

9. Any integer  $n$  can be written as a product of prime powers. Say  $n = \prod_{i=1}^l p_i^{a_i}$ . Then we can say that  $f(n) = \prod_{i=1}^l f(p_i^{a_i})$ , for any multiplicative function  $f$ , because the prime powers are relatively co-prime. Therefore, such a function is purely determined by its values on prime powers. This concludes the proof.

10. Consider  $g(ab)$  where  $\gcd(a, b) = 1$ . By definition of  $g(n)$ ,  $g(ab) = \sum_{d|ab} f(d)$ . But since  $\gcd(a, b) = 1$ , any divisor of  $ab$  can be written in the form  $d_a d_b$  where  $d_a|a$  and  $d_b|b$ . Therefore,  $g(ab) = \sum_{d_a|a, d_b|b} f(d_a d_b)$ . But since  $f$  is multiplicative, we get  $g(ab) = \sum_{d_a|a, d_b|b} f(d_a) f(d_b) = (\sum_{d_a|a} f(d_a)) (\sum_{d_b|b} f(d_b)) = g(a) g(b)$ , which implies that  $g$  too is multiplicative. This concludes the proof.

11. Consider  $\mu(ab)$  where  $\gcd(a, b) = 1$ . If either  $a$  or  $b$  is not square free, then both  $\mu(ab) = 0 = \mu(a)\mu(b)$ . If  $a$  and  $b$  are square free, then  $ab$  is square free as well because  $\gcd(a, b) = 1$ . And  $\mu(ab)$  is 1 or  $-1$  depending on whether the parities of  $\mu(a)$  and  $\mu(b)$  match or not. Therefore  $\mu(ab) = \mu(a)\mu(b)$ , proving that  $\mu(n)$  is a multiplicative function.

It is easy to see that if  $f(n)$  is multiplicative, then both  $nf(n)$  and  $f(n)/n$  are multiplicative. We'll combine these lemmas with the result of problem 10 to conclude that the function  $f(n) = n \sum_{d|n} \mu(d)/d$  is also a multiplicative function.

Now, by the results of problem 9, the values of any multiplicative function are purely determined by its values on prime powers. So, we'll evaluate  $f(p^k)$  for some prime  $p$  and some non-negative integer  $k$ . If we find that this value matches the value of  $\phi(p^k)$  for all  $k$ , then we can infer that  $\phi(n) = f(n) = n \sum_{d|n} \mu(d)/d$ .

$$\begin{aligned} f(p^k) &= p^k \sum_{d|p^k} \mu(d)/d \\ f(p^k) &= p^k \sum_{i \in \{0 \dots k\}} \mu(p^i)/p^i \\ f(p^k) &= p^k ((1/p^0) + (-1/p^1) + 0 + 0 + \dots) \\ f(p^k) &= p^k (1 - 1/p) \end{aligned}$$

But we already know that  $\phi(p^k) = p^k(1 - 1/p)$ . This concludes the proof.

15a. Imagine a function  $f(n) = 1$  for all values of  $n$ . Then, we can say that  $\nu(n) = \sum_{d|n} 1 = \sum_{d|n} f(d)$ . Thus, by the Mobius inversion theorem,  $\sum_{d|n} \mu(n/d) \nu(d) = f(n) = 1$ . This concludes the proof.

15b. Imagine a function  $f(n) = n$  for all values of  $n$ . Then, we can say that  $\sigma(n) = \sum_{d|n} d = \sum_{d|n} f(d)$ . Thus, by the Mobius inversion theorem,

$\sum_{d|n} \mu(n/d)\sigma(d) = f(n) = n$ . This concludes the proof.