

1.

$$Y'(x) = \frac{1}{1+x^2} - 2[Y(x)]^2, 0 \leq x \leq 10, Y(0) = 0$$

şeklinde verilen problemi $h = 0.2, 0.1$ ve 0.05 adım aralıkları verilmiş olsun. Gerçek fonksiyon

$$Y(x) = \frac{x}{1+x^2}$$

olmak üzere bu problemi

- (a) Euler yöntemi ile çözerek x değişkeninin tam sayı değerleri için x, y_n , hata ve bağlı hata miktarlarını gösteren bir tablo oluşturunuz.
- (b) Heun yöntemi ile çözerek x değişkeninin tam sayı değerleri için x, y_n , hata ve bağlı hata miktarlarını gösteren bir tablo oluşturunuz.

Çözüm:

a. Euler formülü

$$y_{n+1} = y_n + hf(x_n, y_n) = y_n + h \left(\frac{1}{1+x_n^2} - 2[y_n]^2 \right)$$

şeklinde oluşturulur. Burada y_0 başlangıç değeri ve x_n değerleri belirlenir.

$$x_0 = 0, x_n = x_0 + hn, (0 \leq x_n \leq 10)$$

$$Y(0) = 0 \Rightarrow y_0 = 0$$

$h = 0.2$ için:

x	y_n	$ Y(x) - y_n $	$ (Y(x) - y_n)/Y(x) $
1.0	0.546604644213360	0.046604644213360	0.093209288426720
2.0	0.406819032556396	0.006819032556396	0.017047581390990
3.0	0.298893681993224	0.001106318006776	0.003687726689253
4.0	0.233156075864659	0.002138041782400	0.009086677575201
5.0	0.190322349384792	0.001985342922901	0.010323783199083
6.0	0.160497698499705	0.001664463662457	0.010264192585154
7.0	0.138630123170914	0.001369876829086	0.009784834493472
8.0	0.121945976977806	0.001130946099117	0.009188937055323
9.0	0.108813517900829	0.000942579660146	0.008587948014666
10.0	0.098215604247748	0.000794296742351	0.008022397097746

$h = 0.1$ için:

\mathbf{x}	$\mathbf{y_n}$	$ \mathbf{Y(x)} - \mathbf{y_n} $	$ (\mathbf{Y(x)} - \mathbf{y_n})/\mathbf{Y(x)} $
1.0	0.522675375110103	0.022675375110103	0.045350750220207
2.0	0.404188667792509	0.004188667792509	0.010471669481273
3.0	0.299769930025394	0.000230069974606	0.000766899915354
4.0	0.234346941396897	0.000947176250162	0.004025499063190
5.0	0.191362709313518	0.000944982994174	0.004913911569706
6.0	0.161348614245190	0.000813547916972	0.005016878821330
7.0	0.139321638729452	0.000678361270548	0.004845437646771
8.0	0.122512762867245	0.000564160209678	0.004583801703637
9.0	0.109283817757716	0.000472279803260	0.004302993763032
10.0	0.098610805366642	0.000399095623457	0.004030865796918

$h = 0.05$ için:

\mathbf{x}	$\mathbf{y_n}$	$ \mathbf{Y(x)} - \mathbf{y_n} $	$ (\mathbf{Y(x)} - \mathbf{y_n})/\mathbf{Y(x)} $
1.0	0.511202462231354	0.011202462231354	0.022404924462708
2.0	0.402271408325142	0.002271408325142	0.005678520812855
3.0	0.299968314535859	0.000031685464141	0.000105618213804
4.0	0.234854463622383	0.000439654024676	0.001868529604874
5.0	0.191849431821231	0.000458260486461	0.002382954529597
6.0	0.161761484588097	0.000400677574065	0.002470845040069
7.0	0.139663351677596	0.000336648322404	0.002404630874312
8.0	0.122795739687043	0.000281183389880	0.002284615042775
9.0	0.109520085175719	0.000236012385257	0.002150335065672
10.0	0.098810120845045	0.000199780145054	0.002017779465042

b. Runge-Kutta yöntemlerinin genel yapısı şu şekildedir.

$$y_{n+1} = y_n + hF(x_n, y_n; h)$$

Heun ise bu yöntemlerin özel bir halidir ve $F(x_n, y_n; h)$ fonksiyonu şu şekilde seçilir.

$$F(x_n, y_n; h) = \frac{1}{2} [f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))]$$

Bu durumda verilen problem için Heun formülü

$$y_{n+1} = y_n + \frac{h}{2} \left[\left(\frac{1}{1 + x_n^2} - 2[y_n]^2 \right) + \left(\frac{1}{1 + (x_n + h)^2} - 2 \left[y_n + h \left(\frac{1}{1 + x_n^2} - 2[y_n]^2 \right) \right]^2 \right) \right]$$

şeklindedir. Burada y_0 başlangıç değeri ve x_n değerleri belirlenir.

$$x_0 = 0, x_n = x_0 + hn, (0 \leq x_n \leq 10)$$

$$Y(0) = 0 \Rightarrow y_0 = 0$$

$h = 0.2$ için:

\mathbf{x}	$\mathbf{y_n}$	$ \mathbf{Y(x) - y_n} $	$ (\mathbf{Y(x) - y_n})/\mathbf{Y(x)} $
1.0	0.491123393740728	0.008876606259272	0.017753212518545
2.0	0.398371373026349	0.001628626973651	0.004071567434129
3.0	0.299910598882150	0.000089401117850	0.000298003726168
4.0	0.235425461766576	0.000131344119518	0.000558212507949
5.0	0.192442715768011	0.000135023460318	0.000702121993656
6.0	0.162270041197313	0.000107879035151	0.000665254050097
7.0	0.140082299404418	0.000082299404418	0.000587852888698
8.0	0.123139588517078	0.000062665440155	0.000509156701255
9.0	0.109804335629412	0.000048238068437	0.000439502401312
10.0	0.099047574755271	0.000037673765172	0.000380505028241

$h = 0.1$ için:

\mathbf{x}	$\mathbf{y_n}$	$ \mathbf{Y(x) - y_n} $	$ (\mathbf{Y(x) - y_n})/\mathbf{Y(x)} $
1.0	0.498125347793041	0.001874652206959	0.003749304413918
2.0	0.399686680579039	0.000313319420961	0.000783298552402
3.0	0.299998657022341	0.000001342977659	0.000004476592196
4.0	0.235332314926103	0.000038197279044	0.000162338435936
5.0	0.192342898885287	0.000035206577595	0.000183074203492
6.0	0.162189467515459	0.000027305353296	0.000168383011994
7.0	0.140020572451687	0.000020572451687	0.000146946083477
8.0	0.123092489868466	0.000015566791542	0.000126480181283
9.0	0.109768039290038	0.000011941729062	0.000108802420346
10.0	0.099019208964651	0.000009307974552	0.000094010542977

$h = 0.05$ için:

\mathbf{x}	$\mathbf{y_n}$	$ \mathbf{Y(x) - y_n} $	$ (\mathbf{Y(x) - y_n})/\mathbf{Y(x)} $
1.0	0.499567179293205	0.000432820706795	0.000865641413589
2.0	0.399930168493392	0.000069831506608	0.000174578766520
3.0	0.300001340548382	0.000001340548382	0.000004468494607
4.0	0.235304010712349	0.000009893065290	0.000042045527481
5.0	0.192316541778077	0.000008849470385	0.000046017246000
6.0	0.162168967738405	0.000006805576243	0.000041967720164
7.0	0.140005110058719	0.000005110058719	0.000036500419423
8.0	0.123080783754021	0.000003860677098	0.000031368001423
9.0	0.109759057028867	0.000002959467892	0.000026964040789
10.0	0.099012207007303	0.000002306017204	0.000023290773764

