Test Exam Solutions

1) The random variables x and y are jointly distributed over the region $0<x<y<1$ as

$$f_{x,y}(x,y) = \begin{cases} kxy^2 & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find k.

$$\iint f_{xy}(x,y)\,dx\,dy = 1.$$

$$\int_0^1\int_0^y kxy^2\,dx\,dy = \int_0^1 k\frac{x^2}{2}y^2\Big|_0^y dy = \int_0^1 \frac{k\,y^4}{2}\,dy$$

$$= \frac{ky^5}{10}\Big|_0^1 = \frac{k}{10} = 1 \qquad k=10 \,//$$

b) $Var[x] = E[x^2] - E[x]^2$

$$E[x] = \int_0^1\int_0^y x\cdot f_{xy}(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^y 10x^2y^2\,dx\,dy = \int_0^1 \frac{10}{3}x^3y^2\Big|_0^y dy$$

$$= \int_0^1 \frac{10}{3}y^5\,dy = \frac{10}{18}y^6\Big|_0^1 = \frac{5}{9} \,//$$

$$E[x^2] = \int_0^1 \int_0^y x^2 \cdot f_{xy}(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^y x^2 \cdot (10xy^2)\, dx\, dy = \int_0^1 \int_0^y 10x^3y^2\, dx\, dy$$

$$= \int_0^1 \frac{10}{4} x^4 y^2 \Big|_0^y dy = \int_0^1 \frac{10}{4} y^6\, dy = \frac{10}{28} y^7 \Big|_0^1 = \frac{5}{14} //$$

$$Var[x] = \frac{5}{14} - \left(\frac{5}{9}\right)^2 = 0{,}0485$$

$$Var[Y] = E[Y^2] - E[Y]^2$$

$$E[Y] = \int_0^1 \int_0^y y \cdot (10xy^2)\, dx\, dy = \int_0^1 \frac{10}{2} x^2 y^3 \Big|_0^y dy$$

$$= \int_0^1 \frac{10}{2} y^5\, dy = \frac{10}{12} y^6 \Big|_0^1 = \frac{5}{6} //$$

$$E[Y^2] = \int_0^1 \int_0^y y^2 (10xy^2)\, dx\, dy = \int_0^1 \frac{10x^2y^4}{2} \Big|_0^y dy$$

$$= \int_0^1 5y^6\, dy = \frac{5}{7} y^7 \Big|_0^1 = \frac{5}{7} //$$

$$Var[Y] = E[Y^2] - E[Y]^2 = \frac{5}{7} - \left(\frac{5}{6}\right)^2 = 0{,}02 //$$

c) $Cov(X,Y) = E[XY] - E[X]E[Y]$

$$E[XY] = \iint xy\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^y xy\left(10xy^2\right) dx\, dy = \int_0^1 \frac{10}{3} x^3 y^3 \Big|_0^y \, dy$$

$$= \int_0^1 \frac{10}{3} y^6\, dy = \frac{10}{21} y^7 \Big|_0^1 = \frac{10}{21}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$\frac{10}{21} - \frac{5}{9} \cdot \frac{5}{6} = 0{,}013 \; //$$

2)

Birth year (x)	Life exp. (y)
1980	75
1990	80
1995	85
2000	90
2010	92
2015	95

$$y_i = \alpha + \beta x_i + e_i$$

$$\beta = \frac{Cov(x,y)}{\sigma_x^2} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}.$$

$$E[x] = \bar{x} = \frac{1}{6} \sum_{i=1}^{6} x_i = 98.3 \Rightarrow \sum_{i=1}^{6} x_i = 590$$

$$E[y] = \bar{y} = \frac{1}{6} \sum_{i=1}^{6} y_i = 86.16 \Rightarrow \sum_{i=1}^{6} y_i = 517$$

$$\sum_{i=1}^{6} x_i^2 = 58.850$$

$$\sum_{i=1}^{6} y_i^2 = 44.839$$

$$\sum_{i=1}^{6} x_i y_i = 51.320$$

$$\beta = \frac{Cov(x,y)}{\sigma_x^2} = \frac{\sum_{i=1}^{6} y_i x_i - \frac{1}{6} \sum_{i=1}^{6} x_i \sum_{i=1}^{6} y_i}{\sum x_i^2 - \frac{1}{6}\left(\sum x_i\right)^2}$$

$$\frac{51.320 - 50838}{58.850 - 58016} = 0,5784$$

$\alpha = \bar{y} - B\bar{x}$

$= 86.16 - 0.57 \cdot (98.3)$

$= 29.34$

$y_i = \underline{0.578\, x_i + 29.34}$

Therefore,

for $x_i = 2003 \rightarrow 103$

$y_i = 0.578 \cdot 103 + 29.34 = \underline{88.874}$