

1a. Let's call the number of solutions for  $n$  seating positions arranged in a straight line as  $S(n)$ . In any such solution, either the  $n$ th seat is chosen or not. The number of solutions for the latter case is  $S(n-1)$ , and that for the former case is  $S(n-2)$  (because when the  $n$ th seat is chosen, the  $(n-1)$ th seat cannot be). Thus,  $S(n) = S(n-1) + S(n-2)$ . For the base case, we can easily see that  $S(0) = 1$  (because the act of not choosing anything is itself a solution) and  $S(1) = 2$  (because we can either choose the only seat or not). Thus, we see that  $S(n)$  has the same recurrence as  $F(n)$ , the Fibonacci sequence, but  $S(0) = F(1)$  and  $S(1) = F(2)$ . Therefore, by induction,  $S(n) = F(n+1)$ . This concludes the proof.

1b. Let's call the number of solutions for  $n$  seating positions arranged in a circle as  $T(n)$ . We can see that almost any solution the problem where seats are arranged in a straight line can be a solution to the case where the seats are arranged in a circle. The only cases we must exclude are those where seat 1 and seat  $n$  are both chosen. For  $n \geq 4$ , the number of such cases is  $S(n-4)$ , since both 1 and  $n$  are chosen, neither seat 2 nor seat  $n-1$  can be. So, the number of solutions for  $n$  seats arranged in a circle is simply  $S(n) - S(n-4) = F(n+1) - F(n-3) = F(n) + F(n-1) - F(n-3) = F(n) + F(n-2)$ . For the cases  $n = 3$  and  $n = 2$ , we can manually verify that this equation holds. This concludes the proof.

2a. Consider the two expressions  $F_n^2 - F_{n+1}F_{n-1}$  and one of similar form obtained by replacing  $n$  with  $n-1$ ,  $F_{n-1}^2 - F_nF_{n-2}$ . We will first prove that these two terms add to 0.

$$\begin{array}{ll}
F_n^2 - F_{n+1}F_{n-1} + F_{n-1}^2 & -F_nF_{n-2} \\
F_n^2 - F_{n-1}(F_{n+1} - F_{n-1}) & -F_nF_{n-2} \\
F_n^2 - F_{n-1}F_n & -F_nF_{n-2} \\
F_n(F_n - F_{n-1}) & -F_nF_{n-2} \\
F_nF_{n-2} & -F_nF_{n-2} \\
0 & 
\end{array}$$

Thus, such expressions have the same magnitude but alternate signs each step along the way (from  $n = 0$  onwards). We can check the base case value of  $F_1^2 - F_2F_0 = -1$  to infer that  $F_n^2 - F_{n+1}F_{n-1} = (-1)^n$ . This concludes the proof.

2b. This one can be derived by repeated application of the recursive definition of the Fibonacci series.

$$\begin{aligned}
F_{n+2} &= F_n + F_{n+1} \\
F_{n+2} &= F_n + F_{n-1} + F_n \\
F_{n+2} &= F_n + F_{n-1} + F_{n-2} + F_{n-1} \\
&\dots \\
F_{n+2} &= \sum_{i=n}^0 F_i + F_1
\end{aligned}$$

This concludes the proof.

2c. We can easily see that  $F_2 = F_0^2 + F_1^2 = 2$  and  $F_3 = F_1(F_0 + F_2) = 3$ . Assume that the given equations are true up to  $2n+1$ . Then, we can derive the following for  $F_{2n+2}$  and  $F_{2n+3}$ .

$$\begin{aligned}
F_{2n+2} &= F_{2n} + F_{2n+1} \\
F_{2n+2} &= F_{n-1}^2 + F_n^2 + F_n(F_{n-1} + F_{n+1}) \\
F_{2n+2} &= F_{n-1}^2 + F_n(F_n + F_{n-1} + F_{n+1}) \\
F_{2n+2} &= F_{n-1}^2 + F_n(2F_{n+1}) \\
F_{2n+2} &= (F_{n+1} - F_n)^2 - 2F_nF_{n+1} \\
F_{2n+2} &= F_n^2 + F_{n+1}^2
\end{aligned}$$

$$\begin{aligned}
F_{2n+3} &= F_{2n+1} + F_{2n+2} \\
F_{2n+3} &= F_n(F_{n-1} + F_{n+1}) + F_n^2 + F_{n+1}^2 \\
F_{2n+3} &= F_n(F_{n-1} + F_{n+1} + F_n) + F_{n+1}^2 \\
F_{2n+3} &= F_n(2F_{n+1}) + F_{n+1}^2 \\
F_{2n+3} &= F_{n+1}(2F_n + F_{n+1}) \\
F_{2n+3} &= F_{n+1}(F_n + F_{n+2})
\end{aligned}$$

This concludes the proof by induction.

7. The inefficient algorithm computes  $F_n$  recursively as  $F_{n-1} + F_{n-2}$ , then each of these recursively without storing intermediate results. Let's call the number of additions required to compute  $F_n$  as  $A_n$ . Clearly,  $A_0 = A_1 = 0$ , since the algorithm uses the base case values of  $F_0 = F_1 = 1$  and does not need to perform

any additions in this case. Further, we can see that  $A_n = A_{n-1} + A_{n-2} + 1$ , because in order to compute  $F_n$  naively, the algorithm would have to compute  $F_{n-1}$  and  $F_{n-2}$  independently, which would respectively take  $A_{n-1}$  and  $A_{n-2}$  additions, and then finally perform an extra addition to sum these two values. It is easy to see via induction that  $A_n = F_n - 1$ . This concludes the proof.