9. Any integer n can be written as a product of prime powers. Say $n = \prod_{i=1}^{l} {p_i}^{a_i}$. Then we can say that $f(n) = \prod_{i=1}^{l} f({p_i}^{a_i})$, for any multiplicative function f, because the prime powers are relatively co-prime. Therefore, such a function is purely determined by its values on prime powers. This concludes the proof.

10. Consider $g(ab)$ where $gcd(a,b) = 1$. By definition of $g(n)$, $g(ab) = \sum_{d|ab} f(d)$. But since $gcd(a,b) = 1$, any divisor of ab can be written in the form $d_a d_b$ where $d_a|a$ and $d_b|b$. Therefore, $g(ab) = \sum_{d_a|a, d_b|b} f(d_a d_b)$. But since f is multiplicative, we get $g(ab) = \sum_{d_a|a, d_b|b} f(d_a)f(d_b) = (\sum_{d_a|a} f(d_a))(\sum_{d_b|b} f(d_b)) = g(a)g(b)$, which implies that g too is multiplicative. This concludes the proof.

11. Consider $\mu(ab)$ where $gcd(a,b) = 1$. If either a or b is not square free, then both $\mu(ab) = 0 = \mu(a)\mu(b)$. If a and b are square free, then ab is square free as well because $gcd(a,b) = 1$. And $\mu(ab)$ is 1 or -1 depending on whether the parities of $\mu(a)$ and $\mu(b)$ match or not. Therefore $\mu(ab) = \mu(a)\mu(b)$, proving that $\mu(n)$ is a multiplicative function.

It is easy to see that if $f(n)$ is multiplicative, then both $nf(n)$ and $f(n)/n$ are multiplicative. We'll combine these lemmas with the result of problem 10 to conclude that the function $f(n) = n\sum_{d|n} \mu(d)/d$ is also a multiplicative function.

Now, by the results of problem 9, the values of any multiplicative function are purely determined by its values on prime powers. So, we'll evaluate $f(p^k)$ for some prime p and some non-negative integer k. If we find that this value matches the value of $\phi(p^k)$ for all k, then we can infer that $\phi(n) = f(n) = n\sum_{d|n} \mu(d)/d$.

$$
\begin{aligned}
f(p^k) &= p^k \sum_{d|p^k} \mu(d)/d \\
f(p^k) &= p^k \sum_{i \in \{0 \cdots k\}} \mu(p^i)/p^i \\
f(p^k) &= p^k((1/p^0) + (-1/p^1) + 0 + 0 + ...) \\
f(p^k) &= p^k(1 - 1/p)
\end{aligned}
$$

Now, let's calculate $\phi(p^k)$ from first principles. Since the only integers in the range $\{1 \cdots p^k\}$ that are not coprime with p^k are the multiples of p, and there are p^{k-1} such multiples, $\phi(p^k) = p^k = p^{k-1} = p^k(1 - 1/p)$. This concludes the proof.

12. From exercises 10 and 11, we have learned that $\mu(n)$ and $\phi(n)$ are multiplicative functions. Therefore, so are $\sum_{d|n} \mu(d)\phi(d)$, $\sum_{d|n} \mu(d)^2\phi(d)^2$ and $\sum_{d|n} \mu(d)/\phi(d)$. This means that for any of these 3 functions, we can com-

pute the value of the function for a given input n by first computing the value of the function on all prime powers involved in n's prime factorization, and then by multipliying these values. In all the equations below, we'll assume $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$.

$$\sum_{d|p^k} \mu(d)\phi(d) = 1(1) + (-1)(p-1) + 0 + 0 + \cdots$$
$$\sum_{d|p^k} \mu(d)\phi(d) = 2 - p$$
$$\therefore \sum_{d|n} \mu(d)\phi(d) = \prod_{i=1}^{k} (2 - p_i)$$

$$\sum_{d|p^k} \mu(d)^2\phi(d)^2 = 1^2(1^2) + (-1)^2(p-1)^2 + 0 + 0 + \cdots$$
$$\sum_{d|p^k} \mu(d)^2\phi(d)^2 = p^2 - 2p + 2$$
$$\therefore \sum_{d|n} \mu(d)^2\phi(d)^2 = \prod_{i=1}^{k} (p_i^2 - 2p_i + 2)$$

$$\sum_{d|p^k} \mu(d)/\phi(d) = 1/1 + (-1)/(p-1) + 0 + 0 + \cdots$$
$$\sum_{d|p^k} \mu(d)/\phi(d) = (p-2)/(p-1)$$
$$\therefore \sum_{d|n} \mu(d)/\phi(d) = \prod_{i=1}^{k} (p_i - 2)/(p_i - 1)$$

15a. Imagine a function $f(n) = 1$ for all values of n. Then, we can say that $\nu(n) = \sum_{d|n} 1 = \sum_{d|n} f(d)$. Thus, by the Mobius inversion theorem, $\sum_{d|n} \mu(n/d)\nu(d) = f(n) = 1$. This concludes the proof.

15b. Imagine a function $f(n) = n$ for all values of n. Then, we can say that $\sigma(n) = \sum_{d|n} d = \sum_{d|n} f(d)$. Thus, by the Mobius inversion theorem, $\sum_{d|n} \mu(n/d)\sigma(d) = f(n) = n$. This concludes the proof.

16. Let's say that $n = p_1^{a_1} p_2^{a_2} \cdots p_l^{a_l}$. Then we know that $\nu(n) = \prod_{i=1}^{l} (a_i + 1)$. We can infer that $\nu(n)$ is odd iff each of the terms on the RHS is odd i.e. if a_i is even i.e. n is a square. This concludes the proof.

17. Let's say that $n = p_1^{a_1} p_2^{a_2} \cdots p_l^{a_l}$. Then we know that $\sigma(n) = \prod_{i=1}^{l} \sum_{j=0}^{a_i} p_i^j$. We can infer that $\sigma(n)$ is odd iff each of the terms on the RHS is odd. i.e. $\sum_{j=0}^{a_i} p_i^j$ is odd. If p_i is odd, then this sum is odd iff a_i is even. If p_i is even (i.e. equal to 2, the only even prime), then this sum is always odd irrespective of whether a_i is even or odd. Thus, the condition is met iff n is either a square or 2 times a square.

25. The Riemann zeta function $\zeta(s)$ is defined as $\zeta(s) = \sum_{i=1}^{\infty}(1/i^s)$. Consider all terms where the denominator's prime factorization contains ${p_1}^{js}$ (where j is obviously a non-negative integer). Group those terms (an ∞ of them) and extract $1/{p_1}^{js}$ as a common factor, wherein the other factor is an infinite sum which we'll denote $\zeta(s, p_1)$. Note that this construction made no assumptions about the actual value of j, so we should be able to independently do this for each non-negative integer j. Thus, we have,

$$\begin{aligned}
\zeta(s) &= (1/{p_1}^{0s})\zeta(s,p_1) + (1/{p_1}^{1s})\zeta(s,p_1) + (1/{p_1}^{2s})\zeta(s,p_1) + \cdots \\
\zeta(s) &= (1 + (1/{p_1}^{s}) + (1/{p_1}^{s})^2 + \cdots)\zeta(s,p_1) \\
\zeta(s) &= (1 - (1/{p_1}^{s}))^{-1}\zeta(s,p_1)
\end{aligned}$$

Now, note that $\zeta(s, p_1)$ has no occurrence of the prime p_1 in it. Aside from that, it is identical to $\zeta(s)$. So, we can perform a similar procedure by extracting out yet another prime p_2 out of $\zeta(s, p_1)$, and repeatedly simplifying the above expression. It is clear that this procedure will help derive the following.

$$\begin{aligned}
\zeta(s) &= (1 - (1/{p_1}^{s}))^{-1}(1 - (1/{p_2}^{2}))^{-1} \cdots \\
\zeta(s) &= \prod_p (1 - (1/p^s))^{-1}
\end{aligned}$$

This concludes the proof.