# Homework 1 KEY

Winter 2023

ANSWER KEY

**Instructions**

- This homework is due in Gradescope on Wednesday Jan 11 by midnight PST.
- Please answer the following questions in the order in which they are posed.
- Don't forget to knit the document frequently to make sure there are no compilation errors.
- When you are done, download the PDF file as instructed in section and submit it in Gradescope.

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**Exercises**

1. (Gizmo) The internal temperature in a gizmo (device) is a random variable X with PDF (in appropriate units)
$$f(x) = 11 \cdot (1-x)^{10}, \quad 0 < x < 1$$

a. The gizmo has a cutoff feature, so that whenever the temperature exceeds the cutoff - call it $k$ - it turns off. It is observed that the gizmo shuts off with probability $10^{-22}$. What is k?

   We have that $P(X > k) = 10^{-22}$.

   First we may find the CDF of the RV X.

$$P(X < x) = \int_0^x 11(1-x)^{10} dx = -(1-x)^{11}|_0^x = 1 - (1-x)^{11}$$

Thus we have that $P(X > k) = 1 - (1 - (1-k)^{11}) = (1-k)^{11}$.

Thus to solve for k we have:

$$(1-k)^{11} = 10^{-22} \Rightarrow 1 - k = 10^{-2} \Rightarrow k = 1 - 10^{-2}$$

Thus we have that cutoff temperature k is 0.99 degrees.

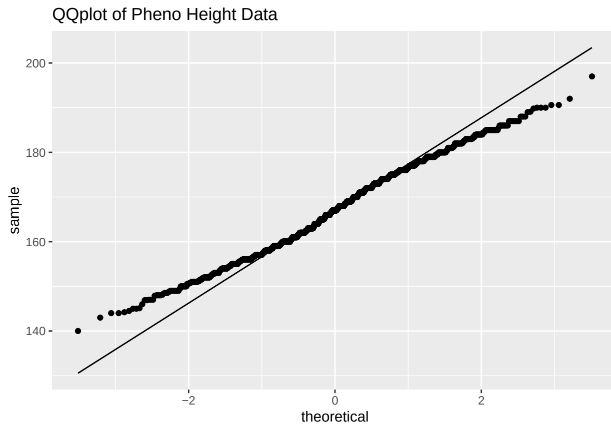
b. Fill in the blank: the number $k$ is the $1 - 10^{-22}$ quantile of the distribution of $X$.

2. Pruim problem 3.46 on page 221 (Please review section 13.2 from notes for how to make QQplots. You can use `group = sex` in the mapping in order to create the conditional plots for part b.)

a)

First we can find the QQplot of all the height data.

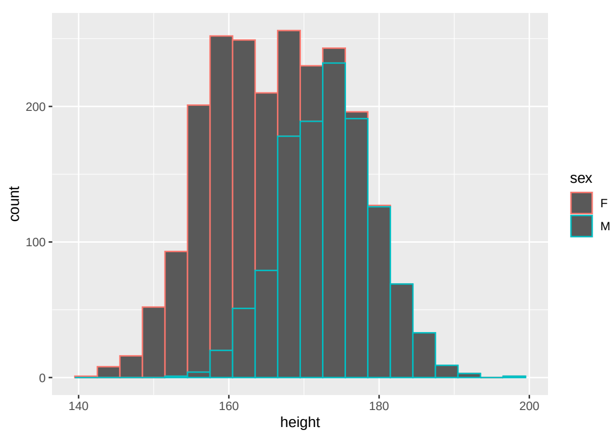
```
ggplot(data=Pheno,mapping = aes(sample=height)) +
  stat_qq(distribution=qnorm,color="black") +
  stat_qqline(distribution=qnorm) +
  labs(title="QQplot of Pheno Height Data")
```




We see that when we construct a normal QQplot for all the data, the data fails to be normal at the tails. It is instructive to take a look at the histogram of the data. As we can see, the shape is bi-modal (two peaks) suggesting that the set of heights are a mixture of two distributions: height distribution of the women is shifted to the left (shorter) compared to the distribution of heights for men.

By combining these data into one distribution and comparing with a single normal density curve, we see that the mean of the normal is somewhere in between the two peaks. The variance is also larger than the variance for either of the two height distributions. This is why the quantiles from the normal are more extreme when compared to the quantiles of the data. The normal quantiles are smaller on the left (points lie above $y = x$ line) and larger on the right (points lie below $y = x$ line).

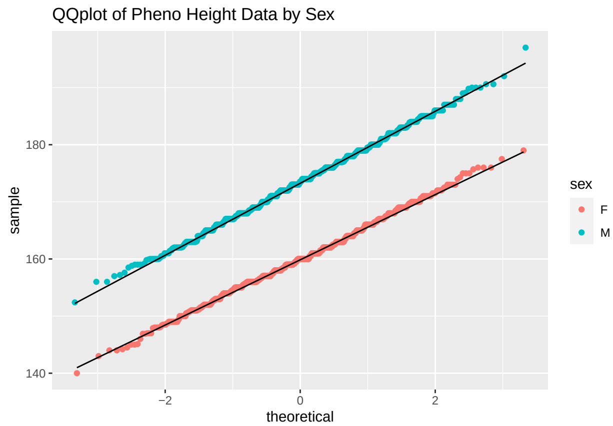
```
ggplot(data = Pheno,
       mapping = aes(x = height, color=sex) )+
  geom_histogram(binwidth=3)
```




b)

Once we group by Sex, however, the height data fits the normal densities quite well as shown by the following QQplots.

```
data(pheno)
ggplot(data=Pheno,mapping = aes(sample=height,group=sex)) +
  stat_qq(distribution=qnorm,aes(color=sex)) +
  stat_qqline(distribution=qnorm) +
  labs(title="QQplot of Pheno Height Data by Sex")
```




3. (CDF method) Suppose $X$ is a gamma random variable with shape parameter $k\,(>0)$ and rate parameter $\lambda\,(>0)$. In other words, the PDF of $X$ is given by:

$$f_X(x) = \frac{\lambda^k}{\Gamma(k)}\, x^{k-1}\, e^{-\lambda\, x}, \qquad x > 0,$$

where $\Gamma(k)$ is the gamma function. Review 13.3 from the notes if you need a reminder.

Define

$$Y = \frac{1}{X}.$$

Show, using the CDF method, that $Y$ has PDF

$$f(y) = \frac{\lambda^k}{\Gamma(k)}\,\frac{1}{y^{k+1}} e^{-\frac{\lambda}{y}} \qquad y > 0$$

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(\frac{1}{X} \le y) \\ &= P(X \ge \frac{1}{y}) \\ &= 1 - P\left(X < \frac{1}{y}\right), \\ &= 1 - F_X\left(\frac{1}{y}\right) \end{aligned}$$

We know by the definition of PDF's that $f_Y(y) = \frac{d}{dy}F_Y(y)$.

Thus, by the chain rule of differentiation we have that:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(1 - F_X(1/y)) = \frac{1}{y^2}f_X(1/y)$$

Plugging in our values to the appropriate densities we thus have:

$$f(y) = \frac{1}{y^2}\frac{\lambda^k}{\Gamma(k)}\, y^{1-k}\, e^{-\lambda\,\frac{1}{y}} = \frac{\lambda^k}{\Gamma(k)}\,\frac{1}{y^{k+1}}e^{-\frac{\lambda}{y}}$$

Since we have that $X > 0$, clearly we have that $Y = \frac{1}{X} > 0$.

This matches the form of the appropriate density.

4. (Beta dist) The beta distribution is a probability distribution that is often used in applications where the random variable is a proportion. The beta PDF depends on two *shape* parameters - call them $\alpha\,(>0)$ and $\beta\,(>0)$ - and is given by:

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\,x^{\alpha-1}\,(1-x)^{\beta-1} \quad 0<x<1.$$

a. Show that the uniform distribution on (0,1) is a special case of the beta when $\alpha=\beta=1$.

Using the fact that $\Gamma(k)=(k-1)!$ when k is a positive integer, we have that our density will be:

$$f(x) = \frac{\Gamma(2)}{\Gamma(1)\,\Gamma(1)}\,x^{1-1}\,(1-x)^{1-1} = \frac{(2-1)!}{0!\,0!} = 1 \quad x>0$$

We see that this is exactly the form uniform density from 0 to 1.

b. The beta distribution offers a very flexible array of shapes for modeling data. Run the following code interactively from your Console to see how it changes with different parameter values. Feel free to try other values for $\alpha$ and $\beta$. Then describe the impact of the parameters $\alpha$ and $\beta$ on the shape. Specifically when is it symmetric? When is it skewed to the right? Skewed to the left? Bowl shaped? (Don't just state the $\alpha, \beta$ values, but rather provide a meaningful summary of the behavior )

We see that for small values (<1) of $\alpha$ we skew to the right. Similarly for small values of $\beta$ we have a skew to the left. When both of these parameters are small we have a bowl shape. When both parameters are greater than 1, we have a bell curve type shape that is shifted towards the the right when $\alpha>\beta$ and vice verse for the opposite case.

c. Since the beta distribution is a valid PDF, it must integrate to 1. In other words, it must be the case that

$$\int\limits_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Use this fact to show that the *rth* moment of the Beta is given by

$$E\left[X^r\right] = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+r)}.$$

We have that:

$$\begin{aligned}
E(X^r) &= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^r x^{\alpha-1}(1-x)^{\beta-1}dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{r+\alpha-1}(1-x)^{\beta-1}dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(r+\alpha)\Gamma(\beta)}{\Gamma(r+\alpha+\beta)} \text{ using the fact that beta densities integrate to one} \\
&= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+r)}
\end{aligned}$$

d. Use your result from part c. to show that

$$\begin{aligned}
\mu = E[X] &= \frac{\alpha}{\alpha+\beta}, \\
\sigma^2 &= Var[X] \\
&= \frac{\mu(1-\mu)}{\alpha+\beta+1}
\end{aligned}$$

We have that:

$$\begin{aligned}
\mu &= E(X^1) \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)} \\
&= \frac{(\alpha+1-1)\times(\alpha+1-2)\times\dots}{(\alpha-1)\times(\alpha-2)\times\dots} \frac{(\alpha+\beta-1)\times(\alpha+\beta-2)\times\dots}{(\alpha+\beta+1-1)\times(\alpha+\beta+1-2)\times\dots} \\
&= \frac{\alpha}{\alpha+\beta}
\end{aligned}$$

We also have that:

$$E(X^2) = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)}$$

Using similar logic as above we have that:

$$E(X^2) = \frac{(\alpha+1)(\alpha)}{(\alpha+\beta+1)(\alpha+\beta)} = \frac{\alpha+1}{\alpha+\beta+1} \times \mu$$

Since we know $Var(X) = E(X^2) - E(X)^2$ we have:

$$\begin{aligned}
Var(X) &= \frac{\alpha+1}{\alpha+\beta+1} \times \mu - \mu^2 = \mu(\frac{\alpha+1}{\alpha+\beta+1} - \mu) \\
&= \mu(\frac{(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta)}{(\alpha+\beta+1)(\alpha+\beta)}) \\
&= \mu(\frac{(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)}) \\
&= \mu(\frac{\alpha+\beta-\alpha}{(\alpha+\beta+1)(\alpha+\beta)}) \\
&= \mu(\frac{\beta}{\alpha+\beta} \times \frac{1}{\alpha+\beta+1}) = \frac{\mu(1-\mu)}{\alpha+\beta+1}
\end{aligned}$$