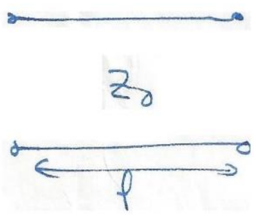
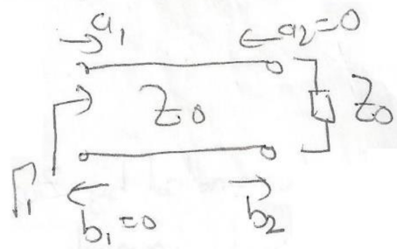
Examples of Scattering Coefficients

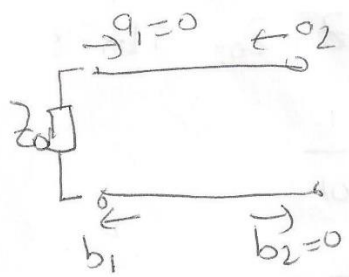
Length of transmission line:



Consider a length, ℓ, of transmission line. From symmetry considerations there is no difference between port 1 and port 2. The reference planes are naturally the ends of the line. Port 2 is terminated by a matched load and a normalised voltage wave (a_1) is assumed to be incident at port 1. This voltage is not reflected because of the continuity of impedance:



$$S_{11} = \frac{b_1}{a_1}\bigg|_{a_2=0} = \Gamma_1 = 0, \quad \Gamma_2 = S_{22} = 0$$



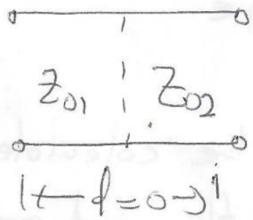
$$S_{12} = \frac{b_1}{a_2}\bigg|_{a_1=0} \qquad b_1 = e^{-\gamma\ell} a_2$$

$$\Rightarrow S_{12} = e^{-\gamma\ell}$$

then (from symmetry)

$$S = \begin{bmatrix} 0 & e^{-\gamma\ell} \\ e^{-\gamma\ell} & 0 \end{bmatrix}$$

Junction with different characteristic impedance



Port 2 is terminated by a matched load which is equal to Z_{02}, and the Z_0 of the line will be Z_{01}. Thus,

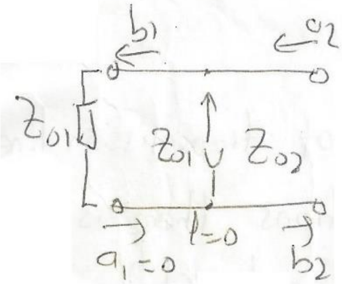
$$S_{11} = \Gamma_1 = \frac{b_1}{a_1}\bigg|_{a_2=0} = \frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}}$$

Similarly

$$S_{22} = \Gamma_2 = \frac{b_2}{a_2}\bigg|_{a_1=0} = \frac{Z_{01}-Z_{02}}{Z_{01}+Z_{02}} = -S_{11}$$

$$S_{12} = \frac{b_1}{a_2}\bigg|_{a_1=0}$$

, we can use the voltage equation of the junction : $V_1 = V_2$



$$V_{a_1} + V_{b_1} = V_{a_2} + V_{b_2}$$

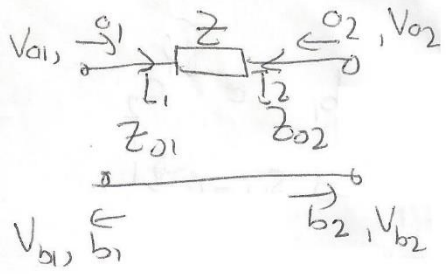
but because of the matched load of port 1, $a_1 = 0$ then

$$V_{b_1} = V_{a_2} + V_{b_2} \Rightarrow \sqrt{Z_{01}}\, b_1 = \sqrt{Z_{02}}\,(a_2 + b_2)$$

$$\frac{b_1}{a_2} = S_{12} = \sqrt{\frac{Z_{02}}{Z_{01}}}\left(1 + \frac{b_2}{a_2}\right) = \sqrt{\frac{Z_{02}}{Z_{01}}}\left(1 + S_{22}\right)$$

$$S_{12} = 2\,\frac{\sqrt{Z_{01}Z_{02}}}{Z_{01}+Z_{02}}$$

A series impedance on the junction



for S_{11}, port 2 is terminated by Z_{02} and the equivalent load impedance at port 2 is $Z + Z_{02}$. Then S_{11} will be,

$$S_{11} = \frac{(Z+Z_{02}) - Z_{01}}{(Z+Z_{02}) + Z_{01}}$$

similarly
$$S_{22} = \frac{(Z+Z_{01}) - Z_{02}}{(Z+Z_{01}) + Z_{02}}$$

For the transfer parameters S_{12} and S_{21} we can use the currents to the impedance Z: $I_1 = -I_2$

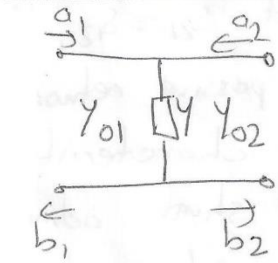
$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} \Rightarrow \frac{V_{a1} - V_{b1}}{Z_{01}} = \frac{V_{b2}}{Z_{02}} \quad (a_2 = 0 \Rightarrow V_{a2} = 0)$$

$$\frac{a_1\sqrt{Z_{01}} - b_1\sqrt{Z_{01}}}{Z_{01}} = \frac{b_2\sqrt{Z_{02}}}{Z_{02}}$$

$$b_2\sqrt{\frac{Z_{01}}{Z_{02}}} = (a_1 - b_1)$$

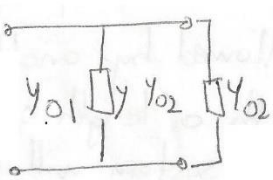
$$S_{21} = \sqrt{\frac{Z_{02}}{Z_{01}}}\,(1 - S_{11}) = \frac{2\sqrt{Z_{01}Z_{02}}}{Z + Z_{01} + Z_{02}} = S_{12}$$

A Shunt Admittance



A commonly occurring waveguide network is formed by a shunt susceptance. To examine such a junction, for generality, the characteristic impedances of the transmission lines on either side of the admittance are different.

Consider a wave incident at port 1 when port 2 is terminated by its characteristic impedance. Then $a_2 = V_{02} = 0$. The scattering coefficient S_{11} is equal to the reflection coefficient Γ_1 of port 1. Thus



$$\Gamma_1 = S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = \frac{Y_{01} - (Y + Y_{02})}{Y_{01} + (Y + Y_{02})}$$

Similarly by terminating port 1 by the transmission line characteristic impedance, we obtain

$$S_{22} = \frac{Y_{02} - (Y + Y_{01})}{Y_{02} + (Y + Y_{01})}$$

The voltage on either side of the admittance Y must be the same, thus $V_1 = V_2$ or $V_{a_1} + V_{b_1} = V_{b_2}$ (when $a_2 = 0 = V_{02}$)

$$\frac{1}{\sqrt{Y_{01}}}(a_1 + b_1) = \frac{1}{\sqrt{Y_{02}}} b_2$$

Dividing through by a_1 and rearranging yields

$$S_{21} = \frac{b_2}{a_1} = \sqrt{\frac{Y_{02}}{Y_{01}}}\,(1 + S_{11}) = \frac{2\sqrt{Y_{01}Y_{02}}}{Y_{01} + Y + Y_{02}}$$

Similarly, $S_{12} = \left.\dfrac{b_1}{a_2}\right|_{a_1=0}$

$$V_1 = V_2 \Rightarrow V_{b1} = V_{a2} + V_{b2} \Rightarrow \frac{b_1}{\sqrt{Y_{01}}} = \frac{1}{\sqrt{Y_{02}}}(a_2+b_2)$$

$$S_{12} = \sqrt{\frac{Y_{01}}{Y_{02}}}\,(1+S_{22}) = \sqrt{\frac{Y_{01}}{Y_{02}}}\;\frac{2Y_{02}}{Y+Y_{01}+Y_{02}} = \frac{2\sqrt{Y_{01}Y_{02}}}{Y+Y_{01}+Y_{02}}$$

Comparing S_{12} and S_{21}, it will be seen that $S_{21} = S_{12}$, thus showing that $S_{ij} = S_{ji}$ for this linear passive network.

When the transmission lines have the same characteristic impedance $Y_0 = Y_{01} = Y_{02}$, then the normalised shunt admittance is y $(y = Y/Y_0)$, and the scattering matrix becomes

$$S = \begin{bmatrix} -\dfrac{y}{y+2} & \dfrac{2}{y+2} \\ \dfrac{2}{y+2} & \dfrac{-y}{y+2} \end{bmatrix}$$

in this case $S_{11} = S_{22}$ and the system is symmetric.