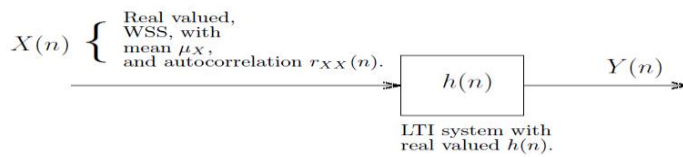


## EHB 315E – Digital Signal Processing

1. What will happen to the mean and autocorrelation of a random process when it goes through an LTI system? Given  $\mu_X$ ,  $r_{XX}(n)$  and  $h(n)$ , can we find  $E[Y(n)]$  and  $r_{YY}(n)$ ?



$$\begin{aligned}
 E[Y(n)] &= E\left[\sum_{k=-\infty}^{\infty} h(k) X(n-k)\right] = \sum_{k=-\infty}^{\infty} h(k) \underbrace{E[X(n-k)]}_{\mu_X} \\
 &= \mu_X \sum_{k=-\infty}^{\infty} h(k) e^{-j0} = \mu_X H(e^{j0})
 \end{aligned}$$

The mean of  $Y(n)$  is constant and it is related to the mean of  $X(n)$  by a scale factor that is frequency response of the filter at  $\omega=0$ .

$$\begin{aligned}
 r_{YX}(n+k, n) &= E[Y(n+k) X(n)] = E\left[X(n+k) \sum_{l=-\infty}^{\infty} h(l) X(n-l)\right] \\
 &= \sum_{l=-\infty}^{\infty} h(n-l) \underbrace{E[X(n+k) X(n-l)]}_{r_{XX}(n+k-l)} = \sum_{l=-\infty}^{\infty} h(n-l) r_{XX}(n+k-l) \\
 &\stackrel{m=n-l}{=} \sum_{m=-\infty}^{\infty} h(m) r_{XX}(n+k-m) = r_{XX}(k) * h(-k)
 \end{aligned}$$

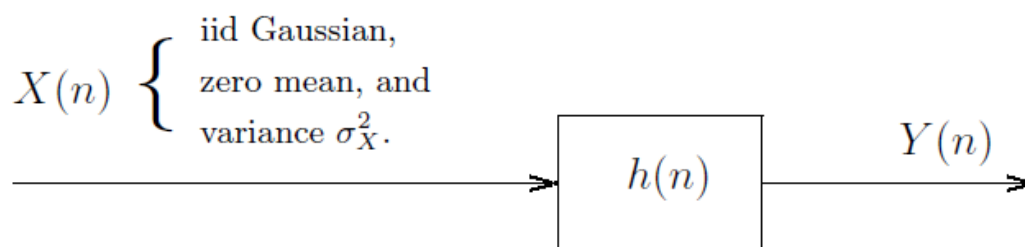
$$\begin{aligned}
 r_{YX}(n+k, n) &= E[X(n+k) X(n)] = E\left[\sum_{l=-\infty}^{\infty} h(l) X(n+k-l) X(n)\right] \\
 &= \sum_{l=-\infty}^{\infty} h(l) \underbrace{E[X(n+k-l) X(n)]}_{r_{XX}(k-l)} = \sum_{l=-\infty}^{\infty} h(l) r_{XX}(k-l) \\
 r_{YX}(k) &= r_{XX}(k) * h(k)
 \end{aligned}$$

$$r_Y[k] = r_{YX}(k) * h(-l) = r_X * h(k) * h(-k)$$

$$\mathcal{F}\{r_Y(n)\} = P_Y(e^{j\omega}) = P_X(e^{j\omega}) |H(e^{j\omega})|^2$$

$$z\{r_Y(n)\} = P_Y(z) = P_X(z) H(z) H(z^{-1})$$

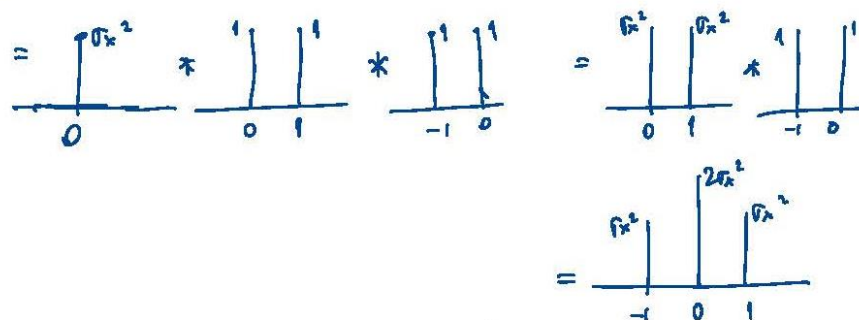
2. Let  $Y(n) = X(n) + X(n-1)$  in the system illustrated in the figure below. Find  $r_{YY}(n)$ .  
I.e., how correlated is the output with itself shifted by some lag value  $n$ ?



$$r_X(m) = E[X(n+m)X(n)]$$

$$= \begin{cases} E[X^2(n)] = \sigma_X^2 & , m=0 \\ E[X(n+m)X(n)] = \underbrace{E[X(n+m)]E[X(n)]}_{\text{since } X \text{ is independent}} = 0 & , m \neq 0 \end{cases} = \sigma_X^2 \delta(m)$$

$$r_Y(k) = r_X(m) * h(m) * h(-m)$$



$$r_X(m) = \sigma_X^2 (\delta(n+1) + 2\delta(n) + \delta(n-1))$$

3. Let  $x[n]$  be the random process that is generated by filtering white noise  $w[n]$  with a first order linear time invariant filter having a system function

$$H(z) = \frac{1}{1 - 0.25z^{-1}}$$

and  $w[n] \sim N(0, \sigma_w^2)$ ,  $\sigma_w^2 = 1$ .

- a) Find the power spectrum of  $x[n]$ ,  $P_X(z)$ .  
b) Find the autocorrelation of  $x[n]$ .

a) The autocorrelation of white noise  $w[n]$ :  $r_w(k) = \sigma_w^2 \delta(k)$

$$P_w(z) = z \sum_{k=-\infty}^{\infty} r_w(k) z^{-k} = z \sum_{k=-\infty}^{\infty} \sigma_w^2 \delta(k) z^{-k} = \sigma_w^2$$

$$P_X(z) = P_w(z) H(z) H(z^{-1}) = \underbrace{\sigma_w^2}_1 \underbrace{\frac{1}{1-0.25z^{-1}}}_{\text{ROC: } |z| > 0.25} \underbrace{\frac{1}{(1-0.25z)}}_{|z| < 4} = \frac{z^{-1}}{(1-0.25z^{-1})(z^{-1}-0.25)} \quad \text{ROC: } 0.25 < |z| < 4$$

b)  $r_x(k) = z^{-k} \sum_{n=-\infty}^{\infty} P_X(z)$

$$P_X(z) = \frac{z^{-1}}{(1-0.25z^{-1})(z^{-1}-0.25)} = \frac{16/15}{(1-0.25z^{-1})} + \frac{4/15}{z^{-1}-0.25}$$

$$P_X(z) = \frac{16/15}{1-0.25z^{-1}} - \frac{16/15}{(1-4z^{-1})}$$

$$r_x(k) = \frac{16}{15} \left(\frac{1}{4}\right)^k u(k) + \left(\frac{16}{15}\right) 4^k u(-k-1) = \frac{16}{15} \left(\frac{1}{4}\right)^{|k|}$$

4. Suppose that we would like to generate a random process having a power spectrum of the form

$$P_x(e^{j\omega}) = \frac{5 + 4 \cos 2\omega}{10 + 6 \cos \omega}$$

by filtering unit variance white noise with a linear shift-invariant filter. Writing  $P_x(e^{j\omega})$  in terms of complex exponentials we have

$$P_x(e^{j\omega}) = \frac{5 + 2e^{j2\omega} + 2e^{-j2\omega}}{10 + 3e^{j\omega} + 3e^{-j\omega}}$$

Replacing  $e^{j\omega}$  by  $z$  gives

$$P_x(z) = \frac{5 + 2(z^2 + z^{-2})}{10 + 3(z + z^{-1})} = \frac{(2z^2 + 1)(2z^{-2} + 1)}{(3z + 1)(3z^{-1} + 1)}$$

Performing the factorization

$$P_x(z) = H(z)H(z^{-1})$$

where

$$H(z) = \frac{2z^2 + 1}{3z + 1} = z \frac{2 \left(1 + \frac{1}{2}z^{-2}\right)}{3 \left(1 + \frac{1}{3}z^{-1}\right)}$$

we see that  $H(z)$  is a stable filter. Since introducing a delay into  $H(z)$  will not alter the power spectrum of the filtered process, we may equivalently use the filter

$$H(z) = \frac{2 \left(1 + \frac{1}{2}z^{-2}\right)}{3 \left(1 + \frac{1}{3}z^{-1}\right)}$$

which is causal and has a unit sample response given by

$$h(n) = \frac{2}{3} \left(-\frac{1}{3}\right)^n u(n) + \frac{1}{3} \left(-\frac{1}{3}\right)^{n-2} u(n-2)$$