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Multidimensional Matrix Mathematics: Algebraic Laws, Part 5 of 6

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***Abstract*—This is the first series of research papers to define multidimensional matrix mathematics, which includes multidimensional matrix algebra and multidimensional matrix calculus. These are new branches of math created by the author with numerous applications in engineering, math, natural science, social science, and other fields. Cartesian and general tensors can be represented as multidimensional matrices or vice versa. Some Cartesian and general tensor operations can be performed as multidimensional matrix operations or vice versa. However, many aspects of multidimensional matrix math and tensor analysis are not interchangeable. Part 5 of 6 describes the commutative, associative, and distributive laws of multidimensional matrix algebra.**

***Index Terms*—multidimensional matrix math, multidimensional matrix algebra, multidimensional matrix calculus, matrix math, matrix algebra, matrix calculus, tensor analysis**

I. Introduction

Part 5 of 6 describes the commutative, associative, and distributive laws of multidimensional matrix algebra.

II. Algebraic Laws of Multidimensional Matrix Algebra

The algebraic laws for classical matrices can be extended to multidimensional matrices. A multidimensional matrix is a concatenation of 1-D submatrices, 2-D submatrices, or both. The operations for addition, subtraction, and multiplication of multidimensional matrices are actually performed on individual 1-D submatrices and 2-D submatrices within the multidimensional matrices. The algebraic laws for classical matrices would apply to each individual 1-D or 2-D submatrix taken separately within a multidimensional matrix. If the algebraic laws for operations on classical matrices apply to each individual 1-D or 2-D submatrix within a multidimensional matrix, then they apply for operations on whole multidimensional matrices too because a multidimensional matrix consists solely of 1-D submatrices, 2-D submatrices, or both.

In the following equations, **A, B,** and **C** are multidimensional matrices and α is a scalar. Furthermore, in the following equations, multiplication refers to the multidimensional matrix product defined above.

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A. Commutative and Noncommutative Laws of Multidimensional Matrix Algebra

The commutative and noncommutative laws below are proven in the next section.

Commutative Law of Multidimensional Matrix Addition

Addition of multidimensional matrices is commutative.

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

$\mathbf{A} - \mathbf{B} = \mathbf{A} + (\mathbf{-B}) = \mathbf{-B} + \mathbf{A}$

Commutative Law of Multidimensional Matrix Multiplication by a Scalar

Multiplication of a multidimensional matrix by a scalar is commutative.

$\alpha\mathbf{A} = \mathbf{A}\alpha$

Noncommutative Law of Multidimensional Matrix Subtraction

However, subtraction of multidimensional matrices is not commutative.

$\mathbf{A} - \mathbf{B}$ does not have to equal $\mathbf{B} - \mathbf{A}$.

Noncommutative Law of Multidimensional Matrix Multiplication

Multiplication of multidimensional matrices is not commutative.

$\mathbf{A} *_{(da,\, db)} \mathbf{B}$ does not have to equal $\mathbf{B} *_{(da,\, db)} \mathbf{A}$.

Noncommutative Law of Multidimensional Matrix Outer Product

Outer product of multidimensional matrices is not commutative.

$\mathbf{A} \otimes \mathbf{B}$ does not have to equal $\mathbf{B} \otimes \mathbf{A}$.

B. Associative and Nonassociative Laws of Multidimensional Matrix Algebra

The associative and nonassociative laws below are proven in the next section.

Associative Law of Multidimensional Matrix Addition

Addition of multidimensional matrices is associative.

$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Associative Law of Multidimensional Matrix Multiplication

Multiplication of multidimensional matrices is associative when the first dimension and second dimension being multiplied in the first multidimensional matrix product are the same as the first dimension and second dimension being multiplied in the second multidimensional matrix product.

The variable *da* below refers to the first dimension being multiplied for the first two multidimensional matrices. The variable *db* below refers to the second dimension being multiplied for the first two multidimensional matrices. Therefore, $db > da$.

$\mathbf{A} *_{(da,\, db)} (\mathbf{B} *_{(da,\, db)} \mathbf{C}) = (\mathbf{A} *_{(da,\, db)} \mathbf{B}) *_{(da,\, db)} \mathbf{C}$

Nonassociative Law of Multidimensional Matrix Multiplication

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Multiplication of multidimensional matrices is not associative when the first dimension and second dimension being multiplied in the first multidimensional matrix product are not the same as the first dimension and second dimension being multiplied in the second multidimensional matrix product.

The variable *dc* below refers to the first dimension being multiplied for the second two multidimensional matrices. The variable *dd* below refers to the second dimension being multiplied for the second two multidimensional matrices. Therefore, $dd > dc$.

When $da \neq dc$ or $db \neq dd$, multidimensional matrix multiplication is not associative.

$\mathbf{A} *_{(da,\,db)} (\mathbf{B} *_{(dc,\,dd)} \mathbf{C})$ does not have to equal $(\mathbf{A} *_{(da,\,db)} \mathbf{B}) *_{(dc,\,dd)} \mathbf{C}$ when $da \neq dc$ or $db \neq dd$

Associative Law of Multidimensional Matrix Outer Product

The outer product of multidimensional matrices is associative.

$\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$

C. Distributive Laws of Multidimensional Matrix Algebra

The distributive laws below are proven in the next section.

Distributive Law of Multiplication over Addition of Multidimensional Matrices

Multiplication is distributive over addition of multidimensional matrices.

$\mathbf{A} *_{(da,\,db)} (\mathbf{B} + \mathbf{C}) = \mathbf{A} *_{(da,\,db)} \mathbf{B} + \mathbf{A} *_{(da,\,db)} \mathbf{C}$
$(\mathbf{A} + \mathbf{B}) *_{(da,\,db)} \mathbf{C} = \mathbf{A} *_{(da,\,db)} \mathbf{C} + \mathbf{B} *_{(da,\,db)} \mathbf{C}$
$\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$

Distributive Law of Outer Product over Addition of Multidimensional Matrices

Outer product is distributive over addition of multidimensional matrices.

$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$
$(\mathbf{B} + \mathbf{C}) \otimes \mathbf{A} = \mathbf{B} \otimes \mathbf{A} + \mathbf{C} \otimes \mathbf{A}$

III. Proofs of Algebraic Laws of Multidimensional Matrix Algebra

A. Proofs of Commutative and Noncommutative Laws of Multidimensional Matrix Algebra

Proof of Commutative Law of Multidimensional Matrix Addition

In proving the commutative property of multidimensional matrix addition, it can be proven that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ as follows:

$\mathbf{A}$ = multidimensional matrix of values $a_{ijk\ldots q}$
$\mathbf{B}$ = multidimensional matrix of values $b_{ijk\ldots q}$
Therefore, $\mathbf{A} + \mathbf{B}$ = multidimensional matrix of values $a_{ijk\ldots q} + b_{ijk\ldots q}$.
And $\mathbf{B} + \mathbf{A}$ = multidimensional matrix of values $b_{ijk\ldots q} + a_{ijk\ldots q}$.
Because $a_{ijk\ldots q}$ and $b_{ijk\ldots q}$ are scalar values, $a_{ijk\ldots q} + b_{ijk\ldots q} = b_{ijk\ldots q} + a_{ijk\ldots q}$.
Therefore, $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

In proving the commutative property of multidimensional matrix addition, it can be proven that $\mathbf{A} - \mathbf{B} = \mathbf{A} + (\mathbf{-B}) = \mathbf{-B} + \mathbf{A}$ as follows:

$\mathbf{A}$ = multidimensional matrix of values $a_{ijk\ldots q}$
$\mathbf{B}$ = multidimensional matrix of values $b_{ijk\ldots q}$
Therefore, $\mathbf{A} - \mathbf{B}$ = multidimensional matrix of values $a_{ijk\ldots q} - b_{ijk\ldots q}$.
And $\mathbf{A} + (\mathbf{-B})$ = multidimensional matrix of values $a_{ijk\ldots q} + (-b_{ijk\ldots q})$.
And $\mathbf{-B} + \mathbf{A}$ = multidimensional matrix of values $-b_{ijk\ldots q} + a_{ijk\ldots q}$.
Because $a_{ijk\ldots q}$ and $b_{ijk\ldots q}$ are scalar values, $a_{ijk\ldots q} - b_{ijk\ldots q} = a_{ijk\ldots q} + (-b_{ijk\ldots q}) = -b_{ijk\ldots q} + a_{ijk\ldots q}$.
Therefore, $\mathbf{A} - \mathbf{B} = \mathbf{A} + (\mathbf{-B}) = \mathbf{-B} + \mathbf{A}$.

Proof of Commutative Law of Multidimensional Matrix Multiplication by a Scalar

The commutative property of multidimensional matrix multiplication by a scalar, $\alpha\mathbf{A} = \mathbf{A}\alpha$, can be proven as follows:

$\mathbf{A}$ = multidimensional matrix of values $a_{ijk\ldots q}$
Therefore, $\alpha\mathbf{A}$ = multidimensional matrix of values $\alpha * a_{ijk\ldots q}$.
And $\mathbf{A}\alpha$ = multidimensional matrix of values $a_{ijk\ldots q} * \alpha$.
Because $a_{ijk\ldots q}$ are scalar values, $\alpha * a_{ijk\ldots q} = a_{ijk\ldots q} * \alpha$.
Therefore, $\alpha\mathbf{A} = \mathbf{A}\alpha$.

Proof of Noncommutative Law of Multidimensional Matrix Subtraction

It can be proven that multidimensional matrix subtraction is not commutative. That is, $\mathbf{A} - \mathbf{B}$ does not have to equal $\mathbf{B} - \mathbf{A}$. This can be proven as follows:

$\mathbf{A}$ = multidimensional matrix of values $a_{ijk\ldots q}$
$\mathbf{B}$ = multidimensional matrix of values $b_{ijk\ldots q}$
Therefore, $\mathbf{A} - \mathbf{B}$ = multidimensional matrix of values $a_{ijk\ldots q} - b_{ijk\ldots q}$.
And $\mathbf{B} - \mathbf{A}$ = multidimensional matrix of values $b_{ijk\ldots q} - a_{ijk\ldots q}$.
Because $a_{ijk\ldots q}$ and $b_{ijk\ldots q}$ are scalar values, $a_{ijk\ldots q} - b_{ijk\ldots q}$ does not have to equal $b_{ijk\ldots q} - a_{ijk\ldots q}$.
Therefore, $\mathbf{A} - \mathbf{B}$ does not have to equal $\mathbf{B} - \mathbf{A}$.

Proof of Noncommutative Law of Multidimensional Matrix Multiplication

It can be proven that multidimensional matrix multiplication is not commutative. That is, $\mathbf{A} *_{(da,\,db)} \mathbf{B}$ does not have to equal $\mathbf{B} *_{(da,\,db)} \mathbf{A}$. This can be proven as follows:

Let $\mathbf{A} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \end{bmatrix} \end{bmatrix}$

Let $\mathbf{B} = \begin{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} \\ \begin{bmatrix} 7 \\ 8 \end{bmatrix} \end{bmatrix}$

$\mathbf{A} *_{(1,\,2)} \mathbf{B} = \mathbf{AB} = \begin{bmatrix} \begin{bmatrix} 17 \end{bmatrix} \\ \begin{bmatrix} 53 \end{bmatrix} \end{bmatrix}$

$\mathbf{B} *_{(1,\,2)} \mathbf{A} = \mathbf{BA} = \begin{bmatrix} \begin{bmatrix} 5 & 10 \\ 6 & 12 \end{bmatrix} \\ \begin{bmatrix} 21 & 28 \\ 24 & 32 \end{bmatrix} \end{bmatrix}$

Therefore, $\mathbf{AB} \neq \mathbf{BA}$.
Therefore, $\mathbf{A} *_{(da,\,db)} \mathbf{B}$ does not have to equal $\mathbf{B} *_{(da,\,db)} \mathbf{A}$.

Proof of Noncommutative Law of Multidimensional Matrix Outer Product

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It can be proven that multidimensional matrix outer product is not commutative. That is, $\mathbf{A} \otimes \mathbf{B}$ does not have to equal $\mathbf{B} \otimes \mathbf{A}$. This can be proven as follows:
Let **A** = multidimensional matrix of values a_{ijk}
Let **B** = multidimensional matrix of values b_{lmn}
Let $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$
Let $\mathbf{D} = \mathbf{B} \otimes \mathbf{A}$
Therefore, **C** = multidimensional matrix of values $c_{ijklmn} = a_{ijk} * b_{lmn}$
And **D** = multidimensional matrix of values $d_{lmnijk} = b_{lmn} * a_{ijk}$
Therefore, c_{ijklmn} does not have to equal d_{lmnijk}.
Therefore, **C** does not have to equal **D**.
Therefore, $\mathbf{A} \otimes \mathbf{B}$ does not have to equal $\mathbf{B} \otimes \mathbf{A}$.

B. Proofs of Associative Laws of Multidimensional Matrix Algebra

Proof of Associative Law of Multidimensional Matrix Addition

The associative property of multidimensional matrix addition, **A + (B + C) = (A + B) + C,** can be proven as follows:
A = multidimensional matrix of values $a_{ijk\ldots q}$
B = multidimensional matrix of values $b_{ijk\ldots q}$
C = multidimensional matrix of values $c_{ijk\ldots q}$
Therefore, **A + (B + C)** = multidimensional matrix of values $a_{ijk\ldots q} + (b_{ijk\ldots q} + c_{ijk\ldots q})$.
And **(A + B) + C =** multidimensional matrix of values $(a_{ijk\ldots q} + b_{ijk\ldots q}) + c_{ijk\ldots q}$.
Because $a_{ijk\ldots q}$, $b_{ijk\ldots q}$, and $c_{ijk\ldots q}$ are scalar values, $a_{ijk\ldots q} + (b_{ijk\ldots q} + c_{ijk\ldots q}) = (a_{ijk\ldots q} + b_{ijk\ldots q}) + c_{ijk\ldots q}$.
Therefore, **A + (B + C) = (A + B) + C.**

Proof of Associative Law of Multidimensional Matrix Multiplication

It can be proven that multidimensional matrix multiplication is associative when the first dimension and second dimension being multiplied in the first multidimensional matrix product are the same as the first dimension and second dimension being multiplied in the second multidimensional matrix product. That is, $\mathbf{A} *_{(da,\, db)} (\mathbf{B} *_{(da,\, db)} \mathbf{C}) = (\mathbf{A} *_{(da,\, db)} \mathbf{B}) *_{(da,\, db)} \mathbf{C}$. This can be proven as follows:
Let $\mathbf{D} = \mathbf{A} *_{(da,\, db)} \mathbf{B}$.
Let $\mathbf{E} = \mathbf{B} *_{(da,\, db)} \mathbf{C}$.
Let $\mathbf{F} = \mathbf{A} *_{(da,\, db)} (\mathbf{B} *_{(da,\, db)} \mathbf{C}) = \mathbf{A} *_{(da,\, db)} \mathbf{E}$.
Let $\mathbf{G} = (\mathbf{A} *_{(da,\, db)} \mathbf{B}) *_{(da,\, db)} \mathbf{C} = \mathbf{D} *_{(da,\, db)} \mathbf{C}$.

Therefore,

$$d_{ijk\ldots q} = \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * b_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$

$$e_{ijk\ldots q} = \sum_x b_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$

$$f_{ijk\ldots q} = \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * e_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$

$$g_{ijk\ldots q} = \sum_x d_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$

Therefore,

$$g_{ijk\ldots q} = \sum_{x1}\left(\sum_{x2} a_{ijk\ldots q \text{ where } x2 \text{ replaces index of } db} * b_{ijk\ldots q \text{ where } x2 \text{ replaces index of } da}\right)_{\text{where } x1 \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x1 \text{ replaces index of } da}$$

$$= \sum_{x1}\sum_{x2}\left(a_{ijk\ldots q \text{ where } x2 \text{ replaces index of } db} * b_{ijk\ldots q \text{ where } x2 \text{ replaces index of } da}\right)_{\text{where } x1 \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x1 \text{ replaces index of } da}$$

$$= \sum_{x1} a_{ijk\ldots q \text{ where } x1 \text{ replaces index of } db}\left(\sum_{x2} b_{ijk\ldots q \text{ where } x2 \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x1 \text{ replaces index of } da}\right)_{\text{where } x1 \text{ replaces index of } da}$$

$$= \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * e_{ijk\ldots q \text{ where } x \text{ replaces index of } da} = f_{ijk\ldots q}$$

Therefore, **G** = **F**.
Therefore, $\mathbf{A} *_{(da,\, db)} (\mathbf{B} *_{(da,\, db)} \mathbf{C}) = (\mathbf{A} *_{(da,\, db)} \mathbf{B}) *_{(da,\, db)} \mathbf{C}$.

Proof of Nonassociative Law of Multidimensional Matrix Multiplication

It can be proven that multidimensional matrix multiplication is not associative when the first dimension and second dimension being multiplied in the first multidimensional matrix product are not the same as the first dimension and second dimension being multiplied in the second multidimensional matrix product. That is, $\mathbf{A} *_{(da,\, db)} (\mathbf{B} *_{(dc,\, dd)} \mathbf{C})$ does not have to equal $(\mathbf{A} *_{(da,\, db)} \mathbf{B}) *_{(dc,\, dd)} \mathbf{C}$ when $da \neq dc$ or $db \neq dd$. This can be proven as follows:

$$\text{Let } \mathbf{A} = \mathbf{B} = \mathbf{C} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \end{bmatrix}$$

$$\mathbf{A} *_{(1,\,2)} (\mathbf{B} *_{(2,\,3)} \mathbf{C}) = \begin{bmatrix} \begin{bmatrix} 143 & 102 \\ 293 & 218 \end{bmatrix} \\ \begin{bmatrix} 637 & 782 \\ 861 & 1058 \end{bmatrix} \end{bmatrix}$$

$$(\mathbf{A} *_{(1,\,2)} \mathbf{B}) *_{(2,\,3)} \mathbf{C} = \begin{bmatrix} \begin{bmatrix} 141 & 166 \\ 409 & 490 \end{bmatrix} \\ \begin{bmatrix} 437 & 518 \\ 833 & 1002 \end{bmatrix} \end{bmatrix}$$

Therefore, $\mathbf{A} *_{(1,\,2)} (\mathbf{B} *_{(2,\,3)} \mathbf{C}) \neq (\mathbf{A} *_{(1,\,2)} \mathbf{B}) *_{(2,\,3)} \mathbf{C}$.

Proof of Associative Law of Multidimensional Matrix Outer Product

The associative property of multidimensional matrix outer product, $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$, can be proven as follows:
The indices of multidimensional matrix **A** are *ia*, *ja*, *ka*, . . ., *qa*. The indices of multidimensional matrix **B** are *ib*, *jb*, *kb*, . .

., *qb*. The indices of multidimensional matrix **C** are *ic*, *jc*, *kc*, . . ., *qc*.
A = multidimensional matrix of values $a_{ia,\, ja,\, ka,\, \ldots,\, qa}$
B = multidimensional matrix of values $b_{ib,\, jb,\, kb,\, \ldots,\, qb}$
C = multidimensional matrix of values $c_{ic,\, jc,\, kc,\, \ldots,\, qc}$
Let **D** = **A** ⊗ **B**
Let **E** = **B** ⊗ **C**
The indices of multidimensional matrix **D** are *id*, *jd*, *kd*, . . ., *qd*. The indices of multidimensional matrix **E** are *ie*, *je*, *ke*, . . ., *qe*.
D = multidimensional matrix of values $d_{id,\, jd,\, kd,\, \ldots,\, qd} = a_{ia,\, ja,\, ka,\, \ldots,\, qa} * b_{ib,\, jb,\, kb,\, \ldots,\, qb}$ where the indices of each element $d_{id,\, jd,\, kd,\, \ldots,\, qd}$ are determined by a concatenation of the indices for $a_{ia,\, ja,\, ka,\, \ldots,\, qa}$ followed by the indices for $b_{ib,\, jb,\, kb,\, \ldots,\, qb}$.
E = multidimensional matrix of values $e_{ie,\, je,\, ke,\, \ldots,\, qe} = b_{ib,\, jb,\, kb,\, \ldots,\, qb} * c_{ic,\, jc,\, kc,\, \ldots,\, qc}$ where the indices of each element $e_{ie,\, je,\, ke,\, \ldots,\, qe}$ are determined by a concatenation of the indices for $b_{ib,\, jb,\, kb,\, \ldots,\, qb}$ followed by the indices for $c_{ic,\, jc,\, kc,\, \ldots,\, qc}$.
Let **F** = **A** ⊗ (**B** ⊗ **C**) = **A** ⊗ **E.**
Let **G** = (**A** ⊗ **B**) ⊗ **C** = **D** ⊗ **C.**
The indices of multidimensional matrix **F** are *if*, *jf*, *kf*, . . ., *qf*. The indices of multidimensional matrix **G** are *ig*, *jg*, *kg*, . . ., *qg*.
F = multidimensional matrix of values $f_{if,\, jf,\, kf,\, \ldots,\, qf} = a_{ia,\, ja,\, ka,\, \ldots,\, qa} * e_{ie,\, je,\, ke,\, \ldots,\, qe}$ where the indices of each element $f_{if,\, jf,\, kf,\, \ldots,\, qf}$ are determined by a concatenation of the indices for $a_{ia,\, ja,\, ka,\, \ldots,\, qa}$ followed by the indices for $e_{ie,\, je,\, ke,\, \ldots,\, qe}$.
G = multidimensional matrix of values $g_{ig,\, jg,\, kg,\, \ldots,\, qg} = d_{id,\, jd,\, kd,\, \ldots,\, qd} * c_{ic,\, jc,\, kc,\, \ldots,\, qc}$ where the indices of each element $g_{ig,\, jg,\, kg,\, \ldots,\, qg}$ are determined by a concatenation of the indices for $d_{id,\, jd,\, kd,\, \ldots,\, qd}$ followed by the indices for $c_{ic,\, jc,\, kc,\, \ldots,\, qc}$.
Therefore, **F** = multidimensional matrix of values $f_{if,\, jf,\, kf,\, \ldots,\, qf} = a_{ia,\, ja,\, ka,\, \ldots,\, qa} * b_{ib,\, jb,\, kb,\, \ldots,\, qb} * c_{ic,\, jc,\, kc,\, \ldots,\, qc}$ where the indices of each element $f_{if,\, jf,\, kf,\, \ldots,\, qf}$ are determined by a concatenation of the indices for $a_{ia,\, ja,\, ka,\, \ldots,\, qa}$ followed by the indices for $b_{ib,\, jb,\, kb,\, \ldots,\, qb}$ and then $c_{ic,\, jc,\, kc,\, \ldots,\, qc}$.
And **G** = multidimensional matrix of values $g_{ig,\, jg,\, kg,\, \ldots,\, qg} = a_{ia,\, ja,\, ka,\, \ldots,\, qa} * b_{ib,\, jb,\, kb,\, \ldots,\, qb} * c_{ic,\, jc,\, kc,\, \ldots,\, qc}$ where the indices of each element $g_{ig,\, jg,\, kg,\, \ldots,\, qg}$ are determined by a concatenation of the indices for $a_{ia,\, ja,\, ka,\, \ldots,\, qa}$ followed by the indices for $b_{ib,\, jb,\, kb,\, \ldots,\, qb}$ and then $c_{ic,\, jc,\, kc,\, \ldots,\, qc}$.
Therefore, **F** = **G.**
Therefore, **A** ⊗ **(B** ⊗ **C)** = **(A** ⊗ **B)** ⊗ **C.**

C. Proofs of Distributive Laws of Multidimensional Matrix Algebra

Proof of Distributive Law of Multiplication over Addition of Multidimensional Matrices

In proving that multiplication is distributive over addition of multidimensional matrices, it can be proven that **A** $*_{(da,\, db)}$ **(B** + **C)** = **A** $*_{(da,\, db)}$ **B** + **A** $*_{(da,\, db)}$ **C** as follows:
Let **M1** = **B** + **C**
Let **M2** = **A** $*_{(da,\, db)}$ **M1** = **A** $*_{(da,\, db)}$ **(B** + **C)**
Let **N1** = **A** $*_{(da,\, db)}$ **B**
Let **N2** = **A** $*_{(da,\, db)}$ **C**
Let **N3** = **N1** + **N2** = **A** $*_{(da,\, db)}$ **B** + **A** $*_{(da,\, db)}$ **C**

$$m1_{ijk\ldots q} = b_{ijk\ldots q} + c_{ijk\ldots q}$$
$$m2_{ijk\ldots q} = \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * m1_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$
$$= \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * \left(b_{ijk\ldots q \text{ where } x \text{ replaces index of } da} + c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}\right)$$
$$n1_{ijk\ldots q} = \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * b_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$
$$n2_{ijk\ldots q} = \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$
$$n3_{ijk\ldots q} = n1_{ijk\ldots q} + n2_{ijk\ldots q}$$
$$= \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * b_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$
$$+ \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$
$$= \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * \left(b_{ijk\ldots q \text{ where } x \text{ replaces index of } da} + c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}\right)$$

Therefore, $m2_{ijk\ldots q} = n3_{ijk\ldots q}$
Therefore, **M2** = **N3**
Therefore, **A** $*_{(da,\, db)}$ **(B** + **C)** = **A** $*_{(da,\, db)}$ **B** + **A** $*_{(da,\, db)}$ **C**

In proving that multiplication is distributive over addition of multidimensional matrices, it can be proven that **(A** + **B)** $*_{(da,\, db)}$ **C** = **A** $*_{(da,\, db)}$ **C** + **B** $*_{(da,\, db)}$ **C** as follows:
Let **M1** = **A** + **B**
Let **M2** = **M1** $*_{(da,\, db)}$ **C** = **(A** + **B)** $*_{(da,\, db)}$ **C**
Let **N1** = **A** $*_{(da,\, db)}$ **C**
Let **N2** = **B** $*_{(da,\, db)}$ **C**
Let **N3** = **N1** + **N2** = **A** $*_{(da,\, db)}$ **C** + **B** $*_{(da,\, db)}$ **C**

$$m1_{ijk\ldots q} = a_{ijk\ldots q} + b_{ijk\ldots q}$$
$$m2_{ijk\ldots q} = \sum_x m1_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$
$$= \sum_x \left(a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} + b_{ijk\ldots q \text{ where } x \text{ replaces index of } db}\right) * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$
$$n1_{ijk\ldots q} = \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$
$$n2_{ijk\ldots q} = \sum_x b_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$
$$n3_{ijk\ldots q} = n1_{ijk\ldots q} + n2_{ijk\ldots q}$$

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$$= \sum_x a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da} + \sum_x b_{ijk\ldots q \text{ where } x \text{ replaces index of } db} * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$
$$= \sum_x \left(a_{ijk\ldots q \text{ where } x \text{ replaces index of } db} + b_{ijk\ldots q \text{ where } x \text{ replaces index of } db}\right) * c_{ijk\ldots q \text{ where } x \text{ replaces index of } da}$$

Therefore, $m2_{ijk\ldots q} = n3_{ijk\ldots q}$
Therefore, **M2** = **N3**
Therefore, $(\mathbf{A} + \mathbf{B}) *_{(da,\, db)} \mathbf{C} = \mathbf{A} *_{(da,\, db)} \mathbf{C} + \mathbf{B} *_{(da,\, db)} \mathbf{C}$

It can be proven that multiplication by a scalar is distributive over addition of multidimensional matrices, $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$, as follows:
A = multidimensional matrix of values $a_{ijk\ldots q}$
B = multidimensional matrix of values $b_{ijk\ldots q}$
Then **A** + **B** = multidimensional matrix of values $a_{ijk\ldots q} + b_{ijk\ldots q}$
Therefore, $\alpha(\mathbf{A} + \mathbf{B})$ = multidimensional matrix of values $\alpha a_{ijk\ldots q} + \alpha b_{ijk\ldots q}$
Then $\alpha\mathbf{A}$ = multidimensional matrix of values $\alpha a_{ijk\ldots q}$
And $\alpha\mathbf{B}$ = multidimensional matrix of values $\alpha b_{ijk\ldots q}$
Therefore, $\alpha\mathbf{A} + \alpha\mathbf{B}$ = multidimensional matrix of values $\alpha a_{ijk\ldots q} + \alpha b_{ijk\ldots q}$
Therefore, $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$

Proof of Distributive Law of Outer Product over Addition of Multidimensional Matrices

In proving that outer product is distributive over addition of multidimensional matrices, it can be proven that $\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$ as follows:
The indices of multidimensional matrix **A** are *ia, ja, ka, . . . qa*. The indices of multidimensional matrix **B** are *ib, jb, kb, . . . qb*. The indices of multidimensional matrix **C** are *ic, jc, kc, . . . qc*.
A = multidimensional matrix of values $a_{ia, ja, ka, \ldots qa}$
B = multidimensional matrix of values $b_{ib, jb, kb, \ldots qb}$
C = multidimensional matrix of values $c_{ic, jc, kc, \ldots qc}$
Let **M1** = **B** + **C**
Let $\mathbf{M2} = \mathbf{A} \otimes \mathbf{M1} = \mathbf{A} \otimes (\mathbf{B} + \mathbf{C})$
The indices of multidimensional matrix **M1** are *im1, jm1, km1, . . . qm1*. The indices of multidimensional matrix **M2** are *im2, jm2, km2, . . . qm2*.
Therefore, **M1** = multidimensional matrix of values $m1_{im1, jm1, km1, \ldots qm1} = b_{ib, jb, kb, \ldots qb} + c_{ic, jc, kc, \ldots qc}$.
Therefore, **M2** = multidimensional matrix of values $m2_{im2, jm2, km2, \ldots qm2} = a_{ia, ja, ka, \ldots qa} * m1_{im1, jm1, km1, \ldots qm1}$ where the indices of each element $m2_{im2, jm2, km2, \ldots qm2}$ are determined by a concatenation of the indices for $a_{ia, ja, ka, \ldots qa}$ followed by the indices for $m1_{im1, jm1, km1, \ldots qm1}$.
Therefore, **M2** = multidimensional matrix of values $m2_{im2, jm2, km2, \ldots qm2} = a_{ia, ja, ka, \ldots qa} * (b_{ib, jb, kb, \ldots qb} + c_{ic, jc, kc, \ldots qc})$ where the indices of each element $m2_{im2, jm2, km2, \ldots qm2}$ are determined by a concatenation of the indices for $a_{ia, ja, ka, \ldots qa}$ followed by the indices for $b_{ib, jb, kb, \ldots qb}$ or $c_{ic, jc, kc, \ldots qc}$.
Because $a_{ia, ja, ka, \ldots qa}$, $b_{ib, jb, kb, \ldots qb}$, and $c_{ic, jc, kc, \ldots qc}$ are scalar values, $m2_{im2, jm2, km2, \ldots qm2} = a_{ia, ja, ka, \ldots qa} * (b_{ib, jb, kb, \ldots qb} + c_{ic, jc, kc, \ldots qc}) = a_{ia, ja, ka, \ldots qa} * b_{ib, jb, kb, \ldots qb} + a_{ia, ja, ka, \ldots qa} * c_{ic, jc, kc, \ldots qc}$.
Therefore, **M2** = multidimensional matrix of values $m2_{im2, jm2, km2, \ldots qm2} = a_{ia, ja, ka, \ldots qa} * b_{ib, jb, kb, \ldots qb} + a_{ia, ja, ka, \ldots qa} * c_{ic, jc, kc, \ldots qc}$ where the indices of each element $m2_{im2, jm2, km2, \ldots qm2}$ are determined by a concatenation of the indices for $a_{ia, ja, ka, \ldots qa}$ followed by the indices for $b_{ib, jb, kb, \ldots qb}$ or are determined by a concatenation of the indices for $a_{ia, ja, ka, \ldots qa}$ followed by the indices for $c_{ic, jc, kc, \ldots qc}$.
Let $\mathbf{N1} = \mathbf{A} \otimes \mathbf{B}$
Let $\mathbf{N2} = \mathbf{A} \otimes \mathbf{C}$
Let $\mathbf{N3} = \mathbf{N1} + \mathbf{N2} = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$
The indices of multidimensional matrix **N1** are *in1, jn1, kn1, . . . qn1*. The indices of multidimensional matrix **N2** are *in2, jn2, kn2, . . . qn2*. The indices of multidimensional matrix **N3** are *in3, jn3, kn3, . . . qn3*.
Therefore, **N1** = multidimensional matrix of values $n1_{in1, jn1, kn1, \ldots qn1} = a_{ia, ja, ka, \ldots qa} * b_{ib, jb, kb, \ldots qb}$ where the indices of each element $n1_{in1, jn1, kn1, \ldots qn1}$ are determined by a concatenation of the indices for $a_{ia, ja, ka, \ldots qa}$ followed by the indices for $b_{ib, jb, kb, \ldots qb}$.
Therefore, **N2** = multidimensional matrix of values $n2_{in2, jn2, kn2, \ldots qn2} = a_{ia, ja, ka, \ldots qa} * c_{ic, jc, kc, \ldots qc}$ where the indices of each element $n2_{in2, jn2, kn2, \ldots qn2}$ are determined by a concatenation of the indices for $a_{ia, ja, ka, \ldots qa}$ followed by the indices for $c_{ic, jc, kc, \ldots qc}$.
Therefore, **N3** = multidimensional matrix of values $n3_{in3, jn3, kn3, \ldots qn3} = a_{ia, ja, ka, \ldots qa} * b_{ib, jb, kb, \ldots qb} + a_{ia, ja, ka, \ldots qa} * c_{ic, jc, kc, \ldots qc}$ where the indices of each element $n3_{in3, jn3, kn3, \ldots qn3}$ are determined by a concatenation of the indices for $a_{ia, ja, ka, \ldots qa}$ followed by the indices for $b_{ib, jb, kb, \ldots qb}$ or are determined by a concatenation of the indices for $a_{ia, ja, ka, \ldots qa}$ followed by the indices for $c_{ic, jc, kc, \ldots qc}$.
Therefore, **M2** = **N3.**
Therefore, $\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$.

In proving that outer product is distributive over addition of multidimensional matrices, it can be proven that $(\mathbf{B} + \mathbf{C}) \otimes \mathbf{A} = \mathbf{B} \otimes \mathbf{A} + \mathbf{C} \otimes \mathbf{A}$ as follows:
The indices of multidimensional matrix **A** are *ia, ja, ka, . . . qa*. The indices of multidimensional matrix **B** are *ib, jb, kb, . . . qb*. The indices of multidimensional matrix **C** are *ic, jc, kc, . . . qc*.
A = multidimensional matrix of values $a_{ia, ja, ka, \ldots qa}$
B = multidimensional matrix of values $b_{ib, jb, kb, \ldots qb}$
C = multidimensional matrix of values $c_{ic, jc, kc, \ldots qc}$
Let **M1** = **B** + **C**
Let $\mathbf{M2} = \mathbf{M1} \otimes \mathbf{A} = (\mathbf{B} + \mathbf{C}) \otimes \mathbf{A}$
The indices of multidimensional matrix **M1** are *im1, jm1, km1, . . . qm1*. The indices of multidimensional matrix **M2** are *im2, jm2, km2, . . . qm2*.
Therefore, **M1** = multidimensional matrix of values $m1_{im1, jm1, km1, \ldots qm1} = b_{ib, jb, kb, \ldots qb} + c_{ic, jc, kc, \ldots qc}$.
Therefore, **M2** = multidimensional matrix of values $m2_{im2, jm2, km2, \ldots qm2} = m1_{im1, jm1, km1, \ldots qm1} * a_{ia, ja, ka, \ldots qa}$ where the indices of each element $m2_{im2, jm2, km2, \ldots qm2}$ are determined by a concatenation of the indices for $m1_{im1, jm1, km1, \ldots qm1}$ followed by the indices for $a_{ia, ja, ka, \ldots qa}$.
Therefore, **M2** = multidimensional matrix of values $m2_{im2, jm2, km2, \ldots qm2} = (b_{ib, jb, kb, \ldots qb} + c_{ic, jc, kc, \ldots qc}) * a_{ia, ja, ka, \ldots qa}$ where the indices of each element $m2_{im2, jm2, km2, \ldots qm2}$ are determined by a concatenation of the indices for $b_{ib, jb, kb, \ldots qb}$ or $c_{ic, jc, kc, \ldots qc}$ followed by the indices for $a_{ia, ja, ka, \ldots qa}$.
Because $a_{ia, ja, ka, \ldots qa}$, $b_{ib, jb, kb, \ldots qb}$, and $c_{ic, jc, kc, \ldots qc}$ are scalar values, $m2_{im2, jm2, km2, \ldots qm2} = (b_{ib, jb, kb, \ldots qb} + c_{ic, jc, kc, \ldots qc}) * a_{ia, ja, ka, \ldots qa} = b_{ib, jb, kb, \ldots qb} * a_{ia, ja, ka, \ldots qa} + c_{ic, jc, kc, \ldots qc} * a_{ia, ja, ka, \ldots qa}$.

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Therefore, **M2** = multidimensional matrix of values $m2_{im2, jm2, km2, \ldots qm2} = b_{ib, jb, kb, \ldots qb} * a_{ia, ja, ka, \ldots qa} + c_{ic, jc, kc, \ldots qc} * a_{ia, ja, ka, \ldots qa}$ where the indices of each element $m2_{im2, jm2, km2, \ldots qm2}$ are determined by a concatenation of the indices for $b_{ib, jb, kb, \ldots qb}$ followed by the indices for $a_{ia, ja, ka, \ldots qa}$ or are determined by a concatenation of the indices for $c_{ic, jc, kc, \ldots qc}$ followed by the indices for $a_{ia, ja, ka, \ldots qa}$.

Let **N1** = **B** ⊗ **A**

Let **N2** = **C** ⊗ **A**

Let **N3** = **N1** + **N2** = **B** ⊗ **A** + **C** ⊗ **A**

The indices of multidimensional matrix **N1** are *in1, jn1, kn1, . . . qn1*. The indices of multidimensional matrix **N2** are *in2, jn2, kn2, . . . qn2*. The indices of multidimensional matrix **N3** are *in3, jn3, kn3, . . . qn3*.

Therefore, **N1** = multidimensional matrix of values $n1_{in1, jn1, kn1, \ldots qn1} = b_{ib, jb, kb, \ldots qb} * a_{ia, ja, ka, \ldots qa}$ where the indices of each element $n1_{in1, jn1, kn1, \ldots qn1}$ are determined by a concatenation of the indices for $b_{ib, jb, kb, \ldots qb}$ followed by the indices for $a_{ia, ja, ka, \ldots qa}$.

Therefore, **N2** = multidimensional matrix of values $n2_{in2, jn2, kn2, \ldots qn2} = c_{ic, jc, kc, \ldots qc} * a_{ia, ja, ka, \ldots qa}$ where the indices of each element $n2_{in2, jn2, kn2, \ldots qn2}$ are determined by a concatenation of the indices for $c_{ic, jc, kc, \ldots qc}$ followed by the indices for $a_{ia, ja, ka, \ldots qa}$.

Therefore, **N3** = multidimensional matrix of values $n3_{in3, jn3, kn3, \ldots qn3} = b_{ib, jb, kb, \ldots qb} * a_{ia, ja, ka, \ldots qa} + c_{ic, jc, kc, \ldots qc} * a_{ia, ja, ka, \ldots qa}$ where the indices of each element $n3_{in3, jn3, kn3, \ldots qn3}$ are determined by a concatenation of the indices for $b_{ib, jb, kb, \ldots qb}$ followed by the indices for $a_{ia, ja, ka, \ldots qa}$ or are determined by a concatenation of the indices for $c_{ic, jc, kc, \ldots qc}$ followed by the indices for $a_{ia, ja, ka, \ldots qa}$.

Therefore, **M2** = **N3.**

Therefore, **(B** + **C)** ⊗ **A** = **B** ⊗ **A** + **C** ⊗ **A.**

IV. Conclusion

Part 5 of 6 described the commutative, associative, and distributive laws of multidimensional matrix algebra.

Part 6 of 6 describes the solution of systems of linear equations using multidimensional matrices. Also, part 6 of 6 defines the multidimensional matrix calculus operations for differentiation and integration.

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