Theorems for lossless reciprocal 3-ports

There are three theorems which follow as a result of the lossless properties of reciprocal 3-ports. The first two relate to the effect of a short-circuit placed in one arm of a 3-port.

I. *A position can be found for the short-circuit for which there is no transmission of energy between the other two arms.*

II. *If the 3-port is symmetrical about the arm containing a short-circuit, a position of the short-circuit can be found for which there is complete transmission between the other two arms without loss or reflection.*

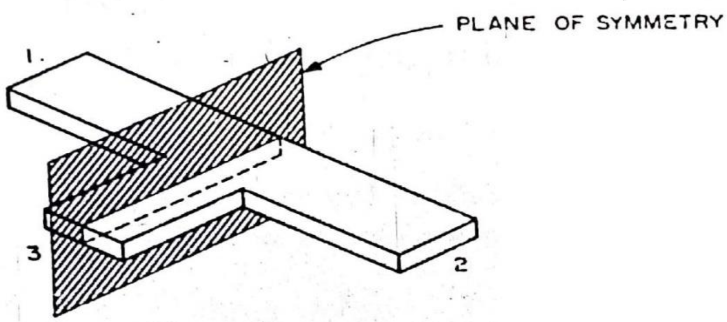
III. *A lossless 3-port cannot be completely non-reflecting.* That is, we cannot have

$$S_{11} = S_{22} = S_{33} = 0.$$

Symmetrical, lossless, reciprocal 3-ports

In the analysis of 3-ports, one needs in general nine complex parameters or eighteen real ones to characterize circuit behavior at one frequency. It is well to reduce the number of independent parameters right at the start, if possible, by making use of any simplifying assumptions that may be justified.

1. An example is the symmetrical, lossless, reciprocal H-plane tee shown in Fig. below Waveguide leads 1 and 2 are identical, as follows from the symmetry property, although number 3 could be different from numbers 1 and 2.



An H-plane tee showing the significant plane of symmetry.

Note that the symmetry and reciprocity properties reduce the number of parameters from nine to four as follows:

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \rightarrow \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{11} & S_{13} \\ S_{13} & S_{13} & S_{33} \end{bmatrix}$$

One of the lossless conditions from eqn. (2.165) is

$$1 - |S_{33}|^2 = 2\,|S_{13}|^2.$$

If we include a symmetrical lossless tuning element in arm 3 to make $S_{33} = 0$, then

$$|S_{13}| = \frac{1}{\sqrt{2}}.$$

An additional lossless condition is

$$\bar{S}_{11}S_{13} + \bar{S}_{21}S_{23} + \bar{S}_{31}S_{33} = 0.$$

But as a result of the above assumptions,

$$S_{11} = -S_{12}$$

Another lossless condition requires that

$$1-|S_{11}|^2 = |S_{21}|^2+S_{31}|^2$$

or

$$1-|S_{12}|^2 = |S_{21}|^2+\tfrac{1}{2} \quad \text{or} \quad |S_{12}| = \tfrac{1}{2}.$$

Finally, if we choose the locations of the terminal surfaces so that all of the elements are real, we can write the scattering matrix for the H-plane tee as follows:

$$S = \tfrac{1}{2}\begin{bmatrix} -1 & 1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{bmatrix}.$$

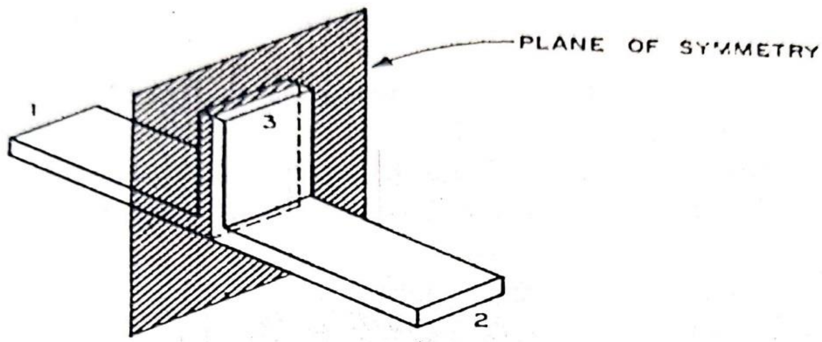
2. A rather different scattering matrix is obtained for the H-plane tee if one assumes that the symmetrical lossless tuning element in arm 3 is adjusted to make S_{11} rather than S_{33} vanish. In this case, it happens that $S_{13}=0$, and

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We know from theorem II that such an adjustment is possible.

This is a limiting case, since arm 3 is completely decoupled, and we have only 2 ports electrically coupled.

3. A similar argument may be used to obtain the scattering matrix of an E-plane tee shown in Fig. below



An E-plane tee showing the significant plane of symmetry.

The symmetry and reciprocity properties reduce the number of parameters from nine to four

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \rightarrow \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{11} & -S_{13} \\ S_{13} & -S_{13} & S_{33} \end{bmatrix}$$

If a lossless tuner located so as to be symmetrical about the above plane is adjusted so that $S_{33} = 0$, the lossless condition

$$|S_{13}|^2 + |S_{23}|^2 = 1 - |S_{33}|^2$$

requires that

$$|S_{13}| = \frac{1}{\sqrt{2}},$$

as for the H-plane tee. The additional lossless condition

$$S_{11}S_{13} + S_{21}S_{23} + S_{31}S_{33} = 0$$

requires that

$$S_{11} = S_{21}.$$

Also, the condition

$$1 - |S_{11}|^2 = |S_{21}|^2 + |S_{31}|^2$$

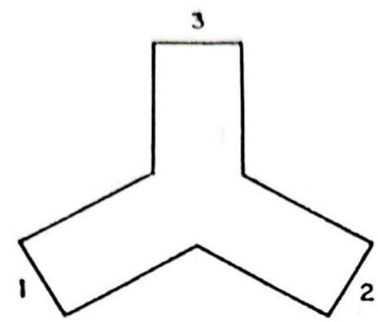
requires that

$$|S_{12}| = \tfrac{1}{2}.$$

Again, choosing the terminal surfaces in arms 1 and 2 so that all of the elements of the scattering matrix are real,

$$S = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{bmatrix}.$$

4. A symmetrical H-plane Y-junction has a threefold axis, three planes of symmetry and straight waveguide leads 120 degrees apart in space as shown in Fig. below.



A symmetrical H-plane Y-junction.

Symmetry reduces the number of parameters from nine to two

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \rightarrow \begin{bmatrix} S_{11} & S_{12} & S_{12} \\ S_{12} & S_{11} & S_{12} \\ S_{12} & S_{12} & S_{11} \end{bmatrix}$$

One of the lossless conditions requires that

$$1-|S_{11}|^2 = 2|S_{12}|^2,$$

and another requires that

$$S_{12}(S_{11}+S_{12}) = -S_{12}S_{11}.$$

If we choose the locations of the terminal surfaces so that S_{12} is real, then

$$S_{11}+S_{11} = -S_{12}, \qquad \cos\psi_{11} = -\frac{\sqrt{(1-|S_{11}|^2)}}{2\sqrt{2}\,|S_{11}|},$$

and

$$\frac{1}{3} \leqslant |S_{11}| \leqslant 1.$$

The scattering matrix is

$$S = \begin{bmatrix} \sqrt{(1-2S_{12}^2)}e^{j\psi_{11}} & S_{12} & S_{12} \\ S_{12} & \sqrt{(1-2S_{12}^2)}e^{j\psi_{11}} & S_{12} \\ S_{12} & S_{12} & \sqrt{(1-2S_{12}^2)}e^{j\psi_{11}} \end{bmatrix}.$$

A lossless tuning element having the symmetry of the waveguide junction and located in its center could presumably produce the variation of $|S_{11}|$ between 1/3 and 1, but it is not possible to reduce $|S_{11}|$ below 1/3. Thus the VSWR of one arm with the other two arms terminated in non-reflecting loads cannot be less than 2.00.

5. A lossless tuning element located in one of the arms, say arm 3, could be adjusted to make $S_{33} = 0$, but some of the symmetry of the junction would be lost. If the tuning element were symmetrically located in arm 3, then the scattering matrix would have the form

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{11} & S_{13} \\ S_{13} & S_{13} & 0 \end{bmatrix}.$$

This is of the same form as the first H-plane tee which was considered and the same results would apply.

It is not possible to tune for $S_{11} = S_{22} = S_{33} = 0$, in any lossless reciprocal 3-port, as was noted in the theorems.

3-port, one arm terminated

A 3-port with one arm terminated in a load so that only two arms are available for connection to sources and loads is essentially a 2-port. The parameters of the 2-port may be found in terms of those of the 3-port and the reflection coefficient of the fixed termination.

If the load of reflection coefficient Γ_{L3} terminates arm 3, then $a_3 = \Gamma_{L3} b_3$, and the scattering equations for the 3-port are

$$\begin{aligned} b_1 &= {}^3S_{11}a_1 + {}^3S_{12}a_2 + {}^3S_{13}\Gamma_{L3}b_3, \\ b_2 &= {}^3S_{21}a_1 + {}^3S_{22}a_2 + {}^3S_{23}\Gamma_{L3}b_3, \\ b_3 &= {}^3S_{31}a_1 + {}^3S_{32}a_2 + {}^3S_{33}\Gamma_{L3}b_3, \end{aligned}$$

where the front superscript identifies the scattering coefficient as that of the 3-port. Solving to obtain the scattering equations of the 2-port, we obtain

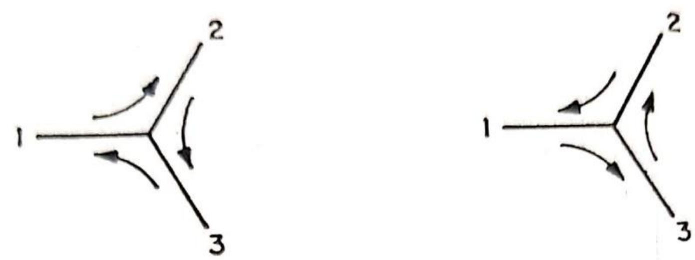
$$\begin{aligned} b_1 &= \left({}^3S_{11} + \frac{{}^3S_{13}\,{}^3S_{31}\Gamma_{L3}}{1 - {}^3S_{33}\Gamma_{L3}}\right) a_1 + \left({}^3S_{12} + \frac{{}^3S_{13}\,{}^3S_{32}\Gamma_{L3}}{1 - {}^3S_{33}\Gamma_{L3}}\right) a_2, \\ b_2 &= \left({}^3S_{21} + \frac{{}^3S_{23}\,{}^3S_{31}\Gamma_{L3}}{1 - {}^3S_{33}\Gamma_{L3}}\right) a_1 + \left({}^3S_{22} + \frac{{}^3S_{23}\,{}^3S_{32}\Gamma_{L3}}{1 - {}^3S_{33}\Gamma_{L3}}\right) a_2. \end{aligned}$$

It is seen that if either the load is non-reflecting ($\Gamma_{L3} = 0$) or the third arm is decoupled ($S_{23} = S_{31} = 0$) the scattering coefficients of the 2-port become those of the 3-port or

$${}^2S_{11} = {}^3S_{11}, \quad {}^2S_{12} = {}^3S_{12}, \quad {}^2S_{21} = {}^3S_{21}, \quad \text{and} \quad {}^2S_{22} = {}^3S_{22}.$$

Non-reciprocal 3-ports

An interesting non-reciprocal 3-port is the circulator, which is shown in two forms in **the figure below.**



Schematic diagrams for lossless, non-reciprocal 3-port circulators.

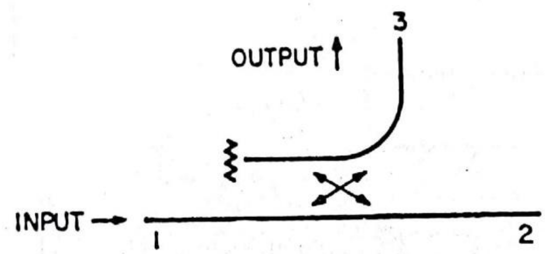
The scattering matrices corresponding to the ideal forms of the two circulators are

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The ideal circulator is lossless ($S^*S = 1$).

The directional coupler, 3-port

Actually, directional couplers are often considered as a class of lossless 4-ports, but if one arm is internally terminated so as to be not available for connection, it is then a 3-port. As shown in Fig. below the side arm (arm 3) to which detectors may be connected couples mainly to the incident wave in arm 1. The coupling ratio is the ratio of the incident input power to the power coupled out to a non-reflecting detector, when the main arm of the coupler is terminated by a non-reflecting load.



Schematic diagram for a directional coupler connected as a 3-port.

When $a_2 = a_3 = 0$ (non-reflecting loads), the coupling is

$$C = 10 \log_{10} \left| \frac{a_1}{b_3} \right|^2 \frac{Z_{03}}{Z_{01}} = 10 \log_{10} \left[\frac{Z_{03}}{|S_{31}|^2 Z_{01}} \right],$$

or if the 3-port is a reciprocal one,

$$C = -10 \log_{10} |S_{13} S_{31}|.$$

Another important property of the directional coupler is the ratio of the emergent power coupled out for a given incident power input to arm 1, to the power coupled out for the same incident power to arm 2, assuming non-reflecting loads on all arms except those connected to sources. This ratio expressed in decibels is called the directivity D. It may be written

$$D = 10 \log_{10}\left[\left(\frac{|{}^{f}b_3|^2}{|a_1|^2}\cdot\frac{Z_{01}}{Z_{03}}\right)_{a_2 = a_3 = 0}\left(\frac{|a_2|^2}{|{}^{b}b_3|^2}\cdot\frac{Z_{03}}{Z_{02}}\right)_{a_1 = a_3 = 0}\right],$$

where ${}^{f}b_3$ indicates coupling to the "forward" wave, and ${}^{b}b_3$ indicates coupling to the "backward" wave, the amplitudes a_1 of the forward, and a_2 of the backward wave being equal. It follows then that the directivity is

$$D = 10 \log_{10}\left[\frac{Z_{01}}{Z_{02}}\cdot\left|\frac{S_{31}}{S_{32}}\right|^2\right],$$

or if the 3-port is a reciprocal one, the directivity can be written

$$D = 10 \log_{10}\left|\frac{S_{13}S_{31}}{S_{23}S_{32}}\right|.$$

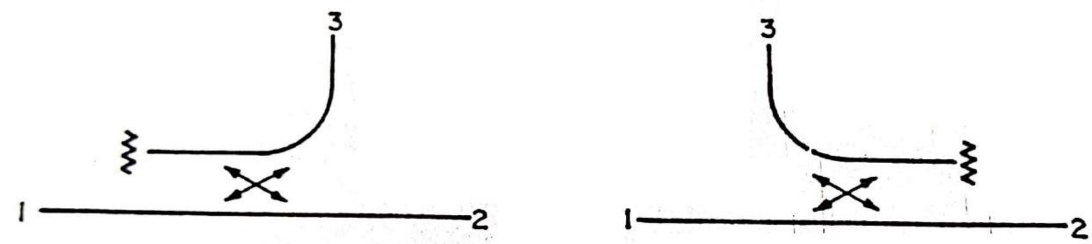
In case that $Z_{01}=Z_{02}=Z_{03}$, reciprocity implies that $S_{31}=S_{13}$, and $S_{32} = S_{23}$, or

$$D = 20 \log_{10}\left|\frac{S_{31}}{S_{32}}\right|.$$

It may be more convenient to measure the power transmitted thru the main arm than the incident power. The directivity might then be defined as the ratio of emergent powers coupled out of the side arm for the same powers transmitted thru the main arm. This would lead to an equation for directivity similar to eqn. above except that the ratio of Z_{02} to Z_{01} would appear in place of the ratio of Z_{01} to Z_{02}.

Note that the ideal circulator can be considered as a directional coupler for which the coupling ratio is unity and the directivity infinite. Thus the ideal circulator couples all of the energy out of the main arm into the side arm.

Two representations for ideal 3-port directional couplers are shown below:



Schematic diagrams for ideal 3-port directional couplers.

The scattering matrices corresponding to the 3-ports shown in the figure above are,

$$S = \begin{bmatrix} 0 & \sqrt{(1-c^2)} & c \\ \sqrt{(1-c^2)} & 0 & 0 \\ c & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & \sqrt{(1-c^2)} & 0 \\ \sqrt{(1-c^2)} & 0 & c \\ 0 & c & 0 \end{bmatrix}.$$

Observe that the scattering matrix of an ideal 3-port directional coupler is not unitary. Although ideal in concept, it contains an internal termination and is not lossless.