$\int$ Limits
$\sum$ Discretizations
$\blacksquare$ + Rectangles

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Outline

An implementation of the numerical examples can be found in the accompanying Python notebook `sinsm2023.ipynb`

Riemann Integration Revisited

Partitions

Let $a, b \in \mathbb{R}$ and $a < b$. A finite collection of points $\{x_k\}_{k=0}^{n}$ is called a **partition** of the closed interval

$$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

if the following inequalities hold:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

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if the following inequalities hold:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The **norm** of a partition is the largest length of the subinterval induced by it, i.e.

$$\|\{x_k\}_{k=0}^n\| = \max_{k=1,\ldots,n} (x_k - x_{k-1}).$$

Example. The collection $P := \{0, \frac{1}{3}, \frac{1}{2}, 1\}$ is a partition of $[0, 1]$ with norm

$$\|P\| = \max\left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2}.$$

Bounded Functions

A function $f : [a, b] \to \mathbb{R}$ is said to be **bounded** if there exist $l, u \in \mathbb{R}$ such that

$$l \leq f(x) \leq u \qquad \text{for all } x \in [a, b].$$

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Example. The function $f : [0,2] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{1}{4} & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \dfrac{1}{\sqrt{x}} & \text{if } \frac{1}{2} < x \leq 2, \end{cases}$$

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Example. Every continuous function on a closed interval is bounded.

Extreme Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous, then there exist $y, z \in [a, b]$ such that $f(y) \leq f(x) \leq f(z)$ for every $x \in [a, b]$.

Riemann Sums

Given a function $f : [a, b] \to \mathbb{R}$, a partition $\{x_k\}_{k=0}^{n}$ of $[a, b]$, and a collection of representative points $\{x_k^*\}_{k=1}^{n}$ with $x_k^* \in [x_{k-1}, x_k]$ for $k = 1, 2, \dots, n$, a sum of the form

$$\sum_{k=1}^{n} f(x_k^*)(x_k - x_{k-1})$$

is called a **Riemann Sum**.

Riemann Integrable Functions

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** if the limit of Riemann sums as the norm of the partition vanishes exists, i.e. there exists a $I \in \mathbb{R}$ such that

$$\lim_{\|\{x_k\}_{k=0}^{n}\| \to 0} \sum_{k=1}^{n} f(x_k^*)(x_k - x_{k-1}) = I.$$

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Equivalently, given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every partition $\{x_k\}_{k=0}^n$ of $[a, b]$ such that $\|\{x_k\}_{k=0}^n\| < \delta$ and every collection of representative points $\{x_k^*\}_{k=1}^n$ with $x_k^* \in [x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$, we have

$$\left| \sum_{k=1}^{n} f(x_k^*)(x_k - x_{k-1}) - I \right| < \varepsilon.$$

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The number I is called the **Riemann integral** of f and is denoted by

$$I := \int_a^b f(x)\, dx.$$

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- The function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 1$ is x is rational and $f(x) = 0$ if x is irrational is **not Riemann integrable** since for any partition of $\{x_k\}_{k=0}^n$ of $[a,b]$

$$\sum_{k=1}^{n} f(x_k^*)(x_k - x_{k-1}) = 1 \qquad \text{if } x_k^* \text{ is rational,}$$

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- $\|\{x_k\}_{k=0}^n\| \to 0$ versus passing $n \to \infty$ in $\{x_k\}_{k=0}^n\|$?

Illustrations and Numerical Experiments

An Algorithm for Computing Sums

Given a list of numbers $s_0, s_1, \ldots, s_n$, the sum $\sum_{k=0}^{n} s_k$ can be computed as follows:

```
Algorithm: Sum(s0, s1, ..., sn)
Input: Numbers s0, s1, ..., sn
Output: The sum s0 + s1 + ... + sn

  1. s = 0                                    // initialize sum to 0
  2. for k = 0, 1, ..., n do                  // loop over all terms
  3.     s = s + sk                           // update value of sum
  4. end for                                  // end of the for loop
  5. return s                                 // return value of sum
```

An Algorithm for Computing Riemann Sums

The following algorithm computes the Riemann sum

$$\sum_{k=1}^{n} f(x_k^*)(x_k - x_{k-1}) \tag{RS}$$

using a random partition and random points as representatives:

Algorithm: RiemannSum(f, a, b, n)

Input: A function f, left endpoint a, right endpoint b, and n as the number of points in the partition

Output: The Riemann sum (RS)

```
1. select random points x0 < x1 < ... < xn on [a, b]
2. set x0 = a and xn = b
3. set rs = 0
4. for k = 1, ..., n do
5.     select a random point xk* on [x(k-1), xk]
6.     rs = rs + [f(xk*) × (xk - x(k-1))]
7. end for
8. return rs
```

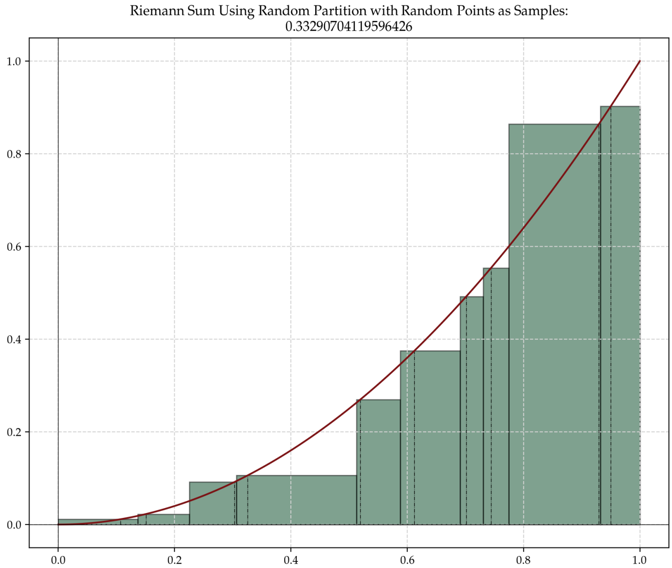
Riemann Sums with Random Partitions

Example 1. Calculate the Riemann sums for the function $f(x) = x^2$ for $0 \leq x \leq 1$ using random partitions $\{x_k\}_{k=0}^{n}$ and with the following type of representative points:

- **a.** left endpoint $\quad x_k^* = x_{k-1}$
- **b.** right endpoint $\quad x_k^* = x_k$
- **c.** midpoint $\quad x_k^* = (x_{k-1} + x_k)/2$
- **d.** random point $\quad x_{k-1} \leq x_k^* \leq x_k$

Use $n = 10^N$ for $N = 1, 2, 3, 4, 5, 6$.

A Riemann Sum with Random Partition



Riemann Sums with Uniform Partitions

A partition $\{x_k\}_{k=0}^{n}$ of $[a, b]$ is called **uniform** if

$$x_k = a + kh \qquad \text{for } k = 0, 1, \dots, n \qquad \text{where } h = \frac{b-a}{n}$$

Specific versions of Riemann Sums:

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Specific versions of Riemann Sums:

- Left endpoints as representatives

$$\sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) = h \sum_{k=1}^{n} f(a + (k-1)h) \tag{RSUL}$$

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- Right endpoints as representatives

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- Right endpoints as representatives

$$\sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) = h \sum_{k=1}^{n} f(a + kh) \tag{RSUR}$$

- Midpoints as representatives

$$\sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)(x_k - x_{k-1}) = h \sum_{k=1}^{n} f\left(a + \frac{(2k-1)h}{2}\right) \tag{RSUM}$$

An Algorithm for Computing RSUM

Algorithm: `RiemannSumUniformMid(f, a, b, n)`

Input: A function f, left endpoint a, right endpoint b, and n as the number of points in the partition

Output: The Riemann sum (RSUM)

```
1. set h = (b-a)/n
1. set x = a - h/2
3. set rs = 0
4. for k = 1, ..., n do
5.     x = x + h
6.     rs = rs + f(x)
7. end for
8. return h×rs
```

Riemann Sums with Uniform Partitions

Example 2. Repeat **Example 1** using uniform partitions.

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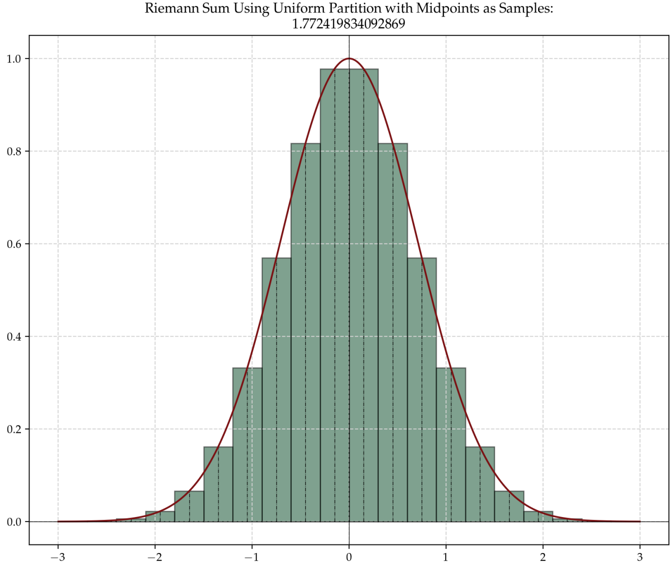
Example 3. Compute the Riemann sums for $f(x) = e^{-x^2}$ for $-3 \leq x \leq 3$ using uniform partitions and midpoints as representatives with $n = 10^N$ for $N = 1, 2, 3, 4, 5, 6$.

A Riemann Sum with Uniform Partition



Box, Midpoint, Trapezoidal, and Simpson Rules

Box and Midpoint Rules

For the rest of the lecture, we use a uniform partition $\{x_k\}_{k=0}^{n}$ of $[a,b]$ where

$$x_k = a + kh \qquad \text{for } k = 0, 1, \ldots, n \qquad \text{where } h = \frac{b-a}{n}.$$

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$$x_k = a + kh \qquad \text{for } k = 0, 1, \ldots, n \qquad \text{where } h = \frac{b-a}{n}.$$

Specific versions of Riemann Sums:

- **Box Rule (Left Endpoint)**

$$h \sum_{k=1}^{n} f(x_{k-1})$$

- **Box Rule (Right Endpoint)**

$$h \sum_{k=1}^{n} f(x_k)$$

- **Midpoint Rule**

$$h \sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)$$

Quadrature Rules

Let $f : [a, b] \to \mathbb{R}$ be continuous, $\{z_k\}_{k=0}^n$ be a partition of $[a, b]$, and $\{w_k\}_{k=0}^n$ be a finite sequence of real numbers. A formula of the form

$$\sum_{k=0}^{n} w_k f(z_k)$$

is called a **quadrature rule** for the integral $\int_a^b f(x)\, dx$.

The collections $\{z_k\}_{k=0}^n$ and $\{w_k\}_{k=0}^n$ are called **quadrature nodes** and **quadrature weights**, respectively.

Box Rule (Right Endpoint)

Example 4. Consider the function $f(x) = x^4 - 3x + 1$ for $-2 \leq x \leq 2$.

a. Use the box rule with right endpoints to compute the Riemann sums (S_N) for $n = 2^N$ for $N = 0, 1, \ldots, 15$.

b. Plot the norms of partitions versus the Riemann sums in log scales.

c. Denote the **numerical absolute errors** by $e_N := |S_N - S_{15}|$. Compute the following log error ratios:

$$\frac{\log(e_{N-1}/e_N)}{\log(2)} \qquad \text{for } N = 1, 2, \ldots, 14$$

Plot the norms of partitions versus the **log error ratios** in log scales. Discuss your observations.

d. Repeat items **a**, **b**, and **c** using left endpoints and midpoints.

Order of Convergence

Let $I(h)$ be an approximation of I where $h > 0$ is the norm of some partition and

$$e(h) := |I - I(h)|$$

be the absolute error. Suppose that for some constant $C > 0$ and $p > 0$ (**order of convergence**)

$$e(h) \approx Ch^p.$$

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$$e(h) \approx Ch^p.$$

Let $\eta > 0$ be the norm of another partition so that $e(\eta) \approx C\eta^p$. Then,

$$\frac{e(h)}{e(\eta)} \approx \left(\frac{h}{\eta}\right)^p \qquad \Longrightarrow \qquad p \approx \frac{\log(e(h)/e(\eta))}{\log(h/\eta)}.$$

If $\eta = h/2$ (**bisection**), then

$$p \approx \frac{\log(e(h)/e(h/2))}{\log(2)}.$$

Linear Convergence of Box Rules

Error with a single subinterval. By **Taylor's Theorem**, for each $x \in [a,b]$

$$f(x) = f(b) + f'(\xi)(x-b) \qquad \text{for some } \xi = \xi(x) \in (a,b)$$

$$E[a,b] := \int_a^b f(x)\,dx - f(b)(b-a) = \int_a^b f'(\xi)(x-b)\,dx.$$

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If $|f'(x)| \leq M_1$ for every $x \in [a,b]$, then

$$|E[a,b]| \leq \int_a^b M_1(x-b)\,fx = \frac{M_1}{2}(b-a)^2.$$

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Cumulative errors for all subintervals. Write

$$\begin{aligned} I - I(h) &:= \int_a^b f(x)\,dx - h\sum_{k=1}^n f(x_k) \\ &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x)\,dx - \sum_{k=1}^n f(x_k)(x_k - x_{k-1}) = \sum_{k=1}^n E[x_{k-1}, x_k] \end{aligned}$$

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By the triangle inequality and $hn = b - a$, we have

$$e(h) \leq Ch \qquad \text{where } C := \frac{M_1(b-a)}{2}.$$

Quadratic Convergence of Midpoint Rule

Error with a single subinterval. Let $m = \frac{a+b}{2}$. For each $x \in [a,b]$

$$f(x) = f(m) + f'(\xi)(x-m) + \frac{1}{2} f''(\xi)(x-m)^2 \qquad \text{for some } \xi = \xi(x) \in (a,b)$$

$$E[a,b] := \int_a^b f(x)\,dx - f(m)(b-a) = \frac{1}{2}\int_a^b f''(\xi)(x-m)^2\,dx.$$

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If $|f''(x)| \leq M_2$ for every $x \in [a,b]$, then

$$|E[a,b]| \leq \frac{1}{2}\int_a^b M_2(x-m)^2 fx = \frac{M_2}{24}(b-a)^3.$$

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Cumulative errors for all subintervals. If $m_k = \frac{x_{k-1}+x_k}{2}$, then

$$I - I(h) := \int_a^b f(x)\,dx - h\sum_{k=1}^n f(m_k) = \sum_{k=1}^n E[x_{k-1}, x_k]$$

As before, this implies

$$e(h) \le Ch^2 \qquad \text{where } C := \frac{M_2(b-a)}{24}.$$

Trapezoidal Rule

Trapezoidal Rule on a singe interval. Let $\ell : [a,b] \to \mathbb{R}$ be the linear function passing through the points $(a,f(a))$ and $(b,f(b))$, i.e.

$$\ell(x) = \frac{f(b) - f(a)}{b-a}(x-a) + f(a).$$

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Composite Trapezoidal Rule. If $\{x_k\}_{k=0}^n$ is a uniform partition of $[a,b]$ with norm $h = (b-a)/n$, then

$$\begin{aligned}
\int_a^b f(x)\,dx &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x)\,dx \\
&\approx \sum_{k=1}^n \frac{x_k - x_{k-1}}{2}(f(x_k) + f(x_{k-1})) \\
&= h\left(\frac{1}{2}f(a) + \sum_{k=1}^{n-1} f(a+kh) + \frac{1}{2}f(b)\right).
\end{aligned}$$

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\int_a^b f(x)\,dx &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x)\,dx \\
&\approx \sum_{k=1}^n \frac{x_k - x_{k-1}}{2}(f(x_k) + f(x_{k-1})) \\
&= h\left(\frac{1}{2}f(a) + \sum_{k=1}^{n-1} f(a + kh) + \frac{1}{2}f(b)\right).
\end{aligned}$$

Example 5. Repeat **Example 4** using the composite trapezoidal rule.

Quadratic Convergence of Trapezoidal Rule

Error with a single subinterval. Let $g(x) = f(x) - \ell(x)$. Then, $g(a) = g(b) = 0$. By **Rolle's Theorem**, there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$.

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$$\begin{aligned} E[a,b] &:= \int_a^b f(x)\,dx - \frac{b-a}{2}(f(a) + f(b)) \\ &= \int_a^b g(x)\,dx = \int_a^b \int_a^x \int_\xi^y g''(z)\,dz\,dy\,dx \end{aligned}$$

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Note that $g'' = f''$. If $|f''(x)| \leq M_2$ for every $x \in [a, b]$, then

$$
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Cumulative errors for all subintervals. As in the case of the midpoint rule

$$e(h) \leq Ch^2 \qquad \text{where } C := \frac{M_2(b-a)}{6}.$$

Simpson Rule

Simpson Rule on a singe interval. Let $q : [a,b] \to \mathbb{R}$ be the quadratic function passing through the points $(a, f(a))$, $(b, f(b))$, and $(m, f(m))$ where $m = \frac{a+b}{2}$. Then,

$$\int_a^b f(x)\,dx \approx \int_a^b q(x)\,dx = \frac{b-a}{6}(f(a) + 4f(m) + f(b)).$$

Composite Simpson Rule. If $\{x_k\}_{k=0}^n$ is a uniform partition of $[a,b]$ with norm $h = (b-a)/n$, then

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \sum_{k=1}^n \frac{x_k - x_{k-1}}{6}(f(x_{k-1}) + f(x_{k-1/2}) + f(x_k)) \\ &= \frac{h}{6}\sum_{k=1}^n \left(f(a+(k-1)h) + f\left(a + \frac{(2k-1)h}{2}\right) + f(a+kh)\right).\end{aligned}$$

Example 6. Repeat **Example 4** using the composite Simpson rule.

It can be shown that the absolute error satisfies

$$e(h) \leq Ch^4.$$

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- Multi-Dimensional Problems:
  - Iterated Integrals
  - Integrals on General Domains Using Polygonal Shapes (e.g. triangles, quadrilaterals, hexagons)

Thank you for your attention!