Homework 1

KEY

Exercises

1.

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{4}{3-x-y}}{\binom{9}{3}}, \quad x = 0,1,2,3, \; y = 0,1,2 \;, x+y \leq 3$$

The denominator represents the total number of different samples that can be formed when we draw 3 chips from the 9 chips in the urn. Since the draws are made randomly, each sample is **equally likely**. Hence the probability of observing a particular event is simply the ratio of the number of samples which satisfy the event divided by $\binom{9}{3}$.

In order to find the numerator, we need to find the number of samples that have x white and y blue. We can approach this using the **multiplication rule**. In other words, the decision is made in a series: first, we count the number of ways we can draw x white chips. This is $\binom{3}{x}$. For each of these ways, we then count the number of ways we can draw y blue chips. This is $\binom{2}{y}$. And similarly for red. Multiplying these out gives the number of samples that satisfy the event of interest.

2.

a. We have that each $U_i \sim Unif(0,1)$. By properties of the Uniform distribution we know that $P(U_i < u) = u$ for Uniform(0,1) random variables.

Thus the probability that a given U_i is under .331 is $\pi_1 = .331$. Similarly, $\pi_2 = P(.331 < U_i < .820) = .820 - .331 = 0.489$. And $\pi_3 = P(U_i > .820) = 1 - P(U_i < .82) = .18$.

The random variables $\langle X, Y \rangle$ follow a trinomial distributon since there are a fixed number of independent trials, each resulting in one of three possible outcomes with probabilities π_1, π_2, π_3 respectively.

To find $P(X = 354, Y = 492)$ we can use the dmultinom function in R.

```
p2_prob <- dmultinom(c(354,492,183), prob =  c(.331,.489,.18))

p2_prob
```

```
## [1] 0.0006064605
```

So the probability is 6×10^{-4}.

b. We have by properties of the trinomial distributions that:

$$Y \sim Binom(n = 1029, \pi_2 = .489)$$

So we have that $E(Y) = n\pi_2 = 503.181$ and $SD(Y) = \sqrt{n\pi_2(1-\pi_2)} = 16.035133$

3.

$$
\begin{aligned}
f_2(y) &= \sum_{x=1}^{y} \left(\frac{\lambda}{2}\right)^y \frac{e^{-y}}{x!(y-x)!} \\
&= \left(\frac{\lambda}{2}\right)^y e^{-\lambda} \sum_{x=0}^{y} \frac{1}{x!(y-x)!} \\
&= \left(\frac{\lambda}{2}\right)^y \frac{e^{-\lambda}}{y!} \sum_{x=0}^{y} \frac{y!}{x!(y-x)!} \\
&= \left(\frac{\lambda}{2}\right)^y \frac{e^{-\lambda}}{y!} \sum_{x=0}^{y} \binom{y}{x} 1^x 1^{y-x} \\
&= \left(\frac{\lambda}{2}\right)^y \frac{e^{-\lambda}}{y!} \times (1+1)^y \text{using the binomial theorem} \\
&= \left(\frac{\lambda}{2}\right)^y \frac{e^{-\lambda}}{y!} \times (2)^y \\
&= \frac{\lambda^y e^{-\lambda}}{y!} \quad \text{for y=0,1,2,3,...}
\end{aligned}
$$

We see that $Y \sim Pois(\lambda)$.

4.

	y						
x	0	1	2	3	4	5	6
4			0.0882		0.0027	0	
5			0.1029				
6			0.108045				
$f_2(y)$			0.299145				

a. We have that $f(x, y) = f(y|x) \times f(x)$

Thus for the first 3 entries in the second column (i, iv, v) we are simply calculating

$$f(x, y = 2) = f(y = 2|x) \times \frac{1}{3}$$

For x=4,5,6.

For entry ii we are doing the same calculation with x=4, and y=4. For entry iii, we see that it is impossible to get y=5 hits with x=4 at bats, so the probability is 0. For the final entry vi, we simply add the entries in that column (i,iv,v) to find the marginal distribution of y at y=2.

b. We have by the definition of conditional PMFs:

$$f(x|y = 2) = \frac{f(x, y = 2)}{f(y = 2)}$$

$$
\begin{aligned}
f(x|y = 2) &= \frac{f(x, y = 2)}{f_2(y = 2)} = \frac{f(y = 2|x) \cdot f(x)}{f_2(y = 2)}, \\
&= \frac{\frac{1}{3} \cdot \binom{x}{2} 0.3^2 (1 - 0.3)^{x-2}}{0.2991}, \quad x = 4, 5, 6.
\end{aligned}
$$