Ayrık-Zamanlı Fourier Dönüşümü

Dönüşüm çifti

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \\ X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \end{aligned}}$$

Örnek:

$$x[n] = a^n u[n], \qquad |a| < 1.$$

İşaretinin Fourier dönüşümünü bulalım.

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}. \end{aligned}$$

Örnek:

$$x[n] = a^{|n|}, \qquad |a| < 1.$$

şeklinde verilen işaretin Fourier dönüşümünü bulalım.

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}. \end{aligned}$$

$m = -n$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{m=1}^{\infty} (ae^{j\omega})^m.$$

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} \\ &= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}. \end{aligned}$$

Kaynak: Oppenheim, Willsky “Signals and Systems”

Dönüşümün Varlık koşulu:

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty,$$

Örnek:

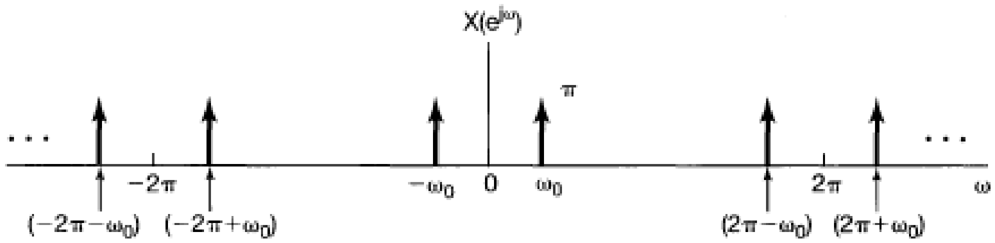
$$x[n] = \delta[n].$$

$$X(e^{j\omega}) = 1.$$

Örnek:

$$x[n] = \cos \omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n},$$

$$\omega_0 = \frac{2\pi}{5}.$$



$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right).$$

$$X(e^{j\omega}) = \pi\,\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\,\delta\left(\omega + \frac{2\pi}{5}\right), \qquad -\pi \le \omega < \pi,$$

Örnek:

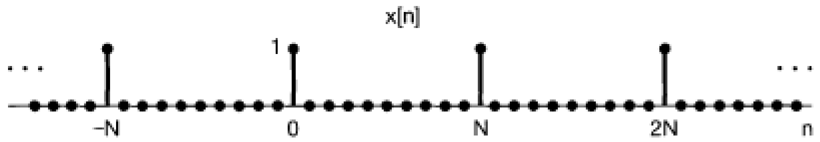
$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN],$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}.$$

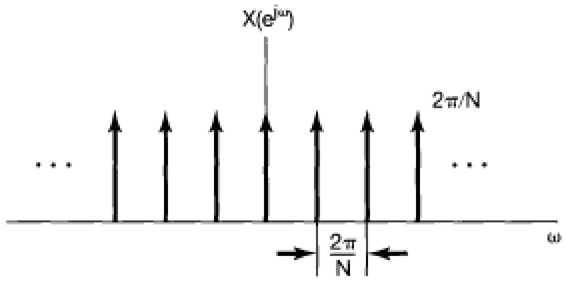
$$0 \le n \le N-1,$$

$$a_k = \frac{1}{N}.$$

$$X(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{N}\right),$$



(a)



(b)

Ayrık-Zamanlı Fourier Dönüşümünün Özellikleri

$$X(e^{j\omega}) = \mathfrak{F}\{x[n]\},$$

$$x[n] = \mathfrak{F}^{-1}\{X(e^{j\omega})\},$$

$$x[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}).$$

Periyodiklik:

$$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega}).}$$

Kaynak: Oppenheim, Willsky “Signals and Systems”

Lineerlik:

$$x_1[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

$$x_2[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X_2(e^{j\omega}),$$

$$\boxed{a x_1[n] + b x_2[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} a X_1(e^{j\omega}) + b X_2(e^{j\omega}).}$$

Zamanda ve frekansta Öteleme:

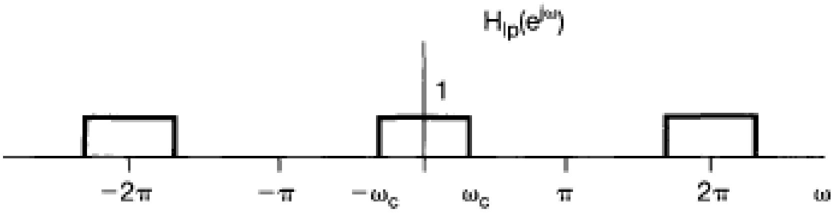
$$x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}),$$

$$\boxed{x[n-n_0] \stackrel{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})}$$

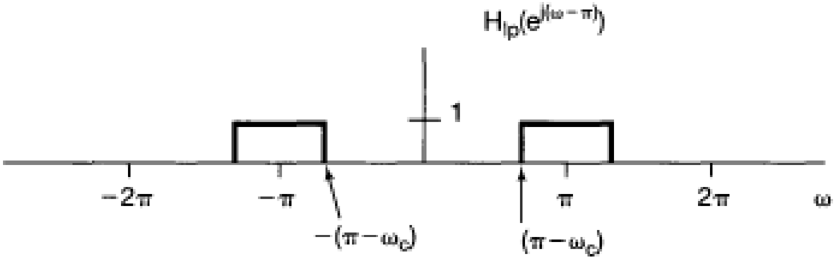
$$\boxed{e^{j\omega_0 n} x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).}$$

Örnek:

AGS-YGS (Alçak geçiren- Yüksek geçiren süzgeç) farkı



(a)



(b)

Süzgeçler arasında frekans ve zaman domeninde

$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}),$$

$$\begin{aligned} h_{hp}[n] &= e^{j\pi n} h_{lp}[n] \\ &= (-1)^n h_{lp}[n]. \end{aligned}$$

ilişkisi vardır.

Eşlenik Alma:

$$x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}),$$

$$\boxed{x^*[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X^*(e^{-j\omega}).}$$

Reel x[n] için

$$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{real}].}$$

Kaynak: Oppenheim, Willsky “Signals and Systems”

Fark Alma:

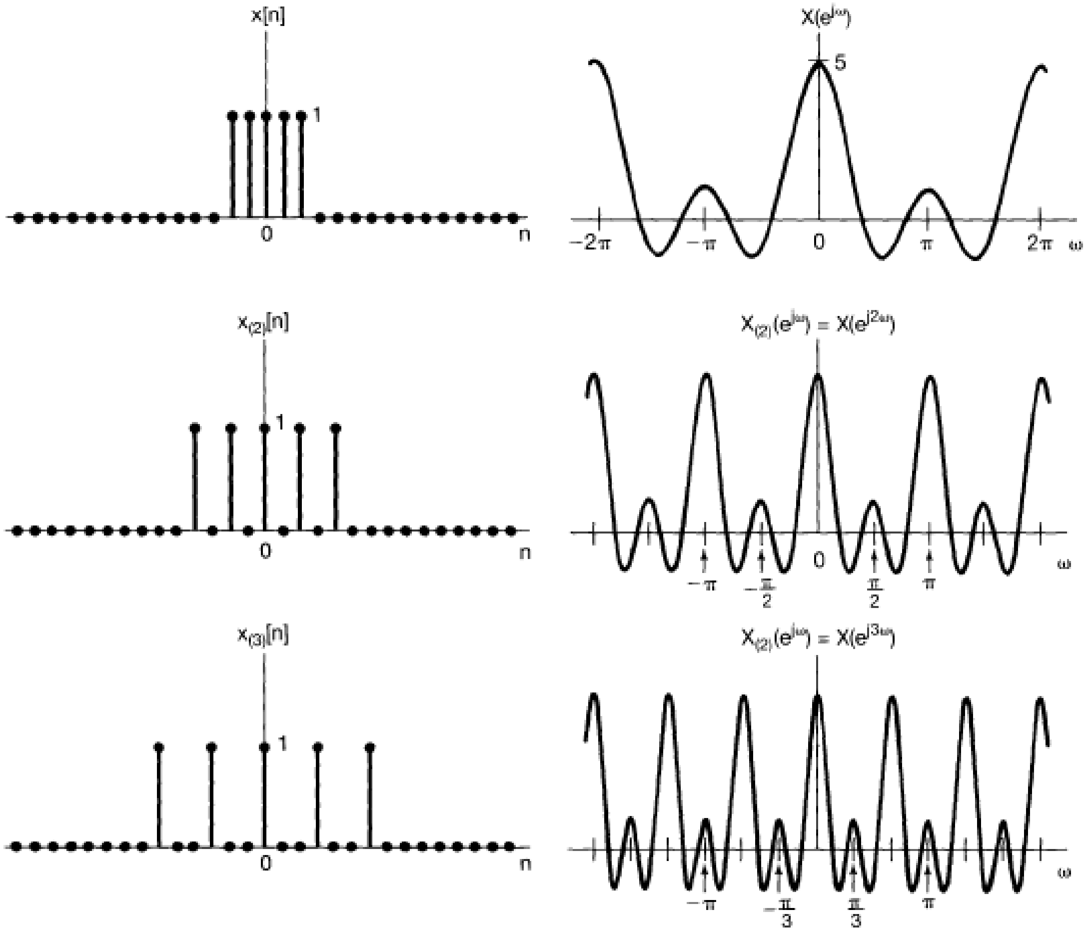
$$\boxed{x[n] - x[n-1] \overset{\mathfrak{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega}).}$$

Zamanda Katlama:

$$\boxed{x[-n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{-j\omega}).}$$

Zamanda Ölçekleme:

$$x(at) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$



Kaynak: Oppenheim, Willsky “Signals and Systems”

Parseval Bağıntısı:

$$\boxed{\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega.}$$

Konvolüsyon:

$$y[n] = x[n] * h[n],$$

$$\boxed{Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}),}$$

Örnek:

$$h[n] = \delta[n - n_0].$$

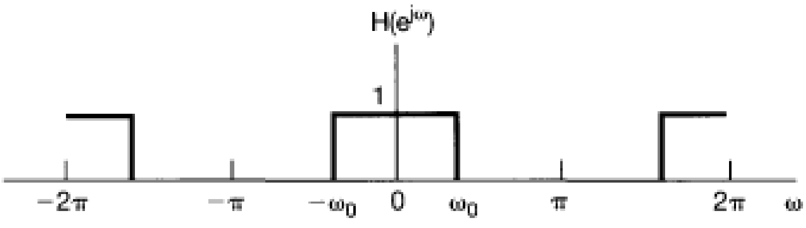
$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0]e^{-j\omega n} = e^{-j\omega n_0}.$$

Herhangi bir giriş işareti için sistem çıkışı

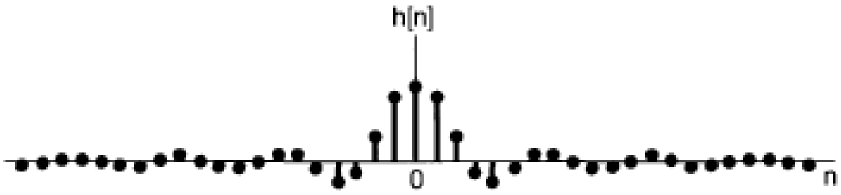
$$Y(e^{j\omega}) = e^{-j\omega n_0}X(e^{j\omega}).$$

Örnek:

Aşağıda frekans cevabı verilen AGS nin impuls cevabını bulalım.



(a)



(b)

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$
$$= \frac{\sin \omega_c n}{\pi n},$$

Örnek:

$$h[n] = \alpha^n u[n],$$

$$x[n] = \beta^n u[n],$$

şeklinde impuls cevabı ve giriş işareti verilen sistemin çıkışını bulalım.

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}},$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}.$$

Rezidü yöntemi ile,

$$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}.$$

$$A = \frac{\alpha}{\alpha-\beta}, \quad B = -\frac{\beta}{\alpha-\beta}.$$

$$\begin{aligned} y[n] &= \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n] \\ &= \frac{1}{\alpha-\beta}[\alpha^{n+1}u[n] - \beta^{n+1}u[n]]. \end{aligned}$$

Çarpım Özelliği:

$$\boxed{Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.}$$

Kaynak: Oppenheim, Willsky “Signals and Systems”

Fourier Dönüşümünün Özellikleri

Property	Aperiodic Signal	Fourier Transform
	$x[n]$ $y[n]$	$\left.\begin{array}{l}X(e^{j\omega})\\Y(e^{j\omega})\end{array}\right\}$ periodic with period 2π
Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansion	$x_{(k)}[n]=\begin{cases}x[n/k], & \text{if } n=\text{multiple of } k\\ 0, & \text{if } n\neq\text{multiple of } k\end{cases}$	$X(e^{jk\omega})$
Convolution	$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differencing in Time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases}X(e^{j\omega})=X^*(e^{-j\omega})\\ \Re e\{X(e^{j\omega})\}=\Re e\{X(e^{-j\omega})\}\\ \Im m\{X(e^{j\omega})\}=-\Im m\{X(e^{-j\omega})\}\\ \lvert X(e^{j\omega})\rvert=\lvert X(e^{-j\omega})\rvert\\ \sphericalangle X(e^{j\omega})=-\sphericalangle X(e^{-j\omega})\end{cases}$
Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even-odd Decomposition of Real Signals	$x_e[n]=\mathcal{E}v\{x[n]\}\quad [x[n]\text{ real}]$ $x_o[n]=\mathcal{O}d\{x[n]\}\quad [x[n]\text{ real}]$	$\Re e\{X(e^{j\omega})\}$ $j\Im m\{X(e^{j\omega})\}$

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty}|x[n]|^2=\frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2d\omega$$

Kaynak: Oppenheim, Willsky “Signals and Systems”

Fourier Dönüşüm Çiftleri

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$, $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & N_1 < \|n\| \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}$, $k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}$, $k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n]$, $\|a\| < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \|n\| \le N_1 \\ 0, & \|n\| > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \|\omega\| \le W \\ 0, & W < \|\omega\| \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n]$, $\|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n]$, $\|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

Kaynak: Oppenheim, Willsky “Signals and Systems”

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n] e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ **continuous frequency** **aperiodic in frequency**	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n}$ **discrete time** **aperiodic in time**	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ continuous frequency periodic in frequency

duality

duality

Fark denklemlerinin Fourier Domeninde Modellenmesi

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

Fark denkleminde 2 tarafın da AZFD (Ayrık-Zamanlı Fourier dönüşümü) alınarak, sistemin frekans cevabı

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}.$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega}),$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}.$$

Örnek:

Fark denklemi

$$y[n] - ay[n-1] = x[n],$$

şeklinde verilen sistemin impuls cevabını bulalım.

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}.$$

$$h[n] = a^n u[n].$$

Kaynak: Oppenheim, Willsky "Signals and Systems"

Örnek:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

Fark denklemi yukarıda verilen sistemin impuls cevabı

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}.$$

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}.$$

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}.$$

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n].$$

Örnek:

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

girişi için önceki örnekte verilen sistemin çıkışını bulalım.

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \left[\frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}\right]\left[\frac{1}{1 - \frac{1}{4}e^{-j\omega}}\right]$$

$$= \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})^2}.$$

$$Y(e^{j\omega}) = \frac{B_{11}}{1 - \frac{1}{4}e^{-j\omega}} + \frac{B_{12}}{(1 - \frac{1}{4}e^{-j\omega})^2} + \frac{B_{21}}{1 - \frac{1}{2}e^{-j\omega}},$$

$$Y(e^{j\omega}) = -\frac{4}{1 - \frac{1}{4}e^{-j\omega}} - \frac{2}{(1 - \frac{1}{4}e^{-j\omega})^2} + \frac{8}{1 - \frac{1}{2}e^{-j\omega}}.$$

$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\} u[n].$$