Homework 8

Significance Testing

Instructions

Please answer the following questions in the order in which they are posed. Add a few empty lines below each and write your answers there. **Focus on answering in complete sentences and show work whether we ask for it or not**. You will also need scratch paper/pen to work out the answers before typing it.

For help with formatting documents in RMarkdown, please consult R Markdown: The Definitive Guide. Another option is to search using Google.

Exercises

1. (NHL) After a 2010 NHL playoff win in which Detroit Red-Wings wingman Henrik Zetterberg scored two goals in a 3-0 win over the Phoneix Coyotes. Detroit coach Mike Babcock said "He's been really good at playoff time each and every year. *He seems to score at a higher rate in playoffs compared to the regular season."*

   In 506 **regular season games**, Zetterberg scored 206 goals (goal scoring rate of $\frac{206}{506} = 0.407$). In 89 **playoff games**, he scored 44 goals (goal scoring rate of $\frac{44}{89} = 0.494$) Clearly, he has a higher goal scoring rate in the playoffs, but can it be explained by the vagaries of random chance? Or is the difference statistically significant?

   Let X denote the number of goals he scores in the 89 playoff games. Assume

   $$X \sim Pois(89 \times \lambda_0)$$

   where λ_0 is his goal scoring rate per game during the playoffs.

   a. State the null and alternative hypothesis.

   $H_0 : \lambda_0 = .407$, $H_1 : \lambda_0 > .407$

   b. Calculate the P-value and summarize what you learn from it. Show your code for calculating the P-value. (You do not need to show the entire Poisson distribution)

   We have that Zetterberg scored 44 goals in the 89 post season games. We see that more goals scored indicates more support for the alternative hypothesis. Thus our p-value, the probability of seeing our observed data or more extreme under the null, will be the $P(X \geq 44)$ when we have that $\lambda_0 = .407$. We can calculate this below:

```
null_lam <- .407
1-ppois(43,lambda = null_lam*89 )
```

```
## [1] 0.1153466
```

   c. Suppose Coach Babcock had said "He's a really different player in the playoffs". How would your answers to a and b change? Show your code for re-calculating the P-value.

If Coach had said he is a different player then we would consider the hypotheses:

$H_0 : \lambda_0 = .407$ and $H_1 : \lambda_0 \neq .407$

Thus we will consider for our p-value, the probability of seeing our data or more extreme in both directions under the null. We know one direction will be $P(X \geq 44)$. We may now examine the Poisson pmf to understand where the probability of goals scored is lower than the probability of 44 goals scored on the left side of the distribution.

```
pois_prob <- tibble(
     x = 0:44,
     f = dpois(x, lambda = 0.407*89),
     less_than_P_44 = dpois(x, lambda = 0.407*89) < dpois(44,lambda = .407*89)
)

pois_prob %>% filter(less_than_P_44 == TRUE) %>% arrange(desc(x))
```

```
## # A tibble: 29 x 3
##        x        f less_than_P_44
##    <int>    <dbl> <lgl>
##  1    28 0.0273   TRUE
##  2    27 0.0211   TRUE
##  3    26 0.0157   TRUE
##  4    25 0.0113   TRUE
##  5    24 0.00779  TRUE
##  6    23 0.00516  TRUE
##  7    22 0.00328  TRUE
##  8    21 0.00199  TRUE
##  9    20 0.00115  TRUE
## 10    19 0.000637 TRUE
## # ... with 19 more rows
```

We see 28 is the largest value such that the probability is lower than seeing 44 goals. Thus our p-value will be:

$$p_value = P(X \geq 44) + P(X \leq 28)$$

We have this as:

```
ppois(28,.407*89)+ (1-ppois(43,.407*89))
```

```
## [1] 0.2113784
```

2. (Sign test) Suppose $Y_1, Y_2, \ldots, Y_{19}$ are independent random variables drawn from some distribution, and we are interested in the parameter $\pi_0 = P(Y < 0)$ More precisely, we want to test $H_0 : \pi_0 = 0.4$ against $H_1 : \pi_0 < 0.4$ at the 5% level.

   A reasonable test statistic would be X, the number of negative observations in the sample. That is,

$$X = X_1 + X_2 + \cdots + X_{19}$$

   where

$$X_i = \begin{cases} 1 & Y_i < 0 \\ 0 & otherwise \end{cases}$$

   a. What is the sampling distribution of X if the null hypothesis is true? Give a brief justification for your answer.

   We have that each Y is independent and identically distributed, meaning that each X_i is independent and has the same probability of equaling one for every X_i. We also have a fixed number of trials with a binary outcomes. Thus X will be a binomial distribution. Under the null we will have:

$$X \sim Binom(n = 29, \pi_0 = .4)$$

b. For what values of X would you reject the null hypothesis? Support your answer showing code/output as necessary.

We would reject the null when the probability of seeing X or more extreme (in this case smaller X since the alternative is one sided) is less than 0.05. We may examine the Binomial CDF to find the threshold value for X:

```
binom_tib <- tibble(
  x = 0:19,
  cdf = pbinom(x,19,.4),
  less_than_05 = cdf<.05
)
binom_tib
```

```
## # A tibble: 20 x 3
##        x       cdf less_than_05
##    <int>     <dbl> <lgl>
##  1     0 0.0000609 TRUE
##  2     1 0.000833  TRUE
##  3     2 0.00546   TRUE
##  4     3 0.0230    TRUE
##  5     4 0.0696    FALSE
##  6     5 0.163     FALSE
##  7     6 0.308     FALSE
##  8     7 0.488     FALSE
##  9     8 0.667     FALSE
## 10     9 0.814     FALSE
## 11    10 0.912     FALSE
## 12    11 0.965     FALSE
## 13    12 0.988     FALSE
## 14    13 0.997     FALSE
## 15    14 0.999     FALSE
## 16    15 1.00      FALSE
## 17    16 1.00      FALSE
## 18    17 1.00      FALSE
## 19    18 1.00      FALSE
## 20    19 1         FALSE
```

Examining the table above, we would reject the null hypothesis at the 5% level if we observed $X \leq 3$.

c. What is the Type I error probability of your test? Support your answer showing code/output as necessary.

Our type 1 error will be the $P(X \leq 3)$ since that is the rejection region we are considering. We have this as:

```
pbinom(3,19,.4)
```

```
## [1] 0.02295932
```

d. Calculate the Type II error probability when $\pi_0 = 0.2$. Support your answer showing code/output as necessary.

We know we will fail to reject the null when $X > 3$. Thus the type 2 error will be $P(X > 3)$ when we have that $\pi_0 = .2$. We have this as:

```
pbinom(3,19,.2,lower.tail=F)
```

```
## [1] 0.5449113
```

3. (Sign test again) Suppose in problem 2,you know that Y is normally distributed with mean μ_0 and (known) standard deviation $\sigma_0 = 1$.

   a. Re-state the null and alternative hypothesis from problem 2 in terms of claims about μ_0.

We know from Problem 3, the null states $P(Y < 0) = .4$. Thus we must find a μ_0 so that the $P(Y < 0) = .4$ when $Y \sim N(\mu_0, 1)$. We may now find this μ_0:

$$P(Y < 0) = P(\frac{Y - \mu_0}{1} < \frac{0 - \mu_0}{1}) = .4$$

So we have that $-\mu_0$ is the .4 quantile of a N(0,1) distribution. Thus using qnorm we have that:

$\mu_0 = 0.2533471$.

We may test this value as well to check that the probability of $Y < 0$ is equal to .4:

```
mu_0_null <- -qnorm(.4)
pnorm(0,mean = mu_0_null, 1)
```

```
## [1] 0.4
```

For the alternative, we are testing $P(Y < 0) < .4$. This implies the Y's take larger values, and thus have a greater mean.

Thus we are testing the hypotheses:

$H_0 : \mu_0 = -qnorm(.4) = 0.2533$ and $H_1 : \mu_0 > -qnorm(.4) = 0.2533$

   b. We are still interested in performing the test at the 5% level, but we will now use the sample mean, $\bar{Y}$, as our statistic. What is the sampling distribution of $\bar{Y}$ if the null hypothesis is true? Give a brief justification for your answer.

We have that each Y_i is independent and normally distributed. Thus using the properties of normal distributions we have under the null:

$$\frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y} \sim N(\mu_0 = -qnorm(.4), \sigma = \frac{1}{\sqrt{n}})$$

   c. For what values of $\bar{Y}$ will you reject the null hypothesis? Support your answer showing code/output as necessary.

We reject the null for large values $\bar{Y}$. Thus we reject if the observed data is greater than the 95th percentile of the null sampling distribution. So we will reject when $\bar{y}$ is greater than:

```
n <- 19
crit_val <- qnorm(.95, mean = -qnorm(.4),sd=1/sqrt(n))
crit_val
```

```
## [1] 0.6307024
```

   d. What is the Type I error probability of your test? Briefly explain why.

For continuous distributions the type I error rate is equal to α. So in this case the Type 1 error rate is .05 .

   e. Calculate the Type II error probability when $\pi_0 = 0.2$. Support your answer showing code/output as necessary. (You will need to figure out what μ_0 value to evaluate the Type II error probability at so it corresponds to a $\pi_0 = 0.2$.)

We have that if $\pi_0 = .2$ then using the same process as above, we will have that under the alternative, $\mu_0 = -qnorm(.2) = 0.8416212$.

Thus the type 2 error rate will be the probability we reject the null (our observed data is less than the critical value found above) when we have the alternative distribution with the mean we just calculated. We have this as:

```
alt_mean <- -qnorm(.2)
pnorm(crit_val, mean = alt_mean, sd = 1/sqrt(n))
```

```
## [1] 0.17895
```