Problem Section 4

KEY

Learning Outcomes

The problems are designed to build conceptual understanding and problem-solving skills. The emphasis is on learning to find, evaluate and build confidence. The specific tasks include:

- Derive and calculate the method of moments estimator as well as its estimated standard error
- Be able to verify unbiasedness and consistency of an estimator
- Be able to "unbias" a biased estimator which is off by a multiplicative constant
- Compare unbiased estimators using their variance
- Compare biased estimators using the mean squared error
- Back up and support work with relevant explanations

Exercises

1.

a. We are estimating one parameter, so we must first find the expectation of our RV X, to find the method of moments estimator.

We have:

$$E[X] = 0 \times \left(\frac{1}{3} - \theta_0\right) + 1 \times \left(\frac{1}{3}\right) + 2 \times \left(\frac{1}{3} + \theta_0\right) = \frac{1}{3} + \frac{2}{3} + 2\theta_0 = 1 + 2\theta_0$$

Now, the methods of moments estimate is the value of θ_0 which solves the equation

$$E\left[X\right] = \bar{x}.$$

That is, we are solving for θ_0 in the equation:

$$1 + 2\,\theta_0 = \bar{x}$$

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This gives:

$$\hat{\theta_0}^{mom} = \frac{\bar{x} - 1}{2}$$

In part b and c, we will now study $\hat{\theta_0}^{mom}$ as an estimator, and thus we will replace $\bar{x}$ with $\bar{X}$.

b. An estimator $\hat{\theta}_0$ is said to be unbiased for the true (but unknown) value of a parameter, θ_0, if

$$E\left[\hat{\theta}_0\right] = \theta_0.$$

In order to find the expected value of our estimator, we will first calculate $E\left[\bar{X}\right]$.

Using the properties of linearity of expectation and because each X has the same distribution:

$$E\left[\bar{X}\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}E\left[\sum_{i=1}^{n} X_i\right] = E\left[X_1\right]$$

We have that $E\left[X_1\right] = 1 + 2\theta_0$. Thus for our estimator $\hat{\theta}_0^{mom}$:

$$E\left[\hat{\theta}_0^{mom}\right] = E\left[\frac{\bar{X}-1}{2}\right] = \frac{E\left[\bar{X}\right]-1}{2} = \frac{2\theta_0}{2} = \theta_0$$

Thus our estimator is in fact unbiased for θ_0.

c. The variance of our estimator is:

$$Var\left(\hat{\theta}_0^{mom}\right) = Var\left(\frac{\bar{X}-1}{2}\right) = \frac{1}{4}Var\left(\bar{X}\right)$$

Now we may find $Var(\bar{X})$ using the fact that each variable is independent and follows the same distribution:

$$\begin{aligned}
Var(\bar{X}) &= Var(\frac{1}{n}\sum_{i=1}^{n} X_i) \\
&= \frac{1}{n^2}Var(\sum_{i=1}^{n} X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \\
&= \frac{1}{n}Var(X_1)
\end{aligned}$$

Now we may find $Var(X) = E[X^2] - (E[X])^2$.

We have for $E[X^2]$:

$$\begin{aligned}
E[X^2] &= 0^2 \times \left(\frac{1}{3} - \theta_0\right) + 1^2 \times \left(\frac{1}{3}\right) + 2^2 \times \left(\frac{1}{3} + \theta_0\right) \\
&= \frac{1}{3} + \frac{4}{3} + 4\theta_0 \\
&= \frac{5}{3} + 4\theta_0
\end{aligned}$$

Thus we have that:

$$Var(X) = \frac{5}{3} + 4\theta_0 - [1 + 2\theta_0]^2 = \frac{2}{3} - 4\theta_0^2$$

Thus:

$$Var(\hat{\theta}_0^{mom}) = \frac{Var(\bar{X})}{4} = \frac{\frac{2}{3} - 4\theta_0^2}{4n}$$

We see as n goes to infinity, $Var(\hat{\theta}_0^{mom}) \Rightarrow 0$. Since our estimator is also unbiased, the fact that it has a variance that goes to zero is *sufficient* to prove that it is consistent.

d. We have for this sample that $\bar{x} = \frac{5}{9}$. Therefore we have our estimate $\hat{\theta}_0^{mom}$ as -0.2222.

We know that the standard error of an estimator is simply its standard deviation (SD):

$$SD(\hat{\theta}_0^{mom}) = \sqrt{Var(\hat{\theta}_0^{mom})} = \sqrt{\frac{\frac{2}{3} - 4\theta_0^2}{4n}}$$

Thus using our equations derived above, and our sample values we can estimate the standard error of our estimator as:

$$\hat{SD}(\hat{\theta}_0^{mom}) = \sqrt{\frac{\frac{2}{3} - 4\hat{\theta_0}^2}{4 * 9}}$$

Plugging in our value for $\hat{\theta}_0^{mom}$ we have $\widehat{SD}(\hat{\theta}_0^{mom}) = 0.1142$

2.

a. Again, in order to show an estimator is unbiased, we need to examine its expected value.

The following shows why $\hat{\mu}_1$ is an unbiased estimator of μ.

$$\begin{aligned} E[\hat{\mu}_1] &= E\left[\frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3\right], \\ &= \frac{1}{4}E[X_1] + \frac{1}{2}E[X_2] + \frac{1}{4}E[X_3], && \text{(linearity of expectation)} \\ &= \frac{1}{4}\mu + \frac{1}{2}\mu + \frac{1}{4}\mu, \\ &= \mu. \end{aligned}$$

A similar proof will show that $\hat{\mu}_2$ is also an unbiased estimator of μ. In fact, any linear combination of the X_i, $\sum_{i=1}^n a_i X_i$ will be an unbiased estimator of μ so long as $\sum\limits_{i=1}^{n} a_i = 1$.

b. In order to answer questions about efficiency, we need to examine the variance of the estimators.

$$\begin{aligned} Var[\hat{\mu}_1] &= Var\left[\frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3\right], \\ &= \frac{1}{16}Var[X_1] + \frac{1}{4}Var[X_2] + \frac{1}{16}Var[X_3], && \text{(independence)} \\ &= \frac{3\sigma^2}{8}. \end{aligned}$$

A similar derivation will show that $Var[\hat{\mu}_2] = \frac{3\sigma^2}{9}$. Hence $\hat{mu}_2$ is the more efficient estimator of μ. It makes better use of the information in the data and is more precise.

In fact, $\hat{\mu}_2$ is the sample mean estimator and it has the smallest variance in the class of linear estimators.

3.

a. Create an unbiased estimator of θ_0 based on X_{max}. Call this estimator $\hat{\theta}_0$.

Hint: an estimator can be based on random variables and numbers like n. The only thing off limits are unknown parameters

For our estimator, $\hat{\theta}_0$ to be unbiased, we must have that $E\left[\hat{\theta}_0\right] = \theta_0$. From the info in the problem we know:

$$E[X_{max}] = \frac{n}{n+1}\theta_0 \Rightarrow E\left[\frac{n+1}{n}X_{max}\right] = \theta_0$$

Thus, the estimator $\frac{n+1}{n}X_{max}$ is unbiased.

b. We wish to find $Var(\hat{\theta}_0) = Var(\frac{n+1}{n}X_{max})$.

By properties of Variance we know:

$$\begin{aligned} Var\left[\frac{n+1}{n}X_{max}\right] &= \left(\frac{n+1}{n}\right)^2 Var(X_{max}) \\ &= \left(\frac{n+1}{n}\right)^2 \frac{n\theta_0^2}{(n+2)(n+1)^2} \\ &= \frac{\theta_0^2}{n(n+2)} \end{aligned}$$

Since we know that $\frac{n+1}{n} > 1$ we have that $Var(\hat{\theta}_0) = \left(\frac{n+1}{n}\right)^2 Var(X_{max}) > Var(X_{max})$

c. We have the bias of $Bias(X_{max}) = \frac{n}{n+1}\theta_0 - \theta_0 = \frac{-\theta_0}{n+1}$

So putting this together with the Variance of X_{max} we derived above we have that:

$$MSE(X_{max}) = \frac{n\theta_0^2}{(n+2)(n+1)^2} + \frac{\theta_0^2}{(n+1)^2}$$

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Since we know from part a that $\hat{\theta_0}$ is unbiased (bias = 0), this means that
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$$MSE(\hat{\theta}_0) = Var(\hat{\theta}_0) = \frac{\theta_0^2}{n(n+2)}.$$

d. To compare the MSE's we can simply analyze the ratio $\frac{MSE(X_{max})}{MSE(\hat{\theta}_0)}$.

Thus we are analyzing the ratio:

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\begin{aligned}
\frac{\frac{n\theta_0^2}{(n+2)(n+1)^2} + \frac{\theta_0^2}{(n+1)^2}}{\frac{\theta_0^2}{n(n+2)}} &= \frac{\frac{n}{(n+2)(n+1)^2} + \frac{1}{(n+1)^2}}{\frac{1}{n(n+2)}} \\
&= \frac{\frac{2n+2}{(n+2)(n+1)^2}}{\frac{1}{n(n+2)}} \\
&= \frac{n(2n+2)}{(n+1)^2} \\
&= \frac{2n(n+1)}{(n+1)^2} \\
&= \frac{2n}{n+1}
\end{aligned}
$$

This is larger than 1 whenever $n > 1$. So the MSE of X_{max} is larger than the MSE of $\hat{\theta}_0$.