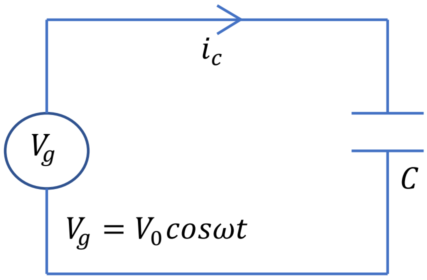
Q1. Current on a capacitance via Maxwell's equations

For the given circuit,



it can be written from the circuit theory,

$$i_c = C\frac{dV_c}{dt} = C\frac{dV_g}{dt} = -\omega C V_0 sin\omega t$$

And, also it is known that for a parallel plate capacitor,

$$|\boldsymbol{E}| = \frac{V_c}{d}$$

Then, find the same current expression by the help of Maxwell's equations.

<u>**A:**</u>

The current on capacitor is displacement current, which means the current originated from variation of electric field with time ($\partial \boldsymbol{E}/\partial t \neq 0$). According to Ampere's Law,

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} \quad [A/m^2]$$

Then, $\partial \boldsymbol{D}/\partial t$ term indicates displacement current density. Thefore, displacement current,

$$i_c = \int_S \frac{\partial \boldsymbol{D}}{\partial t} \cdot d\boldsymbol{s} \quad [A]$$

It can be said that $\boldsymbol{D}$ and $d\boldsymbol{s}$ vectors are in same direction (in direction of normal vectors of the capacitor plates); therefore, dot product of direction vectors of these vectors are equal to 1. And, also it is known that, $\boldsymbol{D} = \epsilon \boldsymbol{E}$. Therefore,

$$i_c = \int \epsilon \frac{\partial E}{\partial t} ds = \int \frac{\epsilon}{d}\frac{\partial V_c}{\partial t} ds = \int \frac{\epsilon}{d}(-\omega V_0 sin\omega t) ds = \frac{\epsilon S}{d}(-\omega V_0 sin\omega t) = -C\omega V_0 sin\omega t$$

Here, S indicates area of capacitor plates. And, $\epsilon S/d$ is a well-known expression for capacitance of a parallel plate capacitor.

Q2. Field vectors and fundamental parameters based on the wave equation

Show that the electric field vector $\boldsymbol{E} = \boldsymbol{e}_x \cos(\omega t - kz)$ satisfies the wave equation in sourceless and free space medium. Find corresponding magnetic field vector by the help of Maxwell's equations. Show that the magnetic field vector also satisfies wave equation.

A:

For an isotropic homogenous media, wave equation can be written as,

$$\nabla^2 \boldsymbol{E} - \epsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \sigma\mu \frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{\epsilon}\nabla\rho + \mu \frac{\partial \boldsymbol{J_v}}{\partial t}$$

If there is no source,

$$\rho = 0$$

$$\boldsymbol{J}_v = 0$$

And, free space is a lossless media,

$$\sigma = 0$$

And, also known that,

$$v = \frac{1}{\sqrt{\epsilon\mu}}$$

For free space, $\epsilon_r = 1, \mu_r = 1$. Then, $\epsilon = \epsilon_r\epsilon_0 = \epsilon_0$, $\mu = \mu_r\mu_0 = \mu_0$,

$$\frac{1}{\sqrt{\epsilon_0\mu_0}} = c$$

which is speed of light. Laplacian operator,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 \boldsymbol{E} = \Delta \boldsymbol{E} = \nabla^2 E_x \boldsymbol{e_x} + \nabla^2 E_y \boldsymbol{e_y} + \nabla^2 E_z \boldsymbol{e_z}$$

Then,

$$\nabla^2 \boldsymbol{E} - \epsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = 0$$

$$\Rightarrow \quad \frac{\partial^2 \cos(\omega t - kz)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \cos(\omega t - kz)}{\partial t^2} = 0$$

$$\Rightarrow \quad -k^2 \cos(\omega t - kz) + \frac{1}{c^2}\omega^2 \cos(\omega t - kz) = 0$$

It is known that for a lossless medium,

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

Then,

$$\Rightarrow\ -k^2\cos(\omega t - kz) + k^2\cos(\omega t - kz) = 0$$

The equation has been satisfied.

From the Faraday's Law,

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} = -\mu\frac{\partial\boldsymbol{H}}{\partial t}$$

And, it is known that curl of a vector can be written as,

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \boldsymbol{e}_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \boldsymbol{e}_y\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \boldsymbol{e}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

Then,

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{e}_y\frac{\partial E_x}{\partial z} = \boldsymbol{e}_y k\sin(\omega t - kz)$$

$$\Rightarrow\quad \boldsymbol{H} = -\frac{k}{\mu}\int\sin(\omega t - kz)\,\boldsymbol{e}_y dt = \frac{k}{\omega\mu_0}\cos(\omega t - kz)\,\boldsymbol{e}_y = \frac{\omega/c}{\omega\mu_0}\cos(\omega t - kz)\,\boldsymbol{e}_y$$

$$= \frac{1}{\frac{1}{\sqrt{\epsilon_0\mu_0}}\mu_0}\cos(\omega t - kz)\,\boldsymbol{e}_y = \sqrt{\frac{\epsilon_0}{\mu_0}}\cos(\omega t - kz)\,\boldsymbol{e}_y$$

Here,

$$\frac{|\boldsymbol{E}|}{|\boldsymbol{H}|} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \eta_0 = Z_0 = 120\pi \cong 377$$

which indicates wave impedance of free space as expected.

Wave equation for **E** and **H** vectors are similar for source free media. Then,

$$\nabla^2\boldsymbol{H} - \epsilon_0\mu_0\frac{\partial^2\boldsymbol{H}}{\partial t^2} = 0 \quad \Rightarrow \sqrt{\frac{\epsilon_0}{\mu_0}}\frac{\partial^2\cos(\omega t - kz)}{\partial z^2} - \frac{1}{c^2}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{\partial^2\cos(\omega t - kz)}{\partial t^2}$$

$$= -k^2\cos(\omega t - kz) + \frac{\omega^2}{c^2}\cos(\omega t - kz) = 0$$

Also, we have seen that magnetic field vector satisfies the wave equation.

In addition, it can be seen that direction of electric field vector is $+x$, direction of magnetic field vector is $+y$, and direction of propagation is $+z$. We can see that right-hand rule has been satisfied as expected for a plane wave.

Q3. Derivation of wave equation and its plane wave solution

(a) Derive time-dependent wave equation for magnetic field vector from Maxwell's equations. (b) Derive phasor domain wave equation (Helmholtz equation) for electric field vector from phasor domain equivalents of Maxwell's equations. (c) Find general solution for Helmholtz equation if only one component of electric field is different than zero.

A:

a) We know that,

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \sigma \boldsymbol{E} + \boldsymbol{J_v}$$

Then,

$$\Rightarrow \quad \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{H}) = \epsilon \frac{\partial (\boldsymbol{\nabla} \times \boldsymbol{E})}{\partial t} + \sigma (\boldsymbol{\nabla} \times \boldsymbol{E}) + \boldsymbol{\nabla} \times \boldsymbol{J_v}$$

And, it is known that,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{H}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{H}) - \boldsymbol{\nabla}^2 \boldsymbol{H}$$

And,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad \Rightarrow \quad \mu(\boldsymbol{\nabla} \cdot \boldsymbol{H}) = 0$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t}$$

Then,

$$-\nabla^2 \boldsymbol{H} = -\epsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \sigma\mu \frac{\partial \boldsymbol{H}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{J_v} \quad \Rightarrow \quad \boldsymbol{\nabla}^2 \boldsymbol{H} - \epsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \sigma\mu \frac{\partial \boldsymbol{H}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{J_v}$$

b) Phasor domain for $e^{-j\omega t}$ time dependency,

$$\boldsymbol{\nabla} \times \boldsymbol{E}(r;t) = -\frac{\partial \boldsymbol{B}(r;t)}{\partial t} \quad \rightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{E}(r) = j\omega \boldsymbol{B}(r)$$

$$\boldsymbol{\nabla} \times \boldsymbol{H}(r;t) = \frac{\partial \boldsymbol{D}(r;t)}{\partial t} + \boldsymbol{J}(r;t) \quad \rightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{H}(r) = -j\omega \boldsymbol{D}(r) + \boldsymbol{J}(r)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{D}(r;t) = \rho(r;t) \quad \rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{D}(r) = \rho(r)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}(r;t) = 0 \quad \rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{B}(r) = 0$$

Then,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = j\omega\mu(\boldsymbol{\nabla} \times \boldsymbol{H}) \quad \Rightarrow \quad \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \boldsymbol{\nabla}^2 \boldsymbol{E} = j\omega\mu(-j\omega \boldsymbol{D} + \sigma \boldsymbol{E} + \boldsymbol{J_v})$$

$$\Rightarrow \quad \frac{1}{\epsilon} \boldsymbol{\nabla} \rho - \boldsymbol{\nabla}^2 \boldsymbol{E} = j\omega\mu(-j\omega\epsilon \boldsymbol{E} + \sigma \boldsymbol{E} + \boldsymbol{J_v}) = \omega^2 \mu\epsilon \boldsymbol{E} + j\omega\mu\sigma \boldsymbol{E} + j\omega\mu \boldsymbol{J_v}$$

$$\Rightarrow \quad \nabla^2 \boldsymbol{E} + (\omega^2 \mu\epsilon + j\omega\mu\sigma)\boldsymbol{E} = -\frac{1}{\epsilon}\boldsymbol{\nabla}\rho + j\omega\mu\boldsymbol{J_v}$$

$$k_c = \sqrt{\omega^2 \mu\epsilon + j\omega\mu\sigma} = \beta + j\alpha$$

Here, k_c is called as complex wavenumber, which is a complex number for a lossy media ($\sigma \neq 0$).

c) In free space, k_c is real. Also, if $\boldsymbol{E} = \boldsymbol{e_z} E_z(x)$, the Helmholtz equation is reduced to,

$$\frac{\partial^2 E_z(x)}{\partial x^2} + k^2 E_z(x) = 0$$

We can consider that we want to find field solution when the source is located at infinity; therefore, it has no effect on the solution region we are interested. This also indicates ideal condition for a plane wave. We can say that a function in form of e^{rx} satisfies the equation,

$$r^2 e^{rx} + k^2 e^{rx} = 0 \;\; \Rightarrow \;\; (r^2 + k^2) e^{rx} = 0 \;\; \Rightarrow \;\; r = \pm jk$$

Then, two possible solution are e^{jkx} and e^{-jkx}. We can also say that any linear combination of these two function is also solution of the equation. Therefore, the general solution can be written as follows:

$$E_z(x) = Ae^{jkx} + Be^{-jkx}$$

In time domain,

$$E_z(x,t) = Re\{E_z(x)e^{-j\omega t}\} = A \cdot \cos(\omega t - kx) + B \cdot \cos(\omega t + kx)$$

We can say that first function indicates a plane wave propagating in $+x$ direction, and second function is in $-x$ direction. The coefficients A and B are determined by boundary and/or initial conditions given for a specific problem.

Consider the wave $A \cdot \cos(\omega t - kx)$. The phase term is constant for a plane wave,

$$\omega t - kx = \text{constant}$$

Therefore,

$$\omega - k\frac{dx}{dt} = 0 \;\; \Rightarrow \;\; \frac{dx}{dt} = \frac{\omega}{k} = v$$

For a lossless media, $k = \beta = \omega\sqrt{\mu\epsilon}$; therefore, phase velocity is given by,

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}$$

Or,

$$\lambda = v \cdot T \;\; \Rightarrow \;\; v = \frac{\lambda}{T} = \lambda \cdot f = \frac{2\pi}{\beta}\frac{\omega}{2\pi} = \frac{\omega}{\beta}$$

Q4. Electric field of a TEM wave and corresponding magnetic field vector

Electric field component of an EM wave propagating in free space has been given as,

$$\boldsymbol{E}(x,t) = 5\cos(3\pi 10^8 t + \beta x)\,\boldsymbol{e}_z \quad [V/m]$$

Determine,

a) Direction of propagation and frequency,
b) Phase constant and wavelength,
c) Corresponding H vector,

for fiven electric field vector.

A:

a) For increasing t, it can be seen that the wave is shifted toward $-x$ direction. Then, direction of propagation is $-\boldsymbol{e}_x$. And,

$$\omega = 3\pi \times 10^8 \;\Rightarrow\; f = \frac{3\pi \times 10^8}{2\pi} = 1.5 \times 10^8\ Hz$$

b) Complex wavenumber can be written as,

$$k = \beta + j\alpha = \sqrt{\omega^2 \epsilon\mu + j\sigma\mu\omega}$$

In free space, $\sigma = 0$, $\epsilon_r = 1$, $\mu_r = 1$. Then,

$$\Rightarrow\; k = \beta = \omega\sqrt{\epsilon_0\mu_0} = \frac{\omega}{c} = \frac{3\pi \times 10^8}{3 \times 10^8} = \pi$$

And,

$$k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda} \;\Rightarrow\; \lambda = 2\ m$$

c) In phasor domain, it can be written,

$$\boldsymbol{H} = \frac{1}{\eta}\boldsymbol{n} \times \boldsymbol{E}$$

where n denotes unit vector in direction of propagation. Then,

$$\boldsymbol{E}(x) = 5e^{-j\beta x}\boldsymbol{e}_z \;\Rightarrow\; \boldsymbol{H} = \frac{1}{\eta}(-\boldsymbol{e}_x \times \boldsymbol{e}_z)|\boldsymbol{E}| = \frac{\boldsymbol{e}_y|\boldsymbol{E}|}{\eta_0} = \boldsymbol{e}_y\frac{1}{24\pi}e^{-j\beta x}$$

$$\Rightarrow\; \boldsymbol{H}(x,t) = \frac{1}{24\pi}\cos(3\pi 10^8 t + \pi x)\,\boldsymbol{e}_y$$

Note that,

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi 10^{-7}}{\frac{1}{36\pi}10^{-9}}} = 120\pi$$