# Homework 2

Winter 2024

KEY

2024-01-15

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**Instructions**

- This homework is due in Gradescope on Wednesday Jan 24 by midnight PST. There is a 15 minute grace period and submissions made during this time will not be marked as late. Any work submitted past this period is considered late.
- Please answer the following questions in the order in which they are posed.
- Don't forget to (i) make a local copy this document for your work and to (ii) knit the document frequently to make sure there are no compilation errors.
- When you are done, download the PDF file as instructed in section and submit it in Gradescope.

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**Exercises**

1. (Joint PMF) The random variables $X$ and $Y$ are independent, each taking the values 1, 2 or 3. Complete the following table of the joint PMF. Show your work for each entry below the table.

| $y$ \ $x$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 0.03 | 0.04 | 0.03 |
| 2 | 0.15 | (b) | (c) |
| 3 | (a) | (d) | (e) |

We wish to find the joint PMF $f(y, x) = P(X = x, Y = y)$. Since these are independent RV's we know $P(X = x, Y = y) = P(X = x) \times P(Y = y) \;\; \forall\, x, y$.

From our table we can extract that $P(Y = 1) = .03 + .04 + .03 = .1$.

This implies that $P(X = 1) = P(X = 1, Y = 1)/P(Y = 1) = .03/.1 = .3$.

Similarly we have, $P(X = 2) = .4$, and $P(X = 3) = .3$.

Using this information we know that for (a), we must have that $.03 + .15 + (a) = P(X = 1) = .3$.

This gives us that (a) = .12.

We also see that $P(X = 1, Y = 2) = 0.15 = P(X = 1) \times P(Y = 2)$ giving us that $P(Y = 2) = .5$.

Thus all in all so far we have that:

$P(Y=1)=.1$

$P(Y=2)=.5$

$P(Y=3)=.4$ (since .1+.5+.4 = 1)

$P(X=1)=.3$

$P(X=2)=.4$

$P(X=3)=.3$

From here we see that:

(b) $=P(X=2)\,P(Y=2)=.5\times .4=.2$

(c) $=P(Y=2)\,P(X=3)=.5\times .3=.15$

(d) $=P(Y=3)\,P(X=2)=.4\times .4=.16$

(e) $=P(Y=3)\,P(X=3)=.4\times .3=.12$

2. (Two friends) Two friends - let's call them Owen and Justin - agree to meet at Tully's for coffee. Suppose that the random variables

$$X = \text{the time that Owen arrives at Tully's and}$$

$$Y = \text{the time that Justin arrives at Tully's}$$

are independent uniform random variables on the interval $[5,6]$. (The units are hours after noon)

a. Calculate the probability that both of them arrive before 5:30 PM.

Hint: You want to find $P(X<5.5, Y<5.5)$

We know that the probability that both people arrive before 5:30, means that $X<5.5\cap Y<5.5$. Since these are independent RV's we have that:

$$P(X<5.5\cap Y<5.5)=P(X<5.5)\times P(Y<5.5)=\left(\frac{6-5.5}{6-5}\right)^2=.25$$

So the probability that both of them arrive before 5:30 is .25.

b. Calculate the probability that Owen arrives before Justin. Show your steps.

Here we want to calculate $P(X < Y)$. As we did in problem set 2, this is simply the double integral

$$\begin{aligned} P(X < Y) &= \int_5^6 \int_5^y f(x, y) dx dy \\ &= \int_5^6 (y - 5) \\ &= \frac{y^2}{2} - 5y \Big|_5^6 = \frac{1}{2} \end{aligned}$$

There is a 50% chance that Owen arrives before Justin.

3. (Rounding errors) An individual makes 100 check transactions between receiving his December and his January bank statements. Rather than subtracting the amounts exactly, he rounds off each checkbook entry to the nearest dollar. Let $X_i$ denote the round off error on the *ith* check. A reasonable assumption is that

$$X_i \sim Unif\left(-\frac{1}{2}, \frac{1}{2}\right)$$

independently of each other.

Let $T = X_1 + X_2 + \cdots + X_{100}$. The random variable $T$ denotes the accumulated error after 100 transactions.

a. Find $\mu = E[T]$. Show your steps.

We begin by noting that $E[X_i] = 0$ by the result on uniform random variables which states that if $U \sim Unif(a, b)$ then $E[U] = \frac{a+b}{2}$ and $Var[X_i] = \frac{(b-a)^2}{12}$.

Since $T$ is a sum of 100 random uniform variables, its expected value is (by linearity of expectation)

$$E[T] = E[X_1 + \cdots + X_{100}] = E[X_1] + \cdots + E[X_{100}] = 0$$

b. Find $\sigma = \sqrt{Var[T]}$. Show your steps.

By the result cited in part a. about uniform random variables, we have that $Var(X_i) = \frac{(1/2-(-1/2))^2}{12} = \frac{1}{12}$.

Using the fact that each $X_i$ is independent we have by Theorem 15.1:

$$Var(T) = Var(\sum_{i=1}^{100} X_i) = \sum_{i=1}^{100} Var(X_i) = \frac{100}{12}$$

and therefore

$$\sigma = \sqrt{Var(T)} = \frac{5}{\sqrt{3}}.$$

c. What can you say about $P(|T| \geq 5)$? Show your steps.

Hint: use Chebychev's inequality

Chebychev's inequality states that for a random variable $X$ with mean $\mu$ and standard deviation $\sigma$,

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Using this result for $T$ we have:

$$\begin{aligned} P(|T| \geq 5) &= P(|T-0| \geq 5) \\ &= P\left(|T-0| \geq \underbrace{\frac{5}{\sqrt{3}}}_{\sigma} \times \underbrace{\sqrt{3}}_{k}\right) \\ &\leq \frac{1}{\sqrt{3}^2} \\ &= \frac{1}{3} \end{aligned}$$

Therefore there is a 33 chance that the rounding errors after 100 transactions are within \$5.

4. A gambler plays $n$ hands of poker. If she wins the $k^{th}$ hand she collects $k$ dollars, while if she loses it, she collects nothing. Assume that her chances of winning each hand is constant (equal to $p$).

a. Let the random variable $X_i$ denote her winnings on the $ith$ hand. Write the PMF of $X_i$ by filling in the empty cells in the table below.

| $x$ | $0$ | $i$ |
|---|---|---|
| $f(x)$ | $1-p$ | $p$ |

b. Find $\mu = E[X_i]$.

By definition of the expected value:

$$E(X_k) = k \times p + 0 \times (1-p) = kp$$

c. Let the random variable $T$ denote the total amount she wins in $n$ hands, that is $T = X_1 + X_2 + \cdots + X_n$. Find $E[T]$. State assumptions (if any) you need to make about the $X_i$ in order to do your calculation.

$$\begin{aligned} E(T) &= E(X_1 + \cdots + X_n) \\ &= E(X_1) + \cdots + E(X_n) && \text{(linearity of expectation)} \\ &= p + 2p + \cdots + np \\ &= p(1 + 2 + \cdots + n) \\ &= p\frac{n(n+1)}{2} \end{aligned}$$

We do not need to make any assumptions about the $X_i$ for the expected value calculation.

d. Find $Var[T]$. State assumptions (if any) you need to make about the $X_i$ in order to do your calculation.

For the variance of T, we need to assume each hand is independent and the chances of winning on one do not affect the chances of winning on the other. Under this assumption, we have:

$$\begin{aligned}
Var(T) &= Var(X_1 + \cdots + X_n) \\
&= Var(X_1) + \cdots + Var(X_n) \\
&= p(1-p) + 2^2 p(1-p) + \cdots + n^2 p(1-p) \\
&= p(1-p)(1 + 2^2 + \cdots + n^2) \\
&= p(1-p)\frac{n(n+1)(2n+1)}{6}
\end{aligned}$$