Homework 3

Winter 2024

KEY

2024-02-05

Exercises

1.

a. Assuming that $X_1, X_2, \ldots, X_{30}$ are independent, we have

$$P(max(X_1, \ldots, X_{30}) \leq k) = \left(\frac{k-20}{20}\right)^{30}$$

b. We wish to find the value k such that:

$$P(max(X_1, \ldots, X_{30}) \geq k) = 1 - \left(\frac{k-20}{20}\right)^{30} = .2$$

Solving for k we have:

$$\begin{aligned} 1 - \left(\frac{k-20}{20}\right)^{30} &= .2 \\ \left(\frac{k-20}{20}\right)^{30} &= .8 \\ \frac{k-20}{20} &= (.8)^{1/30} \\ k - 20 &= 20(.8)^{1/30} \\ k &= 20(.8)^{1/30} + 20 \\ &= 39.8517895 \end{aligned}$$

Thus the Levee should be at least 39.8517895 feet tall.

2. The proposition we are trying to prove is $P(k) : S_k = X_1 + X_2 + \cdots + X_k \sim Pois(\sum_{i-1}^{k} \lambda_i)$ for any integer $k = 2, 3, 4, \ldots$

Define our base case for k=2. We will show that if $X_1 \sim Pois(\lambda_1)$ independently of $X_2 \sim Pois(\lambda_2)$ then $X_1 + X_2 \sim Pois(\lambda_1 + \lambda_2)$.

$$
\begin{aligned}
f(s) &= P(X_1 + X_2 = s) \quad\quad s = 0, 1, 2, \dots, \\
&= \sum_{x=-\infty}^{\infty} f_1(x) f_2(s-x) \text{ using convolution} \\
&= \sum_{x=0}^{s} e^{-\lambda_1} \frac{\lambda_1^x}{x!} \, e^{-\lambda_2} \frac{\lambda_2^{s-x}}{(s-x)!} && \text{(by range of } X_1, X_2) \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{s!} \sum_{x=0}^{s} \frac{s!}{x!(s-x)!} \lambda_1^x \lambda_2^{s-x} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{s!} \sum_{x=0}^{s} \binom{s}{x} \lambda_1^x \lambda_2^{s-x} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{s!} (\lambda_1 + \lambda_2)^s \quad \text{binomial theorem}
\end{aligned}
$$

where the last sentence follows from the binomial expansion for $(\lambda_1 + \lambda_2)^s$.

Thus we have for n=2, $X_1 + X_2 \sim Pois(\lambda_1 + \lambda_2)$.

Now for our inductive step assume that for some fixed integer k^* ($k^* \geq 2$), $S_k^* = X_1 + X_2 + \cdots + X_{k^*} \sim Pois(\sum_{i=1}^{k^*} \lambda_i)$. We want to show that this implies

$$
S_{k^*+1} = X_1 + X_2 + \cdots + X_{k^*} + X_{k^*+1} \sim Pois(\sum_{i=1}^{k^*+1} \lambda_i).
$$

But this follows from the base case because $S_{k^*} \sim Pois(\sum_{i=1}^{k^*} \lambda_i)$ independently of $X_{k^*+1} \sim Pois(\lambda_{k^*+1})$. They are independent because S_{k^*} does not involve X_{k^*+1}.

Hence we have proved that the sum of independent Poisson random variables is also Poisson and the rates add.

3. We have that:

$$
Y = X_1 - \frac{1}{3}(X_1 + X_2 + X_3) = \frac{2}{3}X_1 - \frac{1}{3}X_2 - \frac{1}{3}X_3
$$

Since X_1, X_2, X_3 are independent RV's, and Y is a linear combination of Normal RV's, Y is a normal RV.

We have from Theorem 15.1:

$$
E[Y] = E\left[\frac{2}{3}X_1 - \frac{1}{3}X_2 - \frac{1}{3}X_3\right] = \frac{2}{3}E[X_1] - \frac{1}{3}E[X_2] - \frac{1}{3}E[X_3] = \frac{2}{3}\mu - \frac{1}{3}\mu - \frac{1}{3}\mu = 0.
$$

We also have using the independence of the RV's that:

$$
Var(Y) = Var\left(\frac{2}{3}X_1 - \frac{1}{3}X_2 - \frac{1}{3}X_3\right) = \frac{4}{9}Var(X_1) + \frac{1}{9}Var(X_2) + \frac{1}{9}Var(X_3) = \frac{4}{9} + \frac{2}{9} = \frac{6}{9}
$$

Thus we have that:

$$Y \sim N\left(\mu = 0, \sigma = \sqrt{\frac{2}{3}}\right)$$

Thus using pnorm we have that $P(Y > 1.6) = 0.0250218$.

4.

a. Let the random variable C denote the cost associated with the calls received during a 24 hour day. Write C as a function of X and Y.

   We have that $C = 50X + 60V$

b. Since we have that during normal workday hours, we have a rate of 7 calls an hour, we thus have 63 calls per 9 hours. Thus $X \sim Pois(\lambda = 63)$. Similarly in non-workday hours there is a rate of 60 calls per 15 hours. So $Y \sim Pois(\lambda = 60)$.

Using the fact that for a $Pois(\lambda)$ RV the expectation is λ, and linearity of expectation we have:

$$E[C] = E[50X + 60V] = 50E[X] + 60E[V] = 50 \times 63 + 60 \times 60 = \$6750$$

We also know that the Variance of a $Pois(\lambda)$ RV is equal to λ.

If we assume that the calls in workday and non-workday hours are independent we have using the properties of independent RV's for variance:

$$Var(C) = Var(50X + 60V) = 2500Var(X) + 3600Var(V) = 2500 \times 63 + 3600 \times 60 = 3.735 \times 10^5$$