Homework 4

Winter 2024

KEY

2024-02-21

Exercises

1.

a. The method of moments estimate is the value of θ_0 which solves the equation

$$E\left[X\right] = \bar{x}.$$

The expected value calculation is shown below.

$$\begin{aligned} E\left[X\right] &= \int\limits_{-\infty}^{\infty} x \cdot f(x) dx \\ &= \int\limits_{0}^{1} \left(1+\theta_0\right) x^{\theta_0+1}\, dx, \\ &= \left(1+\theta_0\right) \left.\frac{x^{\theta_0+2}}{\theta_0+2}\right|_0^1, \\ &= \frac{1+\theta_0}{2+\theta_0}. \end{aligned}$$

Therefore, the method of moments estimator of θ_0 is the value which solves the equation

$$\begin{aligned} \frac{1+\theta_0}{2+\theta_0} &= \bar{x}, \\ (1+\theta_0) &= \bar{x}(2+\theta_0), \\ 1-2\bar{x} &= \theta_0(\bar{x}-1). \end{aligned}$$

Therefore $\hat{\theta}_0^{mom} = \frac{1-2\,\bar{x}}{\bar{x}-1}$.

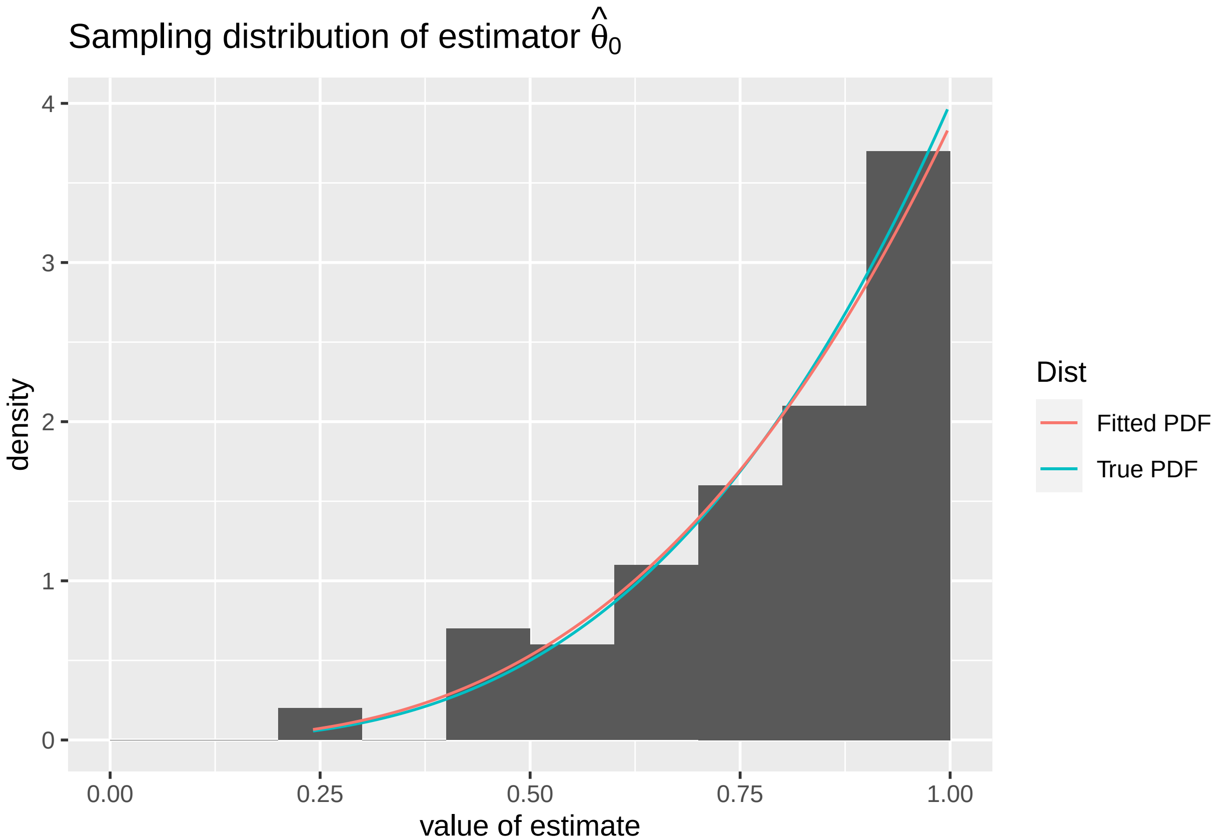
b.

```
set.seed(1131)           # for reproducibility
theta0 <- 3              # specify true value of parameter of PDF
```

```r
#generate 100 x's from the PDF f(x) using the inverse CDF method
#store the x's as a variable in a data frame

sample_df <- tibble(
  x = runif(n = 100, min = 0, max = 1)^(1/(theta0+1) )
       )



mom_df <- sample_df %>% summarise(thetahat =(1-2*mean(x))/(mean(x)-1))

mom_df$thetahat
```

```
## [1] 2.863606
```

```r
ggplot(data=sample_df)+
  geom_histogram(mapping=aes(x=x,
                             y = after_stat(density) ),
                 breaks=seq(0,1,0.1))+
  geom_function(fun=function(x){ (theta0+1)*x^(theta0)},
                mapping=aes(color="True PDF"))+
    geom_function(fun=function(x){
                 (mom_df$thetahat+1)*x^(mom_df$thetahat)},
                mapping=aes(color="Fitted PDF"))+
    labs(title = expression(paste("Sampling distribution of estimator ", hat(theta)[0])),
         x = "value of estimate",
         y = "density",
         color = "Dist")
```



2.

a. Simplifying this expression, we are looking for $E[\frac{X}{n} - \frac{X^2}{n^2}]$

We have for a Binomial RV, X that $E[X] = n\pi_0$ and $Var[X] = n(\pi_0)(1-\pi_0)$. From here we see that:

$$E[X^2] = n(\pi_0)(1-\pi_0) + n^2\pi_0^2 = n\pi_0 - n\pi_0^2 + n^2\pi_0^2 = n\pi_0(1 - \pi_0 + n\pi_0)$$

`Thus we have that:`

$$\begin{aligned} E\left[\frac{X}{n} - \frac{X^2}{n^2}\right] &= \frac{n\pi_0}{n} - \frac{n\pi_0(1-\pi_0+n\pi_0)}{n^2} \\ &= \frac{n^2\pi_0 - n\pi_0 + n\pi_0^2 - n^2\pi_0^2}{n^2} \\ &= \frac{n\pi_0 - \pi_0 + \pi_0^2 - n\pi_0^2}{n} \\ &= \frac{\pi_0(n - 1 + \pi_0 - n\pi_0)}{n} \\ &= \frac{\pi_0((n-1) + (n-1)(\pi_0))}{n} \\ &= \frac{\pi_0(n-1)(1-\pi_0)}{n} \end{aligned}$$

b. An unbiased estimator is one that is equal to the true parameter value on average. In other words, the estimator $\hat{\theta}_0$ for a parameter θ_0 is unbiased if $E\left[\hat{\theta}_0\right] = \theta_0$.

We have:

$$E\left[\frac{X}{n}\left(1-\frac{X}{n}\right)\right] = \frac{(n-1)\,\pi_0\,(1-\pi_0)}{n}$$

`Thus we have that:`

$$E\left[\left(\frac{n}{n-1}\right)\frac{X}{n}\left(1-\frac{X}{n}\right)\right] = \pi_0\,(1-\pi_0)$$

`So we have the unbiased estimator`

$$\left(\frac{n}{n-1}\right)\frac{X}{n}\left(1-\frac{X}{n}\right)$$

3.

a. The estimator is unbiased if $E\left[\hat{\pi}_0^{bayes}\right] = \pi_0$.

We know that $E[X] = n\pi_0$

We have:

$$E[\hat{\pi}_0^{bayes}] = E\left[\frac{X+1}{n+2}\right] = \frac{n\pi_0+1}{n+2}$$

Since this is not equal to π_0, the Bayes estimator is biased. We see that the exact bias is:

$$Bias = E\left[\frac{X+1}{n+2}\right] - \pi_0 = \frac{n\pi_0+1}{n+2} - \pi_0 = \frac{1-2\pi_0}{n+2} \to 0 \text{ as n goes to } \infty$$

Since this bias goes to zero as n goes to infinity, this estimator is asymptotically unbiased.

b. A consistent estimator is one whose distribution is concentrated around the true value for large sample sizes.

We have $Var(\hat{\pi}_0^{bayes}) = Var\left(\frac{X+1}{n+2}\right) = \frac{1}{(n+2)^2}Var(X) = \frac{n(\pi_0)(1-\pi_0)}{(n+2)^2}$. We see as n goes to infinity this variance goes to 0. Since this estimator is also asymptotically unbiased, it is also consistent.

4.

a. We have that

$$E[\hat{\pi}_0^{mom}] = E\left[\frac{X}{n}\right] = \frac{n\pi_0}{n} = \pi_0.$$

Thus this estimator is unbiased, so the MSE will simply equal the variance of the estimator. We have that $Var(\hat{\pi}_0^{mom}) = Var\left(\frac{X}{n}\right) = \frac{Var(X)}{n^2} = \frac{\pi_0(1-\pi_0)}{n}$.

Thus we have that

$$MSE(\hat{\pi}_0^{mom}) = \frac{\pi_0(1-\pi_0)}{n}.$$

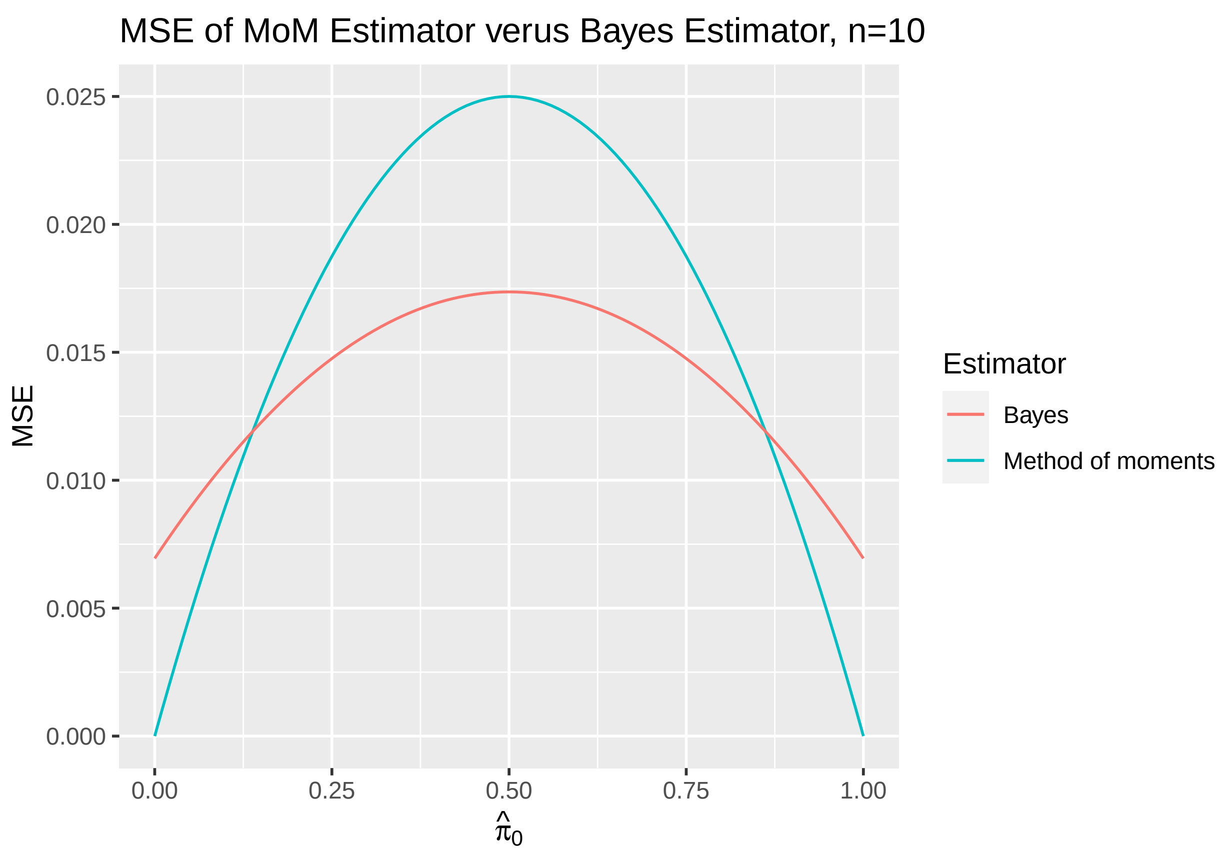
For the Bayes estimator we found that the Bias was equal to $\frac{1-2\pi_0}{n+2}$. We also saw that $Var(\hat{\pi}_0^{bayes}) = \frac{n(\pi_0)(1-\pi_0)}{(n+2)^2}$.

Thus we have that

$$MSE(\hat{\pi}_0^{bayes}) = [\frac{1-2\pi_0}{n+2}]^2 + \frac{n(\pi_0)(1-\pi_0)}{(n+2)^2} = \frac{(1-2\pi_0)^2 + n\pi_0(1-\pi_0)}{(n+2)^2}$$

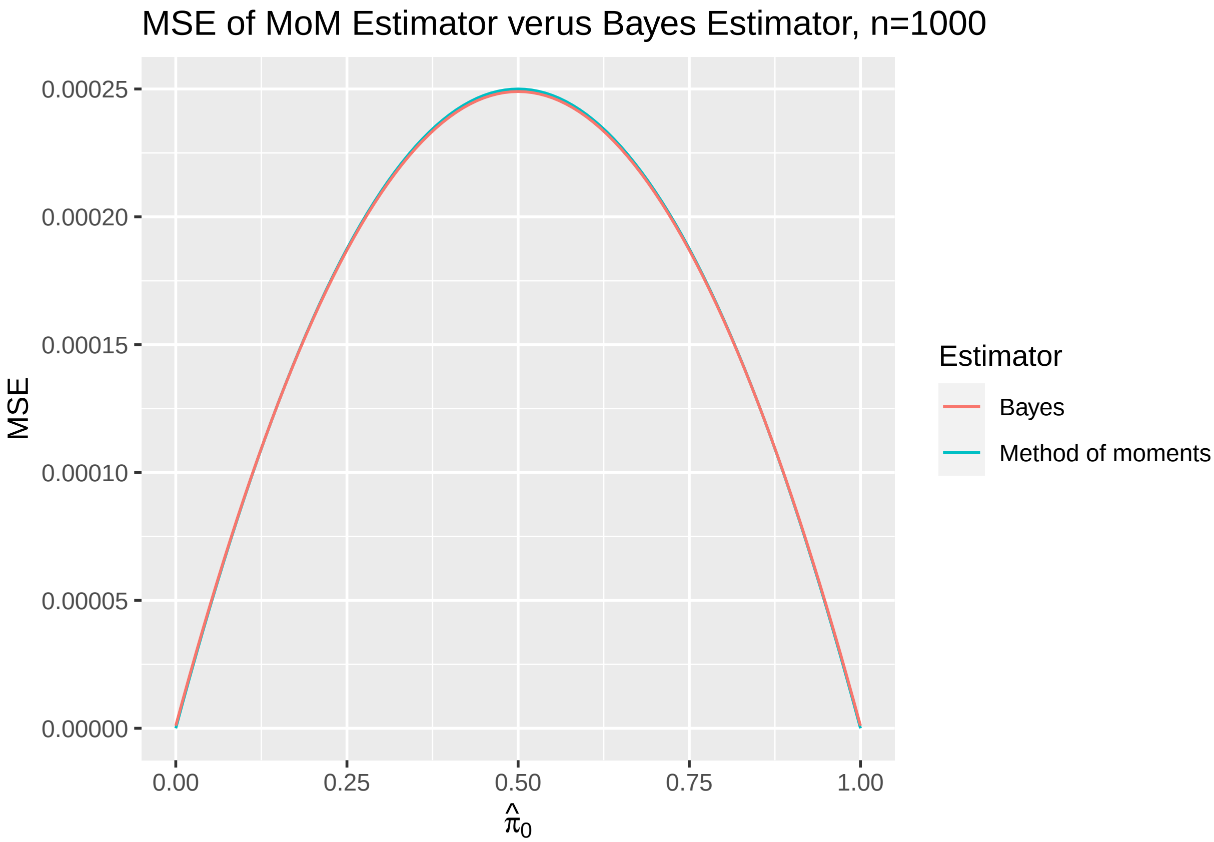
b. For n=10 we have the following plot:

```
ggplot() +
  geom_function( fun = function(n,x){x*(1-x)/n },
                 args = list(n=10),
                 mapping = aes(color = "Method of moments"),
                 xlim=c(0,1) ) +
  geom_function( fun=function(n,x){((1-2*x)^2+n*x*(1-x))/(n+2)^2 },
                 args = list(n = 10),
                 mapping = aes(color = "Bayes"),
                 xlim=c(0,1) )+
  labs(title="MSE of MoM Estimator verus Bayes Estimator, n=10",
       x = expression(hat(pi)[0]),
       y = "MSE",
       color = "Estimator")
```



We see that for small n, the bias added by the Bayes estimator, decreases the variance enough such that the MSE will be lower for values of π_0 away from the tails (the boundary). For extreme values of π_0 however, the MoM estimator performs better.

c.



For large values of n the difference between the MSE's is negligble.

d. What I have learned is that when the sample size is small, there may be differences among estimators with one being preferable over another. However, asymptotically they will perform very similarly.