Conservation Laws

Conservative PDEs are usually derived from constitutive (physical) laws that *conserve* certain quantities **u** e.g. mass, momentum, density, heat, energy, population, particles, cars,...
PDEs in conservative form are so called because conservation laws can always be written in conservative form.

Definition 1.1 Scalar Conservation Law:
$$\frac{\partial}{\partial t}u + \mathrm{div}_{\mathbf{x}}\, f\left(u(\mathbf{x},t),\mathbf{x}\right) = s\left(u(\mathbf{x},t),\mathbf{x},t\right) \qquad \text{in}\ \tilde\Omega := \mathbf{\Omega}\times]0,T[$$
f: flux of conserved quantity u
s: production/source term (1.1)

Definition 1.2 1D Conservation Law:
$$u_t + \frac{\partial}{\partial x} f\left(u(x,t),x\right) = s\left(u(x,t),x,t\right) \qquad \text{in}\ \tilde\Omega := \Omega\times]0,T[\tag{1.2}$$

Definition 1.3 1D inviscide Conservation Law:
$$u_t + f\left(u(x,t),x\right)_x = 0 \qquad \text{in}\ \tilde\Omega := \Omega\times]0,T[\tag{1.3}$$
$$u(0,x) = u_0(x)$$

1. Examples

1.1. Transport Equation

Definition 1.4 Transport Equation $f = au$:
$$u_t + a(x,t)u_x = 0$$
$$u(x,0) = \phi(x) \tag{1.4}$$

1.2. Traffic Flow

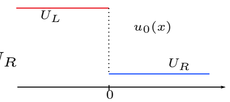
1.3. Burgers Equation

Definition 1.5 (Inviscid) Burgers Equation $f = \left(\frac{u^2}{2}\right)$:
$$u_t + uu_x = 0$$
$$u(x,0) = \phi(x) \tag{1.5}$$

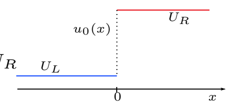
Corollary 1.1 Conservative Formulation [proof 5.1]:
$$u_t + \left(\frac{u^2}{2}\right)_{\mathbf{x}} = 0$$
$$u(x,0) = \Phi(x) \tag{1.6}$$

1.4. Riemann Problem

Definition 1.6 Riemann Problem Shock: Is an initial value problem of a conservation law with picewise initial data with a single discontinuity of the form:
$$u_t + f(u)_x = 0 \qquad (1.7)$$
$$u_0 = \begin{cases} U_R & \text{if } x>0 \\ U_L & \text{if } x<0\end{cases} \quad U_L > U_R \qquad (1.8)$$



Definition 1.7 Riemann Problem Rarefaction: Is an initial value problem of a conservation law with picewise initial data with a single discontinuity of the form:
$$u_t + f(u)_x = 0 \qquad (1.9)$$
$$u_0 = \begin{cases} U_R & \text{if } x>0 \\ U_L & \text{if } x<0\end{cases} \quad U_L < U_R \qquad (1.10)$$



Exploding Gradient Problem

Lemma 1.1 [proof 5.2]
Exploding Gradients:
The Burgers equation with smooth initial data $u_0(x)\in\mathcal{C}^1$ and at least one point x_i s.t. $u_0'(x_i) < 0$ will lead to a discontinuity/shockwave[def. 2.1] at a critical time t_{crit}:
$$\text{if } \exists x_i : u_0'(x_i) < 0 \tag{1.11}$$
$$\Longrightarrow \quad \exists\text{shockwave} \quad \text{at} \quad t_{\text{crit}} = -\frac{1}{\min_{x\in\mathbb{R}} u_0'(x)}$$

Explanation 1.1 (Exploding Gradient Problem).
$u_x \mapsto +\infty$ with time t
thus $u_t + f'(u)u_x = 0$ is meaningless→ Weak Solutions

2. Method Of Characteristics

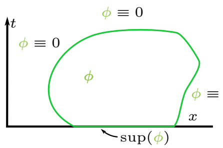
Weak Solutions

Problems

1. Riemann problems[def. 1.6] may lead to discontinuous solutions u − example 5.1.
2. Even smooth initial data may lead to discontinuous solutions u − example 5.2
3. Weak solutions may lead to infinitely many solutions u

Definition 2.1 Shock Waves Γ:
Is a curve $x = \gamma(t)\in\mathcal{C}^1(Rp)$ along which the solution of a conservation law $U\in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ is discontinuous is called a *shock-wave*:
$$U(x,t) = \begin{cases} U^-(x,t) & \text{if } x<\gamma(t) \quad U^-\in\mathcal{C}^1\left(\Gamma^-\right) \\ U^+(x,t) & \text{if } x>\gamma(t) \quad U^+\in\mathcal{C}^1\left(\Gamma^+\right)\end{cases}$$
$$\Gamma := \{(x,t)\in\mathbb{R}\times\mathbb{R}_+ \,|x = \gamma(t)\}$$
$$\Gamma^+ := \{(x,t)\in\mathbb{R}\times\mathbb{R}_+ \,|x > \gamma(t)\} \tag{2.1}$$
$$\Gamma^- := \{(x,t)\in\mathbb{R}\times\mathbb{R}_+ \,|x < \gamma(t)\}$$

Definition 2.2 Test function $\phi\in\mathcal{C}^1_C(\mathbb{R}\times[0,T])$:
Are smooth, compactly supported functions, that are easier to work with.
Idea: use some test functions ϕ that has nicer properties and shift the derivative from u to ϕ by using integration by parts.



Definition 2.3 Weak Solutions [proof 5.3]:
For $u_0 \in L^\infty(\mathbb{R})$, $u : \mathbb{R}\times]0,T[\mapsto\mathbb{R}$ is a weak solution of eq. (1.3) if:
$$\int_{-\infty}^{\infty}\int_0^T \left(u\phi_t + f(u)\phi_x\right) dx\,dt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)\,dx = 0$$
$$\wedge\, u :\in L^\infty(\mathbb{R}\times]0,T[) \qquad \forall\phi\in C_0^\infty(\mathbb{R}\times[0,T[),\ \phi(\cdot,T) = 0 \tag{2.2}$$

Note
Recall L^∞ bounded but not necessarily differentiable functions i.e. step fucntions.

Explanation 2.1. *Derivatives of u are gone ⇒ we do no longer have the exploding gradient problemlemma 1.1.*

1. The Rankine-Hugoniot Condition

Definition 2.4 [example 5.4],[proof 5.4]
Rankine-Hugoniot Condition: Is a condition on the *shock-speed* $s(t) = \gamma'(t)$ of a shock[def. 2.1] i.e. how fast the shock-wave travels:
$$s(t)\left(u^+(t) - u^-(t)\right) = f\left(u^+(t)\right) - f\left(u^-(t)\right) \tag{2.3}$$

Corollary 2.1 Shock Speed: Is the speed of a shock wave[def. 2.1]
$$s(t) = \gamma'(t) = \frac{f\left(u^+(t)\right) - f\left(u^-(t)\right)}{u^+(t) - u^-(t)} \tag{2.4}$$

Theorem 2.1
Necessary Conditions for Weak Solutions and Shocks:
Given a shock wave Γ[def. 2.1] u is a weak solution of eq. (1.2) if and only if:
1. u^- and u^+ are classical solutions of eq. (1.2).
2. the shock speed $s(t) = \gamma'(t)$ satisfies the RH-conditioneq. (2.3) at any discontinuities $x = \gamma(t)$.

1.1. Shock Waves

Definition 2.5 Shock Waves: For conservation laws with convex flux function f and Riemann data [def. 1.6]:
$$u_t + f(u)_x = 0 \tag{2.5}$$
$$u_0 = \begin{cases} U_R & \text{if } x>0 \\ U_L & \text{if } x<0\end{cases} \tag{2.6}$$

check again this solution

Corollary 2.2 Shock Wave Solution:
$$u(x,t) = \begin{cases} 0 & \text{if } x < s(t)t \\ 1 & \text{if } x > s(t)t\end{cases} \tag{2.7}$$

1.2. Rarefaction Waves

1.2.1. Lax-Oleinik Entropy Condition

Proposition 2.1 (Burgers Equation)
Lax-Oleinik Entropy Condition: Characteristics of the Burgers equation have to flow into the shock and not emanate at it:
$$u^-(t) > s(t) > u^+(t) \tag{2.8}$$

Proposition 2.2 (Convex Functions)
Lax-Oleinik Entropy Condition: Characteristics of general *scalar conservation law* with *convex* f should flow into the shock:
$$f'\left(u^-(t)\right) > s(t) > f'\left(u^+(t)\right) \tag{2.9}$$

Explanation 2.2.
- *For an evolution equation the flow of information should come from the initial data.*
- *We want to require that information flows into and not out from a shock.*

Definition 2.6 [proof 5.5]
Rarefaction Wave:
A rarefaction wave is a *self-similar* solutions of the form:
$$u(x,t) = v\left(\frac{x}{t}\right) = \left(f'\right)^{-1}\left(\frac{x}{t}\right) \tag{2.10}$$

Corollary 2.3 [example 5.7]
Rarefaction Solution for the Riemann Problem:
Consider the Riemann problem eq. (5.95), then a solution given by rarefaction wave is given by:
$$u(x,t) = \begin{cases} u_L & x \leqslant f'(u_L)\,t \\ \left(f'\right)^{-1}\left(\frac{x}{t}\right) & \text{if} \quad f'(u_L)\,t < x \leqslant f'(u_R)\,t \\ u_R & x > f'(u_R)\,t\end{cases} \tag{2.11}$$
and satisfies the Lax-entropy condition:

2. Entropy Solutions

The Lax-Olenek entropy condition is based on the heuristic that information emanates from initial date, now we want to derive an entropy condition from a mathematical standpoint.

Proposition 2.3 Viscous Approximation: Is a parabolic *convection-diffusion equation* of the form:
$$u_t^\epsilon + f\left(u_t^\epsilon\right)_x = \epsilon u_{xx}^\epsilon \qquad \epsilon > 0 \tag{2.12}$$

Idea
In the limit $\epsilon\to 0$ we recover the inviscide scalar conservation laweq. (1.3). Thus we can study eq. (2.12) in order to study eq. (1.3).

Definition 2.7 Vanishing Viscosity Solution: Is a weak solution u that is the limit of solutions $u = \lim_{\epsilon\to 0} u^\epsilon$ of the viscous equationeq. (2.12).

Definition 2.8 Entropy Pair (s,q):
The pair (s,q) is called entropy pair, where s is any *strictly convex function*??. Then the entropy pair is defined by the relation:
$$q(u) = \int_0^u f'(\eta)s'(\eta)\,d\eta \quad\Longrightarrow\quad q' = s'f' \tag{2.13}$$
s : entropy function $\qquad q$: entropy flux

Definition 2.9 Entropy Condition [proof 5.6]:
Any vanishing viscosity solution[def. 2.7] u satisfies:
$$s(u)_t + q(u)_x \leqslant 0 \tag{2.14}$$

Corollary 2.4 [proof 5.8]
Kruzkov's Entropy Condition: Is an entropy condition that holds for weak-solutions:
$$\int_{\mathbb{R}}\int_{\mathbb{R}_+} s\left(u(x,t)\right)\phi_t(x,t) + q\left(u(x,t)\right)\phi_x\,dx\,dt + \int_{\mathbb{R}} s(u_0(x))\phi(x,0)\,dx \geqslant 0 \tag{2.15}$$
$$\forall\phi\in\mathcal{C}^1_C\left(\mathbb{R}\times\mathbb{R}_+\right),\ \phi\geqslant 0$$

Definition 2.10 Entropy Solution:
A function $u\in L^\infty(\mathbb{R},\mathbb{R}_+)$ is an entropy solution of the inviscide scalar conservation law eq. (1.3) iff:
1. u is a *weak solution*[def. 2.3] of eq. (1.3).
2. u satisfies the entropy conditioneq. (2.14)/eq. (2.15) for all entropy pairs[def. 2.8] (s,q)

Law 2.1 2nd Laws Of Thermodynamics [proof 5.7]:
The total (mathematical) entropy s decreases in time:
$$\frac{d}{dt}\int_{\mathbb{R}} S\left(u^t(x,t)\right) \qquad \forall\ \text{strict. Convex} \tag{2.16}$$
$$\Longleftrightarrow \quad \int_{\mathbb{R}} S\left(u^\epsilon(x,t)\right)dx \leqslant \int_{\mathbb{R}} S\left(u_0(x)\right)dx \qquad \forall t \tag{2.17}$$

Note: mathematical entropy
The mathematical entropy is defined as the negative physical definition of the entropy $s^{\text{math}} = -s^{\text{phys}} \Rightarrow$ decreases.

2.1. Properties of Entropy Solutions

Property 2.1: Entropy solutions[def. 2.10] for *strictly convex*?? flux function f satisfies the Lax-Oleinik entropy conditioneq. (2.9).

Property 2.2: Entropy solutions are unique.

2.1.1. L^p-bound on entropy solutions

L2-Norm

Property 2.3 L2-Norm:
$$S(u) = u^2 \quad\Longrightarrow\quad \int_{\mathbb{R}}^2 u(x,t)\,dx \leqslant \int_{\mathbb{R}} u_0^2(x)\,dx \qquad \forall t \tag{2.18}$$

L1-Norm

Property 2.4 L1-Norm:
$$S(u) = |u| \quad\Longrightarrow\quad \int_{\mathbb{R}} |u(x,t)|\,dx \leqslant \int_{\mathbb{R}} |u_0(x)|\,dx \qquad \forall t \tag{2.19}$$

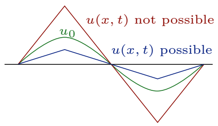
Lp-Norm

Property 2.5 L1-Norm:
$$\|u(\cdot,t)\|_{L^p} \leqslant \|u_0\|_{L^p} \qquad \forall 1\leqslant p\leqslant\infty \tag{2.20}$$

2.1.2. Maximum Principle

Principle 2.1 [proof 5.9]
Maximum Principle: Equation (1.3) attains its maximums on the boundary or its a constant:

$$\max\left(u(x,t)\right) \leqslant \max\left(0, \max u_0(x)\right) \tag{2.21}$$
$$\min\left(u(x,t)\right) \geqslant \min\left(0, \min u_0(x)\right) \tag{2.22}$$



2.1.3. Total Variation Diminishing

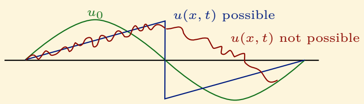
Definition 2.11 Total Variation: If g is differentiable $g \in \mathcal{C}^1([a,b])$ the total variation is defined as:

$$\|g\|_{\mathrm{TV}([a,b])} = \int_a^b \left|\frac{\mathrm{d}g}{\mathrm{d}x}\right| \mathrm{d}x \tag{2.23}$$

Explanation 2.3. *Its a measure on how much a function varies/fluctuates within a interval* $[a,b]$.

Theorem 2.2 [proof 5.10]
Total Variation Diminishing (TVD):
The total variation of an entropy solutions diminished with time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left|u_x^{\epsilon}(\cdot,t)\right| \mathrm{d}x \leqslant 0 \tag{2.24}$$



Corollary 2.5 :

$$\int_{\mathbb{R}} \left|u_x^{\epsilon}(\cdot,t)\right| \mathrm{d}x \leqslant \int_{\mathbb{R}} \left|u_x^0\right| \mathrm{d}x \tag{2.25}$$

Corollary 2.6 [proof 5.11]
Total Variation Diminishing in Time: The total time variation is bounded by the space variation:

$$\int_{\mathbb{R}} \left|u_t^{\epsilon}(\cdot,t)\right| \mathrm{d}x \leqslant C \int_{\mathbb{R}} \left|u_x^{\epsilon}(\cdot,t)\right| \mathrm{d}x \tag{2.26}$$

Finite Volume Methods

From the previous sections we have seen that the solution of conservation laws[def. 1.1] are non-continuous s.t. point values may not be well defined. A solution to this remedy is to work with averages, which are well defined for any integrable function and thus also solutions of conservation laws.

Definition 3.1 Finite Volume Scheme Grid:
Space Discretization $[x_L, x_R]$

$$x_j := x_L + \left(j+\frac{1}{2}\right)\Delta x \qquad \Delta x := \frac{x_R-x_L}{N+1} \tag{3.1}$$

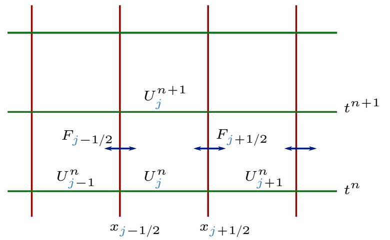
$$= \frac{x_{j-1/2}+x_{j+1/2}}{2} \tag{3.2}$$

$$x_{j\pm\frac{1}{2}} := x_j \pm \Delta x/2 = \begin{cases} x_L + j\Delta x & - \\ x_L+(j+1)\Delta x & + \end{cases} \tag{3.3}$$

$j\in\{1,\dots,N+1\}$

Time Discretization $[0,T]$

$$t^n := n\Delta t \tag{3.4}$$

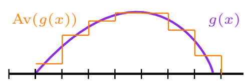


Definition 3.2 Control Volumes/Cells: Are cells defined over the meshpoints x_j of the grid:

$$\mathcal{C}_j := \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right) \tag{3.5}$$

Definition 3.3 Cell Averages: Are averages calculated over the cells[def. 3.2] of a grid[def. 3.1]

$$U_j^n :\approx \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t^n)\,\mathrm{d}x \tag{3.6}$$

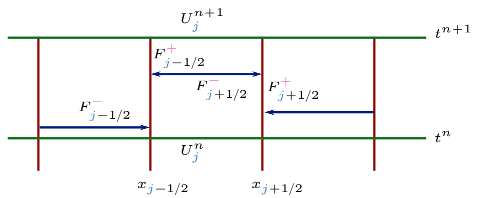


Corollary 3.1 Initial Cell Averages:

$$U_j^0 :\approx \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x)\,\mathrm{d}x \tag{3.7}$$

Definition 3.4 Integrated (Boundary) Fluxes: Is the flux of over left and right boundary of the cells:

$$\bar{F}^{n,\pm}_{j\pm\frac{1}{2}} := \int_{t_n}^{t_{n+1}} f\left(u\left(x^{\pm}_{j\pm\frac{1}{2}}\right), t\right) \mathrm{d}t \tag{3.8}$$



Proposition 3.1 [proof 5.12]
Discontinuous Finite Volume Method (FVM):
discretize conservation laws and calculate cell averages[def. 3.3] iteratively by integrating conservation laws[def. 1.1] over the domain $\left[x_{j-1/2}, x_{j+1/2}\right) \times \left[t^n, t^{n+1}\right]$:

$$U_j^{n+1} = U_j^n - \frac{1}{\Delta x}\left(\bar{F}^{n,-}_{j+\frac{1}{2}} - \bar{F}^{n,+}_{j-\frac{1}{2}}\right) \tag{3.9}$$

Explanation 3.1. *The values of the flux at the boundary points $x_{j\pm1/2}$ may not be continuous, thus we take the values of the fluxes inside the cell over which we are integrating.*

Definition 3.5 [proof 5.17]
Finite Volume Scheme: discretize conservation laws and calculate cell averages[def. 3.3] iteratively by integrating conservation laws[def. 1.1] over the domain $\left[x_{j-1/2}, x_{j+1/2}\right) \times \left[t^n, t^{n+1}\right]$:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F^n_{j+1/2} - F^n_{j-1/2}\right)$$

$$U_j^0 = \frac{1}{\Delta x}\int_{x_{j+1/2}}^{x_{j+1/2}} U_0(x)\,\mathrm{d}x \tag{3.10}$$

Corollary 3.2 [proof 5.13]
Finite Volume/MCC Scheme Incremental Form:

$$U_j^{n+1} = U_j^n + C^n_{j+1/2}\left(U^n_{j+1}-U^n_j\right) - D^n_{j-1/2}\left(U^n_j - U^n_{j-1}\right)$$

For the FVM the coefficients are: (3.11)

$$C^n_{j+1/2} := \frac{\Delta t}{\Delta x}\left(\frac{F(u_j,u_j)-F(u_j,u_{j+1})}{u_{j+1}-u_j}\right) \tag{3.12}$$

$$D^n_{j-1/2} := \frac{\Delta t}{\Delta x}\left(\frac{F(u_j,u_j)-F(u_j,u_{j-1})}{u_j-u_{j-1}}\right) \tag{3.13}$$

if F is lipschitz?? in both arguments this is equal to:

$$C^n_{j+1/2} = -\frac{\Delta t}{\Delta x}\frac{\partial F}{\partial b}\left(u_j^n,\cdot\right) \tag{3.14}$$

$$D^n_{j+1/2} = -\frac{\Delta t}{\Delta x}\frac{\partial F}{\partial a}\left(\cdot,u_j^n\right) \tag{3.15}$$

1. Properties of Schemes

Definition 3.6 General Evolution Equation:

$$U_j^{n+1} = H\left(U^n_{j-p},\dots,U^n_{j+p}\right) \tag{3.16}$$

1.1. Conservative Schemes

Definition 3.7 Conservative Schemes:

$$\sum_j U_j^{n+1} = \sum_j U_j^n \tag{3.17}$$

Corollary 3.3 FVS are conservative: FVM schemes[def. 3.5] are conservative.

add proof – check script

Note

Finite difference schemes are usually not conservative⇒blow up.

1.2. Consistent Schemes

Definition 3.8 Consistent Schemes: A $2p+1$ point scheme

$$F^n_{j+1/2} = F\left(U^n_{j-p+1},\dots,U^n_{j+p}\right) \tag{3.18}$$

is consistent if the Flux function f is consitent with the numerical flux F i.e.:

$$F(U,\dots,U) = f(u) \tag{3.19}$$

Explanation 3.2. *This basically states that if the left and right states are consistent/have the same value then our approximation should do nothing and be equal to the real flux.*

Corollary 3.4 Consitency for FVM:
A FVM[def. 3.5] method is consistent iff for its numerical flux functions it holds that:

$$F(a,a) = f(a) \tag{3.20}$$

Note

Most of the schemes that we see in the next chapter are consistent and conservative.

See also lecture 09 min 30

1.3. Monotonicity Preserving Schemes

Definition 3.9 Monotone Scheme: A numerical scheme[def. 3.6] is monotone if the update function H is non-decreasing in each of its arguments:

$a \mapsto H(a,\dots)$ ↑ when inceas. a and fixing all others
$b \mapsto H(\dots,b\dots)$ ↑ when inceas. b and fixing all others
$c \mapsto H(\dots,c,\dots)$ ↑ when inceas. c and fixing all others (3.21)

if H is Lipschitz continuous this equals to:

$$\frac{\partial H}{\partial a},\frac{\partial H}{\partial b},\frac{\partial H}{\partial c},\dots \geqslant 0 \tag{3.22}$$

Definition 3.10 [example 5.3][proof 5.15]
CFL Condition for FVS :
A FVS[def. 3.5] with *monotone* locally Lipschitz continuous two-point flux $F(a,b)$ has the following CFL (eq. (3.42)) type condition:

$$\left|\frac{\partial F}{\partial a}(v,w)\right| + \left|\frac{\partial F}{\partial b}(u,v)\right| \qquad \forall u,v,w \tag{3.23}$$

$$\leqslant 2\max_{a,b}\left(\left|\frac{\partial F}{\partial a}(v,w)\right|,\left|\frac{\partial F}{\partial b}(u,v)\right|\right)$$

$$\leqslant \frac{\Delta x}{\Delta t}$$

Definition 3.11
Conversation Laws are Monotonicity Preserving:
If U and V are *entropy solutiuons*?? of eq. (1.3) with initial data U_0 and V_0 then it holds:

$$U_0(x) \leqslant V_0(x) \quad \forall x \implies U(x,t) \leqslant V(x,t) \quad \forall x,t \tag{3.24}$$

Corollary 3.5 [proof 5.14]
Monotone Schemes and Monotonicity: Monotone Schemes[def. 3.9] are *monotonicity preserving*[def. 3.11].

Corollary 3.6 [proof 5.15]
Monotone FVM: The FVS[def. 3.5] is montone iff:

1
$a \mapsto F(a,b)$ is **non-decreasing** for fixed a (3.25)
$b \mapsto F(a,b)$ is **non-increasing** for fixed b (3.26)

if F is lipschitz cont.:

$$\frac{\partial F(a,\cdot)}{\partial a} \geqslant 0 \tag{3.27}$$

$$\frac{\partial F(\cdot,b)}{\partial b} \leqslant 0 \tag{3.28}$$

2 it fulfills the CFL-type condition[def. 3.10]

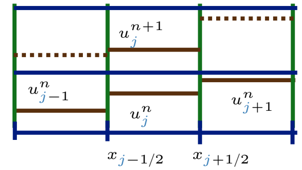
Corollary 3.7
Monotone Consistent Conservative (MCC) Schemes:
MCC schemes satisfy:
1 Entropy Condition
2 Discrete Maximum Principle
3 TVD Property
$\implies$ MCC schemes will converge to the entropy solution as $\Delta x, \Delta t \to 0$

	Monotone	Consistent	Conservative
Godunov	✓	✓	✓
Roe		✓	✓
LxF	✓	✓	✓
EO	✓	✓	✓
Rusanov	✓	✓	✓
Central		✓	✓

1.4. Discrete Maximum Principle

Principle 3.1 Discrete Maximum Principle:

$$\min\left(u^n_{j-1},u^n_j,u^n_{j+1}\right) \leqslant u_j^n \leqslant \max\left(u^n_{j-1},u^n_j,u^n_{j+1}\right) \tag{3.29}$$



1.5. Discrete Total Variation Diminishing

Definition 3.12 Discrete Total Variation: Let g be a function defined on $[a,b]$ then the total variation of g is given by:

$$\|g\|_{\mathrm{TV}([a,b])} = \sup_{\mathcal{P}} \sum_{j=1}^{N-1}\left|g(x_{j+1}) - g(x_j)\right| \tag{3.30}$$

where the supremum is taken over all paritions $\mathcal{P} := \{a = x_1 < x_2 < \dots < x_N = b\}$

Definition 3.13
Discrete Total Variation Diminishing (TVD):

$$\left\|U^{n+1}\right\|_{TV(\mathbb{R})} := \sum_j\left|U^{n+1}_{j+1} - U^{n+1}_j\right| \leqslant \sum_j\left|U^n_{j+1} - U^n_j\right| \tag{3.31}$$

Definition 3.14 Bounded Variation:

$$\|g\|_{\mathrm{BV}([a,b])} = \|g\|_{L^1([a,b])} + \|g\|_{\mathrm{TV}([a,b])} \tag{3.32}$$

Explanation 3.3. *The total variation[def. 2.11] is only a semi-norm as the TV of any constant function is zero.* ⇒ *definition of bounded variation makes this a real norm.*

Definition 3.15
Bounded Variation Function Space BV:

$$\mathrm{BV}(\mathbb{R}) := \left\{g \in L^1(\mathbb{R}) : \|g\|_{\mathrm{BV}(\mathbb{R})} < \infty\right\} \tag{3.33}$$

1.5.1. Harten's Lemma

Lemma 3.1 [proof 5.16]
Harten's Lemma:
A scheme in incremental form eq. (3.71)

$$U_j^{n+1} = U_j^n + C^n_{j+1/2}\left(U^n_{j+1} - U^n_j\right) - D^n_{j-1/2}\left(U^n_j - U^n_{j-1}\right) \tag{3.34}$$

1. with coefficients satisfying: $C^n_{j+1/2}, D^n_{j+1/2} \geqslant 0$ and $C^n_{j+1/2} + D^n_{j+1/2} \leqslant 1$ $\forall n, j$ (3.35)
is TVD eq. (3.31)
2. with coefficients satisfying: $C^n_{j+1/2}, D^n_{j+1/2} \geqslant 0$ and $C^n_{j+1/2} + D^n_{j-1/2} \leqslant 1$ $\forall n, j$ (3.36)

$$\left\|U^{n+1}\right\|_{L^\infty} \leqslant \left\|U^n\right\|_{L^\infty} \qquad \forall n \tag{3.37}$$

Finite Volume Methods Scheme

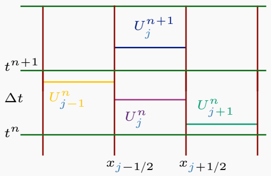
2. Exact Riemann Solvers

2.1. Godunov Method

Problem

The finite volume method?? requires us to calculate the integrated fluxes eq. (3.8) but those depend again on the unknown solution U.

However Gudonuv noticed that the cell **averages** are constant in each cell $\mathcal{C}_j$ for each time level s.t. each *cell interface* $x_{j+1/2}$ defines a Riemann problem.



Definition 3.16 FVM Riemann Problem:

$$U_t + f(U)_x = 0 \tag{3.38}$$

$$U(x,t^n) = \begin{cases} U_j^n & \text{if } x < x_{j+1/2}\\ U_{j+1}^n & \text{if } x > x_{j+1/2}\end{cases} \tag{3.39}$$

Corollary 3.8 Scaled Gudunov Riemann Problem: For $U_j(x,t) = U_j\left(\frac{x-x_{j+1/2}}{t-t^n}\right)$ the Riemann problem[def. 3.16] becomes the standard Riemann problem:

$$u(x,0)=\begin{cases}U_j^n & \text{if } x<0\\ U_{j+1}^n & \text{if } x>0\end{cases}\tag{3.40}$$

Definition 3.17 Godunov Flux:

$$F^n_{j+1/2}\left(U_j^n,U_{j+1}^n\right) = \begin{cases}\min\limits_{U_j^n\leqslant\theta\leqslant U_{j+1}^n} f(\theta) & \text{if } U_j^n \leqslant U_{j+1}^n\\ \max\limits_{U_{j+1}^n\leqslant\theta\leqslant U_j^n} f(\theta) & \text{if } U_j^n > U_{j+1}^n\end{cases}\tag{3.41}$$

Corollary 3.9 Godunov Flux for convex functions: For convex functions f with $\alpha := \min f(\theta)$ it holds:

$$F^n_{j+1/2}\left(U_j^n,U_{j+1}^n\right) = \max\left(f\left(\max\left(U_j^n,\alpha\right)\right), f\left(\min\left(U_{j+1}^n,\alpha\right)\right)\right)$$

add proof

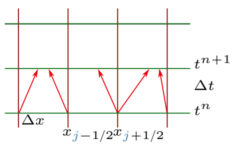
Cons

- Solving Equation (3.41) many times for each timestep can become extremely expensive.

2.1.1. CFL Condition

Definition 3.18 CFL Condition:

$$\max_j |f'\left(U_j^n\right)| \frac{\Delta t}{\Delta x} \leqslant \frac12 \tag{3.42}$$



Explanation 3.4. *Enforces that that neighbouring waves in a cell do not inersect each other:*

$$\text{CFL} := \max_j |f'\left(U_j^n\right)|\Delta t \leqslant \underbrace{\frac12 \Delta x}_{\textit{half the cell width}}\tag{3.43}$$

Corollary 3.10 The CFL condition can be used to calculate Δt:

$$\Delta t = \text{CFL}\frac{\Delta x}{\max_j|f'\left(U_j^n\right)|}\tag{3.44}$$

3. Approximate Riemann Solvers

Solving the exact Riemann problemeq. (3.39) can become very expensive. Thus we want to find an approximate solution by *linearizing* non-linear flux functions f:

$$f(u) = f\left(u_j^n\right) + f'\left(\theta^n_{j+\frac12}\right)\left(u-u_j^n\right) \qquad \theta^n_{j+\frac12}\in\left[u_j^n,u_{j+1}^n\right]$$

$$\Longrightarrow \quad f'(u)_x = f'\left(\theta^n_{j+\frac12}\right) u_x =\approx \hat{A}_{j+\frac12}u_x \tag{3.45}$$

Where $\hat A_{j+\frac12}\left(\theta^n_{j+\frac12}\right) = f'\left(\theta^n_{j+\frac12}\right)$ is a constant state around which the nonlinear flux function is linearized.

The question that remains is at which point $\left(\theta^n_{j+\frac12}\right) \in \left[u_j^n, u_{j+1}^n\right]$ should we evaluate $\hat A_{j+\frac12}$.

Definition 3.19 [Linear Transport Equation] Approximate Riemann Problem:

$$u_t + \hat A_{j+\frac12}u_x = 0 \tag{3.46}$$

$$u(x,t^n) = \begin{cases} u_j^n & \text{if } x < x_{j+1/2}\\ u_{j+1}^n & \text{if } x > x_{j+1/2}\end{cases} \tag{3.47}$$

Definition 3.20 Arithmetic Average:

$$\hat A_{j+\frac12} = f'\left(\frac{u_j^n+u_{j+1}^n}{2}\right)\tag{3.48}$$

3.1. Murman Roe Scheme

Definition 3.21 Roe Average: Directly approximate $f'(u)$ using finite differences:

$$\hat A_{j+\frac12} = \begin{cases}\frac{f\left(u_{j+1}^n\right)-f\left(u_j^n\right)}{u_{j+1}^n-u_j^n} & \text{if } u_{j+1}^n\neq u_j^n\\ f'\left(u_j^n\right) & \text{if } u_{j+1}^n = u_j^n\end{cases}\tag{3.49}$$

Explanation 3.5. *If $u_{j+1}^n = u_j^n$ we don't want to divide by zero.*

Corollary 3.11 Roe Flux: Solving eq. (3.47) with ?? leads to the Roe flux:

$$F^n_{j+1/2} = F^{\text{Roe}}\left(u_j^n,u_{j+1}^n\right) = \begin{cases} f\left(u_j^n\right) & \text{if } \hat A_{j+\frac12}\geqslant 0\\ f\left(u_{j+1}^n\right) & \text{if }\hat A_{j+\frac12}<0\end{cases}\tag{3.50}$$

Pros

- is simpler in comparison to Godunov scheme
- approximates the shock/non-entropy solutions

Cons

- fails at Rarefactions as it does not take into account non-linear bi-directional propagation of information

3.2. Central Schemes

Harten-Lax-van-Lear **1974**

The *Roe-Scheme* fails at resolving rarefaction, this is due to the *linearization* of the *Riemann problem* which leads to a *single wave* solution that travels either to the left or right, depending on the sign of the *Roe average* $\hat A_{j+\frac12}$.

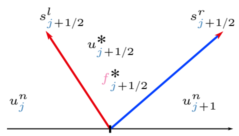
Problem: the exact solution for a rarefaction can lead to waves traveling in both directions.

Idea: approximate the solution by two waves traveling in opposite directions with speeds s_j^r and s_j^l.

Definition 3.22 [proof 5.18]
Central Schemes:

$$F^n_{j+1/2} = F\left(u_j^n,u_{j+1}^n\right) = f^*_{j+1/2}$$

+FVM eq. (3.10)



$$f^*_{j+1/2} = \frac{s^r_{j+1/2}f\left(u_j^n\right) - s^l_{j+1/2}f\left(u^n_{j+1/2}\right) + s^r_{j+1/2}s^l_{j+1/2}\left(u^n_{j+1}-u^n_j\right)}{s^r_{j+1/2}-s^l_{j+1/2}}\tag{3.51}$$

The left $s^l_{j+1/2}$ and right $s^r_{j+1/2}$ speeds have to be specified and depend on the scheme.

Corollary 3.12 $-s^l_{j+1/2} = s^r_{j+1/2} =: s_{j+1/2}$
Symmetric Waves: For anti-symmetric speeds we obtain:

$$f^*_{j+1/2} = \frac{f\left(u_j^n\right)-f\left(u^n_{j+1/2}\right)}{2} - \frac{s_{j+1/2}}{2}\left(u^n_{j+1}-u^n_j\right)\tag{3.52}$$

3.2.1. Lax-Friedrichs Scheme

Definition 3.23 Lax Friedrichs Scheme: Chooses the wave speeds s.t. waves from neighboring Riemann problems do not interact with each other:

$$s^l_{j+1/2} = -\frac{\Delta x}{2\Delta t} \qquad s^r_{j+1/2} = \frac{2\Delta x}{\Delta t}\tag{3.53}$$

with eq. (3.52) it follows:

$$F^n_{j+1/2} = F^{\text{LxF}}\left(u_j^n,u_{j+1}^n\right)\tag{3.54}$$

$$= \frac{f\left(u_j^n\right)-f\left(u^n_{j+1/2}\right)}{2} - \frac{\Delta x}{2\Delta t}\left(u^n_{j+1}-u^n_j\right)$$

Explanation 3.6. *LxF makes sure that waves do not interfere with each other, that is each wave can maximally travel a distance of* $\Delta x = \left|\frac{\Delta t}{s^l_{j+1/2}}\right|$ *i.e. to the next interface until we the next time point.*

Pros

- Easy to implement

Cons

- Does not take into account the local speeds
- Is not the most accurate
- Uses always an additional unnecessary grid point

3.2.2. Rusanov Scheme

Definition 3.24 Rusanov/Local-Lax-Friedrichs Scheme: Takes also into account the local speeds of the waves:

$$s_{j+1/2} = \max\left(|f'\left(u_j^n\right)|, |f'\left(u_{j+1}^n\right)|\right)\tag{3.55}$$

with eq. (3.52) and $s^r_{j+1/2} = s_{j+1/2} = -s^l_{j+1/2}$ it follows:

$$F^n_{j+1/2} = F^{\text{Rus}}\left(u_j^n,u_{j+1}^n\right)\tag{3.56}$$

$$= \frac{f\left(u_j^n\right)-f\left(u^n_{j+1/2}\right)}{2} \tag{3.57}$$

$$-\frac{\max\left(|f'\left(u_j^n\right)|,|f'\left(u_{j+1}^n\right)|\right)}{2}\left(u^n_{j+1}-u^n_j\right)$$

3.2.3. Enquist-Osher Flux

Definition 3.25
Engquist Osher Scheme:
Is related to[def. 3.24] but is kind of a continuous version:

$$F^n_{j+1/2} = F^{\text{EO}}\left(u_j^n,u_{j+1}^n\right)\tag{3.58}$$

$$= \frac{f\left(u_j^n\right)-f\left(u^n_{j+1/2}\right)}{2} - \frac12\int_{u_j^n}^{u_{j+1}^n}\left|f'(\theta)\right|\mathrm{d}\theta$$

Corollary 3.13 Engquist Oshner for Convex Functions:

For convex functions f with a single minimum $\alpha := \min f(\theta)$ it holds:

$$F^{\text{EO}}\left(u_j^n,u_{j+1}^n\right) = f^+\left(u_j^n\right)+f^-\left(u_{j+1}^n\right)\tag{3.59}$$

$$f^+(u) := f\left(\max\left(u,\alpha\right)\right)$$

$$f^-(u) := f\left(\min\left(u,\alpha\right)\right)$$

add convex cases from script

4. Higher Order Schemes

Goal

Design higher-order (2^{nd})-order schemes which are stable:
- TVD
- Max Principle

and reduce the error/are more accurate.

Definition 3.26 Truncation Error τ:
The truncation error w.r.t. $u_j^{n+1} = H\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right)$ is defined as:

$$\tau := u\left(x_j, t^{n+1}\right) - H\left(u\left(x_{j-1}, t^n\right), u\left(x_j, t^n\right), u\left(x_{j+1}, t^n\right)\right) \tag{3.60}$$

Definition 3.27 Order of Scheme:
The order of a scheme is defined:

$$q: \quad \max_{j,n}\left|\tau_j^n\right| \leqslant C\Delta t^{q+1} \tag{3.61}$$

4.1. Lax-Wendroff Scheme 1961

Definition 3.28 [proof 5.19]
Lax-Wendroff Scheme:

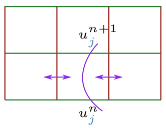
$$\begin{aligned} u_j^{n+1} = u_j^n &- \frac{\Delta t}{2\Delta x}\left(f\left(u_{j+1}^n\right) - f\left(u_{j-1}^n\right)\right) \\ &+ \frac{\Delta t^2}{2\Delta x^2}\Big[a_{j+1/2}^n\left(f\left(u_{j+1}^n\right) - f\left(u_j^n\right)\right) \\ &- a_{j-1/2}^n\left(f\left(u_j^n\right) - f\left(u_{j-1}^n\right)\right)\Big] \end{aligned} \tag{3.62}$$

$$f'(u)\left(x_{j+1/2}\right) =: a_{j+1/2}^n = f'\left(\frac{u_j^n + u_{j+1}^n}{2}\right)$$

Corollary 3.14 As a Finite Volume Scheme:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right)$$

$$F_{j+1/2}^n = F_{j+1/2}^n\left(u_j^n, u_{j+1}^n\right)$$



$$= \frac{f\left(u_j^n\right) + f\left(u_{j+1}^n\right)}{2} - \frac{\Delta t}{\Delta x}a_{j+1/2}^n\left(f\left(u_{j+1}^n\right) - f\left(u_j^n\right)\right)$$

Pros
- Formally 2^{nd}-order accurate
- Is Consistent
- Conservative

Cons
- Comes with oscillations
- Not monotone
- Not TVD
- No discrete maximumprinciple

Why Oscillations only on one (lhs) side of the shock Lecture 11 min 41

5. REA-Algorithms

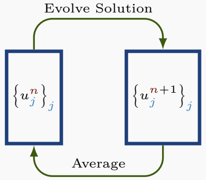
5.1. Reconstruction

Definition 3.29 Averaging Operator:

$$\operatorname{Avg}(g) = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} g(x)\,dx \qquad \text{if } x_{j-1/2} \leqslant x \leqslant x_{j+1/2} \tag{3.63}$$

Interpretation of Gurdonuv Type Schemes

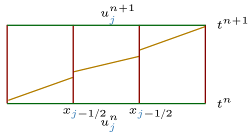
$$\text{Given } \left\{u_j^n\right\}_j \xrightarrow[\text{Rieman Problem Sol.}]{\text{Evolve (approx.)}} \text{Average}$$



$$\left\{u_j^n\right\}_j \leftarrow \operatorname{Avg}\left(\left\{u_j^{n+1}\right\}_j\right)$$

Definition 3.30 Reconstruction:
Replacing cell-averages[def. 3.3] by *piecewise-linears*:

$p^n(x) = p_j^n$
if $x_{j-1/2} \leqslant x \leqslant x_{j+1/2}$
$p_j^n(x) := a_j^n x + b^n$



Definition 3.31 REA Algorithm:
R–E–A–R–E–A–R–E–A (3.64)

1. Reconstruction: at time t^n we know the approximate cell averages u_j^n and realize them by some functions:
$$u(x, t^n) = p_j^n(x) \qquad x_{j-1/2} \leqslant x \leqslant x_{j+1/2}$$
2. Evolution: the reconstruction function is evolved in time by solving the Riemann problem either exactly or approximately:
$$u(x, t^n) \overset{\text{evolve}}{\mapsto} u(x, t^{n+1})$$
3. Averaging: we average the solutions at the next time step t^{n+1} over each control volume.

Corollary 3.15 Evolution is TVD: We have seen that all Riemann solver (apart from Roe-Scheme) are TVD[def. 3.13]

Corollary 3.16 Averaging is TVD:
Given a function $f \in \operatorname{Lip}(\Omega)$ then it holds that the average is TVD[def. 3.13]:

$$\operatorname{TV}\left(\operatorname{Av}(f)\right) \leqslant \operatorname{TV}(f) \qquad \operatorname{Av}(f)_j := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} f(x)\,dx \tag{3.65}$$

Lemma 3.2 Piecewise Constant Averaging:
If we replace the exact solutions with piecewise constant averages then it holds for the error:

$$\|g - \operatorname{Avg}(g)\|_{L^1} \leqslant C\Delta X = \mathcal{O}(\Delta x) \qquad g \in L^1(\Omega) \tag{3.66}$$

Definition 3.32 Generalized Riemann Problem:

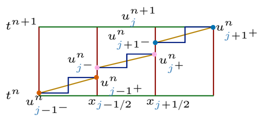
$$u_t + f(u)_x = 0 \tag{3.67}$$
$$u(x, t^n) = p^n(x) \tag{3.68}$$

Cons
- Hard to solve exactly! (except for $f(u) = au$)

5.2. Approximate Reconstruction

Definition 3.33 Approximate Reconstruction:
Approximate *piecewise-linears* of the cell-averages[def. 3.3] by two a simpler problem:

$$p^n(x) = \left\{p_j^n(x)\right\}_j$$
$$p_j^n(x) = a_j^n x + b_j^n$$
$$u_{j+}^n = p_j^n\left(x_{j+1/2}\right)$$
$$u_{j-}^n = p_j^n\left(x_{j-1/2}\right)$$



Corollary 3.17 Linear Approximate Reconstruction:

$$u_{j\pm}^n \overset{\text{eq. (3.10)}}{=} p_j^n\left(x_{j\pm1/2}\right) = \underbrace{u_j^n}_{\text{midpoint}} \pm \overbrace{\frac{\Delta x}{2}}^{\text{distance to boundary}} \sigma_j^n \tag{3.69}$$

Definition 3.34 FVM Evolution and Averaging:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F\left(u_{j+}^n, u_{j+1-}^n\right) - F\left(u_{j-1+}^n, u_{j-}^n\right)\right) \tag{3.70}$$

Corollary 3.18 [proof 5.22]
FVM Evolution and Averaging in Incremental Form:

$$U_j^{n+1} = U_j^n + C_{j+1/2}^n\left(U_{j+1}^n - U_j^n\right) - D_{j-1/2}^n\left(U_j^n - U_{j-1}^n\right)$$

$$c_{j+1/2}^n = \frac{\Delta t}{\Delta x}\frac{f\left(u_{j+}^n, u_{j-}^n\right) - f\left(u_{j+}^n, u_{j+1-}^n\right)}{u_{j+1}^n - u_j^n}$$

$$d_{j+1/2}^n = \frac{\Delta t}{\Delta x}\frac{f\left(u_{j+1+}^n, u_{j+1-}^n\right) - f\left(u_{j+}^n, u_{j-1-}^n\right)}{u_{j+1}^n - u_j^n} \tag{3.71}$$

Lemma 3.3
TVD REA Scheme:
A FVM REA[def. 3.31] scheme is TVD iff *construction*, *averaging* and *evolution* are all TVD.

We know that evolution[cor. 3.15] and averaging[cor. 3.16] is TVD thus we need to find a *reconstruction* that is TVD.

Lemma 3.4 [proof 5.23]
TVD REA scheme:
A REA[def. 3.31] scheme is TVD iff:
1. eq. (3.85) satisfies the CFL condition eq. (3.23)
2. $T_1, T_2 \geqslant 0$
3. $T_1 + T_2 \leqslant 2$

$$T_1 := \frac{U_{j+1-}^n - U_{j-}^n}{U_{j+1}^{n+1} - U_j^n} \qquad T_2 := \frac{U_{j+1+}^n - U_{j+}^n}{U_{j+1}^{n+1} - U_j^n} \tag{3.72}$$

5.2.1. Constraints

1. Conservation:
$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} p_j^n\,dx = u_j^n$$
$$\overset{\text{proof 5.20}}{\Longrightarrow} \int_D p^n(x)\,dx = \int_D u_0(x)\,dx$$
$$\overset{\text{proof 5.21}}{\Longrightarrow} p_j^n = u_j^n + \sigma_j^n\left(x - x_j\right)$$
2. TVD: how would we choose the *slope* σ_j^n?
Obvious choices would be:
   - Forward Differences:
   $$\sigma_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x}$$
   - Backward Differences:
   $$\sigma_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x}$$
   - Central Differences:
   $$\sigma_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

Problem: schemes using this slopes are unstable, satisfy neither TVD nor-discrete maximum principle preserving.

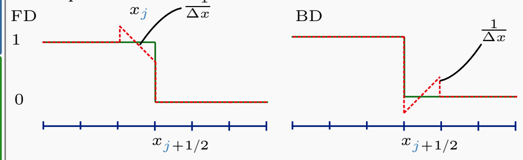
understand/ask upwind scheme and FTCS lecture 12 min 13

5.3. Limiters

We have seen that schemes using simple finite differences for the reconstructions slope σ_j^n are unstable and we know that the evolution and averaging operations are TVD[cor. 3.15] thus we need to ensure that the reconstruction is TVD as well:

$$\operatorname{TV}\left(p^n\right) \leqslant \operatorname{TV}\left(u^n\right)$$

The problem is that schemes using simple finite differences for the slope are not TVD due to discontinuities.



5.3.1. Minmod Limiter

Definition 3.35 Minmod Limiter: Compare the upwind- and downwind slope and checks if they have the same sign. If yes, it sets the slope to the smaller one otherwise it sets the slope to zero.

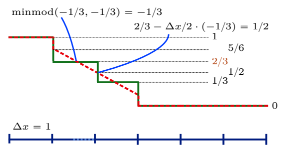
$$\sigma_j^n = \operatorname{minmod}\left(\frac{u_{j+1}^n - u_j^n}{\Delta x}, \frac{u_j^n - u_{j-1}^n}{\Delta x}\right) \tag{3.73}$$

$$\operatorname{minmod}\left(a_1, \ldots, a_n\right) \tag{3.74}$$
$$= \begin{cases} \operatorname{sign}(a_1)\min_{1\leqslant k\leqslant n}\left(|a_k|\right) & \text{if } \operatorname{sign}(a_1) = \cdots = \operatorname{sign}(a_n) \\ 0 & \text{otherwise} \end{cases}$$

Corollary 3.19 [proof 5.25]
Minmod is TVD:

If the reconstruction p^n uses a min-mod limiter, then:

$$\operatorname{TV}\left(p^n\right) \leqslant \operatorname{TV}\left(u^n\right) \tag{3.75}$$



5.3.2. Superbee Limiter

Definition 3.36 [Roe 1981]
Superbee Limiter:

$$\sigma_j^n = \operatorname{maxmod}\left(\sigma_j^L, \sigma_j^R\right) \tag{3.76}$$
$$\sigma_j^L = \operatorname{minmod}\left(\frac{u_{j+1}^n - u_j^n}{\Delta x}, 2\frac{u_j^n - u_{j-1}^n}{\Delta x}\right)$$
$$\sigma_j^R = \operatorname{minmod}\left(2\frac{u_{j+1}^n - u_j^n}{\Delta x}, \frac{u_j^n - u_{j-1}^n}{\Delta x}\right)$$

$$\operatorname{maxmod}\left(a_1, \ldots, a_n\right) \tag{3.77}$$
$$= \begin{cases} \operatorname{sign}(a_1)\max_{1\leqslant k\leqslant n}\left(|a_k|\right) & \text{if } \operatorname{sign}(a_1) = \cdots = \operatorname{sign}(a_n) \\ 0 & \text{otherwise} \end{cases}$$

Corollary 3.20 Superbee is TVD: If the reconstruction p^n uses a superbee-mod limiter, then:

$$\operatorname{TV}\left(p^n\right) \leqslant \operatorname{TV}\left(u^n\right) \tag{3.78}$$

add proof

5.3.3. MC Limiter

Definition 3.37 [Vanleer 1987]
Monotonized Central (MC):

$$\sigma_j^n = \operatorname{minmod}\left(2\frac{u_{j+1}^n - u_j^n}{\Delta x}, 2\frac{u_j^n - u_{j-1}^n}{\Delta x}, \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right)$$

$$\operatorname{minmod}\left(a_1, \ldots, a_n\right) \tag{3.79}$$
$$= \begin{cases} \operatorname{sign}(a_1)\min_{1\leqslant k\leqslant n}\left(|a_k|\right) & \text{if } \operatorname{sign}(a_1) = \cdots = \operatorname{sign}(a_n) \\ 0 & \text{otherwise} \end{cases}$$

Corollary 3.21 MC is TVD: If the reconstrution p^n uses a mc-mod limiter, then:

$$\mathrm{TV}\left(p^n\right) \leqslant \mathrm{TV}\left(u^n\right) \tag{3.80}$$

add proof

5.4. TVD REA Schemes

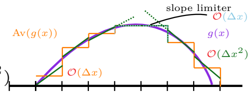
Lemma 3.5 [example 5.25],[proof 5.24]
TVD FVM REA Scheme: A three point FVM REA[def. 3.31] scheme is TVD iff:

1. eq. (3.85) satisfies the CFL condition eq. (3.23)
2. and the following condition:

$$-2 \leqslant \frac{\delta_{j+1}^n - \delta_j^n}{u_{j+1}^n - u_j^n} \leqslant 2 \qquad \delta_j := \sigma_j^n \Delta x \tag{3.81}$$

Proposition 3.2 Order of Accuracy:
Given $g(x) \in \mathcal{C}^2$ and g is monotone (no extreme) and not slope limited then it holds for[def. 3.34]:

$$\|g(x) - p_n(x)\|_{L^\infty} \approx \mathcal{O}(\Delta x^2) \tag{3.82}$$



If we require TVD slope limiters however we will have again be of first order accuracy at the regions of slope limiters/local extrema:

$$\|g(x) - p_n(x)\|_{L^\infty} \approx \mathcal{O}(\Delta x) \tag{3.83}$$

lecture 14 min 40 exam question understand!

5.5. Higher Order Time Schemes

5.5.1. Semi-Discrete Schemes

Definition 3.38 [example 5.8]
Semi-Discrete FVM: Is a discrete time-continuous but space-discrete formulation of[def. 3.34]:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathscr{L}(\mathbf{u}) \tag{3.84}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} u_j(t) &= \mathscr{L}(\mathbf{u}_j) \qquad \text{rate of change} \\ &=: -\frac{1}{\Delta x}\left(F\left(u_{j+}^n, u_{j+1-}^n\right) - F\left(u_{j-1+}^n, u_{j-}^n\right)\right)\end{aligned} \tag{3.85}$$

Definition 3.39 Semi-discrete Cell Averages:

$$U_j^n :\approx \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t)\,\mathrm{d}x \tag{3.86}$$

Definition 3.40 Strong Stability Preserving (SSP) Runge-Kutta Methods: Are Runge-Kutta methods that preserve the TVD propertyeq. (3.31).

Summary what we need

1. Mesh/Grid
2. Numerical Flux $F(u, v)$ (consistent/monotone)
3. Reconstruction: given $\{u_j\}$ output $\left\{u_j^{\pm}\right\}$

$$u_j^{\pm} = u_j \pm \sigma_j \frac{\Delta x}{2}$$

4. Slope Limiters for the slope σ_j
5. SSP-RK scheme

add standard RK method

Heun's Method

Definition 3.41 [proof ??]
Heun's Method (SSP-RK2):
Applies forward Euler twice and averages them to obtain a 2nd-order method:

$$\mathbf{U}^* = \mathbf{U}^n + \Delta t\, \mathscr{L}\left(\mathbf{U}^n\right) \tag{3.87}$$

$$\mathbf{U}^{**} = \mathbf{U}^* + \Delta t\, \mathscr{L}\left(\mathbf{U}^*\right) \tag{3.88}$$

$$\mathbf{U}^{n+1} = \frac{\mathbf{U}^n + \mathbf{U}^{**}}{2} \tag{3.89}$$

Systems of Conservation Laws

Definition 4.1 **Systems of Conservation Law:**
$$\mathbf{u}_t + f(\mathbf{u}(\mathbf{x},t),\mathbf{x})_{\mathbf{x}} = s(\mathbf{u}(\mathbf{x},t),\mathbf{x},t) \quad \text{in } \tilde{\Omega} := \Omega\times]0,T[\tag{4.1}$$

1. Linear System of Conservation Laws

Definition 4.2 [examples 5.9 and 5.10 and ??]
Linear System of Conservation Laws:
$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_{\mathbf{x}} = s(\mathbf{u}(\mathbf{x},t),\mathbf{x},t) \quad \text{in } \tilde{\Omega} := \Omega\times]0,T[$$
$$\mathbf{u} = [u_1 \quad u_2 \cdots\cdots u_m]^\intercal \qquad \mathbf{u} = [f_1 \quad f_2 \cdots\cdots f_m]^\intercal \tag{4.2}$$

Corollary 4.1
Linear Sys. of Cons. Laws with Variable Coefficients:
$$\mathbf{u}_t + (\mathbf{A}(\mathbf{x},t)\mathbf{u})_{\mathbf{x}} = s(\mathbf{u}(\mathbf{x},t),\mathbf{x},t) \quad \text{in } \tilde{\Omega} := \Omega\times]0,T[\tag{4.3}$$

Corollary 4.2 [proof 5.28]
Linearizing Systems of Conservation Laws:
Equation (4.1) can be linearized into eq. (4.2).

Check where to put this

Definition 4.3
Discrete Total Variation Diminishing (TVD): Hyperbolic linear systems of conservation laws
$$\left\|\mathbf{U}^{n+1}\right\|_{TV(\mathbb{R})} := \sum_j \left\|\mathbf{U}^{n+1}_{j+1} - \mathbf{U}^{n+1}_j\right\| \leqslant \sum_j \left\|\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right\|$$
$$\leqslant \sum_j \sum_p^m \left|U^{p,n}_{j+1} - U^{p,n}_j\right| \tag{4.4}$$

1.1. Types of Linear Systems

Definition 4.4 **Hyperbolic System:**
The linear systemeqs. (4.2) and (4.3) are called *hyperbolic* if the matrix $\mathbf{A}$ is diagonalizable and has m real eigenvalues:
$$\operatorname{spectrum}(\mathbf{A})(x,t) = \{\lambda(\mathbf{x},t)_1,\ldots,\lambda(\mathbf{x},t)_m\} \in \mathbb{R} \qquad \forall \mathbf{x}, t \tag{4.5}$$

Corollary 4.3 Strictly Hyperbolic System:
The linear systemeqs. (4.2) and (4.3) is called *strictly hyperbolic* if it is *hyperbolic*[def. 4.4] and all eigenvalues are distinct:
$$\text{eq. } (4.5) \qquad + \qquad \lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_m \tag{4.6}$$

1.2. Decoupling

Proposition 4.1 [proof 5.29]
Decoupled hyperbolic lin. Cons. Law:
Hyperbolic linear systems of conservation laws[def. 4.2] can be decoupled into m linear equations:
$$\mathbf{W}_t + \Lambda\mathbf{W}_x = 0 \quad\Longleftrightarrow\quad W^p_t + \lambda_p W^p_x = 0 \quad \forall p = 1,\ldots,m$$
$$\mathbf{W} = \mathbf{R}^{-1}\mathbf{U} \qquad \mathbf{R} = [\mathbf{r}_1\cdots\cdots\mathbf{r}_p] \qquad \mathbf{A}\mathbf{r}_j = \lambda_j\mathbf{r} \tag{4.7}$$

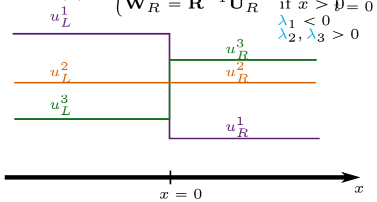
Corollary 4.4 [proof 4.1]
Solution of hyp. lin. cons. laws:
$$W^p(x,t) = W^p_0(x-\lambda_p t) \qquad \mathbf{W}_0(x) = \mathbf{R}^{-1}\mathbf{U}_0(x) \tag{4.8}$$
$$\mathbf{U}(x,t) = \mathbf{R}\mathbf{W}(x,t) \tag{4.9}$$

Proof 4.1 Solution of hyp. lin. cons. law:

add method of characteristics lin. transport equation

1.2.1. Riemann Problems

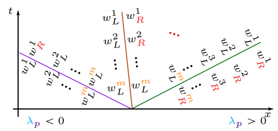
Definition 4.5 **Decoupled Riemann Problem:** Splits the original Riemann data in multiple problems:
$$\mathbf{W}_t + \Lambda\mathbf{W}_x = 0$$
$$\mathbf{W}_0(x) = \begin{cases}\mathbf{W}_L = \mathbf{R}^{-1}\mathbf{U}_L & \text{if } x < 0\\ \mathbf{W}_R = \mathbf{R}^{-1}\mathbf{U}_R & \text{if } x > 0 = 0\end{cases} \tag{4.10}$$



Corollary 4.5
Riemann Problem for hyp. lin. cons. law: The solution of a Riemann problem of a hyperbolic[def. 4.4] linear conservation laweq. (4.10) is given by:
$$W^p(x,t) = W^p_0(x-\lambda_p t) = \begin{cases}W^p_L & \text{if } \lambda_p t < 0\\ W^p_R & \text{if } \lambda_p t > 0\end{cases} \tag{4.11}$$

understand better where this comes from, probably RH condition

Corollary 4.6 [proof 5.30]
Jumps: The Riemann problem of a linear system of conservation laws[cor. 4.5] decomposed into m jumps s.t. we obtain m waves/solutions:
$$\mathbf{U}_R - \mathbf{U}_L = \sum_{p=1}^m \alpha^p r_p \tag{4.12}$$



α^p: strength of the p-th wave
r_p: direction of the characteristics

Explanation 4.1.
- *λ_p speed of the wave*
- *$\lambda_p t$ is called the p-th wave*

Lecture 15 end and script add example wave equation and solve

1.3. FVM Scheme

FVM Scheme

① **Reconstruction:**
$$\mathbf{U}(x,t^n) = p^n_j(x) \overset{i.e.}{=} \begin{cases}\mathbf{U}^n_j & \text{p.w. const}\\ \mathbf{U}^n_j = \mathbf{U}^n_j \pm \frac{\Delta x}{2}\sigma^n_j & \text{linear}\end{cases}$$
$$x_{j-1/2} \leqslant x \leqslant x_{j+1/2}$$

② **Evolution:** by solving Riemann problems:
$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = 0$$
$$\mathbf{U}(x,t^n) = \begin{cases}\mathbf{U}^n_j & \text{if } x < x_{j+1/2}\\ \mathbf{U}^n_{j+1} & \text{if } x > x_{j+1/2}\end{cases}$$

③ **Averaging:**
$$\mathbf{U}^{n+1}_j = \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \mathbf{U}(x,t^{n+1})\,\mathrm{d}x$$
$$\mathbf{U}^{n+1}_j = \mathbf{U}^n_j - \frac{\Delta t}{\Delta x}\left(\mathrm{F}^n_{j+1/2} - \mathrm{F}^n_{j-1/2}\right)$$
$$\mathrm{F}^n_{j\pm 1/2} = \mathrm{F}\left(\mathbf{U}^n_j, \mathbf{U}^n_{j+1}\right) = \mathbf{A}_{j\pm1/2}(x_{j\pm1/2},t^n) \text{ [proof 5.33]}$$

1.3.1. CFL Condition

Definition 4.6 **CFL Condition System of Cons. Laws:**
The wave speed is given by $\lambda_{\max} := \max_{1\leqslant p\leqslant m}|\lambda_p|$ s.t. it follows from eq. (3.42):
$$\lambda_{\max} \leqslant \frac{\Delta x}{\Delta t}\frac{1}{2} \tag{4.13}$$

1.3.2. Exact Fluxes

Godunov Flux

Definition 4.7 [proof 5.31]
Godunov Flux:
$$\mathrm{F} = \mathbf{A}\mathbf{U}_{j+1/2} \tag{4.14}$$
$$= \frac{1}{2}\mathbf{A}\left(\mathbf{U}^n_j + \mathbf{U}^n_{j+1}\right) - \frac{1}{2}\mathbf{R}|\Lambda|\mathbf{R}^{-1}\left(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right)$$

Property 4.1 [proof 5.32]
Total Variation Bounded (TVB): Godunov flux for systems of scalar conservation laws is total variation bounded:
$$\mathrm{TV}(\mathbf{U}^{n+1}) \leqslant \|\mathbf{R}\|\|\mathbf{R}^{-1}\|\mathrm{TV}(\mathbf{U}^n) \tag{4.15}$$

Note
It is not TVD as we do not know what the condition numbers $\|\mathbf{R}\|\|\mathbf{R}^{-1}\|$ are.

Is Gudonov Flux called Godunov flux for linear systems and for non-linear systems Roe flux?

Godunov Flux is the

1.3.3. Approximate Fluxes

Central Fluxes

Definition 4.8 **Lax Friedrichs Scheme:** Chooses the wave speeds s.t. waves from neighboring Riemann problems do not interact with each other:
$$s^l_{j+1/2} = -\frac{\Delta x}{2\Delta t} \qquad s^r_{j+1/2} = \frac{2\Delta x}{\Delta t} \tag{4.16}$$
with eq. (3.52) it follows:
$$\mathrm{F}^n_{j+1/2} = \mathrm{F}^{\mathrm{LxF}}\left(\mathbf{U}^n_j, \mathbf{U}^n_{j+1}\right) \tag{4.17}$$
$$= \frac{1}{2}\mathbf{A}\left(\mathbf{U}^n_j + \mathbf{U}^n_{j+1}\right) - \frac{\Delta x}{2\Delta t}\left(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right)$$

Definition 4.9
Rusanov/Local-Lax-Friedrichs Scheme:
Takes into account the local speeds λ_p of the waves (and not only the grid):
$$s_{j+1/2} = \max|\Lambda| \tag{4.18}$$
with eq. (3.52) and $s^r_{j+1/2} = s_{j+1/2} = -s^l_{j+1/2}$ it follows:
$$\mathrm{F}^n_{j+1/2} = \mathrm{F}^{\mathrm{Rus}}\left(u^n_j, u^n_{j+1}\right) \tag{4.19}$$
$$= \frac{1}{2}\mathbf{A}\left(\mathbf{U}^n_j + \mathbf{U}^n_{j+1}\right) \tag{4.20}$$
$$- \frac{\lambda_{\max}}{2}\left(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right)$$

2. Higher Order Schemes

Goal
Design a 2nd-order TVB-stable scheme.

2.1. Reconstruction

Whats the difference

Definition 4.10 **Conservative Variables:** Are the variables $\mathbf{U}$ used to write a system in conservative form.

Definition 4.11 **Primitive Variables:** are called the Characteristic Variables.

Definition 4.12 **Characteristic Variabels:** $\mathbf{W} = \mathbf{R}^{-1}\mathbf{U}$ are called the Characteristic Variables.

Definition 4.13 **Primitive Reconstruction:** Apply limiterssection 3 componentwise to the primitve variables $\mathbf{U}_j$.

Pros	Cons
• Easy to apply	• Does not necessarily lead to TVBProperty 4.1 stable reconstruction section 1 scheme.

Definition 4.14 **Characteristic Reconstruction:** Apply limiterssection 3 componentwise to the characteristic variables $\mathbf{W}_j$.
$$\gamma^n_j = \operatorname{limiter}\left(\mathbf{W}^n_{j-1}, \mathbf{W}^n_j, \mathbf{W}^n_{j+1}\right) \quad\Longrightarrow\quad \sigma^n_j = \mathbf{R}\gamma^n_j \tag{4.21}$$

Corollary 4.7 : Scheme ③[def. 3.34] is:
- is 2nd-order accurate in space formally.
- is TVB-stable if σ is defined by Characteristic reconstruction.

2.2. Higher Order in Time

Proposition 4.2
Heun's Method for Systems of Conservation Laws:
Given a system of conservation laws the following scheme:
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}_j(t) = -\frac{\Delta t}{\Delta x}\left(\mathrm{F}\left(\mathbf{U}^n_{j+}, \mathbf{U}^n_{j+1-}\right) - \mathrm{F}\left(\mathbf{U}^n_{j-1+}, \mathbf{U}^n_{j-}\right)\right)$$
$$- \frac{\Delta t}{\Delta x}\left(\mathrm{F}_{j+1/2}(t) - \mathrm{F}_{j-1/2}\right) =: \mathscr{L}(\mathbf{U}(t))_j$$
$$\mathbf{U}^+_j = p(x_{j+1/2}) \qquad \mathbf{U}^-_j = p(x_{j-1/2})$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}(t) = \mathscr{L}(\mathbf{U}(t)) \qquad \mathbf{U}(t) := [\cdots \quad \mathbf{U}_{j-1} \quad \mathbf{U}_j \quad \mathbf{U}_{j+1} \quad \cdots]$$
with Heun's Method[def. 3.41] is 2nd-order in time.

Non-Linear Systems of Conservation Laws

Definition 5.1
Nonlinear Systems of Conservation Laws:

$$\partial_t \mathbf{U} + \partial_x f(\mathbf{U}) = \mathbf{0} \qquad \mathbf{U}:\mathbb{R}\times\mathbb{R}_+ \to \mathcal{U}\in\mathbb{R}^m$$
$$\mathbf{U}(x,0) = \mathbf{U}_0(x) \qquad \mathbf{U}\in L^\infty(\mathbb{R}\times[0,T];\mathcal{U}) \qquad f:\mathcal{U}\to\mathbb{R}^m \text{ (nonlinear)} \tag{5.1}$$

Definition 5.2 **Admissible Set** $\mathcal{U}$**:**
Is the domain of admissible values that make sense in a physical way.

Definition 5.3 **j-th Wave Family:** The j-th wave family of nonlinear systems of conservation laws[def. 5.1] is defined as the eigenvalue-eigenvector pair of the Jaccobian $f'(\mathbf{U})$:

$$\{\lambda_j(\mathbf{U}), \mathbf{r}_j(\mathbf{U})\} \tag{5.2}$$

Definition 5.4 [example 5.12]
Hyperbolic Nonlinear Systems of Conservation Laws:
A nonlinear scalar conservation laweq. (5.1) is *hyperbolic* if the Jaccobian?? $f'(\mathbf{U})$ has:

1. *real eigenvalues* $\iff$ spectrum $(f'(\mathbf{U})) \in \mathbb{R}$:
$$\lambda\left(f'(\mathbf{U})\right) = \{\lambda_1(\mathbf{U}) \leqslant \lambda_2(\mathbf{U}) \leqslant \ldots \leqslant \lambda_m(\mathbf{U})\} \in \mathbb{R}$$
2. Linearly independent eigenvectors:
$$r_1(\mathbf{U}), r_2(\mathbf{U}), \ldots, r_m(\mathbf{U}) \tag{5.3}$$

Definition 5.5 [example 5.13]
Strictly Hyperbolic Non. Lin. Sys. of Conservation Laws: Is a hyperbolic Nonlinear Systems of Conservation Laws with distinct *real eigenvalues*:

$$\lambda\left(f'(\mathbf{U})\right) = \{\lambda_1(\mathbf{U}) < \lambda_2(\mathbf{U}) < \ldots < \lambda_m(\mathbf{U})\} \in \mathbb{R}$$

Corollary 5.1 Diagonalizability: A Hyperbolic Nonlinear System of Conservation laws has a diagonalizable Jacobian matrix $f'(\mathbf{U})$:

$$f'(\mathbf{U}) = \mathbf{R}(\mathbf{U})\,\Lambda(\mathbf{U})\mathbf{R}(\mathbf{U})^{-1} \tag{5.4}$$
$$\Lambda(\mathbf{U}) := \operatorname{diag}(\lambda_1(\mathbf{U}), \ldots, \lambda_m(\mathbf{U}))$$
$$\mathbf{R}(\mathbf{U}) := [\mathbf{r}_1(\mathbf{U}) \cdots\cdots \mathbf{r}_m(\mathbf{U})]$$

Definition 5.6 [example 5.12]
Genuinely Nonlinear Wave Family: A *hyperbolic systems*[def. 5.4] j^{th}*-wave family* is *genuinely nonlinear* iff:

$$\nabla\lambda_j(\mathbf{U})\cdot\mathbf{r}_j(\mathbf{U}) \neq 0 \quad \forall\mathbf{U}\in\mathcal{U}, \quad j\in\{1,\ldots,m\} \tag{5.5}$$

Explanation 5.1. *Corresponds to a notion of convexity.*

Definition 5.7
Linearly Degenerat Wave Family: A *hyperbolic systems*[def. 5.4] j^{th}*-wave family* is *linearly degenerated* iff:

$$\nabla\lambda_j(\mathbf{U})\cdot\mathbf{r}_j(\mathbf{U}) = 0 \quad \forall\mathbf{U}\in\mathcal{U}, \quad j\in\{1,\ldots,m\} \tag{5.6}$$

Explanation 5.2. *Linearly to a notion of convexity.*

1. Weak Solutions

Definition 5.8 [proof 5.34]
Weak Solution for 5.1:
$\mathbf{U}\in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ is a weak solution of [def. 5.1] iff:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \mathbf{U}\partial_t\phi + f(\mathbf{U})\,\partial_x\phi + \int_{\mathbb{R}} \mathbf{U}_0(x)\phi(x,t)\,\mathrm{d}x = 0 \tag{5.7}$$
$$\forall\phi\in\mathcal{C}_c^\infty(\mathbb{R}\times[0,\infty))$$

1.1. The Rankine-Hugoniot Condition

Definition 5.9 [proof 5.4]
Rankine-Hugoniot Condition: Is a condition on the *shock-speed* $s(t) = \gamma'(t)$ of a shock[def. 2.1] i.e. how fast the shock-wave travels:

$$s(t)\left(\mathbf{U}^+(t) - \mathbf{U}^-(t)\right) = f\left(\mathbf{U}^+(t)\right) - f\left(\mathbf{U}^-(t)\right) \tag{5.8}$$
$$\mathbf{U}^+ = \lim_{\mathbf{x}\to\gamma^+(t)}\mathbf{U}(\mathbf{x},t) \qquad \mathbf{U}^- = \lim_{\mathbf{x}\to\gamma^-(t)}\mathbf{U}(\mathbf{x},t)$$

Corollary 5.2 Unknowns vs. Equations:
- **Unknown's:** $\mathbf{U}^+, \mathbf{U}^-\in\mathbb{R}^m$, $s(t)\in\mathbb{R} \implies 2m+1$
- **Equations:** $f(\mathbf{U})\in\mathbb{R}^m \iff$ Equation (5.8)$\in\mathbb{R}^m \implies m$

Corollary 5.3 Relationship to Weak Solutions:
If $\mathbf{U}$ is a $\mathcal{C}^1_{pw}$ function with only jump-type discontinuities, the following statemnts are equivilant:
- $\mathbf{U}$ is a weak solution[def. 5.8] of the conservation law[def. 5.1].
- $\mathbf{U}$ is a classical solution whenever it is $\mathcal{C}^1$, and satisfies the Rankine-Hugoniot condition[def. 5.9] across every discontinuity $\mathbf{x}\to\gamma(t)$.

2. Simple Solutions

Definition 5.10
Riemann Problem for Sys. of Non-linear Cons. Laws:

$$\partial_t\mathbf{U} + \partial_x f(\mathbf{U}) = \mathbf{0}$$
$$\mathbf{U}(x,0) = \mathbf{U}_0(x) = \begin{cases} U_R & \text{if } x>0 \\ U_L & \text{if } x<0 \end{cases} \tag{5.9}$$

Recall
For Riemann problems of scalar conservation laws we obtain different solutions:
1. Shock Solutions[cor. 2.2]
2. Rarefaction Solutions[cor. 2.3]

we now study solutions of non-linear systems of conservation laws eq. (5.36).

Definition 5.11 [proof 5.35]
Eigenvalue Problem for Non-lin. sys. of cons. laws: Is the problem we need to solve in order to find solutions to non-linear systems of conservation laws[def. 5.1]:

$$f'(\mathbf{v}(\xi))\,\mathbf{v}'(\xi) = \xi\mathbf{v}'(\xi) \qquad \begin{aligned}\mathbf{v}'(\xi) &= \mathbf{r}_j(\mathbf{v}(\xi)) \\ \xi &= \lambda_j(\mathbf{v}(\xi))\end{aligned} \qquad j\in\{1,\ldots,m\} \tag{5.10}$$

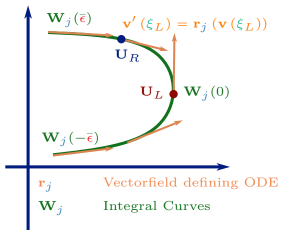
Definition 5.12 [proof 5.36]
Simple ODE: Is the shifted problem eq. (5.75) with initial conditions at zero:

$$\begin{aligned}\mathbf{W}'(\epsilon) &= \mathbf{r}_j(\mathbf{W}(\epsilon)) \\ \mathbf{W}_j(0) &= \mathbf{U}_L\end{aligned} \qquad \epsilon = \xi - \lambda_j(\mathbf{U}_L) \tag{5.11}$$

Note: Piccard-Lindeloef Theorem
Recall from analysis If $\mathbf{r}_p(\mathbf{W}_p(t))$ is Lipschitz continuous?? then eq. (5.11) has a solution for $\epsilon\in[0-\bar{\epsilon}, 0+\bar{\epsilon}]$.

Explanation 5.3 (Integral Curves).

The solution of equation eq. (5.11) is given by integral curves that are tangent to the eigenvectors $\mathbf{r}_p(\mathbf{W}_p(t))$ *of the wave families.*



2.1. Contact Discontinuities

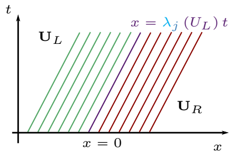
Lemma 5.1 **Existence Contact Discontinuity:**
Let the j-th wave family[def. 5.3] be *linear degenerate*[def. 5.7] and let $\mathbf{U}_L\in\mathcal{U}$. Then by the Piccard-Lindeloef Theorem?? there exists an *integral curve* solving eq. (5.11):

$$\mathcal{C}_j(\mathbf{U}_L) = \left\{\mathbf{W}_j\left(\epsilon^*\right)\in\mathbb{R}^m : \epsilon^*\in[-\bar{\epsilon}, \bar{\epsilon})\right\} \tag{5.12}$$

if $\mathbf{U}_R\in\mathcal{C}_j(\mathbf{U}_L)$ then there exists a *contact discontinuity solution*[def. 5.13] $\mathbf{U}$ to the Riemann problem eq. (5.36).

Definition 5.13 [proof 5.37]
Contact Discontinuity Solution:
If lemma 5.1 is satisfied then the solution of eq. (5.36) is given by:

$$\mathbf{U}(x,t) = \begin{cases}\mathbf{U}_L & \frac{x}{t} < \lambda_j(\mathbf{U}_L) \\ \mathbf{U}_R & \lambda_j(\mathbf{U}_R) < \frac{x}{t}\end{cases} \tag{5.13}$$



Explanation 5.4. *Appear in gas genomics when a with a discontinuity in mass density but not in the pressure or velocity, in comparison to real shocks, which move faster than the gas itself due to a discontinuity in pressure.*

Definition 5.14 [proof 5.38]
Rankine-Hugoniot Condition:
A contact discontinuity solution[def. 5.13] fulfills the Rankine-Hugoniot Condition:

$$f(\mathbf{U}_R) - f(\mathbf{U}_L) = s(\mathbf{U}_R - \mathbf{U}_L) \qquad s := \lambda_j(\mathbf{U}_R) = \lambda_j(\mathbf{U}_L) \tag{5.14}$$

2.2. Rarefactions

Lemma 5.2 **Existence Rarefaction Solution:**
Let the j-th wave family[def. 5.3] be *genuinely nonlinear*[def. 5.6] and let $\mathbf{U}_L\in\mathcal{U}$. Then by the Piccard-Lindeloef Theorem?? there exists an *integral curve* solving eq. (5.11):

$$\mathcal{R}_j(\mathbf{U}_L) = \left\{\mathbf{W}_j\left(\epsilon^*\right)\in\mathbb{R}^m : \epsilon^*\in[0,\bar{\epsilon})\right\} \tag{5.15}$$

if $\mathbf{U}_R\in\mathcal{R}_j(\mathbf{U}_L)$ then there exists a ??[def. 5.15] $\mathbf{U}$ to the Riemann problemeq. (5.36).

Note: Lipschitz Boundaries
We exclude $-\bar{\epsilon}$ i.e. use $[0,\bar{\epsilon})$ as integration boundaries because for the rarefaction solution we have different eigenvalues and in this case the right eigenvalue could be larger than the left eigenvalue, which wouldn't make sense:

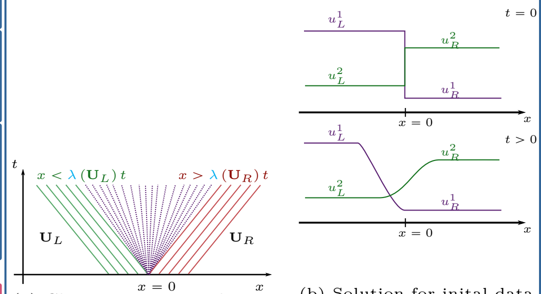
$$\lambda_j(\mathbf{U}_R) = \epsilon + \lambda_j(\mathbf{U}_L) < \lambda_j(\mathbf{U}_L) \quad ↯$$

Proposition 5.1 [proof 5.39]
Rarefaction and GNL wave families: Rarefaction solutions of non-linear systems of conservation laws[def. 5.1] exist if the wave families are *genuinely nonlinear*[def. 5.6]:

$$\nabla\lambda_j(\mathbf{v}(\xi))^\intercal\,\mathbf{r}_j(\mathbf{v}(\xi)) = 1 \qquad \forall j\in\{1,\ldots,m\} \tag{5.16}$$

Definition 5.15 [proof 5.39]
Rarefaction Solution:
If lemma 5.2 is satisfied then the solution of eq. (5.36) is given by:

$$\mathbf{U}(x,t) = \begin{cases}\mathbf{U}_L & \frac{x}{t} < \lambda_j(\mathbf{U}_L) \\ \mathbf{W}_j\left(\frac{x}{t} - \lambda_j(\mathbf{U}_L)\right) & \lambda_j(\mathbf{U}_L) < \frac{x}{t} < \lambda_j(\mathbf{U}_R) \\ \mathbf{U}_R & \lambda_j(\mathbf{U}_R) < \frac{x}{t}\end{cases} \tag{5.17}$$



(a) Characteristics splitting the solution in three regsion. The RH-condition eq. (5.8) is trivaly fulfilled

(b) Solution for inital data and later point; both waves travel with the same speed, one to the right the other to the left.

tell mishra there is a typo in the script UR instead of UL

Note
The eigenvectors $\mathbf{r}_j(\mathbf{v}(\xi))$ for a gnl family can always be rescaled s.t. eq. (5.16) equals to 1.

2.3. Shock Waves

We have seen:
- Smooth genuinely non-linear solutions – Rarefactions
- Discontinuous linear degenerate solutions – Contact Discontinuous

but what about genuinely non-linear discountinuities – real shocks?

Definition 5.16 **Hugoniot Locus:**

$$\mathcal{H}(\mathbf{U}_L) = \Big\{\mathbf{U}_R\in\mathcal{U} : \exists s\in\mathbb{R} \text{ s.t. } f(\mathbf{U}_R) - f(\mathbf{U}_L) = s(\mathbf{U}_R - \mathbf{U}_L)\Big\} \tag{5.18}$$

Notes
- The set of the Hugoniot Locus consist of all $\mathbf{U}_R\in\mathcal{U}$ s.t:
$$\mathbf{U}(x,t) = \begin{cases}\mathbf{U}_L & \frac{x}{t} < s \\ \mathbf{U}_R & s < \frac{x}{t}\end{cases}$$
- The set of contact discontinuities is a subset of the Hugoniot Locus i.e. $\mathcal{C}_j(\mathbf{U}_L)\in\mathcal{H}(\mathbf{U}_L)$

Lemma 5.3 : Assume a strictly hyperbolic[def. 5.5] nonlinear scalar conservation laweq. (5.1) with $\mathbf{U}\in\mathbf{U}_L$ then there exist m curves passing through $\mathbf{U}_L$:

$$\mathcal{H}(\mathbf{U}_L) = \mathcal{H}_1(\mathbf{U}_L)\cup\cdots\cup\mathcal{H}_m(\mathbf{U}_L) \tag{5.19}$$

Definition 5.17 [proof 5.40]
Shock Wave ODE:

$$\mathbf{W}_j'(0) = \mathbf{r}_j(\mathbf{U}_L) \qquad \mathbf{W}_j(0) = \mathbf{U}_L \qquad \forall j = 1,\ldots,m \tag{5.20}$$

2.4. Entrop Conditions

The entropy conditions based on the Lax-Olenek entropy condition must of course also be satisfied for non-linear scalar conservation laws.

Proposition 5.2 Viscous Approximation:
Is a parabolic *convection-diffusion equation* of the form:

$$\begin{aligned}\partial_t \mathbf{U} + \partial_x \mathrm{f}(\mathbf{U}) &= \nu \partial_{xx} \mathbf{U} \\ \mathbf{U}(x,0) &= \mathbf{U}_0(x)\end{aligned} \qquad \begin{aligned}&\mathbf{U}: \mathbb{R}\times\mathbb{R}_+ \to \mathcal{U} \in \mathbb{R}^m \\ &\mathbf{U} \in L^\infty(\mathbb{R}\times[0,T];\mathcal{U}) \\ &\mathrm{f}: \mathcal{U} \to \mathbb{R}^m \quad \text{(nonlinear)}\end{aligned} \tag{5.21}$$

Definition 5.18 Vanishing Viscosity Solution:
In the limit $\epsilon \to 0$ we recover the inviscide non-linear scalar conservation laws. Thus we can study proposition 5.2 for $\epsilon \to 0$ in order to study small scale effects.

Definition 5.19 [examples 5.14 and 5.15]
Entropy Pair (s,q):
The pair (s,q) is called entropy pair, where S is any *strictly convex function*??. Then the entropy pair is defined by the relation:

$$q(\mathbf{U}) = \int_{\mathbf{0}}^{\mathbf{U}} \mathrm{f}'(\gamma)\mathrm{s}'(\gamma)\,\mathrm{d}\gamma \quad\Longrightarrow\quad \mathrm{q}'(\mathbf{U})^\intercal = \mathrm{s}'(\mathbf{U})^\intercal \mathrm{f}'(\mathbf{U}) \tag{5.22}$$

Entropy function $\mathbf{s}$ $\quad \mathbf{s}: \mathcal{U}\subset\mathbb{R}^m \to \mathbb{R}$, strictly convex**??**
Entropy flux $\mathbf{q}$ $\quad \mathbf{q}: \mathcal{U}\subset\mathbb{R}^m \to \mathbb{R}$

Note
For most physical nonlinear hyperbolic systems, there exists only one entropy, whereas for scalar conservation laws there exist a pair for any convex entropy function s.

Definition 5.20 [proof 5.41]
Entropy Condition:
Any vanishing viscosity solution[def. 2.7] u satisfies:

$$\partial_t s(\mathbf{U}) + \partial_x q(\mathbf{U}) \leqslant \mathbf{0} \tag{5.23}$$

Corollary 5.4 similar to [proof 5.8]
Kruzkov's Entropy Condition:
A solution $\mathbf{U}$ ofeq. (5.1) is a weak solution if it satisfies the Kruzkov's Entropy Condition for all entropy pairs[def. 5.19] (s,q):

$$\int_{\mathbb{R}}\int_{\mathbb{R}_+} s(\mathbf{U}(x,t))\,\phi_t(x,t) + q(\mathbf{U}(x,t))\,\phi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} s(\mathbf{U}_0(x))\phi(x,0)\,\mathrm{d}x \geqslant 0 \tag{5.24}$$

$$\forall \phi \in \mathcal{C}_C^1(\mathbb{R}\times\mathbb{R}_+),\ \phi \geqslant 0$$

Definition 5.21 Entropy Solution:
A *weak solution*[def. 2.3] of eq. (5.1) $\mathbf{U} \in L^\infty(\mathbb{R},\mathbb{R}_+)$ is an entropy solution of the inviscide non-linear system of scalar conservation laws eq. (5.1) iff $\mathbf{U}$ satisfies the entropy condition eq. (5.24) for all entropy pairs[def. 2.8] (s,q)

2.4.1. Lax Entropy Condition

Definition 5.22 [proof ??]
Entropy Dissipation: States that the entropy across a discontinuity can only decrease (in a mathematical sense):

$$\left(q\left(U^+\right) - q\left(U^-\right)\right) - s\left(s\left(U^+\right) - s\left(U^-\right)\right) \leqslant 0 \tag{5.25}$$

Definition 5.23 Entropy Solution Equivalence:
Let $\mathbf{U} \in \mathcal{C}^1$ with jump discontinuities across smooth curves, then the following statements are equal:

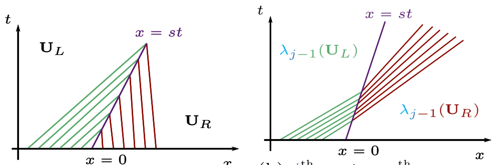
- $\mathbf{U}$ is an entropy solution[def. 5.21] of eq. (5.1)
- $\mathbf{U}$
  - is a classical solution of eq. (5.1), whenever $\mathbf{U} \in \mathcal{C}^1$
  - fulfills the entropy dissipation equation eq. (5.25) for all entropy pair (s,q)

Proposition 5.3 [proof 5.42]
Contact Discontinuity Entropy:
There is no entropy dissipation across *contact discontinuities*[def. 5.13]:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}E(\epsilon) \equiv 0 \qquad\qquad E(\epsilon) \equiv 0 \tag{5.26}$$

Proposition 5.4 [proof 5.43]
Lax Entropy Condition: For *genuinely nonlinear strictly hyperbolic systems*[cor. 4.3] of conservation laws it holds:

$$\lambda_p(\mathbf{U}_R) < s < \lambda_p(\mathbf{U}_L) \tag{5.27}$$
$$\lambda_{p-1}(\mathbf{U}_L) < s < \lambda_{p+1}(\mathbf{U}_R) \tag{5.28}$$



(a) $j^{\text{th}} = p^{\text{th}}$: wave family impingings on shock

(b) $j^{\text{th}} \neq p^{\text{th}}$: wave families go through on shock

Corollary 5.5 : Equation eq. (5.28) can be rewritten as:

$$\lambda_j(\mathbf{U}_L) < s \qquad \lambda_j(\mathbf{U}_R) < s \qquad 1 \leqslant i \leqslant j-1 \tag{5.29}$$

and corresponds to characteristics that have both smaller speeds then the discontinuity.

Lemma 5.4 Lax Entropy Solution:
Let the j-th wave family be *genuinely nonlinear*[def. 5.6] and let $\mathbf{U}_L \in \mathcal{U}$. Then there exists a curve:

$$\mathcal{S}_j(\mathbf{U}_L) = \Big\{\mathbf{W}_j(\epsilon) : \epsilon \in [-\bar{\epsilon}, 0]; \tag{5.30}$$
$$f\left(\mathbf{W}_j(\epsilon)\right) - f(\mathbf{U}_L) = s\left(\mathbf{W}_j(\epsilon) - \mathbf{U}_L\right)\Big\} \tag{5.31}$$

emanating from $\mathbf{U}_L$.
If $\mathbf{U}_R \in \mathcal{S}_j(\mathbf{U}_L)$ then there exists an entropy solution figs. 2a and 2b:

$$\mathbf{U}(x,t) = \begin{cases}\mathbf{U}_L & \frac{x}{t} < s \\ \mathbf{U}_R & s < \frac{x}{t}\end{cases} \tag{5.32}$$

Explanation 5.5. *We require the negative integral curve i.e. $-\bar{<}\epsilon \leqslant 0$ s.t. the entropy condition is fulfiled, which leads in turn to the figures figs. 2a and 2b, depending on the wave family.*

Lemma 5.5 Entropy Solution: Assume a *strictly hyperbolic* non-linear scalar system of conservation laws[def. 5.5] with only *genuinely non-linear* or *linear degenerate* wave families. Then $\mathbf{U}$ is an entropy solution of[def. 5.1] if and only if at every jump $\exists j \in \{1,\ldots,m\}$:

- the j-th wave family is **??** $\Rightarrow$ proposition 5.3 and [def. 5.13].
- the j-th wave family is genuinely nonlinear, and the *Lax entropy condition* holdseqs. (5.27) and (5.28) $\Rightarrow$ lemma 5.4.

2.5. Summary

In the previous section we considered *strictly hyperbolic*[cor. 4.3] Riemann problems for systems of scalar conservation laws[def. 5.10]. We have seen that if *each* wave family is either *linear degenerate*[def. 5.7] or *genuinely-nonlinear*eq. (5.5) then there exist m curves $\mathcal{W}_1(\mathbf{U}_L),\ldots,\mathcal{W}_m(\mathbf{U}_L)$ through $\mathbf{U}_L$ and if $\mathbf{U}_R$ lies in any of these curves then the riemann problem can be solved with a simple solution:

$$\mathcal{W}(\mathbf{U}_L) = \mathcal{W}_1(\mathbf{U}_L) \cup \cdots \cup \mathcal{W}_m(\mathbf{U}_L) \tag{5.33}$$

$$\mathcal{W}(\mathbf{U}_L) = \begin{cases}\mathcal{W}_j = \mathcal{C}_j(\mathbf{U}_L) & \text{if the } j\text{-th wave family is } \textit{linearly degenerate} \\ \mathcal{W}_j = \mathcal{S}_j(\mathbf{U}_L) \cup \mathcal{R}_j(\mathbf{U}_L) & \text{if the } j\text{-th wave family is } \textit{genuinely non-linear}\end{cases}$$

1. If $\mathbf{U}_R \in \mathcal{R}_p(\mathbf{U}_L) \cup \mathcal{C}_j(\mathbf{U}_L)$:
   - If $(\lambda_p, \mathbf{r}_p)$ genuinely nonlinear $\Rightarrow$ rarefaction
   - If $(\lambda_p, \mathbf{r}_p)$ linearly degenerate $\Rightarrow$ contact discontinuity
2. If $\mathbf{U}_R \in \mathcal{H}_p(\mathbf{U}_L)_{[-\bar{\epsilon},0]} \cup \mathcal{C}_j(\mathbf{U}_L) = \mathcal{S}_p(\mathbf{U}_L) \cup \mathcal{C}_j(\mathbf{U}_L)$:
   - If $(\lambda_p, \mathbf{r}_p)$ genuinely nonlinear $\Rightarrow$ shocks
   - If $(\lambda_p, \mathbf{r}_p)$ linearly degenerate $\Rightarrow$ contact discontinuity

Each of the curves $\mathcal{R}_p(\mathbf{U}_L), \mathcal{C}_p(\mathbf{U}_L)$ and $\mathcal{R}_p(\mathbf{U}_L)$ can be paremeterized by some function:

$$\mathbf{W}_j(\mathbf{U}_L, \epsilon) \qquad \epsilon \in \begin{cases}(-\bar{\epsilon}, \bar{\epsilon}) \\ (-\bar{\epsilon}, 0] \\ [0, \bar{\epsilon})\end{cases} \qquad \bar{\epsilon}(\mathbf{U}_L) > 0$$

- *Contact Discontinuity Integral Curves*:
$$\mathcal{C}_j(\mathbf{U}_L) = \left\{\mathbf{W}_j\left(\epsilon^*\right) \in \mathbb{R}^m : \epsilon^* \in [-\bar{\epsilon}, \bar{\epsilon}]\right\}$$
- *Rarefaction Integral Curves* $\mathcal{R}_p(\mathbf{U}_L)$:
$$\mathcal{R}_p(\mathbf{U}_L) = \left\{\mathbf{W}_p(\epsilon) : \frac{\mathrm{d}\mathbf{W}_p(t)}{\mathrm{d}t} = \mathbf{r}_p(\mathbf{W}_p(\epsilon)),\ \mathbf{W}_p(0) = \mathbf{U}_L, \epsilon \in [0,\bar{\epsilon}]\right\}$$
- *Hugoniot Locus*:
$$\mathcal{S}_p(\mathbf{U}_L) = \Big\{\mathbf{W}_j(\epsilon) : \epsilon \in [-\bar{\epsilon}, 0]; \quad f\left(\mathbf{W}_j(\epsilon)\right) - f(\mathbf{U}_L) = s\left(\mathbf{W}_j(\epsilon) - \mathbf{U}_L\right)\Big\}$$

For any $\mathbf{U} \in \mathbf{W}_j(\mathbf{U}_L, \epsilon) \in \mathcal{W}(\mathbf{U}_L)$, there exist then a *simple solution* $\mathbf{u}_j(\mathbf{U}_L, \epsilon; x, t)$ that is either of the formula eqs. (5.13), (5.17) and (5.31) depending whether $\mathbf{U}_R$ lies in $\mathcal{C}_j(\mathbf{U}_L)$, $\mathcal{S}_j$, $\mathcal{R}_j$.

3. General Riemann Problems

What if the wave families of the Riemann problem are neither linear degenerate or genuinely non-linear?

Finite Volume Method

Definition 5.24
FVM RP for Sys. of Non-linear Cons. Laws:

$$\partial_t \mathbf{U} + \partial_x \mathrm{f}(\mathbf{U}) = \mathbf{0}$$
$$\mathbf{U}(x, t^n) = \begin{cases}\mathbf{U}_j & \text{if } x < x_{j+1/2} \\ \mathbf{U}_{j+1} & \text{if } x > x_{j+1/2}\end{cases} \tag{5.34}$$

Definition 5.25
Finite Volume Scheme: For non-linear scalar systems of conservation laws it holds:

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n - \frac{\Delta t}{\Delta x}\left(\mathrm{F}_{j+1/2}^n - \mathrm{F}_{j-1/2}^n\right) \qquad \forall j, n \tag{5.35}$$

$$\mathbf{U}_j^0 = \frac{1}{\Delta x}\int_{x_{j+1/2}}^{x_{j+1/2}} \mathbf{U}_0(x)\,\mathrm{d}x \qquad \mathrm{F}_{j+1/2}^n = \mathrm{f}(\mathbf{u}(0))$$

4. Linearized Riemann Solvers/Roe Schemes

Definition 5.26 [proof 5.44]
Locally Linearized Riemann Problem Approximation:

$$\mathbf{U}_t + \mathbf{A}_{j+1/2}^n \mathbf{U}_x = \mathbf{0}$$
$$\mathbf{U}(x, t^n) = \begin{cases}\mathbf{U}_j & \text{if } x < x_{j+1/2} \\ \mathbf{U}_{j+1} & \text{if } x > x_{j+1/2}\end{cases} \tag{5.36}$$

4.1. Properties of linear Approximations

Property 5.1 Strict Hyperbolicity:
$\mathbf{A}_{j+1/2}^n \in \mathbb{R}^{m\times m}$ should be strictly hyperbolic[cor. 4.3].

Property 5.2 Consistency:
$\mathbf{A}_{j+1/2}^n = \mathbf{A}_{j+1/2}^n\left(\mathbf{u}_j^n, \mathbf{u}_j^{n+1}\right)$ should be consistent:

$$\mathbf{A}_{j+1/2}^n(\mathbf{u},\mathbf{u}) = \mathrm{f}'(\mathbf{u}) \tag{5.37}$$

Explanation 5.6. *If the left and right states are consistent/have the same value then our approximation should do nothing and be equal to the real flux.*

Property 5.3 [proof 5.46]
Roes Criterion: Isolated Discontinuities should be preserved exactly by our approximation:

$$\mathrm{f}\left(\mathbf{u}_{j+1}^n\right) - \mathrm{f}\left(\mathbf{u}_j^n\right) = \mathbf{A}_{j+1/2}^n\left(\mathbf{u}_{j+1}^n - \mathbf{u}_j^n\right) \tag{5.38}$$

4.2. Choices for the linearized flux

4.2.1. Arithmetic Average

Definition 5.27 Arithmetic Average:

$$\mathbf{A}_{j+\frac{1}{2}}^n = \mathrm{f}'\left(\frac{\mathbf{U}_j^n + \mathbf{U}_{j+1}^n}{2}\right) \tag{5.39}$$

Pros	Cons
• Simple • Satisfies eq. (5.37).	• Does not satisfy eq. (5.38).

4.2.2. Roe Matrices

Definition 5.28 [proof 5.45]
Roe Matrices $\mathbf{A}_{j+1/2}^n$:
Are matrices that satisfy the properties 5.1 to 5.3 and **??**

$$\mathbf{A}_{j+1/2}^n = \int_0^1 \mathrm{f}'\left(\mathbf{u}_j^n + \tau\left(\mathbf{u}_{j+1}^n - \mathbf{u}_j^n\right)\right)\mathrm{d}\tau \tag{5.40}$$

Problem
Equation (5.40) is not easy to calculate and in general not possible to calculate in general.

Proposition 5.5 [examples 5.16 and 5.17]
Roe Matrix:
We derive the row matrix by solving eq. (5.38):

$$[[\mathrm{f}]] = \mathbf{A}[[\mathbf{u}]] \iff \mathrm{f}\left(\mathbf{u}_{j+1}^n\right) - \mathrm{f}\left(\mathbf{u}_j^n\right) = \mathbf{A}_{j+1/2}^n\left(\mathbf{u}_{j+1}^n - \mathbf{u}_j^n\right)$$

using a clever change of variables depenindg on the underlying problem:

$$\mathbf{Z}: \mathbf{U} \mapsto \mathbf{Z}(U) \qquad\qquad \mathbf{Z} \in \mathcal{U} \subset \mathbb{R}^m \tag{5.41}$$

Explanation 5.7. *When writing down eq. (5.38) we often arrive at rational equations. By a cleaver change of variables we can transform those equations into polynomial equations, which are much easier to solve.*

Why/Does this automatically satisfy all the equations?

Formula 5.1 Useful Identities:

$$\bar{a} := \frac{a_l + a_r}{2} \qquad [\![a]\!] := a_r - a_l \tag{5.42}$$

$$[\![ab]\!] = \bar{b}[\![a]\!] + \bar{a}[\![b]\!] \tag{5.43}$$

$$[\![a^2]\!] = 2\bar{a}[\![a]\!] \tag{5.44}$$

$$[\![a^4]\!] = 4\bar{a^2}\bar{a}[\![a]\!] \tag{5.45}$$

4.3. Schemes

4.3.1. Roe's Scheme

Definition 5.29 [proof 5.31]
Roe Flux:

$$\begin{aligned} \mathrm{F}^n_{j+1/2} = \mathbf{A}\mathbf{U}_{j+1/2} &= \frac{\mathbf{A}\left(\mathbf{U}^n_j + \mathbf{U}^n_{j+1}\right)}{2} \\ &\quad - \frac{1}{2}\mathbf{R}^n_{j+1/2}\left|\mathbf{\Lambda}^n_{j+1/2}\right|\left(\mathbf{R}^n_{j+1/2}\right)^{-1}\left(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right) \end{aligned} \tag{5.46}$$

$$\mathbf{R}^n_{j+1/2} = \left[\mathbf{r}^{1,n}_{j+1/2} \cdots\cdots \mathbf{r}^{m,n}_{j+1/2}\right]$$

$$\left|\mathbf{\Lambda}^n_{j+1/2}\right| = \left[\left|\lambda^{1,n}_{j+1/2}\right| \cdots\cdots \left|\lambda^{m,n}_{j+1/2}\right|\right]$$

$\mathbf{r}^{p,n}_{j+1/2}, \lambda^{p,n}_{j+1/2}$-pth eienvector pair of $\mathbf{A}^n_{j+1/2}$.

Explanation 5.8.
$\mathrm{F}^n_{j+1/2}$ = *Average Flux/Central Scheme + Numerical Diffusion*

- *Central differences in space is unconditionally unstable.*
- *Diffusion term helps to stabilize the computation*
- *pth-component of Numerical Diffusion* $\propto \left|\lambda^{p,n}_{j+1/2}\right|$
- $\left|\lambda^{p,n}_{j+1/2}\right| \propto$ *average* $\left(\lambda^{p,n}_j, \lambda^{p,n}_{j+1}\right)$ *where* $\lambda^{p,n}_j$*-pth eigenvalue of* $\mathrm{f}'(\mathbf{U}^n_j)$

Definition 5.30 Roe Scheme:

$$\textit{Equation } (5.35) + \textit{Equation } (5.46) \tag{5.47}$$

Pros	Cons
• Great at approximating shocks • Approximates *Linear* systems of conservation laws exactly	• Fails at transonic rarefactions • Computationally expensive as we eigenvaluedecomposition

check if this is true?

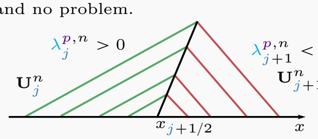
4.3.2. Harten's Entropy Fix

If p-th component of *Numerical Diffusion* in eq. (5.46) is zero that is if $\left|\lambda^{p,n}_{j+1/2}\right| \propto \text{average}\left(\lambda^{p,n}_j, \lambda^{p,n}_{j+1}\right) = 0$, then there exists nothing to stabilize, leading to instability in the p-th component.
When is the p-th component of *Numerical Diffusion* zero?
The problem arises in the p-th component if:

$$\operatorname{sign}\left(\lambda^{p,n}_j\right) \neq \operatorname{sign}\left(\lambda^{p,n}_{j+1}\right) \quad \text{and} \quad \left|\lambda^{p,n}_j\right| \approx \left|\lambda^{p,n}_{j+1}\right|$$

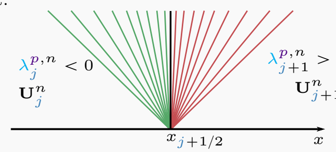
Case I: $\lambda^{p,n}_j > 0$ **and** $\lambda^{p,n}_{j+1} < 0$

By the Lax-entropy condition we obtain a shock wave. Thus information will only be taken from one side thus we have no averaging and no problem.



Case II $\lambda^{p,n}_j < 0$ **and** $\lambda^{p,n}_{j+1} > 0$

Here we cross a zero at some point. Thus the Roe scheme can fail due to averaging of positive and negative eigenvalues s.t. the diffusion becomes zero and we end up with blow up at some point.



Definition 5.31 Roe Flux with Harten's Entropy Fix:
Makes sure that the numerical flux term does not reach zero and thus avoid blow up:

$$\begin{aligned} \mathrm{F}^n_{j+1/2} = \mathbf{A}\mathbf{U}_{j+1/2} &= \frac{\mathbf{A}\left(\mathbf{U}^n_j + \mathbf{U}^n_{j+1}\right)}{2} \\ &\quad - \frac{1}{2}\mathbf{R}^n_{j+1/2}\left|\mathbf{\Lambda}^n_{j+1/2}\right|_\epsilon\left(\mathbf{R}^n_{j+1/2}\right)^{-1}\left(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right) \end{aligned} \tag{5.48}$$

$$\mathbf{R}^n_{j+1/2} = \left[\mathbf{r}^{1,n}_{j+1/2} \cdots\cdots \mathbf{r}^{m,n}_{j+1/2}\right]$$

$$\left|\mathbf{\Lambda}^n_{j+1/2}\right|_\epsilon = \left[\left|\lambda^{1,n}_{j+1/2}\right|_\epsilon \cdots\cdots \left|\lambda^{m,n}_{j+1/2}\right|_\epsilon\right]$$

$\mathbf{r}^{p,n}_{j+1/2}, \lambda^{p,n}_{j+1/2}$-pth eienvector pair of $\mathbf{A}^n_{j+1/2}$.

$$|\lambda|_\epsilon = \begin{cases} |\lambda| & \text{if } |\lambda| \geqslant \epsilon \\ \frac{\lambda^2+\epsilon^2}{2\epsilon} & \text{if } |\lambda| \leqslant \epsilon \end{cases} \qquad |\cdot|_\epsilon : \mathbb{R} \mapsto \mathbb{R} \tag{5.49}$$

Pros	Cons
• Great at approximating shocks • Approximates *Linear* systems of conservation laws exactly	• Computationally expensive as we eigenvaluedecomposition • We do not know the right size for ϵ

check if this is true?

Note
Rarely used in practive nowadays.

5. Central Schemes

Ami (H)arten–Peter (L)ax–Bram van (L)eer 1779-80

We have seen that the Roe schemeeq. (5.46) can very expensive. Another idea by is to approximate the m waves/discontinuities by only $2 \leqslant l \leqslant m$ waves/discontinuities and hope that they are enough to approximate our solution.

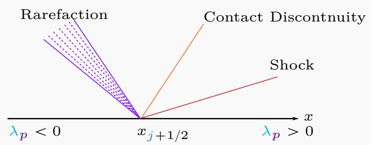


Figure 3: Example of possible waves that we might have to approximate

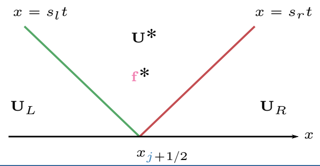
5.1. Two Wave Solver

Definition 5.32 [proof 5.18]
Central Flux:

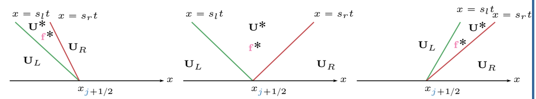
$$\mathrm{F}^n_{j+1/2} = \mathrm{F}\left(\mathbf{U}^n_j, \mathbf{U}^n_{j+1}\right) = \begin{cases} f\left(\mathbf{U}^n_j\right) & \text{if } s^{l,n}_{j+1/2} \geqslant 0 \\ f^*_{j+1/2} & \text{if } s^{l,n}_{j+1/2} < 0 < s^{r,n}_{j+1/2} \\ f\left(\mathbf{U}^n_{j+1}\right) & \text{if } s^{r,n}_{j+1/2} \leqslant 0 \end{cases}$$

$$f^*_{j+1/2} = \frac{s^r_{j+1/2} f\left(\mathbf{U}^n_j\right) - s^l_{j+1/2} f\left(\mathbf{U}^n_{j+1/2}\right) + s^r_{j+1/2} s^l_{j+1/2}\left(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right)}{s^r_{j+1/2} - s^l_{j+1/2}} \tag{5.50}$$

The left $s^{l,n}_{j+1/2}$ and right $s^{r,n}_{j+1/2}$ speeds have to be specified and depend on the scheme.



Explanation 5.9. *Depending on our wave speeds we either take the exact left* $f\left(\mathbf{U}^n_j\right)$*, right* $f\left(\mathbf{U}^n_{j+1}\right)$ *flux or the approximate intermediate flux* $f^*_{j+1/2} \approx f\left(\mathbf{U}^*\right)$ *which is derived/approximated by conservation.*



Corollary 5.6 $-s^l_{j+1/2} = s^r_{j+1/2} =: s_{j+1/2}$
Symmetric Waves:
For anti-symmetric speeds we obtain:

$$f^*_{j+1/2} = \frac{f\left(\mathbf{U}^n_j\right) - f\left(\mathbf{U}^n_{j+1/2}\right)}{2} - \frac{s_{j+1/2}}{2}\left(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right) \tag{5.51}$$

5.1.1. Lax-Friedrich's Scheme

Definition 5.33 Lax Friedrichs Scheme:
Chooses the wave speeds s.t. waves from neighboring Riemann problems do not interact with each other:

$$s^{l,n}_{j+1/2} = -\frac{\Delta x}{2\Delta t} \qquad s^{r,n}_{j+1/2} = \frac{2\Delta x}{\Delta t} \tag{5.52}$$

with eq. (5.51) it follows:

$$\begin{aligned} \mathrm{F}^n_{j+1/2} &= \mathrm{F}^{\mathrm{LxF}}\left(\mathbf{U}^n_j, \mathbf{U}^n_{j+1}\right) \\ &= \frac{f\left(\mathbf{U}^n_j\right) - f\left(\mathbf{U}^n_{j+1/2}\right)}{2} - \frac{\Delta x}{2\Delta t}\left(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right) \end{aligned} \tag{5.53}$$

Explanation 5.10. *LxF makes sure that waves do not interfere with each other, that is each wave can maximally travel a distance of* $\Delta x = \left|\frac{\Delta t}{s^l_{j+1/2}}\right|$ *i.e. to the next interface until we the next time point.*

Pros	Cons
• Easy to implement	• Does not take into account the local speeds • Is not the most accurate • Uses always an additional unnecessary grid point

5.1.2. Rusanov Scheme 1954

Definition 5.34
Rusanov/Local-Lax-Friedrichs Scheme:
Takes also into account the local speeds of the waves:

$$s^{r,n}_{j+1/2} = -s^{l,n}_{j+1/2} = \max\left(\max_p\left|\lambda^{n,p}_j\right|, \left|\lambda^{n,p}_{j+1}\right|\right) \tag{5.54}$$

$\lambda^{p,n}_j / \lambda^{p,n}_{j+1}$ is the p-th eigenvalue of $\mathrm{f}'\left(\mathbf{U}^n_j\right)/\mathrm{f}'\left(\mathbf{U}^n_{j+1}\right)$ with eq. (5.51) and $s^r_{j+1/2} = s_{j+1/2} = -s^l_{j+1/2}$ it follows:

$$\begin{aligned} \mathrm{F}^n_{j+1/2} &= \mathrm{F}^{\mathrm{Rus}}\left(\mathbf{U}^n_j, \mathbf{U}^n_{j+1}\right) \\ &= \frac{f\left(\mathbf{U}^n_j\right) - f\left(\mathbf{U}^n_{j+1/2}\right)}{2} \\ &\quad - \frac{1}{2}\max\left(\max_p\left|\lambda^{n,p}_j\right|, \left|\lambda^{n,p}_{j+1}\right|\right)\left(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right) \end{aligned} \tag{5.55}$$

Pros	Cons
• Easy to implement • Takes into account local information	• Is still a symmetric scheme i.e. problem when all waves go in one direction/are unidirectional.

5.1.3. HLL

Definition 5.35
HLL original:
Approximates the wave cone to capture everything:

$$s^{l,n}_{j+1/2} = \min\left(\lambda^{1,n}_j, \lambda^{1,n}_{j+1}\right) \qquad s^{r,n}_{j+1/2} = \max\left(\lambda^{m,n}_j, \lambda^{m,n}_{j+1}\right) \tag{5.56}$$

Pros	Cons
• Takes into account local information • No longer symmetric, thus can capture unidirectional wavs	• Is still an approximation consisting just of three waves i.e. already for three waves it will no longer model the middle wave.

5.1.4. Einfeldt

Definition 5.36
Einfeldt Scheme:
Is a more refined version of the HLL scheme:

$$s^{l,n}_{j+1/2} = \min\min_p\left(\lambda^{p,n}_j, \hat{\lambda}^{p,n}_{j+1}\right) \tag{5.57}$$

$$s^{r,n}_{j+1/2} = \max\max_p\left(\lambda^{p,n}_j, \hat{\lambda}^{p,n}_{j+1}\right) \tag{5.58}$$

$\hat{\lambda}^{p,n}_{j+1}$ is the p-th eigenvalue of the Roe-matrix $\mathbf{A}^2_{j+1/2}$ (?? 4.2.2).

Pros	Cons
• Takes into account local information • No longer symmetric, thus can capture unidirectional wavs	• Is still an approximation consisting just of three waves i.e. already for three waves it will no longer model the middle wave.

5.2. Three Wave Solver

For many problems such as the euler equation, the general solution may depend on three different types of solution waves-fig. 3 thus two wave solver may be not accurate to capture such solutions.

5.2.1. HLL-3/HLLC Solver

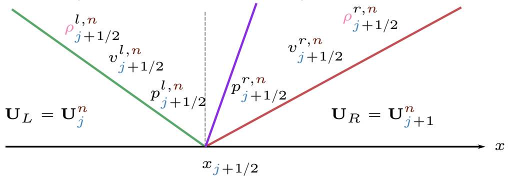
Definition 5.37 [proof 5.47]

HLL-3/HLL-C(enter):

$$F^{n}_{j+1/2} = F\left(\mathbf{U}^{n}_{j}, \mathbf{U}^{n}_{j+1}\right) = \begin{cases} F\left(\mathbf{U}^{n}_{j}\right) & \text{if} \quad 0 < s^{l,n}_{j+1/2} \\ F^{l,n}_{j+1/2} & \text{if} \quad s^{l,n}_{j+1/2} \leqslant 0 < s^{*,n}_{j+1/2} \\ F^{r,n}_{j+1/2} & \text{if} \quad s^{*,n}_{j+1/2} \leqslant 0 < s^{r,n}_{j+1/2} \\ F\left(\mathbf{U}^{n}_{j}\right) & \text{otherwise} \end{cases} \tag{5.59}$$

$$F^{\theta,n}_{j+1/2} = F\left(\mathbf{U}^{n}_{j+k}\right) + s^{\theta,n}_{j+1/2}\left(\mathbf{U}^{\theta,n}_{j+1/2} - \mathbf{U}^{n}_{j+k}\right) \qquad k \in \{0, 1\}$$

$s_l = s^{l,n}_{j+1/2}$ $\quad s_m = s^{m,n}_{j+1/2} = v^{*}$ $\quad s_r = s^{r,n}_{j+1/2}$



$x(t) = s^{l,n}_{j+1/2} \cdot t$ $\qquad x(t) = s^{m,n}_{j+1/2} \cdot t$ $\qquad x(t) = s^{r,n}_{j+1/2} \cdot t$

$$v^{*,n}_{j+1/2} = s^{m,n}_{j+1/2}$$

$$\frac{\rho^{j}_{j+1} v^{n}_{j+1}\left(s^{r,n}_{j+1/2} - v^{n}_{j+1}\right) - \rho^{n}_{j} v^{n}_{j}\left(s^{l,n}_{j+1/2} - v^{n}_{j}\right) - \left(p^{j}_{j} - p^{n}_{j}\right)}{\rho^{n}_{j+1}\left(s^{r,n}_{j+1/2} - v^{n}_{j+1}\right)\rho^{n}_{j}\left(s^{l,n}_{j+1/2} - v^{n}_{j}\right)}$$

$$\rho^{l,n}_{j+1/2} = \frac{\rho^{n}_{j}(v^{n}_{j} - s^{l,n}_{j+1/2})}{(v^{*}_{j+1/2} - s^{l,n}_{j+1/2})} \quad \rho^{r,n}_{j+1/2} = \frac{\rho^{n}_{j+1}(v^{n}_{j+1} - s^{r,n}_{j+1/2})}{(v^{*}_{j+1/2} - s^{r,n}_{j+1/2})}$$

$$p^{*,n}_{j+1/2} = p^{n}_{j+k} + \rho^{n}_{j+k}\left(v^{n}_{j+k} - v^{*,n}_{j+1/2}\right)\left(v^{n}_{j+k} - s^{\alpha,n}_{j+1/2}\right)$$

Note

The third component of the RH condition will in general not be satisfied and we define the flux over either of the intermediate components $F^{\theta,n}_{j+1/2} \approx F\left(\mathbf{U}^{\theta,n}_{j+1/2}\right)$

watch last 20 min of lecture and add Runge Kutta method

6. Proofs

Proof 5.1 Conservative Form Burgers Equation[cor. 1.1]:

$$\frac{\partial}{\partial x}\frac{1}{2}u(\mathbf{x},t)^2 = \frac{2}{2}u(\mathbf{x},t)_{\mathbf{x}}u(\mathbf{x},t)$$

Proof 5.2
Evolution of Spatial Gradients along Characteristics:

$$u_t + uu_x = 0$$
$$u(x,0) = u_0(x)$$

Consider the problem for solving for the spatial gradients $v := u_x$:

$$\frac{\partial}{\partial x}(\cdot) \Rightarrow \qquad (u_x)_t + u(u_x)_x + u_x \cdot u_x = 0$$

$$\boxed{\begin{aligned} v_t + uv_x &= \underline{-v^2}\\ v(x,0) &= v_0(0) = u_0'(x)\end{aligned}} \tag{5.60}$$

ODEs u $\quad \frac{dx}{dt} = \underline{u}(x(t),t)$

$$\frac{dv(x(t),t)}{dt} \overset{\text{C.R.}}{=} = v_t + v_x\frac{dx}{dt} = \underline{v_x + v_t u} = -v^2$$

ODEs v $\quad \left.\begin{array}{l}\frac{dv}{dt} = -v^2\\ v(0) = v_0\end{array}\right\} \quad v(t) = \frac{v_0(x)}{1+v_0(x)t} = \frac{u_0'(x)}{1+u_0'(x)t}$

add integration of ODE

$$\text{If } \begin{cases}u_0'(t) > 0\\ u_0'(t) < 0\end{cases} \Rightarrow \begin{cases} v(t) & \text{well behaved}\\ v(t)\to\infty & \text{as } t \to -\frac{1}{u_0'(x)}\end{cases}$$

Thus soon as we have a negative gradient for the initial data we will run into blow up at some time.

Proof 5.3 Weak Solution??: We first multiply by a test function $\phi \in \mathcal{C}_0^1(\mathbb{R}\times\mathbb{R}_+)$ and integrate over space and time:

$$\overbrace{\int_{-\infty}^{\infty}\underbrace{\int_0^\infty u_t\phi\,dt}_{I_{1a}}dx}^{I_1} + \overbrace{\int_0^\infty\underbrace{\int_{-\infty}^\infty f(u)_x\phi\,dx}_{I_{2a}}dt}^{I_2} = 0$$

$$I_{1a}: \quad \int_0^{+\infty} u_t\phi \overset{??}{=} u(x,\infty)\underbrace{\phi(x,\infty)}_{\equiv 0} - \underbrace{u(x,0)}_{u_0(x)}\phi(x,0) \qquad \partial\Omega$$
$$- \int_0^\infty u\phi_t\,dt$$

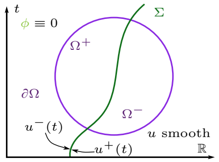
$$I_1 = -\int_{-\infty}^\infty\int_0^{+\infty} u\phi_t\,dt - \int_{-\infty}^\infty u_0(x)\phi(x,0)\,dx$$

$$I_{2a}: \quad \int_{-\infty}^{+\infty} f(u)_x\phi\,dx \overset{??}{=} f(u(\infty,t))\underbrace{\phi(\infty,t)}_{\equiv 0}$$
$$- f(u(-\infty,t))\underbrace{\phi(-\infty,t)}_{\equiv 0} - \int_{-\infty}^\infty f(u)\phi_x\,dx$$

$$I_2 = \int_0^{+\infty}\int_{-\infty}^\infty f(u)\phi_x\,dx\,dt$$

$$\int_{-\infty}^{\infty}\int_0^{\infty}(u\phi_t + f(u)\phi_x)\,dx\,dt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)\,dx = 0$$

Proof 5.4 Rankine-Hugoniot Condition[def. 2.4]: Lets consider a shock-wave[def. 2.1]/discontinuity given by a curve:



$$\Sigma = \{(x,t)\in(\mathbb{R}\times\mathbb{R}_+) : x = \sigma(t)\}$$
$$\Sigma = (\sigma(t),t)\ \forall t$$
$$\text{s.t.} \quad u^\pm(t) := \lim_{h\to 0} u(\sigma(t) \pm ht)$$
$$u^+(t) \neq u^-(t)$$

Now we choose a test function $\phi \in \mathcal{C}_C^1(\Omega)$ and $\sup(\phi)\subset\Omega$. We know that u is a *weak solution* of $\Omega \subseteq \mathbb{R}\times\mathbb{R}_+$:

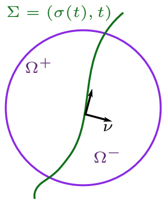
$$\int_\Omega(u\phi_t + f(u)\phi_x)\,dx\,dt + \int_{\mathbb{R}} u_0(x)\underbrace{\phi(x,0)}_{\equiv 0}dx = 0$$

$$\int_\Omega(u\phi_t + f(u)\phi_x)\,dx\,dt = 0$$

$$\underbrace{\int_{\Omega_-}(u\phi_t + f(u)\phi_x)\,dx\,dt}_{I_1} + \underbrace{\int_{\Omega_+}(u\phi_t + f(u)\phi_x)\,dx\,dt}_{I_2} = 0$$

using I.B.P. and the fact that $\phi\equiv 0$ on $\partial\Omega$ we obtain:

$$I_1 = \int_{\Omega_-}\operatorname{grad}\phi\begin{bmatrix}f(u)\\ u\end{bmatrix}d\Omega$$
$$\overset{??}{=} -\int_{\Omega_-}\operatorname{div}_{x,t}\begin{bmatrix}f(u)\\ u\end{bmatrix}\phi\,d\Omega + \int_{\partial\Omega_-}\begin{bmatrix}f(u^+)\\ u^+\end{bmatrix}\boldsymbol{\nu}\phi\,d\Sigma$$
$$= -\int_{\Omega_-}(u_t + f(u)_x)\phi\,dx\,dt$$
$$+ \int_\Sigma\left(u^+(t)\phi\nu_t^+ + f\left(u^+(t)\right)\phi\nu_x^+\right)\phi\,d\Sigma$$



Where the line measure of the "inner boundary" is given by σ and the unit normal of the line is given by:

$$\text{Tangent} = \begin{pmatrix}\sigma'(t)\\ 1\end{pmatrix}$$

$$\nu = \begin{pmatrix}\nu_x\\ \nu t\end{pmatrix} = \begin{pmatrix}\frac{-1}{\sqrt{1+\sigma'(t)}}\\ \frac{\sigma'(t)}{\sqrt{1+\sigma'(t)}}\end{pmatrix}$$

$-\nu$ is the unit normal vector of Ω^+ s.t. it follows:

$$I_1 + I_2 = -\int_{\Omega_-\cup\Omega_+}\overbrace{(u_t + f(u)_x)}^{=0}\phi\,dx\,dt$$
$$+ \int_\Sigma\Big[(u^+(t) - u^+(t))\nu_t$$
$$+ (f(u^+(t)) - f(u^-(t)))\nu_x\Big]\phi(\sigma(t),t)\,d\Sigma \qquad \forall\phi$$
$$\Rightarrow \quad \left(u^+(t) - u^+(t)\right)\nu_t + f\left(u^+(t)\right) - f\left(u^-(t)\right)\nu_x = 0$$
$$\frac{\sigma'(t)}{1+\sigma'(t)}\left(u^+(t) - u^+(t)\right)$$
$$- \frac{1}{1+\sigma'(t)}f\left(u^+(t)\right) - f\left(u^-(t)\right) = 0$$
$$f\left(u^+(t)\right) - f\left(u^-(t)\right) = \sigma'(t)\left(u^+(t) - u^+(t)\right)$$

Proof 5.5 Rarefaction Waves: The solution of the conservation laweq. (1.2) is invariant to the scaling of the input parameters:

$$u(x,t) \text{ solves eq. (1.2)}$$
$$\Longrightarrow \quad w(x,t) := u(\lambda x, \lambda t) \text{ solves eq. (1.2)} \qquad \lambda \neq 0$$

thus it is natural to assume self-similarity – i.e. a solution $v(\xi)$ that only depends on the ration $\xi := x/t$:

$$u(x,t) = v\left(\frac{x}{t}\right) = v\xi$$

$$\xi_t = \frac{-x}{t^2} \qquad \xi_x = \frac{1}{t}$$
$$u_t = V'(\xi)\xi_t = V'(\xi)\frac{-x}{t^2} \qquad u_x = V'(\xi)\xi_x = V'(\xi)\frac{1}{t}$$
$$f(u)_x = f'(u)u_x = f'(v(\xi))\,v'(\xi)\xi_x = \frac{1}{t}f'(v(\xi))\,v'(\xi)$$

Plug it into eq. (1.2): $\quad 0 = \underline{u_t} + \underline{f(u)_x} = \underline{u_t} + \underline{f'(u)u_x}$

$$0 = \underline{V'(\xi)\frac{-x}{t^2}} + \underline{\frac{1}{t}f'(V(\xi))\,V'(\xi)}$$
$$= \underline{V'(\xi)\frac{-\xi}{t}} + \underline{\frac{1}{t}f'(V(\xi))\,V'(\xi)} \qquad \Big|\cdot t$$
$$\Longrightarrow \quad \left(f'(V(\xi)) - \xi\right)V' = 0$$

Thus either $V' = 0$ or in the non-trivial case it follows that for convex f (invertible):

$$\left(f'(V(\xi)) - \xi\right)V' = 0 \qquad \Big|/V'$$
$$f'(V(\xi)) = \xi$$
$$V\left(\frac{x}{t}\right) = \left(f'\right)^{-1}\left(\frac{x}{t}\right) \tag{5.61}$$

From this it follows that:

$$u(x,t) := \left(f'\right)^{-1}\left(\frac{x}{t}\right) \quad \text{is a smooth solution of eq. (1.2)}$$

Proof 5.6 Entropy Condition: We first multiply eq. (2.12) by $s'(u^\epsilon)$:

$$S'(u^\epsilon)u_t^\epsilon + S'(u^\epsilon)f'(u^\epsilon)u_x^\epsilon = \epsilon S'(u^\epsilon)u_{xx}^\epsilon$$
$$\Rightarrow \quad \partial_t S(u^\epsilon) + q'(u^\epsilon)u_x^\epsilon = \underline{\epsilon S'(u^\epsilon)u_{xx}^\epsilon}$$

with $S(u)_{xx} = \left(S'(u)u_x\right)_x \overset{P.R.}{=} S''(u)u_x^2 + \underline{S'(u)u_{xx}}$

$$\Rightarrow \quad \partial_t S(u^\epsilon) + \partial_x q(u^\epsilon) = \epsilon S(u^\epsilon)_{xx} - \epsilon\underbrace{S''(u^\epsilon)}_{\geqslant 0 \text{ convex}}\underbrace{(u_x^\epsilon)^2}_{\geqslant 0}$$

$$\Rightarrow \quad \boxed{\partial_t S(u^\epsilon) + \partial_x q(u^\epsilon) \leqslant \epsilon S(u^\epsilon)_{xx}} \tag{5.62}$$

thus the vanishing viscosity solution $u = \lim_{\epsilon\to0}u^\epsilon$ satisfieseq. (2.14).

Proof 5.7 2nd law of thermodynamicslaw 2.1:
Integrate eq. (5.62) in space:

$$\int_{\mathbb{R}}\partial_t S(u^\epsilon)\,dx + \int_{\mathbb{R}}\partial_x q(u^\epsilon)\,dx \leqslant \epsilon\int_{\mathbb{R}} S(u^\epsilon)_{xx}\,dx$$
$$\partial_t\int_{\mathbb{R}} S(u^\epsilon)\,dx + \underbrace{\left[q(u^\epsilon(\infty,t)) - q(u^\epsilon(-\infty,t))\right]}_{0}$$
$$\leqslant \epsilon\underbrace{\left[s(u^\epsilon(\infty,t))_x - s(u^\epsilon(-\infty,t))_x\right]}_{0}$$

Note
$u^\epsilon(\infty,t) = u^\epsilon(-\infty,t)$ for periodic B.C. or zero otherwise.

Proof 5.8 Entropy Condition for Distributions[cor. 2.4]: Let $\phi\in\mathcal{C}_C^1(\mathbb{R}\times\mathbb{R}_+)$, $\phi \geqslant 0$. Integrate eq. (5.62) and multiply it by ϕ:

$$\int_{\mathbb{R}\times\mathbb{R}_+} S(u^\epsilon)_t\phi + q(u^\epsilon)_x\phi\,dx\,dt \leqslant \epsilon\underbrace{\int_{\mathbb{R}_+}\int_{\mathbb{R}} S(u^\epsilon)_{xx}\phi\,dx\,dt}_{I_c}$$

$$I_c \overset{??}{=} \underbrace{\phi(x,t)s(u^\epsilon)_x\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty}\phi_x(x,t)s(u^\epsilon)_x\,dx$$
$$\overset{??}{=} \underbrace{\phi(x,t)_x s(u^\epsilon)\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty}\phi_{xx}(x,t)s(u^\epsilon)\,dx$$

$$\overbrace{\int_{-\infty}^{\infty}\underbrace{\int_0^\infty S(u^\epsilon)_t\phi\,dt}_{I_{1a}}dx}^{I_1} + \overbrace{\int_0^\infty\underbrace{\int_{-\infty}^\infty q(u^\epsilon)_x\phi\,dx}_{I_{2a}}dt}^{I_2} = 0$$

$$I_{1a}: \quad \int_0^{+\infty} S(u^\epsilon)_t\phi \overset{??}{=} S(u^\epsilon(x,\infty))\underbrace{\phi(x,\infty)}_{\equiv 0}$$
$$- \underbrace{S(u^\epsilon(x,0))}_{S(u_0(x))}\phi(x,0) - \int_0^\infty S(u^\epsilon)\phi_t\,dt$$

$$I_1 = -\int_{\mathbb{R}\times\mathbb{R}_+} S(u^\epsilon)\phi_t\,dt - \int_{-\infty}^\infty S(u_0(x))\phi(x,0)\,dx$$

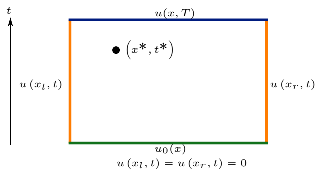
$$I_{2a}: \quad \int_{-\infty}^{+\infty} q(u^\epsilon)_x\phi\,dx \overset{??}{=} q(u^\epsilon(\infty,t))\underbrace{\phi(\infty,t)}_{\equiv 0}$$
$$- q(u^\epsilon(-\infty,t))\underbrace{\phi(-\infty,t)}_{\equiv 0} - \int_{-\infty}^\infty q(u^\epsilon)\phi_x\,dx$$

$$I_2 = -\int_0^{+\infty}\int_{-\infty}^\infty q(u^\epsilon)\phi_x\,dx\,dt$$

$$\Longrightarrow \quad \lim_{\epsilon\to0}\int_{\mathbb{R}\times\mathbb{R}_+}\left(S(u^\epsilon)\phi_t + q(u^\epsilon)\phi_x\right)dx\,dt$$
$$+ \int_{-\infty}^{\infty} S(u_0(x))\phi(x,0)\,dx$$
$$\geqslant \epsilon\underbrace{\int_{\mathbb{R}\times\mathbb{R}_+}\phi_{xx}(x,t)s(u^\epsilon)\,dx\,dt}_{0}$$

Proof 5.9 Maximum Principleprinciple 2.1:

1. Assume eq. (1.3) attains a strict maximum at its interior $\left(x^*, t^*\right)$
$$u_t^\epsilon\left(x^*, t^*\right) \equiv 0 \quad u_x^\epsilon\left(x^*, t^*\right) \equiv 0 \quad u_{xx}^\epsilon\left(x^*, t^*\right) < 0$$
Now define the sum of all the termseq. (2.12), which are supposed to equal zero if u solves this equation:
$$R\left(x^*, t^*\right) := \underbrace{u_t^\epsilon\left(x^*, t^*\right)}_{=0} + f'\left(u^\epsilon\left(x^*, t^*\right)\right)\underbrace{u_x^\epsilon\left(x^*, t^*\right)}_{=0} - \underbrace{\epsilon u_{xx}^\epsilon\left(x^*, t^*\right)}_{<0}$$
But $R\left(x^*, t^*\right) < 0$ and not 0 – a contradiction, thus the maximums cannot be inside the interior.
2. Now assume u attains a strict maximum at $\left(x^*, T\right)$ at the time horizon boundary:
$$u_x^\epsilon\left(x^*, t^*\right) \equiv 0 \qquad u_{xx}^\epsilon\left(x^*, t^*\right) < 0$$
for the time derivative we can define the backward in time derivative:
$$u_t^\epsilon(x, T) = \lim_{h \to 0} \frac{u_t^\epsilon(x, T) - u_t^\epsilon(x, T-h)}{h} > 0$$
Thus $R(x, T) > 0$ again a contradiction. **Note**: as $u_t^\epsilon(x, T-h)$ is inside the interior and we already know that the interior has no maximum.



Proof 5.10 Total Variation Diminishingtheorem 2.2: Lets $v^\epsilon = u_x^\epsilon$ and differentiate eq. (2.12):
$$u_{tx}^\epsilon + \left(f'\left(u^\epsilon\right)u_x^\epsilon\right)_x = \epsilon u_{xxx}^\epsilon$$
$$u_{xt}^\epsilon \overset{??}{=} -f''\left(u^\epsilon\right)\left(u_x^\epsilon\right)^2 - f'\left(u^\epsilon\right)u_{xx}^\epsilon + \epsilon u_{xxx}^\epsilon$$
$$v_t^\epsilon = -f''\left(u^\epsilon\right)\left(v^\epsilon\right)^2 - f'\left(u^\epsilon\right)v_x^\epsilon + \epsilon v_{xx}^\epsilon \tag{5.63}$$
Now we define the test function:
$$\phi(v) = \eta(v) = |v| \qquad \eta'(v) = \operatorname{sign}(v) \qquad \eta''(v) = 2\delta_{\{v=0\}}$$
and multiply eq. (5.63) by $n'\left(v^\epsilon\right)$
$$\begin{aligned}\eta'\left(v^\epsilon\right)v_t^\epsilon &= -f'\left(u^\epsilon\right)\eta'\left(v^\epsilon\right)v_x^\epsilon - f''\left(u^\epsilon\right)\eta'\left(v^\epsilon\right)\left(v^\epsilon\right)^2 \\ &\quad + \epsilon\eta'\left(v^\epsilon\right)v_{xx}^\epsilon \\ \partial_t\left(v^\epsilon\right) &= -f'\left(u^\epsilon\right)\partial_x\left(v^\epsilon\right) - f''\left(u^\epsilon\right)\eta'\left(v^\epsilon\right)\left(v^\epsilon\right)^2 \\ &\quad + \epsilon\eta'\left(v^\epsilon\right)v_{xx}^\epsilon\end{aligned}$$
$$\int_{\mathbb{R}}\partial_t\left(v^\epsilon\right) = -\overbrace{\int_{\mathbb{R}} f'\left(u^\epsilon\right)\partial_x\left(v^\epsilon\right)\mathrm{d}x}^{I)} - \int_{\mathbb{R}} f''\left(u^\epsilon\right)\eta'\left(v^\epsilon\right)\left(v^\epsilon\right)^2\mathrm{d}x + \underbrace{\epsilon\int_{\mathbb{R}}\eta'\left(v^\epsilon\right)v_{xx}^\epsilon\,\mathrm{d}x}_{II)}$$
$$\begin{aligned}I) &\overset{??}{=} f'\left(u^\epsilon\right)\eta\left(v^\epsilon\right)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}}\left(f'\left(u^\epsilon\right)\right)_x\eta\left(u^\epsilon\right)\mathrm{d}x \\ &= \overbrace{f'\left(u^\epsilon\right)\left|u^\epsilon\right|}^{u_x(\partial\Omega, t)=0 \Rightarrow =0}\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} f''\left(u^\epsilon\right)u_x^\epsilon\eta\left(u^\epsilon\right)\mathrm{d}x \\ &= -\int_{\mathbb{R}} f''\left(u^\epsilon\right)v^\epsilon\eta\left(u^\epsilon\right)\mathrm{d}x\end{aligned}$$
$$\begin{aligned}II) &\overset{??}{=} \overbrace{\eta'\left(v^\epsilon\right)v_x^\epsilon\Big|_{-\infty}^{\infty}}^{=0} - \int_{\mathbb{R}}\left(\eta'\left(v^\epsilon\right)\right)_x v_x^\epsilon\,\mathrm{d}x \\ &= -\int_{\mathbb{R}}\eta''\left(v^\epsilon\right)\left(v_x^\epsilon\right)^2\mathrm{d}x\end{aligned}$$
$$\begin{aligned}\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\eta\left(v^\epsilon\right)\mathrm{d}x &= +\int_{\mathbb{R}} f''\left(u^\epsilon\right)v^\epsilon\eta\left(u^\epsilon\right)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}} f''\left(u^\epsilon\right)\eta'\left(v^\epsilon\right)\left(v^\epsilon\right)^2\mathrm{d}x \\ &\quad - \epsilon\int_{\mathbb{R}}\eta''\left(v^\epsilon\right)\left(v_x^\epsilon\right)^2\mathrm{d}x \\ &= +\int_{\mathbb{R}}\underbrace{\left[v^\epsilon\eta\left(u^\epsilon\right) - \eta'\left(v^\epsilon\right)\left(v^\epsilon\right)^2\right]}_{=0} f''\left(u^\epsilon\right)\mathrm{d}x \\ &\quad \underbrace{-2\epsilon\int_{x:v^\epsilon=0}\left(v_x^\epsilon\right)^2\mathrm{d}x}_{\leqslant 0}\end{aligned}$$
Thus it follows that:
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\eta\left(v^\epsilon\right)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left|u_x^\epsilon\right|\mathrm{d}x \leqslant 0$$

Proof 5.11 TVD in time[cor. 2.6]: From eq. (1.3) we have:
$$u_t^\epsilon = -f'\left(u^\epsilon\right)u_x^\epsilon$$
$$\Longrightarrow \quad \left|u_t^\epsilon\right| \overset{??}{\leqslant} \left|f'\left(u^\epsilon\right)\right|\left|u_x^\epsilon\right| \overset{\text{eq. (2.21)}\ f \text{ convex}}{=} C\left|u_x^\epsilon\right|$$
$$\Longrightarrow \quad \int_{\mathbb{R}}\left|u_t^\epsilon(\cdot, t)\right|\mathrm{d}x \leqslant C\int_{\mathbb{R}}\left|u_x^\epsilon(\cdot, t)\right|\mathrm{d}x$$

Proof 5.12 Finite Volume Methods??:
$$\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} \text{eq. (1.2)}$$
$$\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{t^n}^{t^{n+1}} u_t\,\mathrm{d}t\,\mathrm{d}x + \int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} f_x(u)\,\mathrm{d}x\,\mathrm{d}t = 0$$
$$\overset{??}{\Longleftrightarrow} \int_{x_{j-1/2}}^{x_{j+1/2}} U\left(x, t^{n+1}\right)\mathrm{d}x - \int_{x_{j-1/2}}^{x_{j+1/2}} U\left(x, t^n\right)\mathrm{d}x$$
$$= \int_{t^n}^{t^{n+1}} f\left(U\left(x_{j+1/2}, t\right)\right)\mathrm{d}t - \int_{t^n}^{t^{n+1}} f\left(U\left(x_{j-1/2}, t\right)\right)\mathrm{d}t$$
the result follow immediately from the definitions[def's. 3.3, 3.4]

Proof 5.13 FVM Incremental Form[cor. 3.2]:
$$\begin{aligned}&\textit{Equation}\ (3.10) + F\left(u_j, u_j\right) - F\left(u_j, u_j\right) \\ &= U_j^n + \frac{\Delta t}{\Delta x}\left(F\left(u_j, u_j\right) - F_{j+1/2}^n\right) - \frac{\Delta t}{\Delta x}\left(F\left(u_j, u_j\right) - F_{j-1/2}^n\right) \\ &= U_j^n + \frac{\Delta t}{\Delta x}\left(F\left(u_j, u_j\right) - F_{j+1/2}^n\right)\frac{u_{j+1} - u_j}{u_{j+1} - u_j} \\ &\quad - \frac{\Delta t}{\Delta x}\left(F\left(u_j, u_j\right) - F_{j-1/2}^n\right)\frac{u_j, u_{j-1}}{u_j, u_{j-1}}\end{aligned}$$

Proof 5.14 Monotonicity Preserving Schemes[cor. 3.5]: Assume $u_j^n \leqslant v_j^n, \forall j$ and H is monotone:
$$\begin{aligned}u_j^{n+1} &= H\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right) \\ u_j^{n+1} &\overset{eq.\ (3.21)}{\leqslant} H\left(v_{j-1}^n, u_j^n, u_{j+1}^n\right) \\ u_j^{n+1} &\overset{eq.\ (3.21)}{\leqslant} H\left(v_{j-1}^n, v_j^n, u_{j+1}^n\right) \\ u_j^{n+1} &\overset{eq.\ (3.21)}{\leqslant} H\left(v_{j-1}^n, v_j^n, v_{j+1}^n\right) \\ \Longrightarrow \quad u_j^{n+1} &\leqslant v_j^{n+1}\end{aligned}$$

Proof 5.15 Monotone FVS:
$$H(x, y, z) = y - \frac{\Delta t}{\Delta x}\left(F(x, z) - F(x, y)\right)$$
$$\frac{\partial H}{\partial x} = \frac{\Delta t}{\Delta x}\frac{\partial F}{\partial a}(x, y) \overset{\text{mon. non-dec.}}{\geqslant} 0$$
$$\frac{\partial H}{\partial z} = -\frac{\Delta t}{\Delta x}\frac{\partial F}{\partial b}(y, z) \overset{\text{mon. non-inc.}}{\geqslant} 0$$
$$\frac{\partial H}{\partial y} = 1 - \frac{\Delta t}{\Delta x}\frac{\partial F}{\partial a} - \frac{\Delta t}{\Delta x}\frac{\partial F}{\partial b} \overset{!}{\geqslant} 0$$
Thus in order for the last equation to hold we need to enforceeq. (3.23).

Proof 5.16 Harten's Lemma: From eq. (3.71) we can define U_{j+1}
$$u_{j+1}^{n+1} = u_{j+1}^n + C_{j+3/2}^n\left(u_{j+2}^n - u_{j+1}^n\right) - D_{j+1/2}^n\left(u_{j+1}^n - u_j^n\right)$$
From this and eq. (3.71) it follows $u_{j+1}^{n+1} - u_{j+1}^n$:
$$\begin{aligned}u_{j+1}^{n+1} - u_{j+1}^n &= \left(1 - C_{j+1/2}^n - D_{j+1/2}^n\right)\left(u_{j+1}^n - u_j^n\right) \\ &\quad + C_{j+3/2}^n\left(u_{j+2}^n - u_{j+1}^n\right) + D_{j-1/2}^n\left(u_j^n - u_{j-1}^n\right)\end{aligned}$$
Assuming:
$$C_{j+1/2}^n, D_{j+1/2}^n \geqslant 0 \qquad C_{j+1/2}^n + D_{j+1/2}^n \leqslant 1 \qquad \forall j$$
with this and ?? it follows:
$$\begin{aligned}\left|u_{j+1}^{n+1} - u_j^n\right| &\leqslant \overbrace{\left(1 - C_{j+1/2}^n - D_{j+1/2}^n\right)}^{0\leqslant}\underline{\left|u_{j+1}^n - u_j^n\right|} \\ &\quad + C_{j+3/2}^n\left|u_{j+2}^n - u_{j+1}^n\right| + D_{j-1/2}^n\left|u_j^n - u_{j-1}^n\right|\end{aligned}$$
we can analogously define from this:
$$\begin{aligned}\left|u_{j+2}^{n+1} - u_j^{n+1}\right| &\leqslant \left(1 - C_{j+3/2}^n - D_{j+3/2}^n\right)\left|u_{j+2}^n - u_{j+1}^n\right| \\ &\quad + C_{j+5/2}^n\left|u_{j+3}^n - u_{j+2}^n\right| + D_{j+1/2}^n\underline{\left|u_{j+1}^n - u_j^n\right|} \\ \left|u_j^n - u_{j-1}^n\right| &\leqslant \left(1 - C_{j-1/2}^n - D_{j-1/2}^n\right)\left|u_j^n - u_{j-1}^n\right| \\ &\quad + C_{j+1/2}^n\underline{\left|u_{j+1}^n - u_j^n\right|} + D_{j-3/2}^n\left|u_{j-1}^n - u_{j-2}^n\right|\end{aligned}$$
summing this three, solving for $\underline{\left|u_{j+1}^n - u_j^n\right|}$ leads to
$$\sum\left|u_{j+1}^{n+1} - u_j^{n+1}\right| \leqslant \sum\left|u_{j+1}^n - u_j^n\right|$$

Proof 5.17 Godunov Scheme??:
$$\bar{F}_{j+\frac{1}{2}}^{n,\pm} := \int_{t_n}^{t_{n+1}} f\left(u\left(x_{j+\frac{1}{2}}^{\pm}, t\right)\right)\mathrm{d}t$$
Either u

- is continuous at the boundary:
$$u\left(x_{j+\frac{1}{2}}^{+}, t\right) = u\left(x_{j+\frac{1}{2}}^{-}, t\right)$$
- it is discontinuous at the boundary but smoothed out and we are fine
- is a stationary shock at the boundaries $x_{j+1/2}$ i.e. $s(t) = 0$ thus from eq. (2.3) it follows:
$$f\left(u\left(x_{j+\frac{1}{2}}^{-}, t\right)\right) = f\left(u\left(x_{j+\frac{1}{2}}^{+}, t\right)\right)$$

thus it follow that the fluxes over the boundaries are conserved/continuous:
$$\bar{F}_{j+\frac{1}{2}}^{n,+} = \int_{t_n}^{t_{n+1}} f\left(u\left(x_{j+\frac{1}{2}}^{+}, t\right)\right)\mathrm{d}t \tag{5.64}$$
$$= \int_{t_n}^{t_{n+1}} f\left(u\left(x_{j+\frac{1}{2}}^{-}, t\right)\right)\mathrm{d}t = \bar{F}_{j+\frac{1}{2}}^{n,-} = F_{j+\frac{1}{2}} \tag{5.65}$$
Furthermore we assume a *self-similar* solution and want to have the Riemann problem at zero thus we subtract the offset $x_{j+1/2}, t^n$:
$$U_j(x, t) = U_j\left(\frac{x - x_{j+1/2}}{t - t^n}\right) \tag{5.66}$$
Next we are only interested in the flux at the boundary $x_{j+1/2}$ s.t. we obtain:
$$\begin{aligned}F_{j+\frac{1}{2}} &= \int_{t_n}^{t_{n+1}} f\left(u\left(x_{j+\frac{1}{2}}, t\right)\right)\mathrm{d}t \\ &= \int_{t_n}^{t_{n+1}} f\left(U\left(\frac{x_{j+1/2} - x_{j+1/2}}{t - t^n}\right)\right)\mathrm{d}t = \Delta t f\left(U(0)\right)\end{aligned}$$
where U is the solution of the standard Riemann problem:
$$u_t + f(u)_x = 0 \tag{5.67}$$
$$u(x, 0) = \begin{cases} U_j^n & \text{if } x < 0 \\ U_{j+1}^n & \text{if } x > 0\end{cases} \tag{5.68}$$

Proof 5.18 Central Scheme[def. 3.22]:

$$u(x,t)=\begin{cases}u_j^n & \text{if } x<s_{j+1/2}t\\ u_{j+1/2}^n & \text{if } s_{j+1/2}^l t<x<s_{j+1/2}^r t\\ u_{j+1}^n & x>s_{j+1/2}^r t\end{cases}$$

By local conservation using the RH-conditioneq. (2.3) we can determine the middle state:

$$f\left(u_{j+1}^n\right)-f_{j+1/2}^*=s_{j+1/2}^r\left(u_{j+1}^n-u_{j+1/2}^*\right) \tag{5.69}$$

$$f\left(u_j^n\right)-f_{j+1/2}^*=s_{j+1/2}^l\left(u_j^n-u_{j+1/2}^*\right) \tag{5.70}$$

$$eq.\ (5.70)*s_{j+1/2}^l+eq.\ (5.70)/s_{j+1/2}^r$$

$$\Rightarrow s_{j+1/2}^l s_{j+1/2}^R\left(u_{j+1}^n-u_j^n\right)$$
$$=s_{j+1/2}^l f\left(u_{j+1}^n\right)+s_{j+1/2}^R f\left(u_j^n\right)+\left(s_{j+1/2}^l-s_{j+1/2}^R\right)f_{j+1/2}^*$$

Proof 5.19

Based on *Cauchy-Kovalevskaya Procedure*[def. 3.28]: Given

$$u_t+f(u)_x=0$$
$$u(x,0)=u_0$$

Idea: replace temporal derivatives with spatial derivatives:

$$\underline{u_t}=-f(u)_x$$

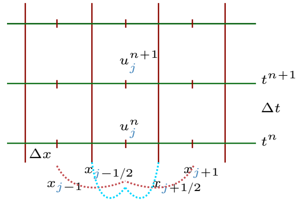
$$\underline{u_{tt}}=-f(u)_{xt}\overset{\text{C.R}}{=}-\left(f'(u)\underline{u_t}\right)_x=\left(f'(u)f(u)_x\right)$$

$\Rightarrow$ finite difference scheme $u_j^n\approx u\left(x_j,t^n\right)$ but we want to find u_j^{n+1}

Idea: use 2^{nd}-order Taylor expansion:

$$u_j^{n+1}\approx u\left(x_j,t^{n+1}\right)=u\left(x_j,t^n+\Delta t\right)$$
$$=u\left(x_j,t^n\right)+\Delta t\underline{u_t}\left(x_j,t^n\right)+\frac{\Delta t^2}{2}\underline{u_{tt}}\left(x_j,t^n\right)$$
$$+\mathcal{O}\left(\Delta t^3\right)$$

This terms can now be approximated using central differences:



$$u_j^{n+1}\approx=u_j^n-\Delta t\underline{f(u)_x\left(x_j,t^n\right)}$$
$$+\frac{\Delta t^2}{2}\underline{\left(f'(u)f(u)_x\right)_x\left(x_j,t^n\right)}$$

$$\underline{f(u)_x\left(x_j,t^n\right)}\approx\frac{f\left(u_{j+1}^n\right)-f\left(u_{j-1}^n\right)}{2\Delta x}$$

$$\underline{\left(f'(u)f(u)_x\right)_x\left(x_j,t^n\right)}\approx$$
$$\approx\frac{f'(u)f(u)_x\left(x_{j+1/2}\right)-f'(u)f(u)_x\left(x_{j+1/2}\right)}{\Delta x}$$

$$\underline{f'(u)}\left(x_{j+1/2}\right)=a_{j+1/2}^n=f'\left(\frac{u_j^n+u_{j+1}^n}{2}\right)$$

$$\underline{f(u)}\left(x_{j+1/2}\right)\approx\frac{f\left(u_{j+1}^n\right)-f\left(u_j^n\right)}{\Delta x}$$

Proof 5.20 Sum of integrals:

todo

Proof 5.21 Conservation and reconstruction: We calculate the flux at the interfaces x_j thus we need to recover the true value:

$$p_j^n(x_j)=u_j^n \tag{5.71}$$

Proof 5.22

FVM Evolution and Averaging Incremental Form[cor. 3.18]:

Add and subtract $F\left(U_{j+}^n,u_{j-}^n\right)$ fromeq. (3.85) and divide and multiply by $U_j^n-U_{j-1}^n$:

$$U_j^{n+1}=U_j^n+$$
$$+\overbrace{\frac{\Delta t}{\Delta x}\left[\frac{F\left(U_{j+}^n,u_{j-}^n\right)-F\left(U_{j+}^n,u_{j+1-}^n\right)}{u_{j+1}^n-u_j^n}\right]}^{C_{j+1/2}^n}\left(U_{j+1}^n-U_j^n\right)$$
$$\underbrace{-\frac{\Delta t}{\Delta x}\left[\frac{F\left(U_{j+1+}^n,u_{j+1-}^n\right)-F\left(U_{j+}^n,u_{j-1-}^n\right)}{u_{j+1}^n-u_j^n}\right]}_{D_{j+1/2}^n}\left(U_j^n-U_{j-1}^n\right)$$

Proof 5.23 TVD FVM scheme[lemma 3.4]: We need to show that evolution and averaging eq. (3.85) is TVD i.e. fullfils hartens lemma eq. (3.35):

$$c_{j+1/2}^n=\frac{\Delta t}{\Delta x}\frac{F\left(u_{j+}^n,\underline{u_{j-}^n}\right)-F\left(u_{j+}^n,\underline{u_{j+1-}^n}\right)}{u_{j+1}^n-u_j^n}$$
$$\overset{\text{Lips. Cont.}}{=}\frac{\Delta t}{\Delta x}\frac{\partial F}{\partial b}\left(u_{j+}^n,\cdot\right)\left(\frac{\underline{u_{j-}^n}-\underline{u_{j+1-}^n}}{u_{j+1}^n-u_j^n}\right)$$
$$:=\frac{\Delta t}{\Delta x}\frac{\partial F}{\partial b}\left(u_{j+}^n,\cdot\right)\cdot(-T_1)\ \overset{\substack{1.T_1\geqslant 0\\ 2.eq.\ (3.28)}}{\geqslant}\ 0$$

$$d_{j-1/2}^n=\frac{\Delta t}{\Delta x}\frac{f\left(\underline{u_{j+1+}^n},u_{j+1-}^n\right)-f\left(\underline{u_{j+}^n},u_{j-1-}^n\right)}{u_{j+1}^n-u_j^n}$$
$$\overset{\text{Lips. Cont.}}{=}\frac{\Delta t}{\Delta x}\frac{\partial F}{\partial a}\left(\cdot,u_{j+1-}^n\right)\left(\frac{\underline{u_{j+1+}^n}-\underline{u_{j+}^n}}{u_{j+1}^n-u_j^n}\right)$$
$$:=\frac{\Delta t}{\Delta x}\frac{\partial F}{\partial a}\left(\cdot,u_{j+1-}^n\right)\cdot(T_2)\ \overset{\substack{1.T_2\geqslant 0\\ 2.eq.\ (3.27)}}{\geqslant}\ 0$$

next wee need to show that $c_{j+1/2}^n+d_{j+1/2}^n\leqslant 1$ of eq. (3.35) if fullfiled:

$$c_{j+1/2}^n+d_{j+1/2}^n=\frac{\Delta t}{\Delta x}\left(-\frac{\partial f}{\partial b}\left(u_{j+}^n,\cdot\right)\right)T_1$$
$$+\frac{\Delta t}{\Delta x}\left(\frac{\partial f}{\partial a}\left(\cdot,u_{j+1-}^n,\cdot\right)\right)T_2$$
$$\leqslant\frac{\Delta t}{\Delta x}\max_{a,b}\left(\left|\frac{\partial F}{\partial a}\right|,\left|\frac{\partial F}{\partial b}\right|\right)(T_1+T_2)$$
$$\overset{\text{eq. (3.23)}}{\leqslant}\frac{1}{2}(T_1+T_2)\leqslant 1$$
$$\Longrightarrow T_1+T_2\leqslant 2$$

Proof 5.24 TVD FVM REA Scheme: Fromeq. (3.69) we know:

$$u_{j+}^n=u_j^n+\frac{\sigma_j^n}{2}\Delta x=u_j^n+\frac{\delta_j^n}{2}$$
$$u_{j-}^n=u_j^n-\frac{\sigma_j^n}{2}\Delta x=u_j^n-\frac{\delta_j^n}{2}$$
$$T_1=\frac{u_{j+1}^n-\frac{\delta_{j-1}^n}{2}-u_j^n+\frac{\delta_j^n}{2}}{u_{j+1}^n-u_j^n}$$
$$=1-\frac{1}{2}\left[\frac{\delta_{j+1}^n-\delta_j^n}{u_{j+1}^n-u_j^n}\right]$$
$$T_2=1+\frac{1}{2}\left[\frac{\delta_{j+1}^n-\delta_j^n}{u_{j+1}^n-u_j^n}\right]$$
$$\Longrightarrow T_1+T_2\equiv 2$$

and the rest follows from the condition that $T_1,T_2\geqslant 0$

Proof 5.25 TVD Minmod Limiter[cor. 3.19],lemma 3.5:

$$\frac{\delta_j^n}{u_{j+1}-u_j^n}=\frac{\Delta x\sigma_j^n}{u_{j+1}-u_j^n}=\frac{\operatorname{minmod}\left(u_{j+1}^n-u_j^n,u_j^n-u_{j-1}^n\right)}{u_{j+1}-u_j^n}$$

$$\operatorname{sign}(u_{j+1}^n-u_j^n)\neq\operatorname{sign}(u_j^n-u_{j-1}^n)\quad\Longrightarrow\sigma_j^n=0$$
$$\operatorname{sign}(u_{j+1}^n-u_j^n)=\operatorname{sign}(u_j^n-u_{j-1}^n)=\pm 1$$
$$\Longrightarrow\quad\frac{\delta_j^n}{u_{j+1}-u_j^n}=\operatorname{minmod}\left(\frac{u_{j+1}-u_j^n}{u_{j+1}-u_j^n},\frac{u_{j+1}-u_j^n}{u_{j+1}-u_j^n}\right)$$
$$=\operatorname{minmod}\Big(1,\underbrace{\frac{u_{j+1}-u_j^n}{u_{j+1}-u_j^n}}_{\geqslant 0}\Big)\leqslant 1$$

$$\Longrightarrow\quad 0\leqslant\frac{\delta_j^n}{u_{j+1}-u_j^n}\leqslant 1\qquad 0\leqslant\frac{\delta_{j+1}^n}{u_{j+1}-u_j^n}\leqslant 1$$

$$\Longrightarrow -1\leqslant\frac{\delta_{j+1}^n-\delta_j^n}{u_{j+1}-u_j^n}\leqslant 1 \tag{5.72}$$

Proof 5.26 Heun's Method TVD[def. 3.41]: We know that F.E. is TVD s.t.

$$\mathrm{TV}\left(U^*\right)\leqslant\mathrm{TV}\left(U^n\right)$$
$$\mathrm{TV}\left(U^{**}\right)\leqslant\mathrm{TV}\left(U^*\right)$$
$$\Longrightarrow\mathrm{TV}\left(U^{**}\right)\leqslant\mathrm{TV}\left(U^*\right)\leqslant\mathrm{TV}\left(U^n\right)$$
$$\mathrm{TV}\left(U^{n+1}\right)=\mathrm{TV}\left(\frac{U^n+U^{**}}{2}\right)$$
$$\overset{\mathrm{TV}(au+bv)\leqslant a\mathrm{TV}(u)+b\mathrm{TV}(v)}{\leqslant}\frac{1}{2}\mathrm{TV}\left(U^n\right)+\frac{1}{2}\mathrm{TV}\left(U^{**}\right)$$
$$\leqslant\frac{1}{2}\mathrm{TV}\left(U^n\right)+\frac{1}{2}\mathrm{TV}\left(U^n\right)=\mathrm{TV}\left(U^n\right)$$
$$\mathrm{TV}\left(U^{n+1}\right)\leqslant\mathrm{TV}\left(U^n\right)$$

Proof 5.27 Heuristic Heun's Method 2nd Order[def. 3.41]: We take a linear ODE:

$$u_t=au\quad\overset{\text{exac. sol}}{\Longrightarrow}\quad u_{n+1}=u_n e^{a\Delta t}$$

with the discretization $U_n:=u(t_n)$ for our scheme it follows:

$$U^*=U_n+a\Delta tU_n$$
$$U^{**}=U^*+a\Delta tU^*$$
$$U_n+a\Delta tU_n+a\Delta tU_n+a^2\Delta t^2U_n$$
$$U_n+2a\Delta tU_n+a^2\Delta t^2U_n$$
$$U_{n+1}=\frac{1}{2}\left(U^n+U^{**}\right)=U_n+a\Delta tU_n+\frac{1}{2}a^2\Delta t^2U_n$$
$$=U_n\left(1+a\Delta t+\frac{1}{2}a^2\Delta t^2\right)$$

for a Taylor expansion of the exact solution it holds:

$$U_{n+1}=u_n e^{a\Delta t}=U_n\left(1+a\Delta t+\frac{1}{2}a^2\Delta t^2+\frac{1}{6}a^3\Delta t^3+\ldots\right)$$

$$\Longrightarrow\quad\tau_n=\left|u_{n+1}-U_{n+1}\right|=\mathcal{O}(\Delta t^3)\quad\Longrightarrow\quad\text{2nd order}$$

Proof 5.28 Linearizing Conservation Laws[cor. 4.2]:

Let $\bar{\mathbf{u}}(\mathbf{x},t)\in\mathbb{R}^m$ a solution of eq. (4.1) and define $\hat{\mathbf{u}}(\mathbf{x},t):=\mathbf{u}-\bar{\mathbf{u}}(\mathbf{x},t)$ s.t.:

$$(\mathbf{u}-\bar{\mathbf{u}}(\mathbf{x},t))_t+(f(\mathbf{u})-f(\bar{\mathbf{u}}))_{\mathbf{x}}=0$$
$$\hat{\mathbf{u}}_t+(f(\mathbf{u})-f(\bar{\mathbf{u}}))_{\mathbf{x}}=0$$

$f(\mathbf{u})-f(\bar{\mathbf{u}}$ can be approximated by a Taylor expansion:

$$\underline{f(\mathbf{u})-f(\bar{\mathbf{u}})}=f'(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}})+\mathcal{O}(\|\mathbf{u}-\bar{\mathbf{u}}\|^2)$$

for small perturbations/step sizes δ $\mathbf{u}+\delta=\bar{\mathbf{u}}$ it holds that $\mathcal{O}(\|\mathbf{u}-\bar{\mathbf{u}}\|^2)\ll 1$:

$$\Longrightarrow\hat{\mathbf{u}}_t+\left(\underline{f'(\bar{u})\hat{u}}\right)_x=:\hat{\mathbf{u}}_t+(\mathbf{A}(\mathbf{x},t)\hat{u}))_x\,0$$

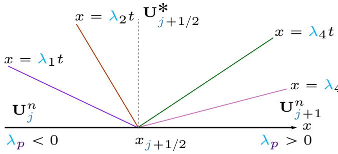
Proof 5.29 Decoupled hyperbolic lin. Cons. Law.??:

$$\mathbf{U}_t+\mathbf{A}\mathbf{U}_x=0$$
$$\mathbf{U}_t+\mathbf{R}\Lambda\mathbf{R}^{-1}\mathbf{U}_x=0\qquad ??$$
$$(\mathbf{R}^{-1}\mathbf{U})_t+\mathbf{R}^{-1}\mathbf{R}\Lambda(\mathbf{R}^{-1}\mathbf{U})_x=0\qquad\text{Multiplying by }\mathbf{R}^{-1}$$
$$\mathbf{W}_t+\Lambda\mathbf{W}_x=0\qquad\mathbf{W}:=\mathbf{R}^{-1}\mathbf{U}_t$$

Proof 5.30 Jump Decomposition[cor. 4.6]:

$$\mathbf{U}_R-\mathbf{U}_L=\mathbf{R}(\mathbf{W}_R-\mathbf{W}_L)=\sum_{p=1}^m\left(W_R^p-W_L^p\right)r_p$$
$$:=\sum_{p=1}^m\alpha^p r_p$$

Proof 5.31 Godunov Flux Systems of Cons. Laws.[def. 4.7]:
Idea we split Equation (4.12):
$$\mathbf{U}_R - \mathbf{U}_L = \sum_{p=1}^{m} \alpha^p r_p$$
into positive and negative jumps:



And then multiply by $\mathbf{A}$:
$$\mathbf{AU}^n_{j+1/2} = \mathbf{AU}_j + \mathbf{A}\sum_{p:\lambda_p<0}^{m} \alpha^p_{j+1/2} r_p \tag{5.73}$$
$$\mathbf{AU}^n_{j+1/2} = \mathbf{AU}_{j+1} - \mathbf{A}\sum_{p:\lambda_p\geqslant 0}^{m} \alpha^p_{j+1/2} r_p \tag{5.74}$$
$$5.73 = \mathbf{AU}_j + \sum_{p:\lambda_p<0}^{m} \alpha^p_{j+1/2}\lambda_p r_p \qquad r_p \text{ eigenv. of } \mathbf{A}$$
$$= \mathbf{AU}_j + \sum_{p=1}^{m} \lambda_p^- \alpha^p_{j+1/2} r_p$$
$$5.74 = \mathbf{AU}_{j+1} - \sum_{p=1}^{m} \lambda_p^+ \alpha^p_{j+1/2} r_p$$
$$\frac{1}{2}(5.73+5.74) = \mathbf{AU}^n_{j+1/2}$$
$$= \frac{1}{2}\left(\mathbf{AU}^n_j + \mathbf{AU}^n_{j+1} - \sum_{p=1}^{m}(\lambda_p^+ - \lambda_p^-)\alpha^p_{j+1/2} r_p\right)$$
$$= \frac{1}{2}\mathbf{A}\left(\mathbf{U}^n_j + \mathbf{U}^n_{j+1}\right) - \frac{1}{2}\sum_{p=1}^{m}|\lambda_p|\alpha^p_{j+1/2} r_p$$
$$= \frac{1}{2}\mathbf{A}\left(\mathbf{U}^n_j + \mathbf{U}^n_{j+1}\right) - \frac{1}{2}\mathbf{R}|\Lambda|\mathbf{R}^{-1}\left(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right)$$

Notes

- $a^+ := \max(a,0)$ $\qquad a^- := \min(a,0)$
- $a = a^+ + a^-$ $\qquad a^- - a^+ = |a|$
- $|\Lambda| = \mathrm{diag}(|\lambda_1|,\dots,|\lambda_m|)$

Proof 5.32 Godunov TVBProperty 4.1:
$$\mathrm{TV}(\mathbf{U}^{n+1}) = \sum_j \left\|\mathbf{U}^{n+1}_{j+1} - \mathbf{U}^{n+1}_j\right\|$$
$$= \sum_j \left\|\mathbf{RW}^{n+1}_{j+1} - \mathbf{RW}^{n+1}_j\right\|$$
$$= \sum_j \left\|\mathbf{R}(\mathbf{W}^{n+1}_{j+1} - \mathbf{W}^{n+1}_j)\right\|$$
$$\leqslant \|\mathbf{R}\|\sum_j \left\|\mathbf{W}^{n+1}_{j+1} - \mathbf{W}^{n+1}_j\right\|$$
we know that w^p solves the linear transport eq. s.t it holds:
$$\sum_j |w^{p,n+1}_{j+1} - w^{p,n+1}_j| \leqslant \sum_j |w^{p,n}_{j+1} - w^{p,n}_j|$$
$$\Longrightarrow \|\mathbf{R}\|\sum_j \left\|\mathbf{W}^{n+1}_{j+1} - \mathbf{W}^{n+1}_j\right\| \leqslant \|\mathbf{R}\|\sum_j \left\|\mathbf{W}^{n}_{j+1} - \mathbf{W}^{n}_j\right\|$$
$$= \|\mathbf{R}\|\sum_j \left\|\mathbf{R}^{-1}\mathbf{U}^n_{j+1} - \mathbf{R}^{-1}\mathbf{U}^n_j\right\|$$
$$\leqslant \|\mathbf{R}\|\|\mathbf{R}^{-1}\|\sum_j \left\|\mathbf{U}^n_{j+1} - \mathbf{U}^n_j\right\|$$

Proof 5.33 Exact Flux for conservation laws:
$$\mathbf{F}^n_{j+1/2} = \mathbf{F}\left(\mathbf{U}^n_j, \mathbf{U}^n_{j+1}\right) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{f}\left(\mathbf{U}(x_{j+1/2},t)\right)\mathrm{d}t$$
$$= \frac{1}{\Delta t}\mathbf{AU}_{j+1/2}\int_{t^n}^{t^{n+1}}\mathrm{d}t = \mathbf{A}_{j+1/2}$$

Proof 5.34 Weaks Solutions[def. 5.8]:
Multiply [def. 5.1] by a test function $\phi \in \mathcal{C}^1_0(\mathbb{R}\times\mathbb{R}_+)$ and integrate over space and time:
$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \phi\partial_t\mathbf{U} + \phi\partial_x\mathbf{f}(\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t = 0$$
exactly as in [proof 5.3] but now with vector valued functions.

Proof 5.35 Eigenvalue Equation Conservation Laws[def. 5.11]:
The solution of the conservation law[def. 5.1] is invariant to the scaling of the input parameters:
$$\mathbf{U}(x,t) \text{ solves eq. (5.1)}$$
$$\Longrightarrow \mathbf{w}(x,t) := \mathbf{U}(\lambda x, \lambda t) \text{ solves eq. (5.1)} \qquad \lambda \neq 0$$
thus it is natural to assume self-similarity – i.e. a solution $\mathbf{v}(\xi)$ that only depends on the ration $\xi := x/t$:
$$\mathbf{U}(x,t) = v\left(\frac{x}{t}\right) = \mathbf{v}(\xi)$$
$$\xi_t = \frac{-x}{t^2} \qquad \xi_x = \frac{1}{t}$$
$$\mathbf{U}_t = \mathbf{v}'(\xi)\xi_t = \mathbf{v}'(\xi)\frac{-x}{t^2} \qquad \mathbf{U}_x = \mathbf{v}'(\xi)\xi_x = \mathbf{v}'(\xi)\frac{1}{t}$$
$$\mathbf{f}(\mathbf{U})_x = \mathbf{f}'(\mathbf{U})\mathbf{U}_x = \mathbf{f}'(\mathbf{v}(\xi))\mathbf{v}'(\xi)\xi_x = \frac{1}{t}\mathbf{f}'(\mathbf{v}(\xi))\mathbf{v}'(\xi)$$
Plug it into ??: $\quad 0 = \partial_t\mathbf{U} + \partial_x\mathbf{f}(\mathbf{U})$
$$0 = \mathbf{v}'(\xi)\frac{-x}{t^2} + \frac{1}{t}\mathbf{f}'(\mathbf{v}(\xi))\mathbf{v}'(\xi)$$
$$= \mathbf{v}'(\xi)\frac{-\xi}{t} + \frac{1}{t}\mathbf{f}'(\mathbf{v}(\xi))\mathbf{v}'(\xi) \qquad \Big|\cdot t$$
$$\Longrightarrow \mathbf{f}'(\mathbf{v}(\xi))\mathbf{v}'(\xi) = \xi\mathbf{v}'(\xi)$$
Thus either $\mathbf{v}(\xi)' = 0$ or in the non-trivial case it follows that $\mathbf{v}(\xi)'$ is an eigenvector of the Jacobian $\mathbf{f}'(\mathbf{v}(\xi))$ with corresponding eigenvalue ξ:
$$\mathbf{f}'(\mathbf{v}(\xi))\mathbf{v}'(\xi) = \xi\mathbf{v}'(\xi) \qquad \begin{aligned}\mathbf{v}'(\xi) &= \mathbf{r}_j(\mathbf{v}(\xi))\\ \xi &= \lambda_j(\mathbf{v}(\xi))\end{aligned} \qquad j \in \{1,\dots,m\} \tag{5.75}$$

Proof 5.36 Simple ODE[def's. 5.12, 5.15]:
From eq. (5.75):
$$\mathbf{v}'(\xi) = \mathbf{r}_j(\mathbf{v}(\xi)) \qquad \xi = \lambda_j(\mathbf{v}(\xi)) \tag{5.76}$$
we see that if:
$\mathbf{v}(\xi_L) = \mathbf{U}_L$ and $\mathbf{v}(\xi_R) = \mathbf{U}_R$ for some $\xi_L, \xi_R \in \mathbb{R}$
then it must hold that:
$$\xi_L = \lambda_j(\mathbf{U}_L) \qquad \xi_R = \lambda_j(\mathbf{U}_R)$$
from which it follows that:
$$\mathbf{U}(x,t) = \begin{cases}\mathbf{U}_L & \frac{x}{t} < \lambda_j(\mathbf{U}_L) = \xi_L\\ \mathbf{v}_j\left(\frac{x}{t}\right) & \lambda_j(\mathbf{U}_L) < \frac{x}{t} < \lambda_j(\mathbf{U}_R)\\ \mathbf{U}_R & \xi_R = \lambda_j(\mathbf{U}_R) < \frac{x}{t}\end{cases} \tag{5.77}$$
now we need to take care of the initial condition.
We know that
$$\mathbf{v}(\xi_L) = \mathbf{U}_L \iff \xi_L = \mathbf{U}_L \tag{5.78}$$
but we do not know what $\xi_L = \lambda_j(\mathbf{U}_L)$ is.
Idea: we re-parameterize eq. (5.75) in terms of a new variable ϵ s.t. eq. (5.78) is satisfied at $\xi = 0$ and $\mathbf{v}(\xi_L)$:
$$\epsilon := \xi_L - \lambda_j(\mathbf{U}_L) \overset{\substack{\text{if } \epsilon = 0\\ \xi_L = \lambda_j(\mathbf{U}_L)}}{\Longrightarrow} \mathbf{W}(\epsilon)\Big|_{\epsilon=0} = U_L$$

Proof 5.37 Contact Discontinuity[def. 5.13]:
We are looking at eq. (5.76) and differentiate $\lambda(\mathbf{W}_j(\epsilon))$:
$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\lambda(\mathbf{W}_j(\epsilon)) = \nabla\lambda(\mathbf{W}_j(\epsilon))\mathbf{W}'_j(\epsilon) = \nabla\lambda(\mathbf{W}_j(\epsilon))\mathbf{r}_j(\mathbf{W}(\epsilon))$$
$$= 0 \qquad \text{(eq. (5.6))}$$
$$\Longrightarrow \int_0^\epsilon \nabla\lambda(\mathbf{W}_j(\epsilon))\,\mathrm{d}\epsilon = 0$$
$$\Longrightarrow \lambda(\mathbf{W}_j) = \lambda(\mathbf{W}_j(0)) \overset{\text{eq. (5.11)}}{=} \lambda(\mathbf{U}_L) \qquad \forall\epsilon \in (-\bar{\epsilon},\bar{\epsilon})$$
We know that $\lambda(\mathbf{W}_j) = \lambda(\mathbf{U}_L)$, thus if $\exists\epsilon \in (-\bar{\epsilon},\bar{\epsilon})$ s.t. $\mathbf{U}_R = \mathbf{W}_j(\epsilon)$ then it holds:
$$\lambda(\mathbf{W}_j) = \lambda(\mathbf{U}_L) = \lambda(\mathbf{U}_R) = \text{const}$$
Thus the middle rarefaction solution in eq. (5.77) cannot exist.

Proof 5.38 RH condition for contact discontinuities[def. 5.14]:
We want to proof a RH condition. From ?? 5.37 we know that:
$$\lambda(\mathbf{W}_j) = \lambda(\mathbf{U}_L) \overset{\text{if } \exists\epsilon:\mathbf{U}_R = \mathbf{W}_j(\epsilon)}{=} \lambda(\mathbf{U}_R) = \text{const}$$
let us differentiate $f(\mathbf{W}_j) - \lambda_j(\mathbf{W}_j)\mathbf{W}_j$:
$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(f(\mathbf{W}_j) - \lambda_j(\mathbf{W}_j)\mathbf{W}_j\right) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(f(\mathbf{W}_j) - \lambda_j(\mathbf{W}_j)\mathbf{W}_j\right)$$
$$= f'(\mathbf{W}_j)\mathbf{W}'_j - \lambda_j(\mathbf{W}_j)\mathbf{W}'_j$$
$$= \left(f'(\mathbf{W}_j) - \lambda_j(\mathbf{W}_j)\right)\mathbf{r}_j$$
$$= \left(\lambda(\mathbf{W}_j) - \lambda_j(\mathbf{W}_j)\right)\mathbf{r}_j$$
$$= \left(\lambda(\mathbf{W}_j) - \lambda_j(\mathbf{W}_j)\right)\mathbf{r}_j = 0 \qquad \forall\epsilon \in (-\bar{\epsilon},\bar{\epsilon})$$
Thus:
$$f(\mathbf{W}_j) - \lambda_j(\mathbf{W}_j)\mathbf{W}_j = \text{const} \qquad \forall\epsilon \in (-\bar{\epsilon},\bar{\epsilon})$$
Thus it must hold that:
$$f(\mathbf{U}_L) - \lambda_j(\mathbf{U}_L)\mathbf{U}_L = f(\mathbf{U}_R) - \lambda_j(\mathbf{U}_R)\mathbf{U}_R$$
$$f(\mathbf{U}_R) - f(\mathbf{U}_L) = s(\mathbf{U}_R - \mathbf{U}_L)$$
$$s := \lambda_j(\mathbf{U}_R) = \lambda_j(\mathbf{U}_L)$$

Proof 5.39
Rarefaction sol. of non-linear sys. of conser. laws$^{prop.\ 5.1}$:
Differentiate eq. (5.75) w.r.t. ξ:
$$\frac{\mathrm{d}}{\mathrm{d}\xi}\xi = \frac{\mathrm{d}}{\mathrm{d}\xi}\lambda_j(\mathbf{v}(\xi))$$
$$= \nabla\lambda_j(\mathbf{v}(\xi))^\intercal\mathbf{v}'(\xi)$$
$$= \nabla\lambda_j(\mathbf{v}(\xi))^\intercal\mathbf{r}_j(\mathbf{v}(\xi)) \qquad \text{(eq. (5.75))}$$
$$= c = 1 \qquad \text{(eq. (5.5) + rescaling } \mathbf{r}_j)$$
Thus in comparison to the contact discontinuity **we do not have** the condition that $\lambda(\mathbf{U}_L) = \lambda(\mathbf{U}_R) = $ const.

Proof 5.40 Shock Wave ODE: We want to find another expression for the shock speed in eq. (5.18). Idea we use the mean value theorem??:

why f' and not f

$$M(\mathbf{U}_L,\mathbf{U}) = \int_0^1 f'(\tau\mathbf{U}_L + (\tau-1)\mathbf{U})\,\mathrm{d}\tau = \frac{f(\mathbf{U}) - f(\mathbf{U}_L)}{\mathbf{U} - \mathbf{U}_L}$$
Thus we obtain the equation:
$$\mathcal{H}(\mathbf{U}_L) = \Big\{\mathbf{U} \in \mathcal{U} : \exists s \in \mathbb{R} \text{ s.t. } M(\mathbf{U}_L,\mathbf{U})(\mathbf{U} - \mathbf{U}_L) = s(\mathbf{U} - \mathbf{U}_L)\Big\} \tag{5.79}$$
Thus we obtain an equation with $m+1$ unknown's $(\mathbf{U}_L, s)$, where $(\mathbf{U} - \mathbf{U}_L)$ must be an eigenvector of $M(\mathbf{U}_L,\mathbf{U})$.
By the *Implicit Function Theorem*?? theorem we know that eq. (5.18) must have m curves $\{\mathbf{W}_j\}_{j=1}^m$:
$$\begin{aligned}f(\mathbf{W}_j(\epsilon)) - f(\mathbf{U}_L) &= s(\mathbf{W}_j(\epsilon) - \mathbf{U}_L)\\ \mathbf{W}_j(0) &= \mathbf{U}_L\end{aligned} \qquad \forall j = 1,\dots,m \tag{5.80}$$
Dividing by ϵ and taking the limit leads to:
$$\lim_{\epsilon\to 0}\quad \frac{f(\mathbf{W}_j(\epsilon)) - f(\mathbf{U}_L)}{\epsilon} = s\frac{(\mathbf{W}_j(\epsilon) - \mathbf{U}_L)}{\epsilon}$$
$$f'(\mathbf{W}_j(0))\mathbf{W}'_j(0) = s\mathbf{W}'_j(0)$$
$$s = \lambda_j(\mathbf{U}_L) \qquad \mathbf{W}'_j(0) = \mathbf{r}_j(\mathbf{U}_L)$$

this explaination is weird in comparison with video lecture

Proof 5.41 Entropy Cond. Non-lin. Systems[def. 5.20]:
Similar to [proof 5.6] but from *strict convexity* it follows that the Hessian?? matrix $\mathbf{s}''(\mathbf{U})$ is positive definite??.

Proof 5.42
Entropy Dissipation Contact Discontinuity[def. 5.22]:
At contact discontinuities it holds:
$$\mathbf{W}'_j(\epsilon) = \mathbf{r}_j(\mathbf{W}(\epsilon)) \qquad \mathbf{W}_j(0) = \mathbf{U}_L$$
$$\lambda_j(\mathbf{W}(\epsilon)) = \lambda_j(\mathbf{U}_L) = s$$
$$E(\epsilon) := q(\mathbf{W}_j(\epsilon)) - q(\mathbf{U}_L) - \lambda_j(s(\mathbf{W}_j(\epsilon)) - s(\mathbf{U}_L))$$
$$E(\epsilon)' = q'(\mathbf{W}_j(\epsilon))\mathbf{W}'_j(\epsilon) - \lambda_j(\mathbf{U}_L)s'(\mathbf{W}_j(\epsilon))\mathbf{W}'_j(\epsilon)$$
$$\mathbf{s}'(\mathbf{W}_j(\epsilon))^\intercal\mathbf{f}'(\mathbf{W}_j(\epsilon))\mathbf{W}'_j(\epsilon) - \lambda_j(\mathbf{U}_L)s'(\mathbf{W}_j(\epsilon))\mathbf{W}'_j(\epsilon)$$
$$\mathbf{s}'(\mathbf{W}_j(\epsilon))^\intercal\left[\mathbf{f}'(\mathbf{W}_j(\epsilon))\mathbf{W}'_j(\epsilon) - \lambda_j(\mathbf{U}_L)\mathbf{W}'_j(\epsilon)\right]$$
$$\mathbf{s}'(\mathbf{W}_j(\epsilon))^\intercal\Big[\underbrace{\mathbf{f}'(\mathbf{W}_j(\epsilon))\mathbf{r}_j(\epsilon) - \lambda_j(\mathbf{U}_L)\mathbf{W}'_j(\epsilon)}_{\text{eigenvalue equation} \Longrightarrow \equiv 0}\Big]$$
Thus it follows that:
$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}E(\epsilon) \equiv \quad\Longrightarrow\quad E(\epsilon) = E(\mathbf{U}_L) \overset{E(\mathbf{U}_L)\equiv 0}{=} \tag{5.81}$$

Proof 5.43
Entropy Dissipation Genuinely Nonlinear[def. 5.22]:
Consider a genuinely non-linear wave family $(\lambda_j, \mathbf{r}_j)$ and define:
$$E(\epsilon) := q(\mathbf{W}_j(\epsilon)) - q(\mathbf{U}_L) - \lambda_j(s(\mathbf{W}_j(\epsilon)) - s(\mathbf{U}_L))$$
together with the RH condition it follows through tedious computation that:
$$E(\epsilon) < 0 \quad \text{for } \epsilon \text{ small} \iff \lambda_j(\mathbf{U}_R) < s < \lambda_j(\mathbf{U}_L)$$
for *strictly hyperbolic systems*[cor. 4.3] one can also deduce for small ϵ that:
$$\lambda_{j-1}(\mathbf{U}_L) < s < \lambda_{j+1}(\mathbf{U}_R) \tag{5.82}$$

Proof 5.44 Locally Linearized Riemann Problem[def. 5.26]: We locally $[\mathbf{U}^n_j, \mathbf{U}^n_{j+1}]$ approximate $\mathbf{f}_x$ using Taylor:
$$\mathbf{f}(\mathbf{u}) \overset{??}{=} \mathbf{f}\left(\mathbf{u}^n_j\right) + \mathbf{f}'(\theta)\left(\mathbf{u} - \mathbf{u}^n_j\right) \qquad \theta \in [\mathbf{U}^n_j, \mathbf{U}^n_{j+1}]$$
$$\mathbf{f}(\mathbf{u}) = \mathbf{f}'(\theta)\mathbf{u}_x \approx \mathbf{A}\left(\mathbf{u}^n_j, \mathbf{u}^n_{j+1}\right)\mathbf{u}_x$$

Proof 5.45 Roe Matrix[def. 5.28]: We use the mean value theorem ?? to relate eq. (5.38) and the RH condition[def. 5.9]:

$$\mathrm{f}\left(\mathbf{U}^n_{j+1/2}\right)-\mathrm{f}\left(\mathbf{U}^n_j\right) = \int_0^1 \mathrm{f}'\left(\mathbf{u}^n_j+\tau\left(\mathbf{u}^n_{j+1}-\mathbf{u}^n_j\right)\right)\left(\mathbf{U}^n_j-\mathbf{U}^n_{j+1}\right)\mathrm{d}\tau$$

$$\mathrm{f}\left(\mathbf{U}^n_{j+1/2}\right)-\mathrm{f}\left(\mathbf{U}^n_j\right) = \mathbf{A}^n_{j+1/2}\left(\mathbf{U}^n_j-\mathbf{U}^n_{j+1}\right)$$

Proof 5.46 Roes Criterion – Property 5.3: We assume that the exact solution of the non-linearized Riemann problem[def. 5.24] is given by a single discontinuity i.e. a *shock wave* or a *contact discontinuity* s.t. the exact solution is given by:

$$\mathbf{U}(x,t)=\begin{cases}\mathbf{U}^n_j & x<x_{j+1/2}+s^n_{j+1/2}(t-t^n)\\ \mathbf{U}^n_{j+1} & x>x_{j+1/2}+s^n_{j+1/2}(t-t^n)\end{cases}$$

and must satisfy the Rankine Heuginote condition??:

$$\mathrm{f}\left(\mathbf{U}^n_{j+1}(t)\right)-\mathrm{f}\left(\mathbf{U}^n_j(t)\right)=s^n_{j+1/2}\left(\mathbf{U}^n_j(t)-\mathbf{U}^n_{j+1}(t)\right)$$

Plugging in Roes Criterioneq. (5.38) leads to:

$$\mathbf{A}^n_{j+1/2}\left(\mathbf{u}^n_j,\mathbf{u}^{n+1}_j\right)=s^n_{j+1/2}\left(\mathbf{U}^n_j(t)-\mathbf{U}^n_{j+1}(t)\right)$$

This implies that $\left(\mathbf{u}^n_j,\mathbf{u}^{n+1}_j\right)$ is an eigenvector of the matrix $\mathbf{A}^n_{j+1/2}$ and $s^n_{j+1/2}$ is the corresponding eigenvalue. Thus in order for equation ?? to hold we need to require:

$$\exists p\in\{1,\ldots,m\}:\quad \begin{aligned} s^n_{j+1/2}&=\lambda^{p,n}_{j+1/2}\\ \left(\mathbf{u}^n_j-\mathbf{u}^{n+1}_j\right)&=\mathbf{r}^{p,n}_{j+1/2}\end{aligned}$$

$$\implies\quad \left(\mathbf{u}^n_j-\mathbf{u}^{n+1}_j\right)\overset{\text{eq. (4.12)}}{=}\sum_{l=1}^m \mathbf{W}^{l,n}_{j+1/2}\mathbf{r}^{l,n}_{j+1/2}\overset{!}{=}\mathbf{r}^{l,p}_{j+1/2}$$

$$\implies\qquad \mathbf{u}\text{ is a solution}$$

don't really get why this makes u a solution/the point

Proof 5.47 HLL-3/HLLC[def. 5.37]:

1. We have seen in example 5.13 that first and third wave families are genuinely non-linear while the second wave family is linear degenerate and thus results in a contact discontinuity.
From this it follows that the pressure and the velocity are constant across the second discontinuity and that only the denity changes:

Why is this true?

$$v^{l,n}_{j+1/2}=v^{r,n}_{j+1/2}=v^{*,n}_{j+1/2}\qquad p^{l,n}_{j+1/2}=p^{r,n}_{j+1/2}=p^{*,n}_{j+1/2}$$

2. Moreover from example 5.13 we also know that the speed of the second contact discontinuity is equal to is eigenvalue which is equal to the velocity:
$$s^{m,n}_{j+1/2}=v^*$$

Thus we can write the euler equations in terms of the conservative variables as:

$$\begin{aligned}\partial_t\rho+\partial_x(\rho v)&=0\\ \partial_t(\rho v)+\partial_x\left(\rho v^2+p\right)&=0\\ \partial_t E+\partial_x\left((E+p)v\right)&=0\end{aligned}\tag{5.83}$$

$$E=\frac{p}{\gamma-1}+\frac{1}{2}\rho v^2\qquad \gamma>1:\ \text{heat capacity ratio}\tag{5.84}$$

The compressible euler equations can be written as conservation law:

$$\mathbf{U}=\begin{pmatrix}\rho\\ m\\ E\end{pmatrix}=\begin{pmatrix}\rho\\ mv\\ \frac{p}{\gamma-1}+\frac{1}{2}\rho v^2\end{pmatrix}\qquad \mathrm{f}(\mathbf{U})=\begin{pmatrix}\rho v\\ \rho v^2+p\\ (E+p)v\end{pmatrix}$$

$$\mathbf{U}^{\alpha,n}_{j+1/2}=\begin{pmatrix}\rho^{\alpha,n}_{j+1/2}\\ \rho^{\alpha,n}_{j+1/2}v^{*,j}_{j+1/2}\\ \frac{p^{*,n}_{j+1/2}}{\gamma-1}+\frac{1}{2}\rho^{\alpha,n}_{j+1/2}\left(v^{*,j}_{j+1/2}\right)^2\end{pmatrix}\qquad \alpha\in\{l,r\}$$

Now we use conservation/the RH condition[def. 5.9]:

$$\mathrm{F}\left(\mathbf{U}^{\theta,n}_{j+1/2}\right)-\mathrm{F}\left(\mathbf{U}^{\theta,n}_{j+k}\right)=s^{\theta,n}_{j+1/2}\left(\mathbf{U}^{\theta,n}_{j+1/2}-\mathbf{U}_{j+k}\right)\qquad k\in\{0,1\}\tag{5.85}$$

We begin with the left l and right r discontinuity for the first component of the Euler equations.

$$\begin{aligned}\rho^{l,n}_{j+1/2}(v^*-s^{l,n}_{j+1/2})&=\rho^n_j(v^n_j-s^{l,n}_{j+1/2})\\ \rho^{r,n}_{j+1/2}(v^*-s^{r,n}_{j+1/2})&=\rho^n_{j+1}(v^n_{j+1}-s^{r,n}_{j+1/2})\end{aligned}\tag{5.86}$$

Thus it follows:

$$\rho^{l,n}_{j+1/2}=\frac{\rho^n_j(v^n_j-s^{l,n}_{j+1/2})}{(v^*_{j+1/2}-s^{l,n}_{j+1/2})}\qquad \rho^{r,n}_{j+1/2}=\frac{\rho^n_{j+1}(v^n_{j+1}-s^{r,n}_{j+1/2})}{(v^*_{j+1/2}-s^{r,n}_{j+1/2})}$$

Next we look use either the left or right discontinuity with the second component of eq. (5.85) and use again the RH condition[def. 5.9]:

$$\begin{aligned}&\rho^{*,n}_{j+1/2}\left(v^{*,n}_{j+1/2}\right)^2+p^{*,n}_{j+1/2}-\rho^n_j\left(v^n_j\right)^2-p^n_j\\ &=s^{m,n}_{j+1/2}\left(\rho^{*,n}_{j+1/2}v^{*,n}_{j+1/2}-\rho^{*,n}_jv^{*,n}_j\right)\end{aligned}$$

With eq. (5.86) we can solve for $p^{*,n}_{j+1/2}$:

$$p^{*,n}_{j+1/2}=p^n_{j+k}+\rho^n_{j+k}\left(v^n_{j+k}-v^{*,n}_{j+1/2}\right)\left(v^n_{j+k}-s^{\alpha,n}_{j+1/2}\right)\qquad \alpha\in\{l,r\}\qquad k\in\{0,1\}$$

next we need to find a find an expression for $v^{*,n}_{j+1/2}$, we do this by using conservation over all three waves:

$$\begin{aligned}\mathrm{F}\left(^n_{j+1}\right)-\mathrm{F}\left(\mathbf{U}^n_j\right)=&s^{r,n}_{j+1}\left(\mathbf{U}^n_{j+1/2}-\mathbf{U}^{r,n}_{j+1/2}\right)\\ &+s^{m,n}_{j+1/2}\left(\mathbf{U}^{r,n}_{j+1/2}-\mathbf{U}^{l,n}_{j+1/2}\right)\\ &+s^{l,n}_{j+1/2}\left(\mathbf{U}^{l,n}_{j+1/2}-\mathbf{U}^n_j\right)\end{aligned}$$

Proof 5.48: we compare the second component:

$$\begin{aligned}&\rho^n_{j+1}v^2_{j+1}+p^n_{j+1}-\rho^{r,n}_{j+1/2}v^*_{j+1/2}-p^{*,n}_{j+1/2}\\ &\rho^{r,n}_{j+1/2}\left(v^{*,n}_{j+1/2}\right)^2+p^{*,n}_{j+1/2}-\rho^{*,n}_{j+1/2}\left(v^{*,n}_{j+1/2}\right)^2-p^{*,n}_{j+1/2}\\ &\rho^{l,n}_{j+1/2}\left(v^{*,n}_{j+1/2}\right)^2+p^{*,n}_j-\rho^n_j\left(v^n_j\right)^2-p^n_j\\ =&s^{r,n}_{j+1/2}\left(\rho^n_{j+1}v^n_{j+1}-\rho^{r,n}_{j+1/2}v^{*,n}_{j+1/2}\right)\\ &v^{*,n}_{j+1/2}\left(\rho^{r,n}_{j+1/2}v^{*,n}_{j+1/2}-\rho^{l,n}_{j+1/2}v^{*,n}_{j+1/2}\right)\\ &s^{l,n}_{j+1/2}\left(\rho^{l,n}_{j+1/2}v^{*,n}_{j+1/2}-\rho^n_{j+1}v^n_{j+1}\right)\end{aligned}$$

From this it follows:

$$\begin{aligned}&s^{r,n}_{j+1/2}\rho^n_{j+1}v^n_{j+1}-s^{r,n}_{j+1/2}\rho^{r,n}_{j+1/2}v^{*,n}_{j+1/2}\\ &+\rho^{r,n}_{j+1/2}\left(v^{*,n}_{j+1/2}\right)^2-\rho^{l,n}_{j+r/2}\left(v^{*,n}_{j+1/2}\right)^2\\ &+s^{l,n}_{j+1/2}\rho^{l,n}_{j+1/2}v^{*,n}_{j+1/2}-s^{l,n}_{j+1/2}\rho^n_jv^n_j\\ &=\rho^n_{j+1}\left(v^n_{j+1}\right)^2+p^n_{j+1}-\rho^n_j\left(v^n_j\right)^2-p^n_j\\ &v^{*,n}_{j+1/2}\rho^{r,n}_{j+1/2}\left(v^{*,n}_{j+1/2}-s^{r,n}_{j+1/2}\right)\\ &-v^{*,n}_{j+1/2}\rho^{l,n}_{j+1/2}\left(v^{*,n}_{j+1/2}-s^{l,n}_{j+1/2}\right)\\ &=\rho^n_{j+1}\left(v^n_{j+1}\right)^2+p^n_{j+1}-\rho^n_j\left(v^n_j\right)^2-p^n_j\\ &-s^{r,n}_{j+1/2}\rho^n_{j+1}v^n_{j+1}+s^{l,n}_{j+1/2}\rho^n_jv^n_j\end{aligned}$$

pluggin in $\rho^{l,n}_{j+1/2}$ and $\rho^{r,n}_{j+1/2}$ on the lhs leads to:

$$v^{*,n}_{j+1/2}\left(\rho^n_{j+1}\left(v^n_{j+1}-s^{r,n}_{j+1/2}\right)-\rho^n_j\left(v^n_j-s^{l,n}_{j+1/2}\right)\right)=\text{——}\shortparallel\text{——}$$

From this it follows:

$$v^{*,n}_{j+1/2}=\frac{\rho^j_{j+1}v^n_{j+1}\left(s^{r,n}_{j+1/2}-v^n_{j+1}\right)-\rho^n_jv^n_j\left(s^{l,n}_{j+1/2}-v^n_j\right)-\left(p^j_j-p^n_j\right)}{\rho^n_{j+1}\left(s^{r,n}_{j+1/2}-v^n_{j+1}\right)\rho^n_j\left(s^{l,n}_{j+1/2}-v^n_j\right)}$$

is f and F here the same?

7. Examples

Example 5.1
Burgers Equation Riemann Problem:

$$u_t+uu_x=0$$

$$u(x,0)=u_0(x)=\begin{cases}1 & \text{if } x<0\\ 0 & \text{if } x>0\end{cases}$$

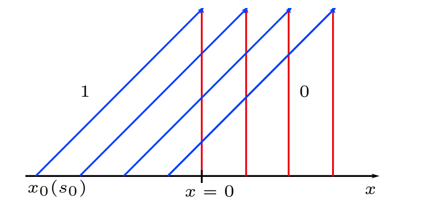
$$\textbf{ODEs}\qquad \frac{\mathrm{d}t}{\mathrm{d}r}=1\Rightarrow\mathrm{d}t=\mathrm{d}r\qquad \frac{\mathrm{d}x}{\mathrm{d}r}=\frac{\mathrm{d}x}{\mathrm{d}t}=u\qquad \frac{\mathrm{d}u}{\mathrm{d}r}=0$$

$$\begin{aligned}\frac{\mathrm{d}u\left(x(t),t\right)}{\mathrm{d}t}&\overset{\text{C.R.}}{=}=u_t\left(x(t),t\right)+u_x\left(x(t),t\right)\frac{\mathrm{d}x(t)}{\mathrm{d}t}\\ &=u_t\left(x(t),t\right)+u_x\left(x(t),t\right)u=0\end{aligned}$$

thus u is constant along the projectd characteristics $x(t)=x(r)$.
Problem: let look at the inital data and the *projected characteristcs*:

$$\left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=0}=u\left(x(t),t\right)\Big|_{t=0}=u_0(x_0)=\begin{cases}1 & \text{if } x<0\\ 0 & \text{if } x>0\end{cases}$$

$$\Rightarrow\begin{cases}\int x(t)\,\mathrm{d}x=\int 1\,\mathrm{d}t & \Rightarrow x(t)=x_0+t\quad \text{if } x<0\\ \int\frac{\mathrm{d}x(t)}{\mathrm{d}t}\,\mathrm{d}t=\int 0\,\mathrm{d}t & \Rightarrow x(t)=x_0\quad \text{if } x>0\end{cases}$$

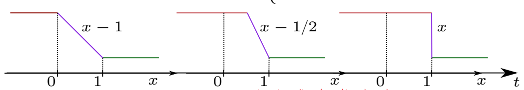


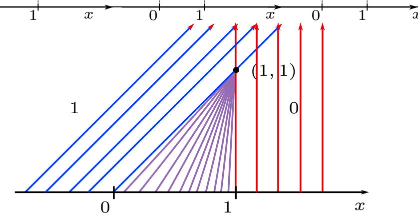
Thus for $x>0$ we have intersecting project. characteristics i.e. a multivalued function that cannot be inverted.

Example 5.2
Burgers Equation Continuous Initial Data:

$$u_t+uu_x=0$$

$$u(x,0)=u_0(x)=\begin{cases}1 & \text{if } x<0\\ 1-x & \text{if } 0\leqslant x\leqslant 1\\ 0 & \text{if } x>0\end{cases}$$





Thus even for smooth initial data we will get intersection after the point $(1,1)$.

Example 5.3 Monotoni-city LxF[def. 3.10]: Consider the LxF scheme[def. 3.23]:

$$F(a,b) = \frac{1}{2}\left(f(a)+f(b)\right) - \frac{\Delta x}{2\Delta x}(b-a)$$

$$\frac{\partial f}{\partial a} = \frac{1}{2}f'(a) + \frac{1}{2}\frac{\Delta x}{\Delta t} = \frac{1}{2}\left(\frac{\Delta x}{\Delta t} + f'(a)\right) \overset{!}{\geqslant} 0$$

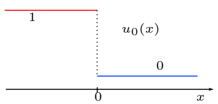
$$\frac{\partial f}{\partial b} = \frac{1}{2}f'(b) - \frac{1}{2}\frac{\Delta x}{\Delta t} = -\frac{1}{2}\left(\frac{\Delta x}{\Delta t} - f'(b)\right) \overset{!}{\geqslant} 0$$

In order for both of this equations to hold it follows the CFL condition:

$$\left|f'(x)\right| \leqslant \frac{\Delta x}{\Delta t}$$

Example 5.4 RK for Riemann Problem[def. 2.4]:

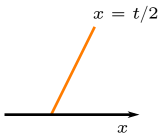
$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{5.87}$$

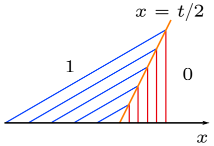


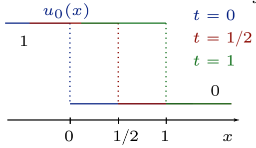
$$u_0 = \begin{cases}1 & \text{if } x<0\\ 0 & \text{if } x>0\end{cases} \tag{5.88}$$

$$s(t) = \sigma'(t) = \frac{f\left(u^-(t)\right) - f\left(u^+(t)\right)}{u^-(t)-u^+(t)} = \frac{f(1)-f(0)}{1-0} = \frac{\frac{1}{2}-0}{1} = \frac{1}{2} \Rightarrow \sigma(t) = \frac{t}{2}$$

$$u = \begin{cases}1 & \text{if } x<\frac{t}{2}\\ 0 & \text{if } x>\frac{t}{2}\end{cases} \tag{5.89}$$



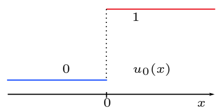




Thus we found a weak solution, where the characteristics are colliding on a traveling discontinuity/shockwave[def. 2.1].

Example 5.5 RK for Riemann Problem emanating:

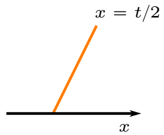
$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{5.90}$$

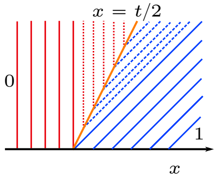


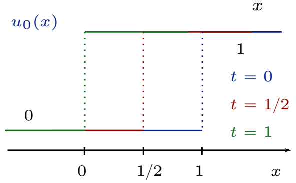
$$u_0 = \begin{cases}0 & \text{if } x<0\\ 1 & \text{if } x>0\end{cases} \tag{5.91}$$

$$s(t) = \sigma'(t) = \frac{f\left(u^-(t)\right) - f\left(u^+(t)\right)}{u^-(t)-u^+(t)} = \frac{f(0)-f(1)}{0-1} = \frac{-\frac{1}{2}-0}{-1} = \frac{1}{2} \Rightarrow \sigma(t) = \frac{t}{2}$$

$$u(x,t) = \begin{cases}0 & \text{if } x<\frac{t}{2}\\ 1 & \text{if } x>\frac{t}{2}\end{cases} \tag{5.92}$$





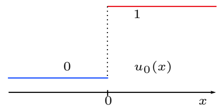


Problem we now get an area with characteristics emanating from the shock, thus we cannot track them back to the initial data.
This region of outflowing characteristics may in fact be filled in several ways seeexample 5.6

Example 5.6 RK for Riemann Problem emanating:

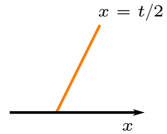
$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{5.93}$$

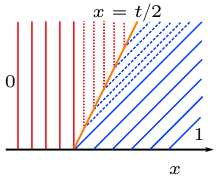
$$u_0 = \begin{cases}0 & x<\frac{1}{4}t\\ \frac{1}{2} \quad \text{if} & \frac{1}{4}t<x<\frac{3}{4}t\\ 1 & x>\frac{3}{4}t\end{cases} \tag{5.94}$$

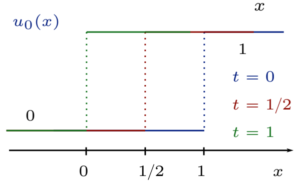


$$s(t) = \sigma'(t) = \frac{f\left(u^-(t)\right) - f\left(u^+(t)\right)}{u^-(t)-u^+(t)} = \frac{f(0)-f(1)}{0-1} = \frac{-\frac{1}{2}-0}{-1} = \frac{1}{2} \Rightarrow \sigma(t) = \frac{t}{2}$$

$$u(x,t) = \begin{cases}0 & \text{if } x<\frac{t}{2}\\ 1 & \text{if } x>\frac{t}{2}\end{cases} \tag{5.95}$$







This solution obviously also fullfils the previous problem. **problem**: we thus can construct arbitrary many weak solutions by using the rh conditioneq. (2.3) with different inermediate states.

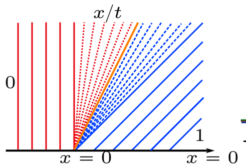
finish example and understand why it also fullfils the previous example

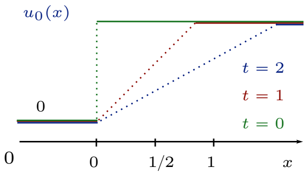
Example 5.7 Riemann Rarefaction[cor. 2.3]:

$$u(x,0) = \begin{cases}0 & \text{if } x<0\\ 1 & \text{if } x>0\end{cases} \qquad u_t\left(\frac{u^2}{2}\right)_x = 0$$

Thus $f'(u) = u \implies (f')^{-1}(u) = u$ s.t.:

$$u(x,t) = \begin{cases}0 & \text{if } x<0\\ \frac{x}{t} & \text{if } x>0\\ 1 & \text{if } x>0\end{cases}$$





- Thus after the after a small time period our solution will be piecewise/lipschitz continious $\implies u^- = u^+ \implies f\left(u^+\right) = f\left(u^-\right) \implies$ RH conditioneq. (2.3) will be automatically satisfied.
- From this it also follows that the Lax-entropy condition?? is fullfiled.
$$f\left(u^+\right) = s(t) = f\left(u^-\right)$$

Example 5.8 $f(u) = au, \quad a>0$
Why do we need Semi-Disc. FVS[def. 3.38]: Consider the upwind flux $F(u,u) = au$ then it follows for the FVM[def. 3.34]

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}\left(u_{j+}^n - u_{j-1+}^n\right) \tag{5.96}$$

and$\sigma_j^n \in$ {minmod,MC,superbee} it follows for the truncation error[def. 3.26]:

$$\|\tau_j^n\| \approx \mathcal{O}(\Delta x^3) + \mathcal{O}(\Delta t^2) \overset{eq.\ (3.43)}{\approx} \mathcal{O}(\Delta x^3) + \mathcal{O}(\Delta x^2)$$

thus schemes seem to be 2nd order may actually be first order due to the time-discretization.

Example 5.9 Wave Equation: The wave equation:

$$\overbrace{u_{tt}}^{\text{acceleration}} - c^2\overbrace{u_{xx}}^{\text{strain}} = 0$$

can be rewritten as a first-order system of equations by using the *change of variables*:

$$v := u_t \qquad w := -cu_x$$

$$u_{tt} - c\left(cu_{xx}\right) = 0 \implies v_t + cw_x$$

we can find a second equations to obtain a system:

$$w_t = -cu_{xt} = -c\left(u_t\right)_x = -cv_x$$

Hence it follows:

$$\begin{matrix} v_t + cw_x = 0 \\ w_t + cv_x = 0\end{matrix} \iff \mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0 \quad \mathbf{u} := \begin{bmatrix}v\\w\end{bmatrix}, \mathbf{A} = \begin{bmatrix}0 & c\\ c & 0\end{bmatrix} \tag{5.97}$$

Example 5.10 Linearized Euler Equations:

check lecture 14 1h and linearize

Example 5.11 Laplace's Equations:

$$\Delta\mathbf{u} = 0 \implies u_{tt} + u_{xx} = 0$$

can be rewritten as a first-order system of equations by using the *change of variables* similary toexample 5.9 but with $c = 1$ and a changed sign:

$$v := u_t \qquad w := u_x$$

$$u_{tt} + u_{xx} = 0 \implies v_t + w_x$$

we can find a second equations to obtain a system:

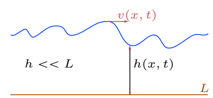
$$w_t = u_{xt} = (u_t)_x = v_x$$

Hence it follows:

$$\begin{matrix} v_t + w_x = 0 \\ w_t - v_x = 0\end{matrix} \iff \mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0 \quad \mathbf{u} := \begin{bmatrix}v\\w\end{bmatrix}, \mathbf{A} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \tag{5.98}$$

Example 5.12 Shallow Water Equations:

$$\begin{aligned}\partial_t h + \partial_x(hv) &= 0\\ \partial_t(hv) + \partial_x\left(\frac{1}{2}gh^2 + hv^2\right) &= 0\end{aligned} \tag{5.99}$$



$v(x,t)$: horizontal velocity of water column at x.
With $m := hv$ eq. (5.112) can be rewritten as non-linear scalar conservation laweq. (5.1):

$$\mathbf{U} = \begin{pmatrix}h\\m\end{pmatrix} \qquad \mathrm{f}(\mathbf{U}) = \begin{pmatrix}m\\ \frac{1}{2}gh^2 + \frac{m^2}{h}\end{pmatrix} \tag{5.100}$$

$$\mathrm{f}'(\mathbf{U}) = \begin{pmatrix}0 & 1\\ gh & \frac{2m}{h}\end{pmatrix} \qquad \left|\begin{pmatrix}0 & 1\\ gh & \frac{2m}{h}\end{pmatrix}\right| = gh$$

$$\lambda_{1/2}(\mathrm{f}'(\mathbf{U})) \overset{\overset{??}{\text{tr}=0}}{=} v \mp c \qquad c := \sqrt{gh}$$

$$\left(\mathrm{f}'(\mathbf{U}) - \lambda_j\right)\mathbf{r}_j(\mathbf{U}) = 0 \implies \mathbf{r}_{1/2}(\mathbf{U}) = \begin{pmatrix}1\\ v\mp c\end{pmatrix}$$

- Assuming that $h>0$ we find that $\mathcal{U} = \left\{(h,m) \in \mathbb{R}^2 : h>0\right\}$ s.t. eq. (5.112) is *hyperbolic*.
- moreover we find that both wave families of eq. (5.112) are *genuinely nonlinear*[def. 5.4]:
$$\nabla\lambda_{1/2}(\mathbf{U})\cdot\mathbf{r}_{1/2}(\mathbf{U}) = \mp\frac{3}{2}\sqrt{\frac{g}{h}}$$

Example 5.13 Compressible Euler Equations:

$$\partial_t \rho + \partial_x(\rho v) = 0 \tag{5.101}$$
$$\partial_t(\rho v) + \partial_x\left(\rho v^2 + p\right) = 0 \tag{5.102}$$
$$\partial_t E + \partial_x\left((E+p)v\right) = 0 \tag{5.103}$$

The pressure p and the total energy E are related by the equation of state:

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2 \qquad \gamma > 1: \text{ heat capacity ratio} \tag{5.104}$$

The compressible euler equations can be written as conservation law:

$$\mathbf{U} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} \qquad \mathbf{f}(\mathbf{U}) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}$$

$$\begin{aligned} \lambda_1 &= v - c \\ \lambda_2 &= v \\ \lambda_3 &= v + c \end{aligned} \qquad c = \sqrt{\frac{\gamma p}{\rho}} \qquad v = \frac{m}{\rho}$$

For non-antimatter the pressure has to be positive thus admissible set is given by:

$$\mathcal{U} = \left\{ (p, m, E) : p > 0 \iff E > \frac{m^2}{2\rho} \right\}$$

and the euler equations are thus a *strictly hyperbolic* system[def. 5.5].

$$\begin{aligned} \mathbf{r}_1 &= \begin{pmatrix} 1 & v - c & H - vc \end{pmatrix}^\intercal \\ \mathbf{r}_2 &= \begin{pmatrix} 1 & v & \frac{v^2}{2} \end{pmatrix}^\intercal \\ \mathbf{r}_3 &= \begin{pmatrix} 1 & v + c & H + vc \end{pmatrix}^\intercal \end{aligned} \qquad H = \frac{E+p}{\gamma} \text{ Enthalpy}$$

The second wave family is *linearly degenerated*:

$$\nabla \lambda_2 \cdot \mathbf{r}_2 = \begin{pmatrix} -\frac{m}{\rho^2} \\ \frac{1}{\rho} \\ 0 \end{pmatrix}^\intercal \mathbf{r}_2 = -\frac{m}{\rho^2} + \frac{v}{\rho} = -\frac{v}{\rho} + \frac{v}{\rho} = 0$$

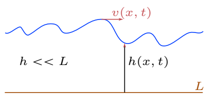
while the first and third wave family are *genuinely non-linear*.

Note

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2 \implies p = ()\gamma - 1)\left(E - \frac{m^2}{2\rho}\right)$$

Example 5.14 Shallow Water Equations Entropy Pair[def. 5.19]:

$$\partial_t h + \partial_x(hv) = 0$$
$$\partial_t(hv) + \partial_x\left(\frac{1}{2}gh^2 + hv^2\right) = 0 \tag{5.105}$$



$v(x,t)$: horizontal velocity of water column at x.
With $m := hv$ eq. (5.112) can be rewritten as non-linear scalar conservation laweq. (5.1):

$$\mathbf{U} = \begin{pmatrix} h \\ m \end{pmatrix} \qquad \mathbf{f}(\mathbf{U}) = \begin{pmatrix} m \\ \frac{1}{2}gh^2 + \frac{m^2}{h} \end{pmatrix} \tag{5.106}$$

We now define the energy of a state $\mathbf{U} \in \mathcal{U} = \left\{(h, m) \in \mathbb{R}^2 : h > 0\right\}$ as the sum of the potential and kinetic energy:

$$s(\mathbf{U}) = \frac{1}{2}gh^2 + \frac{1}{2}hv^2$$

- Assuming that $h > 0$ we see that $s(\mathbf{U})$ is strictly convex is *hyperbolic*.
- if we define $q(\mathbf{U}) = h^2 v + \frac{1}{3}hv^3$ it is straight forward to see that s, q is an entropy pair.

Example 5.15 Compressible Euler Equations[def. 5.19]:

$$\partial_t \rho + \partial_x(\rho v) = 0 \tag{5.107}$$
$$\partial_t(\rho v) + \partial_x\left(\rho v^2 + p\right) = 0 \tag{5.108}$$
$$\partial_t E + \partial_x\left((E+p)v\right) = 0 \tag{5.109}$$

The pressure p and the total energy E are related by the equation of state:

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2 \qquad \gamma > 1: \text{ heat capacity ratio} \tag{5.110}$$

The compressible Euler equations can be written as conservation law:

$$\mathbf{U} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} \qquad \mathbf{f}(\mathbf{U}) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}$$

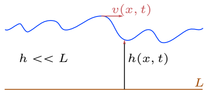
The *thermodynamic entropy* is defined as:

$$s(\mathbf{U}) = -\frac{\gamma S}{\gamma - 1} \qquad S - \log\left(\frac{p}{\rho^\gamma}\right) \text{ specifyc entropy} \tag{5.111}$$

If we define $q(\mathbf{U}) = -\frac{\gamma v S}{\gamma - 1}$ then s, q is an entropy pair.

Example 5.16
Roe Matrix for Shallow Water Equation??:

$$\partial_t h + \partial_x(hv) = 0$$
$$\partial_t(hv) + \partial_x\left(\frac{1}{2}gh^2 + hv^2\right) = 0 \tag{5.112}$$



$v(x,t)$: horizontal velocity of water column at x and the moment is $m = hv$:

$$\mathbf{U} = \begin{pmatrix} h \\ m \end{pmatrix} \qquad \mathbf{f}(\mathbf{U}) = \begin{pmatrix} m \\ \frac{1}{2}gh^2 + \frac{m^2}{h} \end{pmatrix} \tag{5.113}$$

Thus we have:

$$[\![\mathbf{U}]\!] = \begin{pmatrix} [\![h]\!] \\ [\![hv]\!] \end{pmatrix} \qquad [\![\mathbf{f}]\!] = \begin{pmatrix} [\![hv]\!] \\ [\![\frac{1}{2}gh^2 + hv^2]\!] \end{pmatrix} \tag{5.114}$$

From eq. (5.38) it follows:

$$\begin{pmatrix} [\![hv]\!] \\ [\![\frac{1}{2}gh^2 + hv^2]\!] \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} A_{11}[\![h]\!] + A_{12}[\![hv]\!] \\ A_{21}[\![h]\!] + A_{22}[\![hv]\!] \end{pmatrix}$$

We see that $A_{11} = 0$ and $A_{12} = 1$ in order for the first equation to hold.
In order to solve the second rational equation we use the approach proposition 5.5 and define:

$$\begin{aligned} & & z_1 &= \sqrt{h} & z_2 &= \sqrt{hv} \\ &\implies & h &= z_1^2 & hv &= z_1 z_2 \\ &\implies & h^2 &= z^4 & hv^2 &= z_2^2 \end{aligned}$$

Thus it follows for the second equation:

$$\left[\!\!\left[\frac{1}{2}gz_1^4 + z_2^2\right]\!\!\right] \stackrel{!}{=} A_{21}[\![z_1^2]\!] + A_{22}[\![z_1 z_2]\!]$$

Using the identities from proposition 5.5 we obtain:

$$2\bar{z}_2 + 2g\overline{z_1^2}\bar{z}_1[\![z_1]\!] = 2A_{21}\bar{z}_1[\![z_1]\!] + A_{22}\bar{z}_2[\![z_1]\!] + A_{22}\bar{z}_1[\![z_2]\!]$$

By comparing $[\![\cdot]\!]$ terms we find:

$$A_{22}\bar{z}_1 = 2\bar{z}_2 \implies \underline{A_{22}} = \frac{2\bar{z}_2}{\bar{z}_1}$$

$$2A_{21}\bar{z}_1 + A_{22}\bar{z}_2 = 2g\overline{z_1^2}\bar{z}_1 \implies A_{21} = g\overline{z_1^2} - \underline{A_{22}}\frac{\bar{z}_2}{2\bar{z}_1}$$

$$A_{22} = \frac{2\bar{z}_2}{\bar{z}_1} \qquad A_{21} = g\overline{z_1^2} - \left(\frac{\bar{z}_2}{\bar{z}_1}\right)^2$$

Plugging in z_1 and z_2 leads to the Roe matrix:

$$\mathbf{A}^n_{j+1/2} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ g\bar{h} - \hat{v}^2 & 2\hat{v} \end{pmatrix} \tag{5.115}$$

with the Roe Averages defined as:

$$\bar{h} := \frac{h_j^n + h_{j+1}^n}{2} \qquad \hat{v} = \frac{\sqrt{h_j}v_j^n + \sqrt{h_{j+1}^n}v_{j+1}^n}{\sqrt{h_j^n} + \sqrt{h_{j+1}^n}} \tag{5.116}$$

Thus the Roe matrix is exactly equal to the Jaccobian of $\mathbf{f}'(\mathbf{U})$ **but** evaluate at the *Roe Averages*.

Tell Mishra typo in script for second equation $2gz_1z_1$ instead of $z_1z_1^2$

Example 5.17
Roe Matrix for Euler Equation??:

add