RESEARCH STATEMENT FOR GUILLAUME ROY-FORTIN.

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My research interests lie in geometric spectral theory, a field which is at the intersection of analysis, differential geometry and partial differential equations. In my thesis, I focus on the study of eigenfunctions of the Laplace operator, more precisely on the relation between the growth of eigenfunctions and the size of their nodal sets. I will present these results in Section 1. In Section 2, I will describe my plans to further investigate this topic, as well as new related questions regarding the Steklov eigenvalue problem and the bi-Laplacian operator. In the last section, I will briefly discuss another research project that is related to an earlier paper of mine concerning shape optimization of Neumann eigenvalues.

1. GROWTH AND NODAL SETS OF LAPLACE EIGENFUNCTIONS.

Let (M^n, g) be a compact, smooth n-dimensional manifold without boundary and let $\{\phi_\lambda\}$, $\lambda \nearrow \infty$ be any sequence of eigenfunctions of the Laplace-Beltrami operator Δ_g:

$$\Delta_g \phi_\lambda + \lambda \phi_\lambda = 0.$$

It is a common intuition in spectral geometry (see [DF1, DF2]) that the eigenfunctions ϕ_λ roughly behave like a polynomial of degree $\sqrt{\lambda}$. In the case of a complex polynomial, the degree controls both the number of zeros and the growth and it is a central theme of my research to show that a similar link between growth and zero set holds for eigenfunctions.

1.1. **Nodal sets.** Let $Z_\lambda = \{p \in M : \phi_\lambda(p) = 0\}$ be the nodal set of an eigenfunction ϕ_λ. It is a smooth hypersurface away from the singular set $S_\lambda = \{p \in Z_\lambda : \nabla\phi_\lambda(p) = 0\}$. The interest for nodal sets goes all the way back to Chladni's plates and is also relevant in the context of quantum mechanics, where a properly normalized ϕ_λ is the probability density of a free particle and Z_λ can be thought as the set where that particle is least likely to be found. In the past decades, much effort has been put in measuring the size of the nodal set of an eigenfunction ϕ_λ. In fact, Yau conjectured that for a compact, C^∞ n-dimensional manifold M, the size of the nodal set is comparable to the frequency

$$c\lambda^{\frac{1}{2}} \leq \mathcal{H}^{n-1} \leq C\lambda^{\frac{1}{2}},$$

where $\mathcal{H}^{n-1}$ is the codimension-1 Hausdorff measure and c, C are positive constants. In [DF1], Donnelly and Fefferman proved that the conjecture is true in any dimension for real-analytic manifolds (M^n, g). For smooth surfaces, the conjectured lower bound for the size of the nodal set holds (see [Br]), but the current best upper bound is $\lambda^{\frac{3}{4}}$ (see [D, DF2]).

1.2. **Local growth.** Let (X, d) be a metric space and $f \in C^0(X)$ a continuous function. One way to measure, at least locally, the growth of f is the following: given a ball $B \subset X$ and a small scaling parameter $0 < \alpha < 1$, we define the *α-growth exponent* or simply *growth exponent* of f on B by

$$\beta(f, B; \alpha) := \log \frac{\sup_B |f|}{\sup_{\alpha B} |f|},$$

where αB is a ball concentric to B whose radius is scaled by a factor α. The growth exponent generalizes the idea of the degree of a polynomial, as can be seen by the basic example $\beta(x^n, [-1, 1]; \alpha) = n \log \alpha^{-1}$. Given $p \in M$, denote by $B_p(r)$ the metric ball of radius f centred at p. In [DF1], Donnelly and Fefferman provide a universal growth bound for eigenfunctions on smooth, compact manifolds of *any* dimension. Namely, they show that for a small enough radius r and for any $0 < \alpha < 1$, there exists a constant $c = c(\alpha)$ such that

$$\beta(\phi_\lambda, B_p(r); \alpha) \leq c\lambda^{\frac{1}{2}}, \ \forall p \in M.$$

We now restrict our attention to pointwise growth exponents of eigenfunctions on small balls of radius $r = k_0 \lambda^{-\frac{1}{2}}$ by defining

$$\beta_{\lambda,\alpha}(p) := \beta(\phi_\lambda, B_p(r); \alpha) = \log \frac{\sup_{B_p(r)} |\phi_\lambda|}{\sup_{B_p(\alpha r)} |\phi_\lambda|},$$

where k_0 is a fixed positive constant that can only depend on the geometry of (M, g). We finally consider the average of these local growth exponents $\beta_{p,\alpha}(\lambda)$ on the whole surface, by defining the *average local growth* of ϕ_λ by

$$A_\alpha(\lambda) = \frac{1}{\mathrm{Vol}(M)} \int_M \beta_{\lambda,\alpha_0}(p) dV_g(p).$$

1.3. **Main results.** In [R], I prove the following result announced in Section 7.3 of [NPS] which provides a link between the growth and the size of the nodal sets of Laplace eigenfunctions on surfaces:

Theorem 1.3.1. *Let (M, g) be a smooth, connected, compact surface without boundary. There exists a constant $\alpha_0 \in (0, 1)$ such that for $0 < \alpha < \alpha_0$, we have*

$$c_1 \lambda^{\frac{1}{2}} A_{\alpha_0}(\lambda) \leq \mathcal{H}^1(Z_\lambda) \leq c_2 \lambda^{\frac{1}{2}} (A_{\alpha_0}(\lambda) + 1),$$

where c_1, c_2 are positive constants depending only on the geometry of M and α.

Note that by definition of the growth exponents, the lower bound for the size of the nodal set actually holds for any $\alpha \in (0, 1)$. The theorem has the following two interesting immediate consequences:

1. *The average local growth is bounded by a constant on real-analytic surfaces.*
Since Yau's conjecture holds in dimension 2 for real-analytic surfaces, it follows that, in such a setting, the average local growth $A(\lambda)$ of an eigenfunction ϕ_λ is bounded by a constant in the semi-classical limit $\lambda \to \infty$. Combined with the growth bound of Donnelly and Fefferman, this means that one cannot find a sequence of eigenfunctions whose growth exponents on disks of wavelength radius saturate the upper bound $\lambda^{\frac{1}{2}}$ on a set of positive measure. Simply put, on a C^ω pair (M, g), the growth of eigenfunctions at wavelength scale is bounded by a constant almost everywhere.

2. *Function theoretic reformulation of Yau's conjecture for smooth surfaces.*
Going back to the setting of smooth surfaces and since we know that $c\lambda^{\frac{1}{2}} \leq \mathcal{H}^1(Z_\lambda)$, the result implies that Yau's conjecture in dimension 2 is equivalent to the statement $A(\lambda) = O(1)$. As such, I am interested in understanding which sequence of eigenfunctions have growth exponents that can saturate a polynomial type upper bound on a set of positive measure.

A central idea in the proof of Theorem 1.3.1 is the reduction of the Laplace-Beltrami eigenvalue problem on the surface to a planar Schrödinger equation with small, smooth potential of the type $\Delta F + qF = 0$, where here Δ is the usual flat Laplacian. This is done through the use of conformal coordinates and restriction of eigenfunctions to small balls of radius comparable to the wavelength $\lambda^{-\frac{1}{2}}$. That being done, I obtain results which are suitably linking growth exponents and nodal sets of such Schrödinger eigenfunctions F, which are then extended back to the eigenfunctions on the surface. Some of the principal tools used in these proofs are Carleman estimates, classical complex analysis and quasiconformal maps.

2. Ongoing and future projects.

2.1. L^q **growth exponents.** In my past research, the growth of eigenfunctions has been measured through growth exponents defined on small disks on which we have taken the L^∞ norm. Indeed, we recall that

$$\beta_{p,\alpha}(\lambda) = \log \frac{\sup_B |\phi_\lambda|}{\sup_{\alpha B} |\phi_\lambda|},$$

where B is a metric ball of small radius centered at $p \in M$. For $1 \le q \le \infty$, we can define the more general L^q growth-exponent $\beta^q_{p,\alpha}(\lambda)$ of an eigenfunction ϕ_λ in the following way

$$\beta^q_{p,\alpha}(\lambda) := \log \frac{||\phi_\lambda||_{L^q(B)}}{||\phi_\lambda||_{L^q(\alpha B)}},$$

where B is once again a suitably small metric ball centred at p. Notice that $\beta_{p,\alpha}(\lambda) = \beta^\infty_{p,\alpha}(\lambda)$. Consider the average of such quantities on the surface, that is, define

$$A^q_\alpha(\lambda) := \frac{1}{\mathrm{Vol}(M)} \int_M \beta^q_{p,\alpha}(\lambda) dV_g.$$

My current goal is to investigate for which $q \in [1, \infty)$, if any, do we have the following analogue of Theorem 1.3.1:

$$c\lambda^{\frac{1}{2}} A^q_\alpha(\lambda) \le \mathcal{H}^1(Z_\lambda) \le C\lambda^{\frac{1}{2}}(A^q_\alpha(\lambda) + 1).$$

The most interesting case seems to be $q = 2$, where we could benefit from the many known results about L^2 norms of eigenfunctions, such as the Bernstein estimates and various doubling estimates in the spirit of Lin and Donnelly-Fefferman.

2.2. **Gaussian random waves.** A Gaussian random wave of frequency λ is a random function φ_λ defined in the following way :

$$\varphi_\lambda := \sum_{\lambda_j \in (\lambda, \lambda+1]} a_j \phi_j,$$

where ϕ_j are eigenfunctions of the Laplace-Beltrami operator as before and the a_j are independent, identically distributed centered Gaussians of variance chosen so to normalize the expected L^2 norm:

$$\mathbb{E}(||\varphi_\lambda||_{L^2}) = 1.$$

The size of zero sets of such random waves (and their complexification) has been of interest lately, with notably a result from Zelditch (see [Z1]) showing that Yau's conjecture hold in average on manifolds which are either Zoll or aperiodic:

$$c\lambda^{\frac{1}{2}} \le \mathbb{E}(\mathcal{H}^{n-1}(Z_{\varphi_\lambda})) \le C\lambda^{\frac{1}{2}}.$$

I am interested in proving an averaged version of Theorem 1.3.1 for random waves on compact C^∞ surfaces. Doing so, Yau's conjecture in average would become equivalent to show that random waves have bounded growth exponents almost everywhere on balls of wavelength radius.

2.3. **Growth and nodal sets of Laplace eigenfunctions on higher dimensional manifolds.** It is natural to ask whether Theorem 1.3.1 holds for compact, smooth manifolds of dimension $n \geq 3$. It seems reasonable to expect a positive answer: on the one hand, as previously stated, Yau's conjecture on the size of nodal sets is formulated for manifolds of any dimensions. On the other hand, some fundamental results for the growth exponents of eigenfunctions, such as the aforementioned Donnelly-Fefferman growth bound, are known to hold in any dimension. However, the approach I have used relies crucially on the reduction of an eigenfunction ϕ_λ to a planar solution F to a Schrödinger equation, a transformation made possible by the existence of local conformal coordinates, a fact that does not generalize in dimensions $n \geq 3$. One would therefore need to follow a fundamentally different approach to prove a result in the spirit of Theorem 1.3.1 in that setting. One idea I am currently exploring is the following: extend an eigenfunction ϕ_λ on a n-dimensional manifold M to a harmonic function on the $(n+1)$-dimensional product manifold $M \times \mathbb{R}$. Then, study the average local growth and nodal sets of that harmonic function to learn something about similar quantities for the original ϕ_λ. This can be done, for example, through the use of frequency functions estimates for harmonic functions. This approach has been successful in the past in extending to higher dimensions results which were initially proven only in the context of surfaces, see for example [M].

2.4. **Interior nodal lines of Steklov eigenfunctions.** Let Ω be a smooth, compact Riemannian manifold of dimension n with smooth boundary $M = \partial\Omega$. The Steklov eigenvalue problem on Ω looks for functions u harmonic in Ω and satisfying $\frac{\partial u}{\partial \nu} = \sigma u$ on M, where ν is the unit outward normal direction. It gives rise to a discrete spectrum of eigenvalues

$$0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \ldots \nearrow \infty,$$

with corresponding eigenfunctions u_j. The boundary nodal set Z_σ of an eigenfunction u_σ is its zero set on M and has Hausdorff codimension 2. The interior nodal set N_σ is the zero set of u_σ in Ω and has Hausdorff codimension 1. For more background on the Steklov problem, see [PoS] and the references therein. In [Z2], Zelditch proves a conjecture from Bellova and Lin (see [BL]) by showing the following sharp upper bound for boundary nodal sets of Steklov eigenfunctions on real analytic manifolds Ω with real analytic boundary M:

$$\mathcal{H}^{n-2}(Z_\sigma) \leq c\sigma.$$

There does not seem to be any known result for interior nodal sets, in any setting. I plan to investigate the question, first in the setting of Euclidean domains with real analytic boundary. In that context, our first goal is to prove the following analogous result :

$$\mathcal{H}^{1}(N_\sigma) \leq c\sigma.$$

Further goals are obtaining results in the same spirit in the smooth setting.

2.5. **Nodal geometry of vibrating plates.** The study of nodal domains and nodal lines for eigenfunctions of vibrating plates goes back to Chladni's experiments set more than two centuries ago. However, in the mathematical treatment of this problem the following substitution is typically made (though rarely emphasized): instead of studying boundary value problems for vibrating plates (that involves the bi-Laplacian), one considers the corresponding problems for vibrating membranes (involving the usual Laplace operator). The bi-Laplacian Δ^2 is a fourth order operator, and the nodal sets of its eigenfunctions have very different properties compared to those of the Laplacian. There are two main directions in the study of nodal geometry for the bi-Laplacian that I wish to explore.

I. NODAL DOMAIN COUNT. It is well-known that eigenfunctions of the clamped plate problem (i.e. the Dirichlet problem for the bi-Laplacian) may have infinitely many nodal domains - for instance, this is the case for the square plate (see [C]). At the same time, the following result was obtained in [PS]. Let us say that a nodal domain D of an eigenfunction ϕ_λ has depth at least a if $\max_{x\in D}|\phi_\lambda(x)| \geq a$. Then there exists a constant $C > 0$, depending only on the shape of the plate, such that the number of nodal domains of an eigenfunction ϕ_λ of the clamped plate problem that have depth at least a is bounded above by $C\sqrt{\lambda}a^{-1}$. I would like to check whether the power of λ in the estimate above is sharp, and if not - to improve it.

II. SIZE OF NODAL SETS. Another question that connects to some of my previously mentioned areas of research is to find some nontrivial estimate from below for the length of the nodal set of eigenfunctions of vibrating plates. Recall that, for membranes, the sharp lower bound conjectured by Yau has been obtained by Brüning in [Br]. We note that for plates with analytic boundary, upper bounds on the size of nodal sets follow from results of [K], that extend the estimates of Donnelly-Fefferman for Laplace eigenfunctions to the case of higher order operators.

3. OPTIMIZATION OF PLANAR EIGENVALUES.

3.1. **Shape maximization of Neumann eigenvalues.** Consider a bounded Euclidean domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary. The Laplace eigenvalue problem $\Delta u + \lambda u = 0$ on Ω with either Dirichlet or Neumann boundary condition gives rise to a discrete spectrum, which we respectively denote by

$$\sigma_D(\Omega) = \{0 < \lambda_1 < \lambda_2, ...\},\ \ \sigma_N(\Omega) = \{0 = \mu_0 \leq \mu_1 \leq \mu_2, ...\}.$$

When $N = 2$, such a setting is used to model the vibrations of a membrane with either fixed or free boundary. Notice that it is not assumed here that Ω is connected. In the case where Ω is the disjoint union of connected components, the spectrum of Ω is obtained from the ordered union of the spectra of its components. A classical area of investigation is the optimization of eigenvalues in a class of domains with fixed area: which Ω, if any, will minimize or maximize a given Dirichlet or Neumann eigenvalue ? The Dirichlet problem is a minimizing one, whereas the Neumann problem is maximization one. In the planar case and for the first few eigenvalues, for both the Dirichlet and Neumann problems, it is either known or conjectured that the optimal domain is a single disk or a union of disks. A natural question to ask is whether this is true the whole spectrum: are all eigenvalues optimized by disks and union of disks ? Wolf and Keller provided a negative answer for the Dirichlet case and, together with Guillaume Poliquin, we provide a negative answer for the Neumann case:

Theorem. *For $N = 2$, the Neumann eigenvalue μ_{23} is not maximized by a disk or a union of disks.*

The proof relies on a result which roughly says that a disconnected optimal domain has to be built from previous optimal domains. More precisely, if a disconnected domain maximizes a given Neumann eigenvalue μ_n, then its connected components have to be maximizing for some previous eigenvalue μ_k, $k \leq n$. The details can be found in our paper [PR].

3.2. **Neumann minimizing functionals.** In their paper [IV], the authors study variational problems for planar Dirichlet eigenvalues of the following form :

$$\inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^n, T(\Omega) \leq 1\},$$

where T is a non-negative set function defined on the open sets of $\mathbb{R}^n$ with certain specific properties. For example, T could be the Lebesgue measure. They obtain upper bounds for the number of components of the minimizing domain of a given eigenvalue if T satisfies a scaling relation. In light of my previous work, I am interested in investigating whether analogous results hold for the Neumann eigenvalue problem. A trivial upper bound for the number of connected components of a domain maximizing μ_k is k. It would be interesting to get better, non-trivial upper bounds on the number of connected components of a domain maximizing a problem of the following type:

$$\sup\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^n, T(\Omega) \leq 1\},$$

where once again here T is acting on open sets and satisfies a scaling property.

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