

1a. Let's call the number of solutions for n seating positions arranged in a straight line as $S(n)$. In any such solution, either the n th seat is chosen or not. The number of solutions for the latter case is $S(n-1)$, and that for the former case is $S(n-2)$ (because when the n th seat is chosen, the $(n-1)$ th seat cannot be). Thus, $S(n) = S(n-1) + S(n-2)$. For the base case, we can easily see that $S(0) = 1$ (because the act of not choosing anything is itself a solution) and $S(1) = 2$ (because we can either choose the only seat or not). Thus, we see that $S(n)$ has the same recurrence as $F(n)$, the Fibonacci sequence, but $S(0) = F(1)$ and $S(1) = F(2)$. Therefore, by induction, $S(n) = F(n+1)$. This concludes the proof.

1b. Let's call the number of solutions for n seating positions arranged in a circle as $T(n)$. We can see that almost any solution the problem where seats are arranged in a straight line can be a solution to the case where the seats are arranged in a circle. The only cases we must exclude are those where seat 1 and seat n are both chosen. For $n \geq 4$, the number of such cases is $S(n-4)$, since both 1 and n are chosen, neither seat 2 nor seat $n-1$ can be. So, the number of solutions for n seats arranged in a circle is simply $S(n) - S(n-4) = F(n+1) - F(n-3) = F(n) + F(n-1) - F(n-3) = F(n) + F(n-2)$. For the cases $n = 3$ and $n = 2$, we can manually verify that this equation holds. This concludes the proof.

2a. Consider the two expressions $F_n^2 - F_{n+1}F_{n-1}$ and one of similar form obtained by replacing n with $n-1$, $F_{n-1}^2 - F_nF_{n-2}$. We will first prove that these two terms add to 0.

$$\begin{array}{ll}
F_n^2 - F_{n+1}F_{n-1} + F_{n-1}^2 & -F_nF_{n-2} \\
F_n^2 - F_{n-1}(F_{n+1} - F_{n-1}) & -F_nF_{n-2} \\
F_n^2 - F_{n-1}F_n & -F_nF_{n-2} \\
F_n(F_n - F_{n-1}) & -F_nF_{n-2} \\
F_nF_{n-2} & -F_nF_{n-2} \\
0 &
\end{array}$$

Thus, such expressions have the same magnitude but alternate signs each step along the way (from $n = 0$ onwards). We can check the base case value of $F_1^2 - F_2F_0 = -1$ to infer that $F_n^2 - F_{n+1}F_{n-1} = (-1)^n$. This concludes the proof.

2b. This one can be derived by repeated application of the recursive definition of the Fibonacci series.

$$\begin{aligned}
F_{n+2} &= F_n + F_{n+1} \\
F_{n+2} &= F_n + F_{n-1} + F_n \\
F_{n+2} &= F_n + F_{n-1} + F_{n-2} + F_{n-1} \\
&\dots \\
F_{n+2} &= \sum_{i=n}^0 F_i + F_1
\end{aligned}$$

This concludes the proof.

2c. We can easily see that $F_2 = F_0^2 + F_1^2 = 2$ and $F_3 = F_1(F_0 + F_2) = 3$. Assume that the given equations are true for up to $2n+1$. Then, we can derive the following for F_{2n+2} and F_{2n+3} .

$$\begin{aligned}
F_{2n+2} &= F_{2n} + F_{2n+1} \\
F_{2n+2} &= F_{n-1}^2 + F_n^2 + F_n(F_{n-1} + F_{n+1}) \\
F_{2n+2} &= F_{n-1}^2 + F_n(F_n + F_{n-1} + F_{n+1}) \\
F_{2n+2} &= F_{n-1}^2 + F_n(2F_{n+1}) \\
F_{2n+2} &= (F_{n+1} - F_n)^2 - 2F_nF_{n+1} \\
F_{2n+2} &= F_n^2 + F_{n+1}^2
\end{aligned}$$

$$\begin{aligned}
F_{2n+3} &= F_{2n+1} + F_{2n+2} \\
F_{2n+3} &= F_n(F_{n-1} + F_{n+1}) + F_n^2 + F_{n+1}^2 \\
F_{2n+3} &= F_n(F_{n-1} + F_{n+1} + F_n) + F_{n+1}^2 \\
F_{2n+3} &= F_n(2F_{n+1}) + F_{n+1}^2 \\
F_{2n+3} &= F_{n+1}(2F_n + F_{n+1}) \\
F_{2n+3} &= F_{n+1}(F_n + F_{n+2})
\end{aligned}$$

This concludes the proof by induction.