Sürekli Zamanlı Fourier Dönüşümü

Ref: Oppenheim and Willsky, "Signals and Systems"

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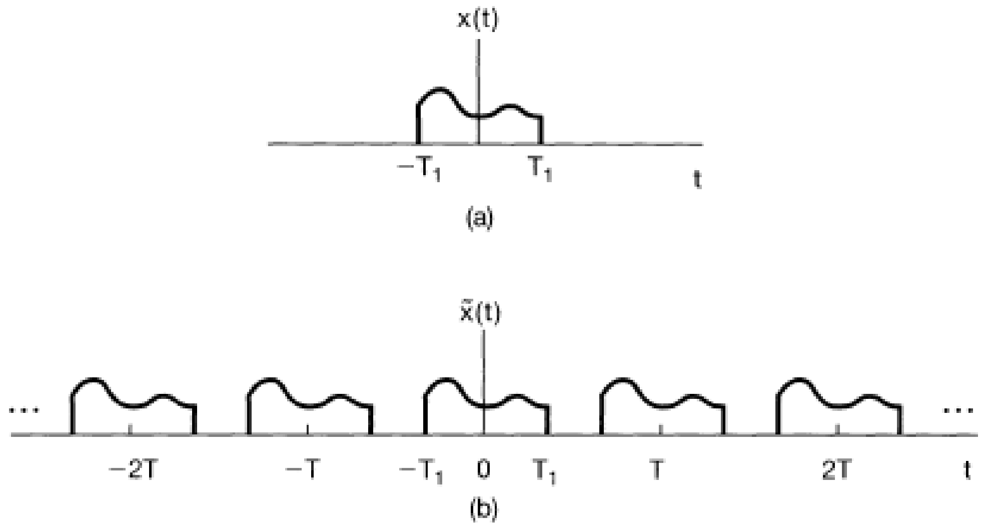


a)Aperiyodik işaret, b) periyodik işaret

Periyodik işareti Fourier Serisine açarsak, Fourier serisi gösterilimi ve Fourier serisi katsayılarının aşağıdaki gibi yazılabileceğini önceki derslerimizde görmüştük.

$-T/2 \leq t \leq T/2$, we have

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}\, dt,$$

$$\omega_0 = 2\pi/T.$$

Temel periyod için, her iki işaret birbirine eşit olduğundan

$$\tilde{x}(t) = x(t) \text{ for } |t| < T/2$$

Fourier serisi katsayısı için

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt.$$

yazılabilir.

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt,$$

Tanımını yapalım. Bu durumda

$$a_k = \frac{1}{T}X(jk\omega_0).$$

yazılabilir. Periyodik işaret Fourier serisi gösterilimi ile

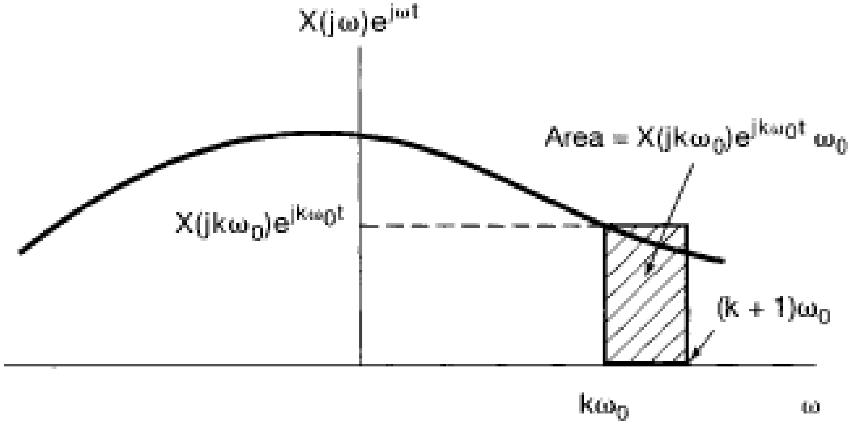
$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t},$$

şeklinde ifade edilebilir

$$2\pi/T = \omega_0,$$

olduğundan

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0.$$



$\omega_0 \to 0$ as $T \to \infty$, için

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega}$$

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.}$$

Sonlu sayıda maksimum ve minimum noktası içermeli

Sonlu sayıda süreksizlik olmalı ve süreksizliklerde sonlu değer almalı

$$\int_{-\infty}^{+\infty} |x(t)|dt < \infty.$$

Örnek :

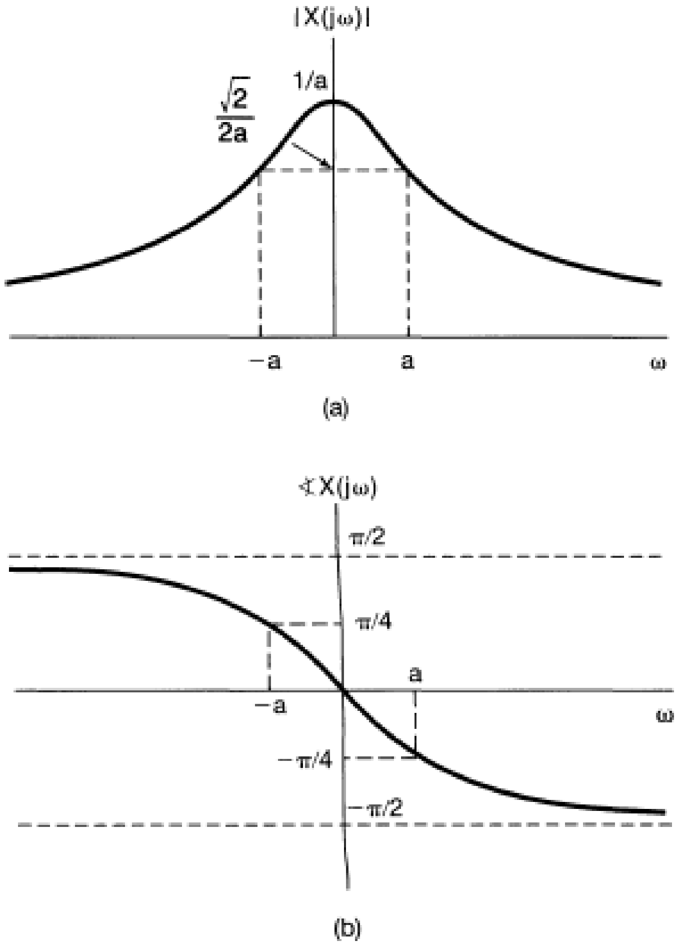
$x(t) = e^{-at}u(t) \quad a > 0.$ şeklinde verilen işaret için Fourier dönüşümü

$$X(j\omega) = \int_0^\infty e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^\infty.$$

$$X(j\omega) = \frac{1}{a+j\omega}, \quad a > 0.$$

Genlik ve faz

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$



a) Genlik, b) faz cevabı

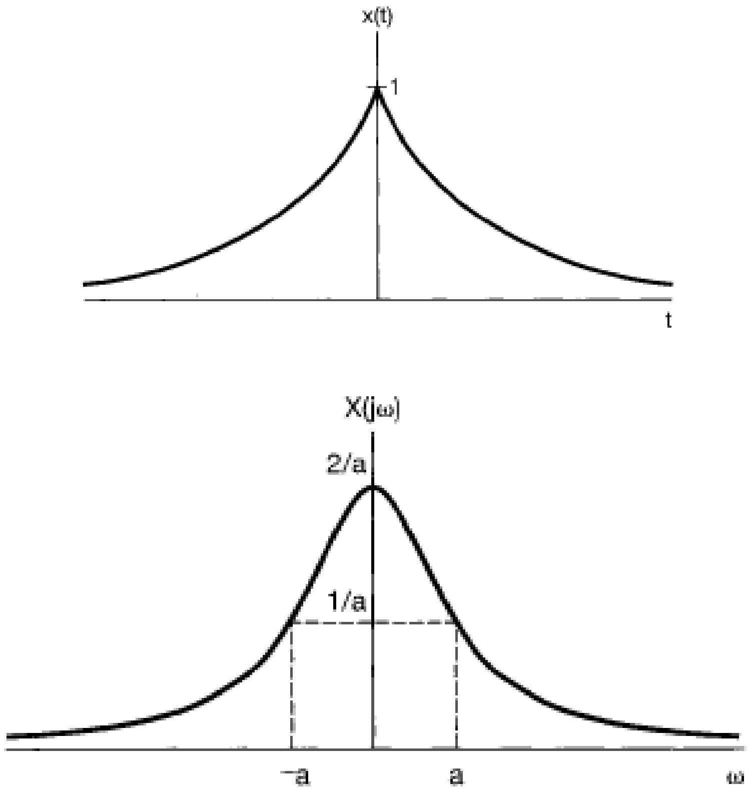
Örnek :

$x(t) = e^{-a|t|}, \quad a > 0.$ şeklinde verilen işaretin Fourier dönüşümü

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}.$$



İşaret ve fourier dönüşümü

Örnek :

$x(t) = \delta(t)$, şeklinde verilen işaretin Fourier dönüşümü

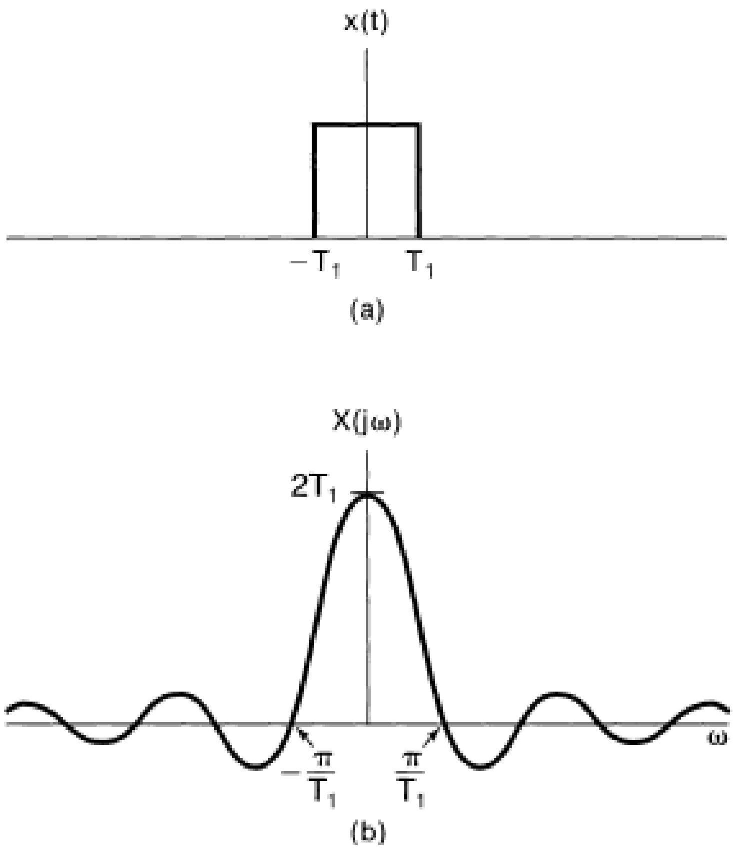
$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t}\, dt = 1.$$

Örnek :

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases},$$

şeklinde verilen işaretin Fourier dönüşümü

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = 2\frac{\sin \omega T_1}{\omega}$$

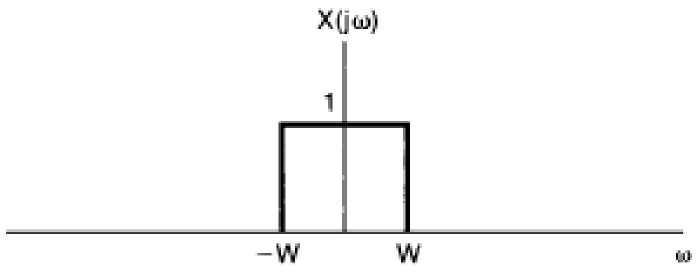


a) İşaret, b) Fourier dönüşümü

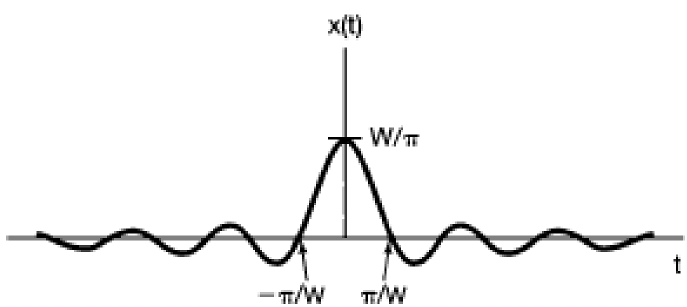
Örnek :

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases},$$

şeklinde verilen Fourier dönüşümüne karşı gelen zaman domeni işareti bulunuz.



(a)



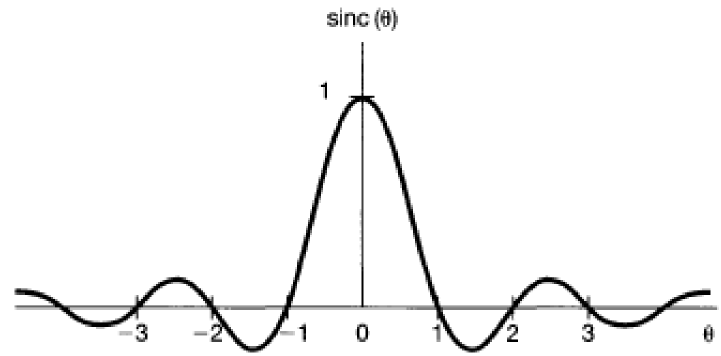
(b)

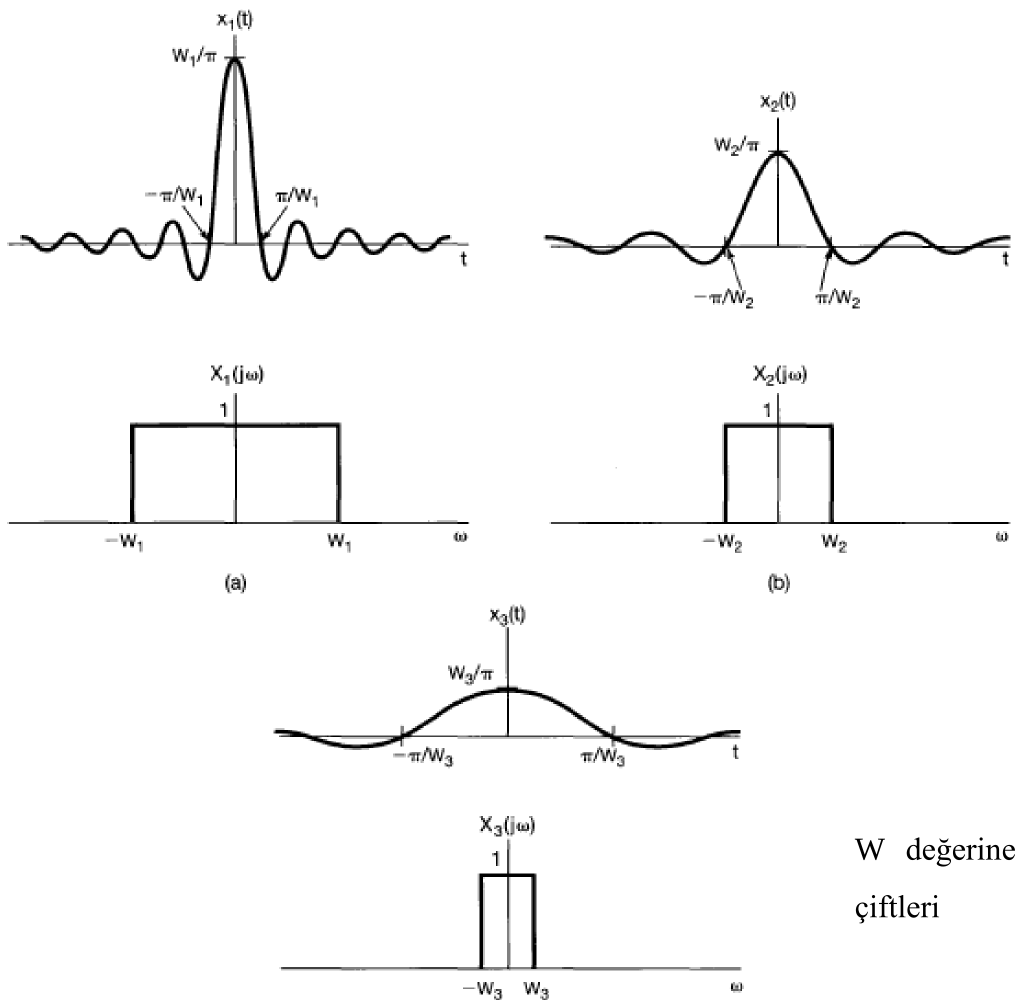
$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{\sin Wt}{\pi t},$$

Sinc işareti: $\mathrm{sinc}(\theta) = \dfrac{\sin \pi\theta}{\pi\theta}.$ şeklinde tanımlanır.

$$\frac{2\sin\omega T_1}{\omega} = 2T_1\ \text{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi}\ \text{sinc}\left(\frac{Wt}{\pi}\right).$$





(a) (b) (c)

W değerine bağlı olarak elde edilen Fourier dönüşüm çiftleri

Fourier Dönüşümünün Özellikleri :

Fourier dönüşüm çifti

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.$$

Lineerlik (Doğrusallık) :

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega)$$

$$y(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} Y(j\omega),$$

$$\boxed{ax(t) + by(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega).}$$

Zamanda Öteleme :

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

$$\boxed{x(t-t_0) \stackrel{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega).}$$

İspat : $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

t yerin $t - t_0$ koyara

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(e^{-j\omega t_0}X(j\omega)\right)e^{j\omega t}d\omega.$$

$$\mathfrak{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega).$$

Zamanda öteleme özelliği genlik spektrumunu

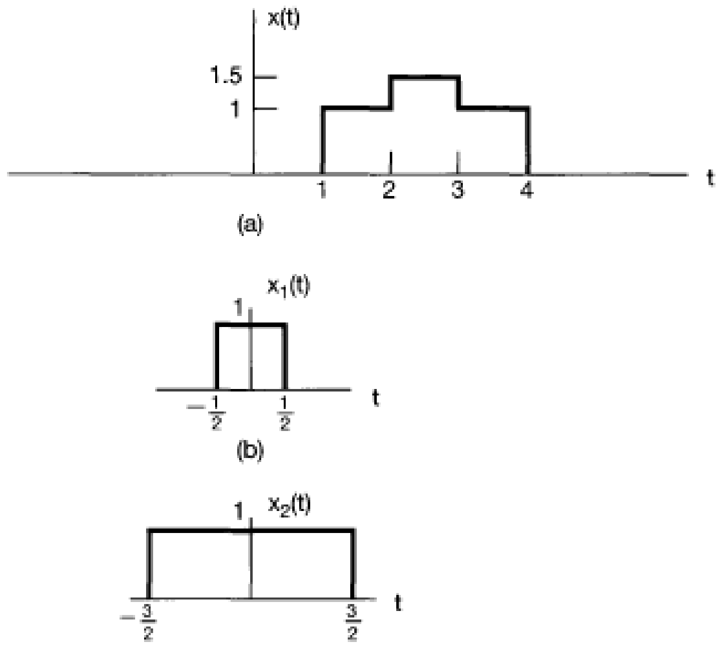
$$\mathfrak{F}\{x(t)\} = X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)},$$

$$\mathfrak{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega) = |X(j\omega)|e^{j[\sphericalangle X(j\omega)-\omega t_0]}.$$

Örnek :

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5),$$

şeklinde verilen işaretin Fourier dönüşümünü bulunuz.



(a)
(b)
(c)

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases},$$

için

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = 2\frac{\sin \omega T_1}{\omega},$$

şeklinde hesaplanmıştı.

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega} \quad \text{and} \quad X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}.$$

şeklinde doğrudan yazılarak,

$$X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}$$

Eşlenik :

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

$$\boxed{x^*(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X^*(-j\omega).}$$

İspat :

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right]^*$$

$$= \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}\,dt.$$

ω by $-\omega$.

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}\,dt.$$

Reel x(t)

için,

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}\,dt = X(j\omega),$$

Örnek :

$$x(t) = e^{-at}u(t),$$

$$X(j\omega) = \frac{1}{a + j\omega}$$

$$X(-j\omega) = \frac{1}{a - j\omega} = X^*(j\omega).$$

Türev ve integral Alma :

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega X(j\omega)e^{j\omega t}d\omega.$$

$$\boxed{\frac{dx(t)}{dt} \stackrel{\mathfrak{F}}{\longleftrightarrow} j\omega X(j\omega).}$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega).}$$

Örnek :

$x(t) = u(t)$. işaretinin Fourier dönüşümünü bulalım

$$g(t) = \delta(t) \overset{\mathfrak{F}}{\longleftrightarrow} G(j\omega) = 1.$$

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega),$$

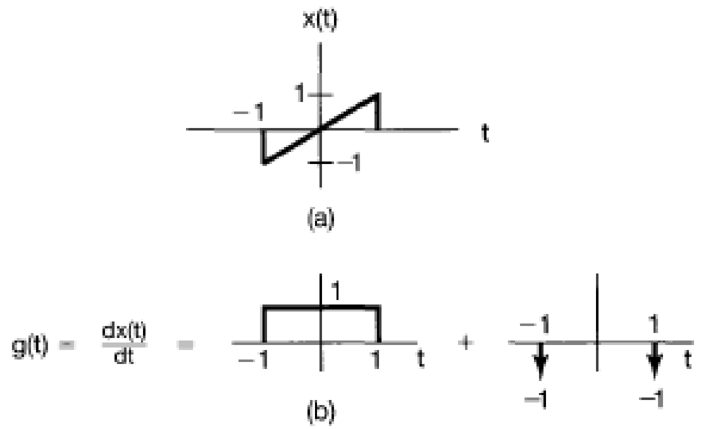
$$G(j\omega) = 1$$

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

$$\delta(t) = \frac{du(t)}{dt} \overset{\mathfrak{F}}{\longleftrightarrow} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1$$

Örnek :

$$g(t) = \frac{d}{dt}x(t).$$



$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}$$

$$G(0) = 0.$$

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega)$$ integrasyon özelliği kullanılarak,

$$X(j\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

Zaman ve frekans Ölçekleme :

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

$$\boxed{x(at) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right),}$$

İspat :

$$\mathfrak{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at) e^{-j\omega t} dt.$$

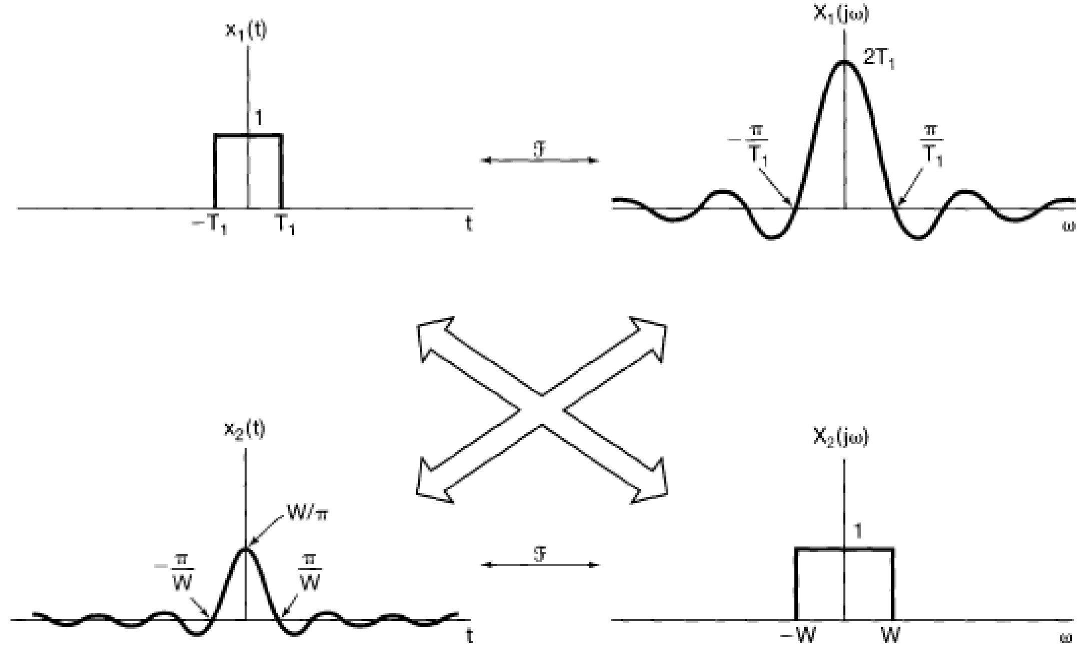
$$\tau = at$$

$$\mathfrak{F}\{x(at)\} = \begin{cases} \dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a > 0 \\ -\dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a < 0 \end{cases}$$

Dualite :

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \overset{\mathfrak{F}}{\longleftrightarrow} X_1(j\omega) = \frac{2 \sin \omega T_1}{\omega},$$ dönüşüm çifti

$$x_2(t) = \frac{\sin Wt}{\pi t} \overset{\mathfrak{F}}{\longleftrightarrow} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}.$$ İlşkisi görülmüştü.



Örnek :

$$g(t) = \frac{2}{1+t^2}.$$

şeklinde verilen işaretin Fourier dönüşümünü dualite özelliğinden yararlanarak hesaplayalım.

$$X(j\omega) = \frac{2}{1+\omega^2}.$$

işaretini düşünelim

$$x(t) = e^{-|t|} \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}.$$

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega.$$

Multiplying this equation by 2π and replacing t by $-t$, we obtain

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega.$$

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt.$$

$$\mathfrak{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}$$

Parseval Bağıntısı :

$$\boxed{\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega.}$$

İspat :

$$\int_{-\infty}^{+\infty}|x(t)|^2dt = \int_{-\infty}^{+\infty}x(t)x^*(t)dt$$

$$= \int_{-\infty}^{+\infty}x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}X^*(j\omega)e^{-j\omega t}d\omega\right]dt.$$

$$\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}X^*(j\omega)\left[\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt\right]d\omega.$$

$$\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega.$$

Konvolüsyon Özelliği :

$$\boxed{y(t) = h(t) * x(t) \overset{\mathfrak{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega).}$$

İspat :

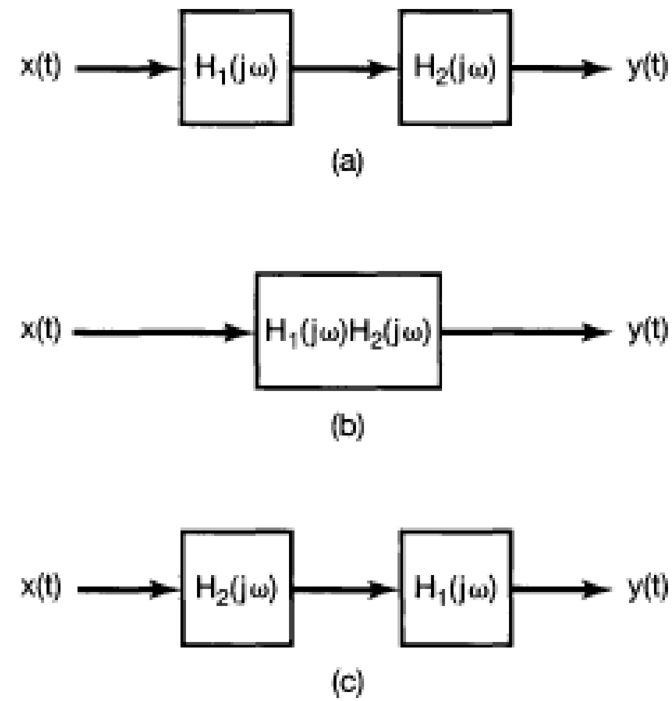
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau.$$

$$Y(j\omega) = \mathfrak{F}\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt.$$

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau = H(j\omega)\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau.$$

$$Y(j\omega) = H(j\omega)X(j\omega).$$



Şekildeki 3 sistem de eşdeğerdir.

$$\int_{-\infty}^{+\infty} |h(t)|dt < \infty.$$

koşulunun sağlanması durumunda (Kararlı sistem) sistemin frekans cevabı $H(j\omega)$ tanımlıdır

.

Örnek :

$$h(t) = \delta(t - t_0).$$

şeklinde impuls cevabı verilen sistemin frekans

$H(j\omega) = e^{-j\omega t_0}$. olarak bulunur. Sistemin çıkışı

$$Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega t_0}X(j\omega).$$

$$y(t) = x(t - t_0).$$

Örnek :

$y(t) = \frac{dx(t)}{dt}$. işareti için türev özelliği kullanılarakFourier dönüşümü

$$Y(j\omega) = j\omega X(j\omega).$$

$$H(j\omega) = j\omega.$$

türev alıcı sistemin frekans cevabı bulunmuş olur.

Örnek :

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

şeklinde verilen sistemin impuls cevabının birim basamak işareti u(t) olduğu görülebilir. Bu durumda integral alıcı sistemin frekans cevabı

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Örnekte verilen işaretin Fourier dönüşümü olarak bulunur.

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(j\omega)\delta(\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega). \end{aligned}$$

Örnek :

$$h(t) = e^{-at}u(t), \quad a > 0,$$

içi

$$x(t) = e^{-bt}u(t), \quad b > 0.$$

n

$y(t) = x(t) * h(t)$ şeklinde tanımlanan çıkış işaretini

$$X(j\omega) = \frac{1}{b + j\omega}$$

$$H(j\omega) = \frac{1}{a + j\omega}.$$

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega},$$

$$Y(j\omega) = \frac{1}{b - a}\left[\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right]$$

Ters Fourier dönüşümü alınarak

$$y(t) = \frac{1}{b - a}[e^{-at}u(t) - e^{-bt}u(t)].$$

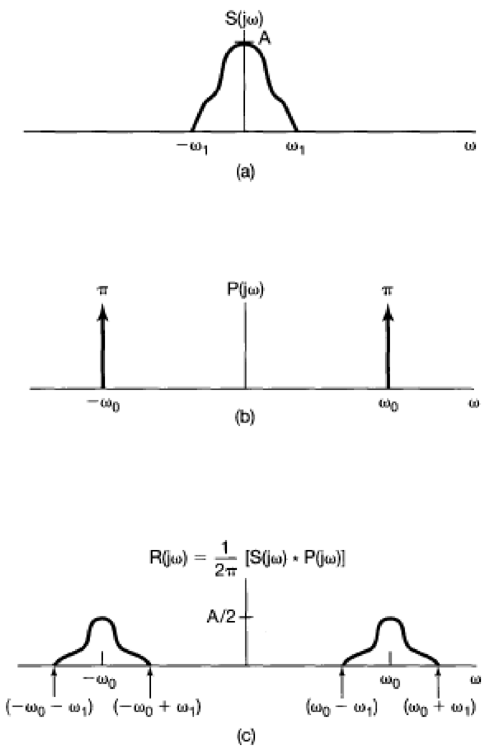
Çarpım Özelliği :

$$\boxed{r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega - \theta))d\theta}$$

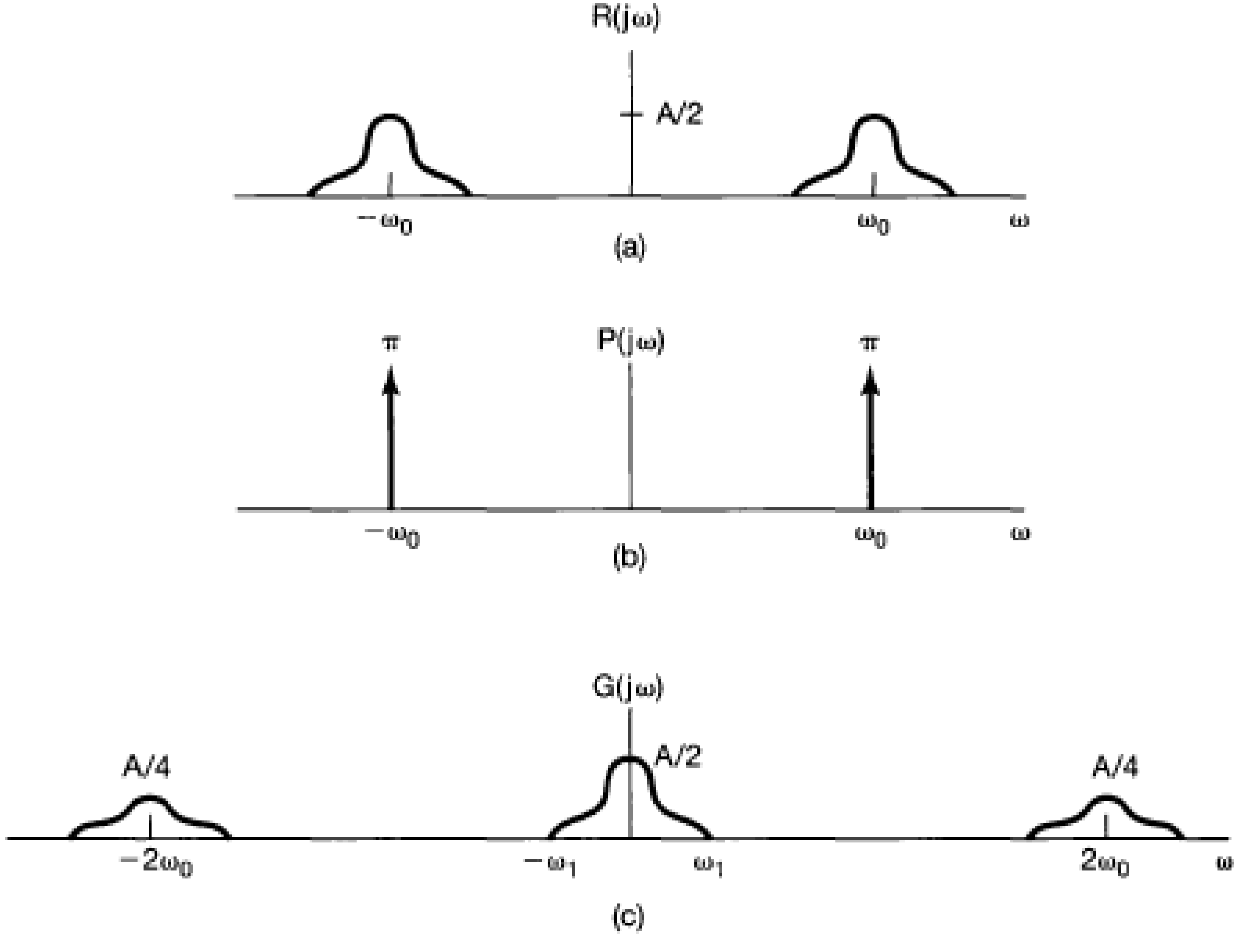
Örnek :

$$p(t) = \cos \omega_0 t.$$

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0),$$



$$
\begin{aligned}
R(j\omega) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta \\
&= \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0)).
\end{aligned}
$$

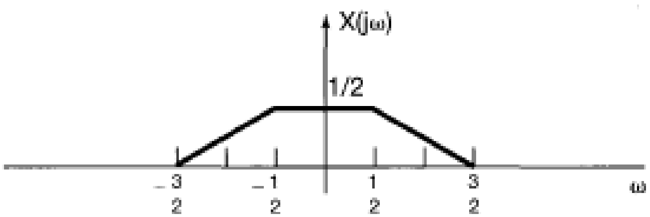


Örnek :

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}.$$

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right).$$

$$X(j\omega) = \frac{1}{2}\mathfrak{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathfrak{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}.$$



Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2 d\omega$$

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \ \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \ \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Periyodik işaretlerin Fourier dönüşümü ;

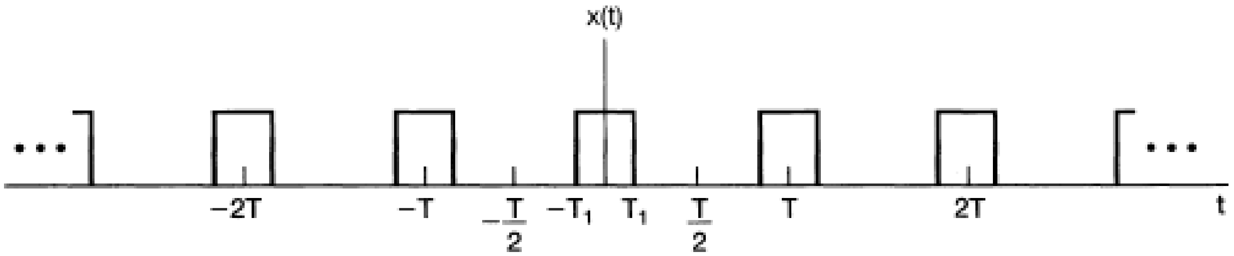
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t},$$

şeklinde Fourier serisine açılan periyodik bir işaret için Fourier dönüşümü , 2 tarafın da dönüşümü alınarak

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0),$$

şeklinde tanımlanabilir.

Örnek :



şeklinde verilen periyodik işaret $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$ için, ve Fourier serisi katsayıları $a_k = \dfrac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$, şeklinde

$T = 4T_1$ için

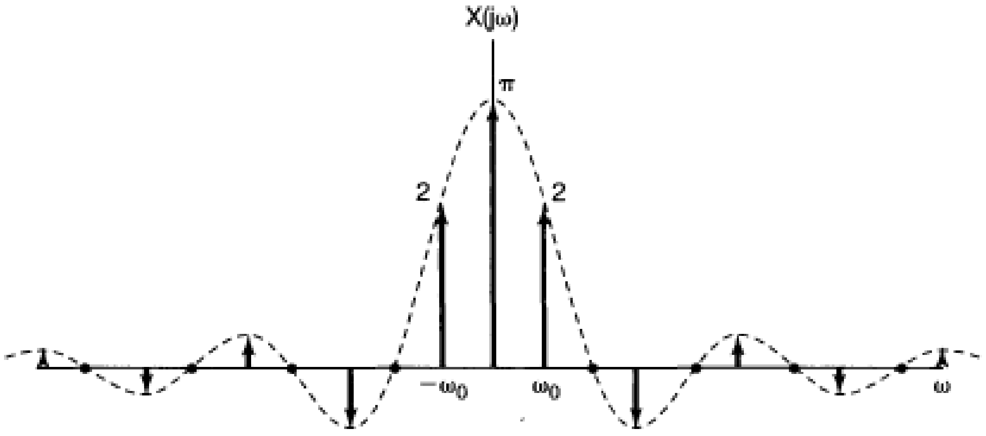
$$a_k = \frac{\sin k\omega_0 T_1}{\pi k},$$

İşaretin Fourier dönüşümü

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0),$$

yazılabilir

.



Örnek :

$x(t) = \sin \omega_0 t$. işareti için Fourier serisi katsayıları

$$a_1 = \frac{1}{2j},$$

$$a_{-1} = -\frac{1}{2j},$$

$$a_k = 0, \qquad k \neq 1 \quad \text{or} \quad -1.$$

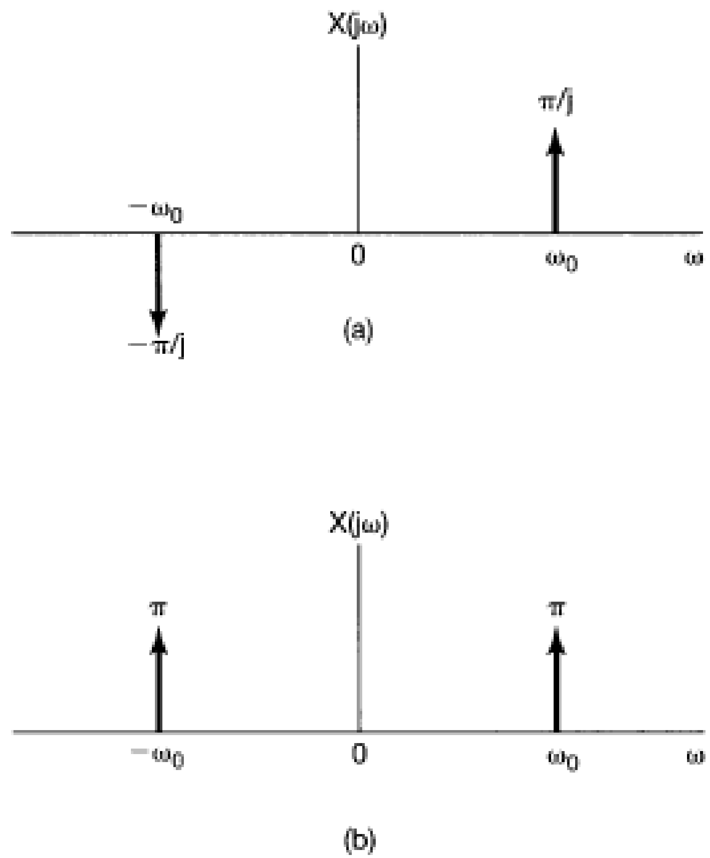
$x(t) = \cos \omega_0 t$, işareti için Fourier serisi katsayıları

$$a_1 = a_{-1} = \frac{1}{2},$$

$$a_k = 0, \qquad k \neq 1 \quad \text{or} \quad -1.$$

bulunmuştu.

Karşı gelen Fourier dönüşümleri

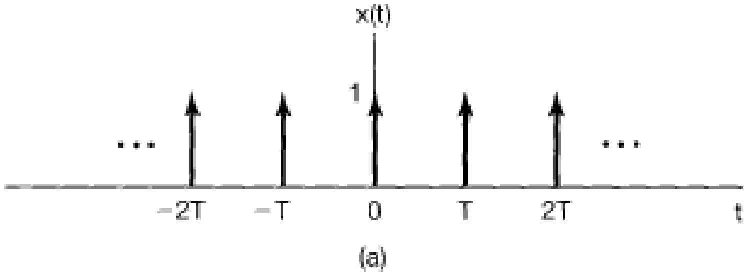


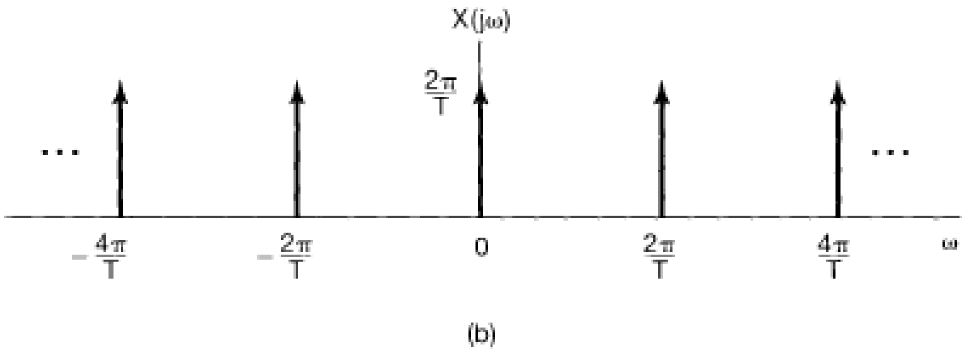
(a) $x(t) = \sin \omega_0 t$; (b) $x(t) = \cos \omega_0 t$.

Örnek :

$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$, işareti için Fourier serisi katsayıları $a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}$.

şeklinde elde edilmişti. İşaretin Fourie dönüşümü $X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$. bulunur.





a) İmpuls(dürtü katarı b) Fourier dönüşümü

LZD sistemlerin frekans domeninde modellenmesi

Fourier dönüşümünün konvolusyon özelliği

$$Y(j\omega) = H(j\omega)X(j\omega),$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)},$$

şeklinde sistem çıkışının bulunabildiği gösterilmişti. Benzer şekilde fark denklemiyle modellenen sistemler

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

2 tarafında Fourier dönüşümü alınarak

$$\mathfrak{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathfrak{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}.$$

Lineerlik özelliğinden yararlanarak

$$\sum_{k=0}^{N} a_k \mathfrak{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathfrak{F}\left\{\frac{d^k x(t)}{dt^k}\right\},$$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega),$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}.$$ bulunur.

Örnek :

$$\frac{dy(t)}{dt} + ay(t) = x(t),$$ şeklinde giriş-çıkış ilşkisi tanımlanan sistem için frekans cevabı $H(j\omega) = \frac{1}{j\omega + a}$.

ve impuls cevabı $h(t) = e^{-at}u(t)$. olarak bulunur.

Örnek :

Fark denklemi $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$. şeklinde verilen LZD sistem için frekans cevabı

$$H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}.$$

bulunur. Sistemin impuls cevabı ters Fourier dönüşümü alınarak elde edilir.

Frekans cevabını çarpanlarına ayırarak $H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}$.

Kısmı kesirlere ayırma (rezidü yöntemi kullanılarak, $G(v) = \frac{A_{11}}{v+1} + \frac{A_{21}}{v+3}$,

$$A_{11} = [(v+1)G(v)]\,|_{v=-1} = \frac{-1+2}{-1+3} = \frac{1}{2},$$

$$A_{21} = [(v+3)G(v)]\,|_{v=-3} = \frac{-3+2}{-3+1} = \frac{1}{2}.$$

$$H(j\omega) = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}.$$

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

Örnek :

$$x(t) = e^{-t}u(t).$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega + 2}{(j\omega+1)(j\omega+3)}\right]\left[\frac{1}{j\omega+1}\right]$$
$$= \frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}.$$

$$Y(j\omega) = \frac{A_{11}}{j\omega+1} + \frac{A_{12}}{(j\omega+1)^2} + \frac{A_{21}}{j\omega+3},$$

$$G(v) = \frac{A_{11}}{v+1} + \frac{A_{12}}{(v+1)^2} + \frac{A_{21}}{v+3},$$

$$\boxed{A_{ik} = \frac{1}{(\sigma_i - k)!}\left[\frac{d^{\sigma_i - k}}{dv^{\sigma_i - k}}[(v-\rho_i)^{\sigma_i}G(v)]\right]\Bigg|_{v=\rho_i}.}$$

$$v = jw \, , G(v)$$

$$A_{11} = \frac{1}{(2-1)!}\frac{d}{dv}[(v+1)^2G(v)]\,|_{v=-1} = \frac{1}{4},$$

$$A_{12} = [(v+1)^2G(v)]\,|_{v=-1} = \frac{1}{2},$$

$$A_{12} = [(v+3)G(v)]\,|_{v=-3} = -\frac{1}{4}.$$

$$A_{11} = \frac{1}{4}, \quad A_{12} = \frac{1}{2}, \quad A_{21} = -\frac{1}{4},$$ bulunarak

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega+1} + \frac{\frac{1}{2}}{(j\omega+1)^2} - \frac{\frac{1}{4}}{j\omega+3}.$$ yazılabilir.

$$e^{-at}u(t), \Re e\{a\} > 0 \qquad \frac{1}{a+j\omega}$$ ve

$$te^{-at}u(t), \Re e\{a\} > 0 \qquad \frac{1}{(a+j\omega)^2}$$

dönüşüm çiftleri kullanılarak

$$y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t).$$