Sürekli Zamanlı Periyodik İşaretler için Fourier Serisi gösterilimi

Özfonksiyonlar

LZD sistemin

$$x(t) = e^{st}$$

girişine cevabını bulalım.

Sistem çıkışı

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}\,d\tau. \end{aligned}$$

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau.$$

$$y(t) = H(s)e^{st},$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau.$$

Benzer şekilde

$$x(t) = a_1e^{s_1t} + a_2e^{s_2t} + a_3e^{s_3t}.$$

Şeklinde verilen giriş işareti için sistem çıkışı

$$\begin{aligned} a_1e^{s_1t} &\longrightarrow a_1H(s_1)e^{s_1t}, \\ a_2e^{s_2t} &\longrightarrow a_2H(s_2)e^{s_2t}, \\ a_3e^{s_3t} &\longrightarrow a_3H(s_3)e^{s_3t}, \end{aligned}$$

$$y(t) = a_1H(s_1)e^{s_1t} + a_2H(s_2)e^{s_2t} + a_3H(s_3)e^{s_3t}.$$

Sonuç olarak,

$$x(t) = \sum_k a_ke^{s_kt},$$

şeklinde modellenen işaret için sistemin cevabı

$$y(t) = \sum_k a_kH(s_k)e^{s_kt}.$$

Fourier serisi gösterilimi

$$x(t) = x(t+T)$$

koşulu tüm T değerleri için sağlayan işaret T (koşulu sağlayan en küçük T) ile periyodiktir.

Temel açısal frekans

$$\omega_0 = 2\pi/T$$

Fourier serisi gösterilimi

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Örnek:

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t},$$

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

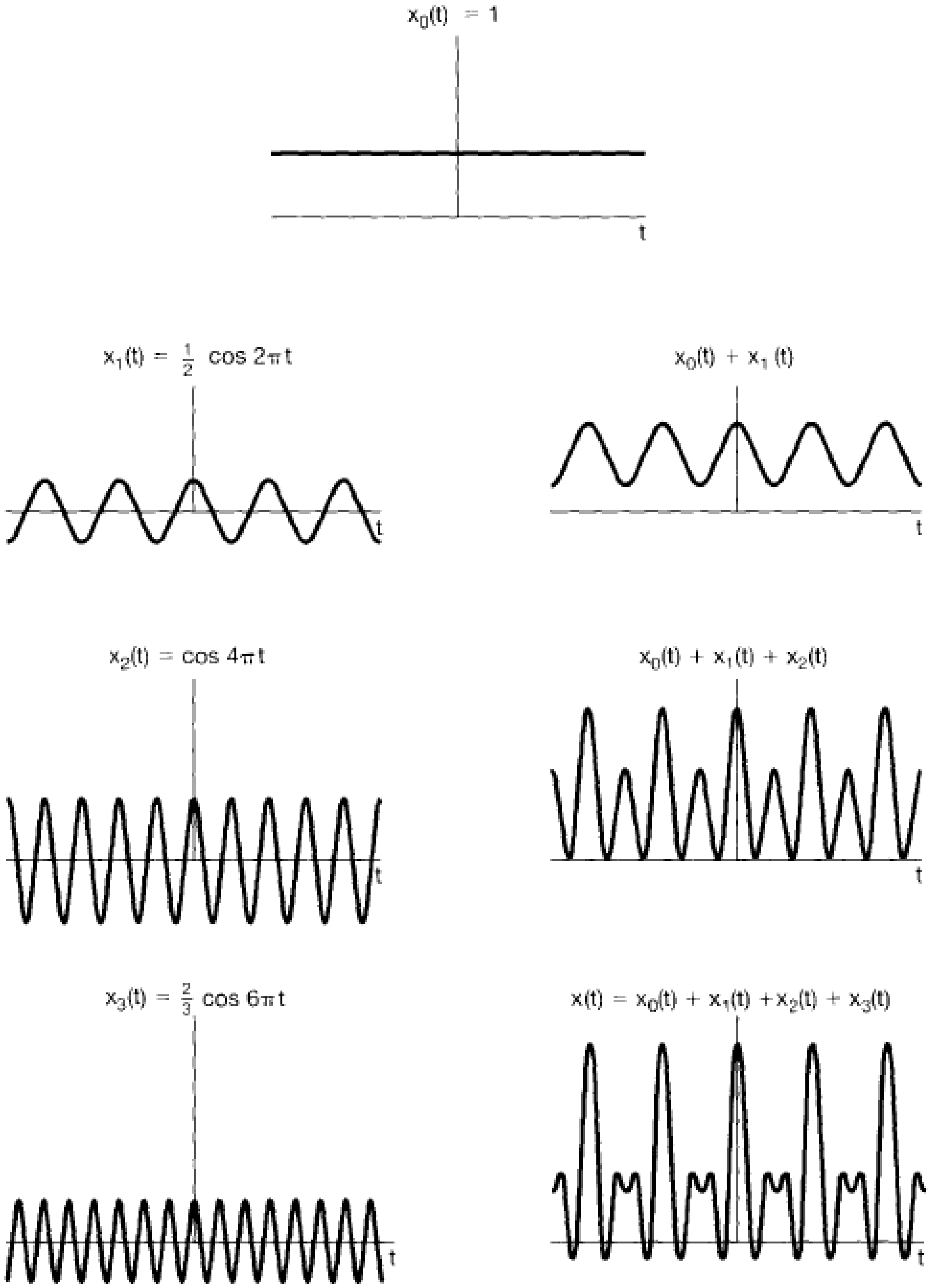
katsayıları verilsin. Temel açısal frekans ω_0 olmak üzere, x(t) işaretini bulunuz.

$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t}).$$

Euler bağıntısından yararlanarak,

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t.$$

Kaynak: Oppenheim, Willsky "Signals and Systems"



Fourier serisi gösterilimi (Fourier Series representation)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t},$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt.$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt,$$

Örnek:

$$x(t) = \sin\omega_0 t,$$

şeklinde verilen işaretin Fourier serisi katsayılarını bulalım. (temel açısal frekans ω_o)

$$\sin\omega_0 t = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}.$$

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$

$$a_k = 0, \qquad k \neq +1 \text{ or } -1.$$

Örnek:

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

şeklinde verilen işaretin Fourier serisi katsayılarını bulalım. (temel açısal frekans ω_o)

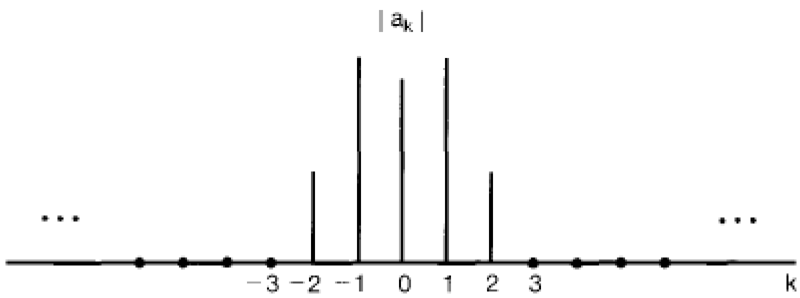
$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}].$$

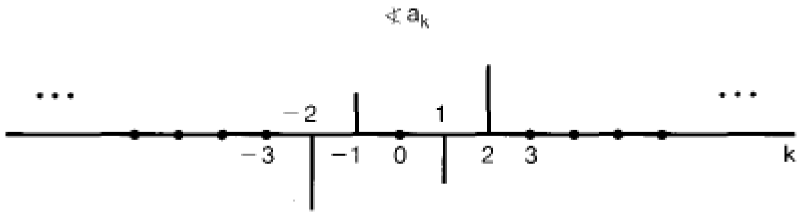
$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Fourier Serisi katsayıları

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),$$
$$a_k = 0, \ |k| > 2.$$



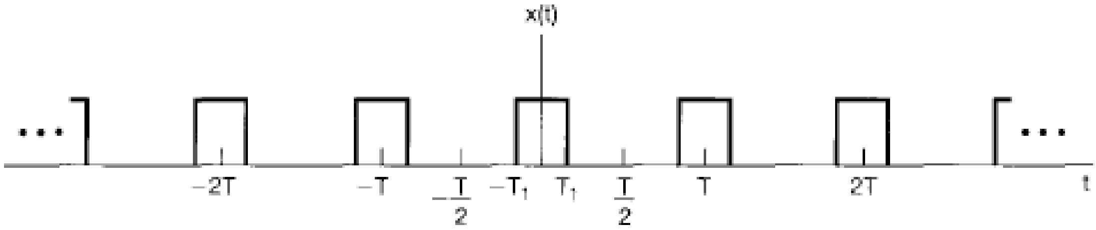


Genlik ve faz spektrumları

Örnek:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases},$$

$\omega_0 = 2\pi/T.$



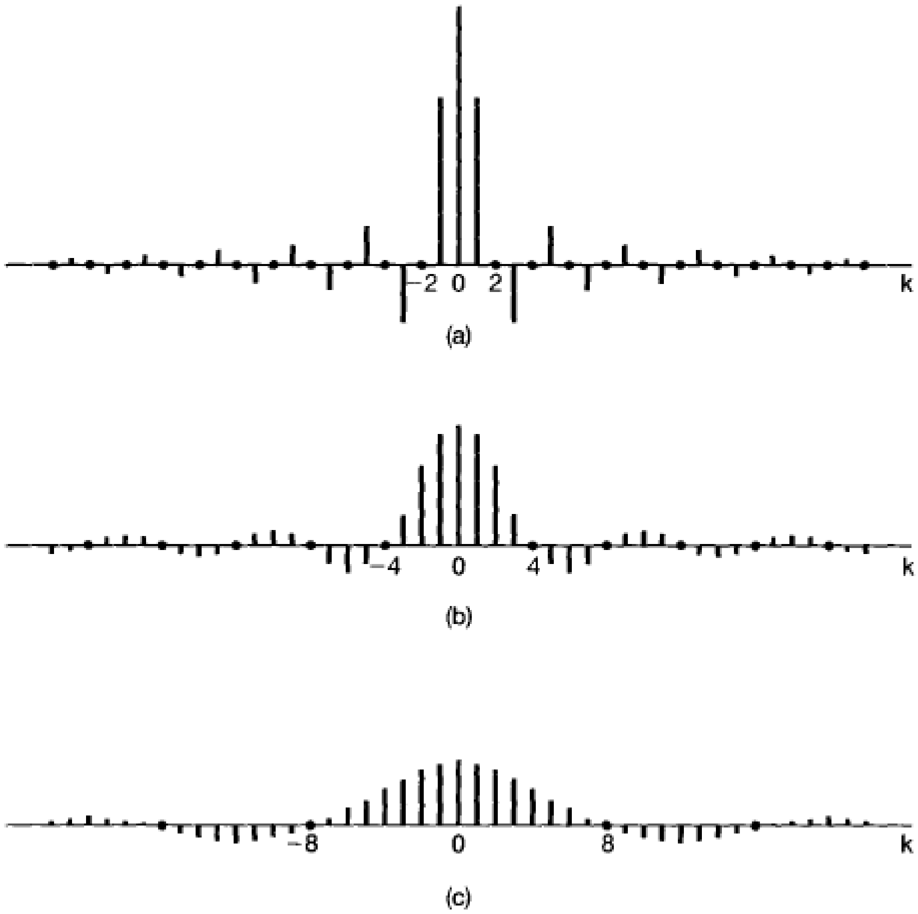
$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1},$$

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right].$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0,$$

$\omega_0 T = 2\pi.$



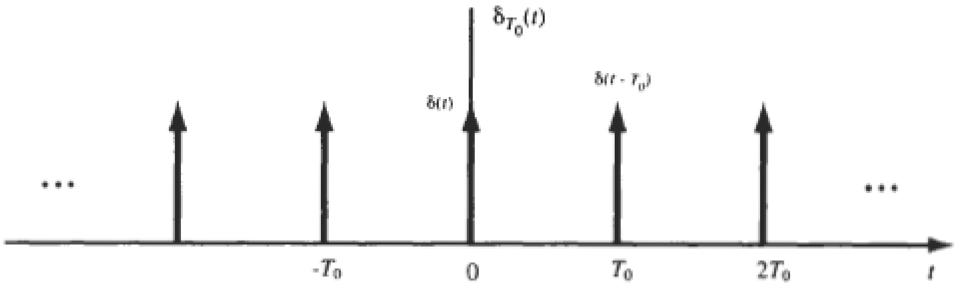
Farklı darbe genişlileri için katsayıların değişimi

(*a*) $T = 4T_1$;

(*b*) $T = 8T_1$; (*c*) $T = 16T_1$.

Örnek: Periyodik impuls dizisi (impuls katarı) için Fourier Serisi katsayılarını bulalım.

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT);$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi t/T} dt = \frac{1}{T}.$$

Fourier Serilerinin yakınsama koşulu (Dirichlet Koşulları)

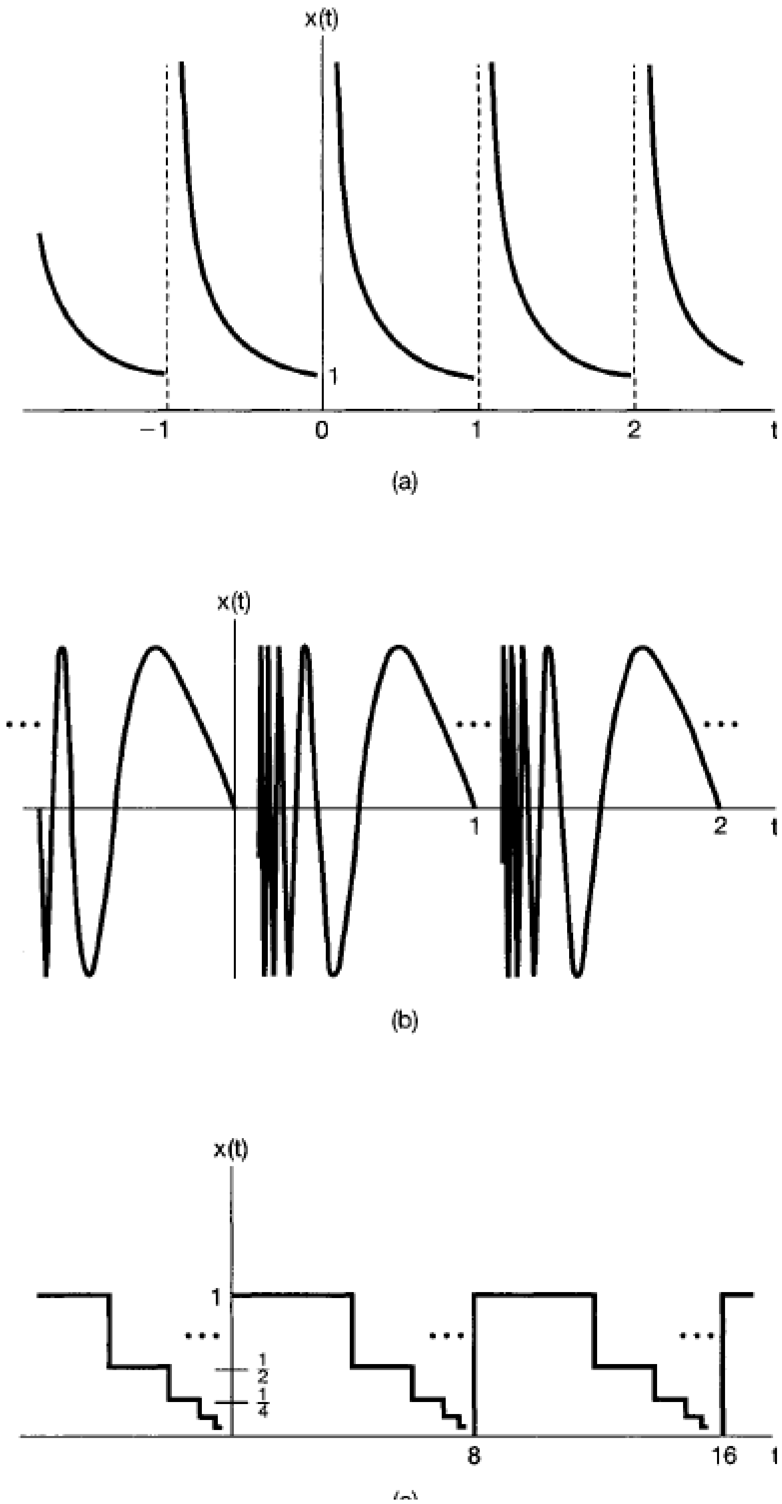
1. İşaret ana periyotta integre edilebilir olmalı

$$\int_T |x(t)|\, dt < \infty.$$

2. Sonlu sayıda maksimum ve minimumum olmalı

3. Sonlu sayıda süreksizlik göstermeli ve bu süreksizliklerde sonlu değer almalı

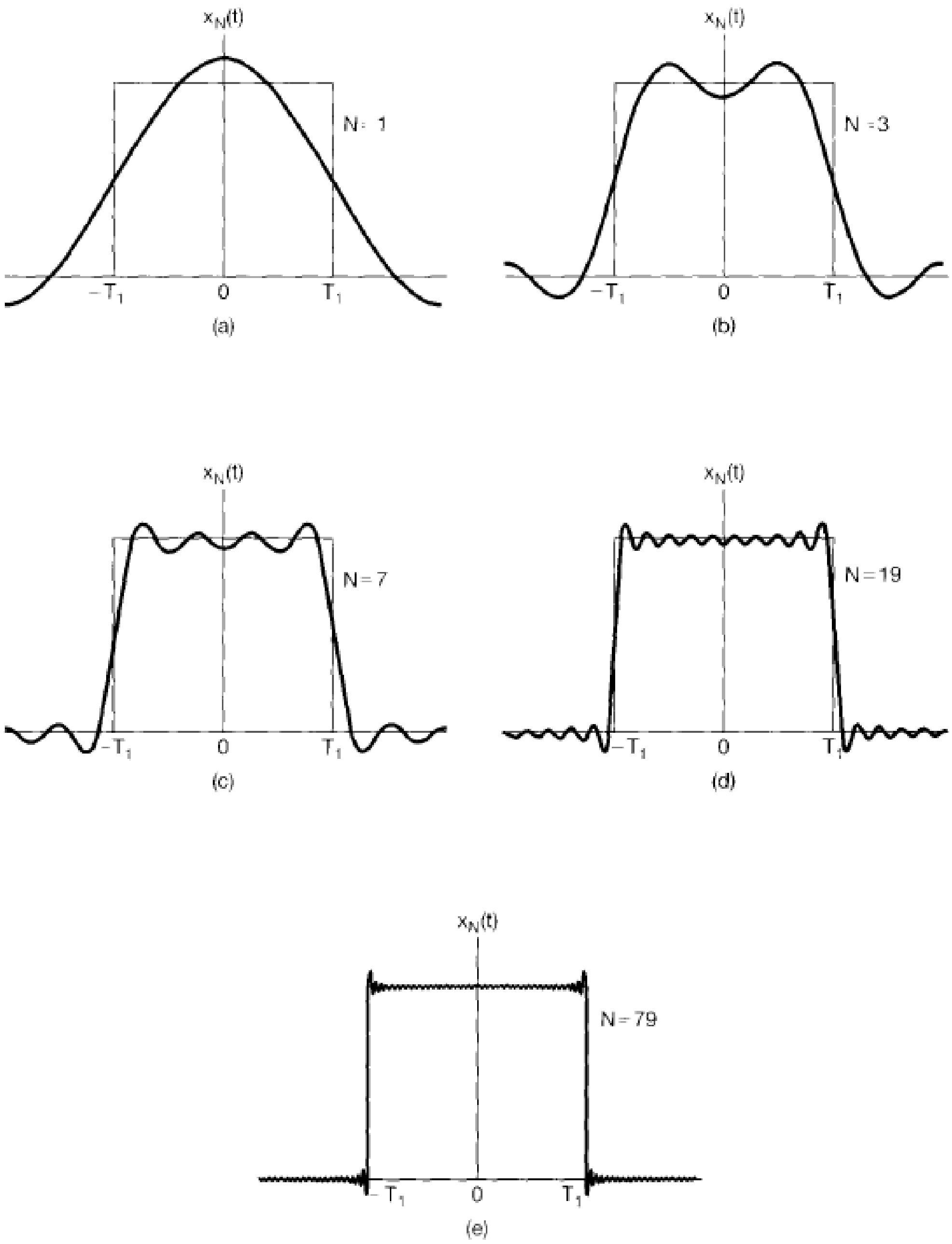
Kaynak: Oppenheim, Willsky “Signals and Systems”



Dirichlet şartlarını sağlamayan işaretler (sırasıyla 1,2 ve 3. Koşullar sağlanmıyor)

Örnek: Gibbs Olayı

Periyodik dikdörtgen dalga işaretinin sonlu sayıda Fourier serisi katsayısı ile ifade edilmesi



Kaynak: Oppenheim, Willsky “Signals and Systems”

Fourier Serilerinin özellikleri

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau) y(t-\tau) d\tau$	$T a_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Zamanda kaydırma (Time shifting)

$$y(t) = x(t - t_0)$$

$$b_k = \frac{1}{T}\int_T x(t - t_0) e^{-jk\omega_0 t} dt.$$

$$\tau = t - t_0$$

$$\frac{1}{T}\int_T x(\tau) e^{-jk\omega_0(\tau + t_0)} d\tau = e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau} d\tau$$
$$= e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_o} a_k,$$

$$x(t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

$$x(t-t_0) \overset{\mathfrak{F}S}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k.$$

Zamanda katlama (Time reversal)

$y(t) = x(-t).$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}.$$

$k = -m.$

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}.$$

$$b_k = a_{-k}.$$

$$x(t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

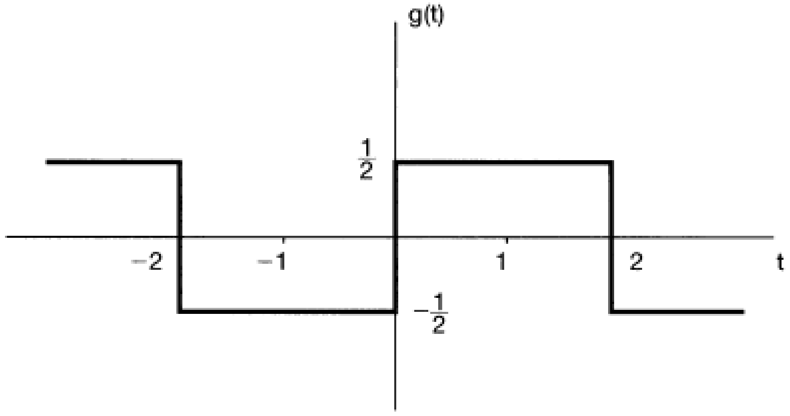
$$x(-t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_{-k}.$$

Parseval Bağıntısı

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2,$$

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2,$$

Örnek:



işareti için Fourier Serisi katsayılarını önceki örnek (periyodik dikdörtgen darbe işareti) ve Fourier katsayılarının özelliklerinden yararlanarak bulalım.

Hatırlatma

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases},$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \qquad k \neq 0,$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$

$T = 4$

$T_1 = 1,$

alınarak

$$g(t) = x(t-1) - 1/2.$$

$$b_k = a_k e^{-jk\pi/2}.$$

ilişkisinden yararlanarak

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - \frac{1}{2}, & \text{for } k = 0 \end{cases},$$

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}.$$