We have that $T = X_1 + \cdots + X_{100}$

We wish to find $P(|T| > 5)$. We take the absolute value because the error could be negative or positive but we wish to find whether it's magnitude is greater than 5.

We see that $E[T] = E[X_1 + \cdots + X_{100}] = E[X_1] + \cdots + E[X_{100}] = 0$ by using linearity of expectation and the fact that $E[X_i] = 0$.

We also see that, using the variance formula for a Uniform RV, that $Var(X_i) = \frac{(1/2-(-1/2))^2}{12} = \frac{1}{12}$.

Using the fact that each error is independent we have:

$Var(T) = Var(\sum_{i=1}^{100} X_i) = \sum_{i=1}^{100} Var(X_i) = \frac{100}{12}$.

Thus we have that $E[T] = 0$ and $SD(T) = \sigma = \sqrt{100/12} = \frac{5}{\sqrt{3}}$

Now we may begin to utilize Chebyshev's inequality.

$$\begin{aligned}
P(|T| > 5) &= P(|T-0| > 5) \\
&= P(|T-0| > \frac{5}{\sqrt{3}}\sqrt{3}) \\
&= P(|T-0| > \sigma\sqrt{3}) \\
&\leq \frac{1}{\sqrt{3}^2} \\
&= \frac{1}{3}
\end{aligned}$$