Sayısal Yöntemler - MAT202 **Dersi veren:** *Prof. Dr. Ali Yapar*
Bahar 2020-2021 **Dersin yardımcısı:** *Araş. Gör. Furkan Şahin*
6. Uygulama 04.05.2021

1.
$$A = \begin{bmatrix} 9 & 1 & 1 & 1 \\ 1 & 8 & 1 & 1 \\ 1 & 1 & 7 & 1 \\ 1 & 1 & 1 & 6 \end{bmatrix}, \; b = \begin{bmatrix} 75 \\ 54 \\ 43 \\ 34 \end{bmatrix}$$

matrisleri verilmiş olsun. $Ax = b$ denklem sisteminde gerçek değerler

$$x = \begin{bmatrix} 7 \\ 5 \\ 4 \\ 3 \end{bmatrix}$$

şeklindedir. Bu denklem sistemini $x^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ başlangıç değerlerini kullanarak $\|x - x^{(k)}\| \leq 0.00005$ hata aralığında

(a) Jacobi yöntemi ile çözünüz.

(b) Gauss-Seidal yöntemi ile çözünüz.

(c) İki yöntemin A katsayılar matrisi için yakınsadığını gösteriniz.

Çözüm:

$$\begin{aligned} 9x_1 + \;\; x_2 + \;\; x_3 + \;\; x_4 &= 75 \\ x_1 + 8x_2 + \;\; x_3 + \;\; x_4 &= 54 \\ x_1 + \;\; x_2 + 7x_3 + \;\; x_4 &= 43 \\ x_1 + \;\; x_2 + \;\; x_3 + 6x_4 &= 34 \end{aligned}$$

a.

<u>Jacobi yöntemi:</u>

$$x_1^{(k+1)} = \frac{1}{9}\left[75 - x_2^{(k)} - x_3^{(k)} - x_4^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{8}\left[54 - x_1^{(k)} - x_3^{(k)} - x_4^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{7}\left[43 - x_1^{(k)} - x_2^{(k)} - x_4^{(k)}\right]$$

$$x_4^{(k+1)} = \frac{1}{6}\left[34 - x_1^{(k)} - x_2^{(k)} - x_3^{(k)}\right]$$

$\underline{k = 0}$:

$$x_1^{(1)} = \frac{1}{9}\left[75 - x_2^{(0)} - x_3^{(0)} - x_4^{(0)}\right] = 8.333333333333332$$

$$x_2^{(1)} = \frac{1}{8}\left[54 - x_1^{(0)} - x_3^{(0)} - x_4^{(0)}\right] = 6.75$$

$$x_3^{(1)} = \frac{1}{7}\left[43 - x_1^{(0)} - x_2^{(0)} - x_4^{(0)}\right] = 6.142857142857143$$

$$x_4^{(1)} = \frac{1}{6}\left[34 - x_1^{(0)} - x_2^{(0)} - x_3^{(0)}\right] = 5.666666666666667$$

$$x - x^{(1)} = \begin{bmatrix} 7 \\ 5 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 8.333333333333332 \\ 6.75 \\ 6.142857142857143 \\ 5.666666666666667 \end{bmatrix} = \begin{bmatrix} -1.333333333333332 \\ -1.75 \\ -2.142857142857143 \\ -2.666666666666667 \end{bmatrix}$$

$$\|x - x^{(k)}\| = \max_{\forall i} |x_i - x_i^{(k)}|$$

$$\Rightarrow \|x - x^{(1)}\| = 2.666666666666667$$

$\underline{k = 1}$:

$$x_1^{(2)} = \frac{1}{9}\left[75 - x_2^{(1)} - x_3^{(1)} - x_4^{(1)}\right] = 6.27116402116402$$

$$x_2^{(2)} = \frac{1}{8}\left[54 - x_1^{(1)} - x_3^{(1)} - x_4^{(1)}\right] = 4.232142857142858$$

$$x_3^{(2)} = \frac{1}{7}\left[43 - x_1^{(1)} - x_2^{(1)} - x_4^{(1)}\right] = 3.178571428571429$$

$$x_4^{(2)} = \frac{1}{6}\left[34 - x_1^{(1)} - x_2^{(1)} - x_3^{(1)}\right] = 2.128968253968254$$

$$x - x^{(2)} = \begin{bmatrix} 7 \\ 5 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 6.27116402116402 \\ 4.232142857142858 \\ 3.178571428571429 \\ 2.128968253968254 \end{bmatrix} = \begin{bmatrix} 0.728835978835980 \\ 0.767857142857142 \\ 0.821428571428571 \\ 0.871031746031746 \end{bmatrix}$$

$$\|x - x^{(2)}\| = 0.871031746031746$$

k	$\mathbf{x_1^{(k)}}$	$\mathbf{x_2^{(k)}}$	$\mathbf{x_3^{(k)}}$	$\mathbf{x_4^{(k)}}$	$\|\mathbf{x}-\mathbf{x^{(k)}}\|$
0	0	0	0	0	7
1	8.33333333	6.75	6.14285714	5.66666667	2.66666667
2	6.27116402	4.23214286	3.17857143	2.12896825	0.87103175
3	7.27336861	5.30266204	4.33824641	3.38635362	0.38635362
4	6.88585977	4.87525392	3.86251653	2.84762049	0.15237951
5	7.04606767	5.0505004	4.05589512	3.0627283	0.0627283
6	6.98120847	4.97941361	3.97724338	2.97458947	0.02541053
7	7.00763928	5.00836984	4.00925549	3.01035576	0.01035576
8	6.99689099	4.99659368	3.99623359	2.99578923	0.00421077
9	7.00126483	5.00138577	4.0015323	3.00171362	0.00171362
10	6.99948537	4.99943616	3.99937654	2.99930285	0.00069715
11	7.00020938	5.00022941	4.00025366	3.00028366	0.00028366
12	6.99991481	4.99990666	3.99989679	2.99988459	0.00011541
13	7.00003466	5.00003798	4.00004199	3.00004696	0.00004696

b.

Gauss-Seidel yöntemi:

$$x_1^{(k+1)} = \frac{1}{9}\left[75 - x_2^{(k)} - x_3^{(k)} - x_4^{(k)}\right]$$
$$x_2^{(k+1)} = \frac{1}{8}\left[54 - x_1^{(k+1)} - x_3^{(k)} - x_4^{(k)}\right]$$
$$x_3^{(k+1)} = \frac{1}{7}\left[43 - x_1^{(k+1)} - x_2^{(k+1)} - x_4^{(k)}\right]$$
$$x_4^{(k+1)} = \frac{1}{6}\left[34 - x_1^{(k+1)} - x_2^{(k+1)} - x_3^{(k+1)}\right]$$

$k = 0$:

$$x_1^{(1)} = \frac{1}{9}\left[75 - x_2^{(0)} - x_3^{(0)} - x_4^{(0)}\right] = 8.333333333333332$$
$$x_2^{(1)} = \frac{1}{8}\left[54 - x_1^{(1)} - x_3^{(0)} - x_4^{(0)}\right] = 5.708333333333334$$
$$x_3^{(1)} = \frac{1}{7}\left[43 - x_1^{(1)} - x_2^{(1)} - x_4^{(0)}\right] = 4.136904761904762$$
$$x_4^{(1)} = \frac{1}{6}\left[34 - x_1^{(1)} - x_2^{(1)} - x_3^{(1)}\right] = 2.636904761904762$$

$$x - x^{(1)} = \begin{bmatrix}7\\5\\4\\3\end{bmatrix} - \begin{bmatrix}8.333333333333332\\5.708333333333334\\4.136904761904762\\2.636904761904762\end{bmatrix} = \begin{bmatrix}-1.333333333333332\\-0.708333333333334\\-0.136904761904762\\0.363095238095238\end{bmatrix}$$

$$\Rightarrow \|x - x^{(1)}\| = 1.333333333333332$$

$\underline{k = 1:}$

$$x_1^{(2)} = \frac{1}{9}\left[75 - x_2^{(1)} - x_3^{(1)} - x_4^{(1)}\right] = 6.946428571428572$$

$$x_2^{(2)} = \frac{1}{8}\left[54 - x_1^{(2)} - x_3^{(1)} - x_4^{(1)}\right] = 5.034970238095239$$

$$x_3^{(2)} = \frac{1}{7}\left[43 - x_1^{(2)} - x_2^{(2)} - x_4^{(1)}\right] = 4.054528061224489$$

$$x_4^{(2)} = \frac{1}{6}\left[34 - x_1^{(2)} - x_2^{(2)} - x_3^{(2)}\right] = 2.994012188208617$$

$$x - x^{(2)} = \begin{bmatrix}7\\5\\4\\3\end{bmatrix} - \begin{bmatrix}6.946428571428572\\5.034970238095239\\4.054528061224489\\2.994012188208617\end{bmatrix} = \begin{bmatrix}0.053571428571428\\-0.034970238095239\\-0.054528061224489\\0.005987811791383\end{bmatrix}$$

$$\|x - x^{(2)}\| = 0.054528061224489$$

$\mathbf{k}$	$\mathbf{x_1^{(k)}}$	$\mathbf{x_2^{(k)}}$	$\mathbf{x_3^{(k)}}$	$\mathbf{x_4^{(k)}}$	$\|\mathbf{x} - \mathbf{x^{(k)}}\|$
0	0	0	0	0	7
1	8.33333333	5.70833333	4.13690476	2.63690476	1.33333333
2	6.94642857	5.03497024	4.05452806	2.99401219	0.05452806
3	6.99072106	4.99509234	4.00288206	3.00188409	0.00927894
4	7.00001572	4.99940227	3.99981399	3.000128	0.00059773
5	7.00007286	4.99999814	3.99997157	2.9999929	0.00007286
6	7.00000415	5.00000392	3.99999986	2.99999868	0.00000415

c.

<u>Jacobi yöntemi için:</u>

$$N = \begin{bmatrix}9&0&0&0\\0&8&0&0\\0&0&7&0\\0&0&0&6\end{bmatrix},\ P = N - A = \begin{bmatrix}0&-1&-1&-1\\-1&0&-1&-1\\-1&-1&0&-1\\-1&-1&-1&0\end{bmatrix}$$

$$M = N^{-1}P = \begin{bmatrix} 0 & -0.1111 & -0.1111 & -0.1111 \\ -0.125 & 0 & -0.125 & -0.125 \\ -0.1429 & -0.1429 & 0 & -0.1429 \\ -0.1667 & -0.1667 & -0.1667 & 0 \end{bmatrix}$$

$$\|M\| = \max_{\forall i} \sum_{\forall j} |m_{ij}|$$

$$\Rightarrow \|M\| = 0.5001 < 1 \rightarrow \text{yakınsar}$$

<u>Gauss-Seidel yöntemi için:</u>

$$N = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 1 & 8 & 0 & 0 \\ 1 & 1 & 7 & 0 \\ 1 & 1 & 1 & 6 \end{bmatrix},\ P = N - A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M = N^{-1}P = \begin{bmatrix} 0 & -0.1111 & -0.1111 & -0.1111 \\ 0 & 0.0139 & -0.1111 & -0.1111 \\ 0 & 0.0139 & 0.0317 & -0.1111 \\ 0 & 0.0139 & 0.0317 & 0.0556 \end{bmatrix}$$

$$\|M\| = 0.3333 < 1 \rightarrow \text{yakınsar}$$