3a. Consider any processing operation that has two types A and B. Now, let's count the number of ways we can pick k objects from n and apply processing operation A on l of the k objects and operation B on the remaining $k-l$ objects. We can count this in two equivalent ways.

- $\binom{n}{k}\binom{k}{l}$, which involves picking k objects from n, and then picking the l from amongst those k.
- $\binom{n}{l}\binom{n-l}{k-l}$, which involves picking l objects from n, and then picking the $k-l$ from amongst the remaining $n-l$.

This concludes the proof.

3b. Let's say we have m boys and n girls to pick a total of k people from. We can count this in two equivalent ways.

- $\binom{m+n}{k}$, which involves picking k objects from amongst the total of $m+n$ people.
- $\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$, which involves counting $k+1$ exclusive possibilities, wherein in each possibility, we pick a certain number i of boys from amongst the m boys, and the remaining $k-i$ required people from the n girls.

This concludes the proof.

3c. This can easily be proven by repeated application of the rule $\binom{n+k+1}{k} = \binom{n+k}{k} + \binom{n+k}{k-1}$, to the last term in this equation all the way down to $\binom{n}{0}$. Induction seems like the cleanest way to setup this proof.

3d. The algebraic proof is based on differentiating the two sides of the binomial expression for $(1+t)^n$ and then replacing t with 1. But this is a bit mundane. So let's give a combinatorial proof instead.

Consider n objects, and say we need to do two things - first pick at least one of them, and then designate a leader from amongst those picked. We can count this in two equivalent ways.

- $\sum_{i=1}^{k}\binom{n}{i}i$, which involves counting k exclusive possibilities, wherein in each possibility, we first pick a certain number i of objects from among the n objects, and then designate one of those i objects as leader.
- $n2^{n-1}$, which involves first picking a leader, and then picking 0 or more objects from the remaining $n-1$ objects.

This concludes the proof.

3e. We'll use the binomial theorem. Consider the expression $(1+a)^n(1-a)^n = (1-a^2)^n$ and its full expansion. We'd like to specifically find the coefficient of the term a^n in the expansion. We can count this in two equivalent ways.

- by picking for each value of k in the range $[0, n]$, a^k from the expansion of $(1 + a)^n$ and multiplying it with a^{n-k} from the expansion of $(1 - a)^n$, thus implying that the desired coefficient would be $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k}$.
- by picking the coefficient of a^n in the expansion of $(1-a^2)^n$, which happens to be 0 in case n is odd, and $(-1)^{n/2}\binom{n}{n/2}$ in case n is even.

This concludes the proof.

6. For any prime p, it is clear that $0^p \equiv 0 \mod p$. Let's assume that r is the largest integer for which we know $r^p \equiv r \mod p$ to be true. Then, the following is true.

$$(1 + r)^p \equiv (1 + r^p) \mod p$$
$$(1 + r)^p \equiv (1 + r) \mod p$$

This concludes the proof by induction.