EHB 315E Digital Signal Processing
Fall 2020
Prof. Dr. Ahmet Hamdi KAYRAN
Res. Asst. Hasan Hüseyin KARAOĞLU

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HOMEWORK 3 - SOLUTIONS

1 [20 pts] The output $y[n]$ of a discrete-time LTI system is found to be $3\left(\frac{1}{4}\right)^n u[n]$ when the input $x[n]$ is $u[n]$

(a) Find the system function, $H(z)$, and determine whether or not the system is stable and/or causal

(b) Draw the block diagram of the system.

(c) Plot the poles and zeros of $H(z)$, and indicate the ROC.

(d) Find the impulse response $h[n]$ of the system.

(e) Write the difference equation that characterizes the system.

(a)

$$\boxed{Y(z) = H(z)X(z)}$$

$$X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

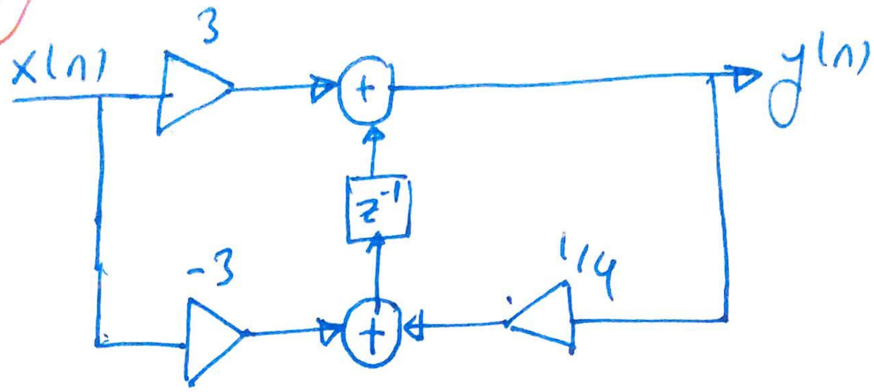
$$Y(z) = \frac{3}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{3}{1-1/4\,z^{-1}}}{\frac{1}{1-z^{-1}}}, \quad |z| > \frac{1}{4}$$

$$= \frac{3(1-z^{-1})}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$$

$\boxed{\text{The system is CAUSAL and STABLE}}$

(b)



(c) one pole and one zero

(p_1) (z_1)

$\boxed{p_1 = \frac{1}{4}}$ $\boxed{z_1 = 1}$

(e)
$$H(z) = \frac{Y(z)}{X(z)} = \frac{3\cdot(1-z^{-1})}{1-\frac{1}{4}z^{-1}}$$

$$Y(z) - \frac{1}{4}Y(z)z^{-1} = 3X(z) - 3z^{-1}X(z)$$

$$y(n) - \frac{1}{4}y(n-1) = 3x(n) - 3x(n-1)$$

(d)
$$H(z) = \underbrace{\frac{3}{1-\frac{1}{4}z^{-1}}} - \underbrace{\frac{3z^{-1}}{1-\frac{1}{4}z^{-1}}}$$

$$h(n) = 3\left(\frac{1}{4}\right)^{n}u(n) - 3\left(\frac{1}{4}\right)^{n-1}u(n-1)$$

2 [20 pts] Use the method of partial fractions to obtain the time-domain signals corresponding to the following z-transforms:

(a) $X(z) = \dfrac{1 + \frac{7}{6}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)}, \quad |z| < \frac{1}{3}$

(b) $X(z) = \dfrac{1 + \frac{7}{6}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)}, \quad \frac{1}{3} < |z| < \frac{1}{2}$

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a)

$$X(z) = \frac{A1}{1 - \frac{1}{2}z^{-1}} + \frac{A2}{1 + \frac{1}{3}z^{-1}}$$

$$A1 = \left[\left(1 - \frac{1}{2}z^{-1}\right)X(z)\right]\Big|_{z=\frac{1}{2}}$$

$$= \frac{1 + \frac{7}{6}(2)}{1 + \frac{1}{3}(2)} = \frac{10/3}{5/3} = 2$$

$$A2 = \left[\left(1 + \frac{1}{3}z^{-1}\right)X(z)\right]\Big|_{z=-\frac{1}{3}}$$

$$A2 = \frac{1 + \frac{7}{6}(-3)}{1 - \frac{1}{2}(-3)} = \frac{-5/2}{5/2} = -1$$

$$X(z) = \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{-1}{1 + \frac{1}{3}z^{-1}}$$

$$\boxed{x[n] = \left[-2\left(\frac{1}{2}\right)^n + \left(-\frac{1}{3}\right)^n\right]u(-n-1)}$$

(b) $X(z) = \dfrac{2}{1-\frac{1}{2}z^{-1}} + \dfrac{-1}{1+\frac{1}{3}z^{-1}}$

$$\boxed{x(n) = -2\left(\frac{1}{2}\right)^n u(-n-1) - \left(-\frac{1}{3}\right)^n u(n)}$$

3 [20 pts] Let $x(n)$ be a sequence with a DTFT $X\left(e^{j\omega}\right)$ For each of the following sequences that are formed from $x(n)$, express the DTFT in terms of $X\left(e^{j\omega}\right)$ by using DTFT formula. (If you don't make use of DTFT definition, your solutions will not be accepted)

(a) $x^*(-n)$

(b) $x(n) * x^*(-n)$

(c) $x(2n+1)$

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a) The DTFT of $x^*(-n)$ is

$$DTFT\left[x^*(-n)\right] = \sum_{n=-\infty}^{\infty} x^*(-n)\, e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} x^*(n)\, e^{jwn}$$

Bringing the conjugate operator outside, we obtain

$$DTFT\left[x^*(-n)\right] = \left[\underbrace{\sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}}_{X(e^{jw})}\right]^*$$

Which leads to the DTFT pair

$$x^*(-n) \xleftrightarrow{F} X^*(e^{jw})$$

(b)

$$\mathrm{DTFT}\left[x(n) * x^*(-n)\right] = X(e^{j\omega})\, X^*(e^{j\omega})$$
$$= |X(e^{j\omega})|^2$$

(c)

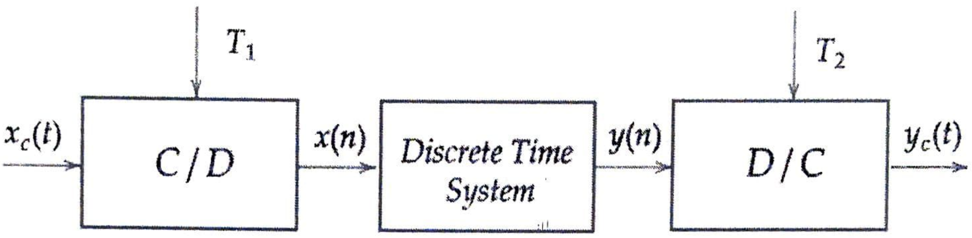
$$\mathrm{DTFT}\left[x(2n+1)\right] = \sum_{n=-\infty}^{\infty} x(2n+1)\, e^{-j\omega n}$$
$$= \sum_{n \text{ odd}} x(n)\, e^{-j\omega n}$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[1-(-1)^n\right] x(n)\, e^{-j\omega n}$$

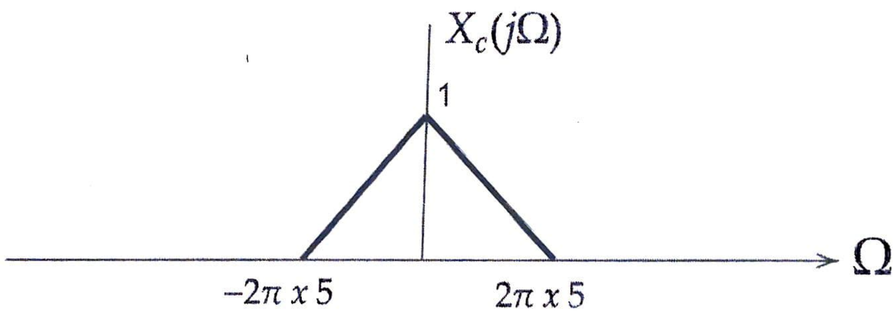
$$= \underbrace{\frac{1}{2} \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}}_{X(e^{j\omega})} - \underbrace{\frac{1}{2} \sum_{n=-\infty}^{\infty} \underbrace{(-1)^n}_{e^{j\pi n}} x(n)\, e^{-j\omega n}}_{X(e^{j(\omega-\pi)})}$$

$$\mathrm{DTFT}\left[x(2n+1)\right] = \frac{1}{2}\left[X(e^{j\omega}) - X(e^{j(\omega-\pi)})\right]$$

4 **[20 pts]** The following system is used to process an analog signal with a discrete-time system.



Suppose that $x_c(t)$ is bandlimited with $X_c(j\Omega) = 0$ for $|\Omega| > 10\pi$ as shown in the figure below



and that discrete-time system is an ideal low-pass filter with a cut-off frequency of $2\pi/3$

(a) Find the Fourier Transform of $y_c(t)$ if the sampling frequencies are $1/T_1 = 1/T_2 = 10$

(b) Repeat for $1/T_1 = 20$ Hz and $1/T_2 = 10$ Hz

(c) Repeat for $1/T_1 = 10$ Hz and $1/T_2 = 20$ Hz

(a) When the sampling frequency of C/D and D/C converters are the same, $x_c(t)$ is bandlimited with $X_c(j\Omega)=0$ for $|\Omega| > \pi/T_1$, this system is equivalent to an analog filter with a frequency response

$$H_c(j\Omega) = \begin{cases} H(e^{j\Omega T_1}) & , |\Omega| < \pi/T_1 \\ 0 & , \text{else} \end{cases}$$

$$\boxed{\begin{matrix} ! \; h(n) \xleftrightarrow{F} H(e^{j\Omega T_1}) \\ h_c(t) \xleftrightarrow{F} H_c(j\Omega) \end{matrix}}$$

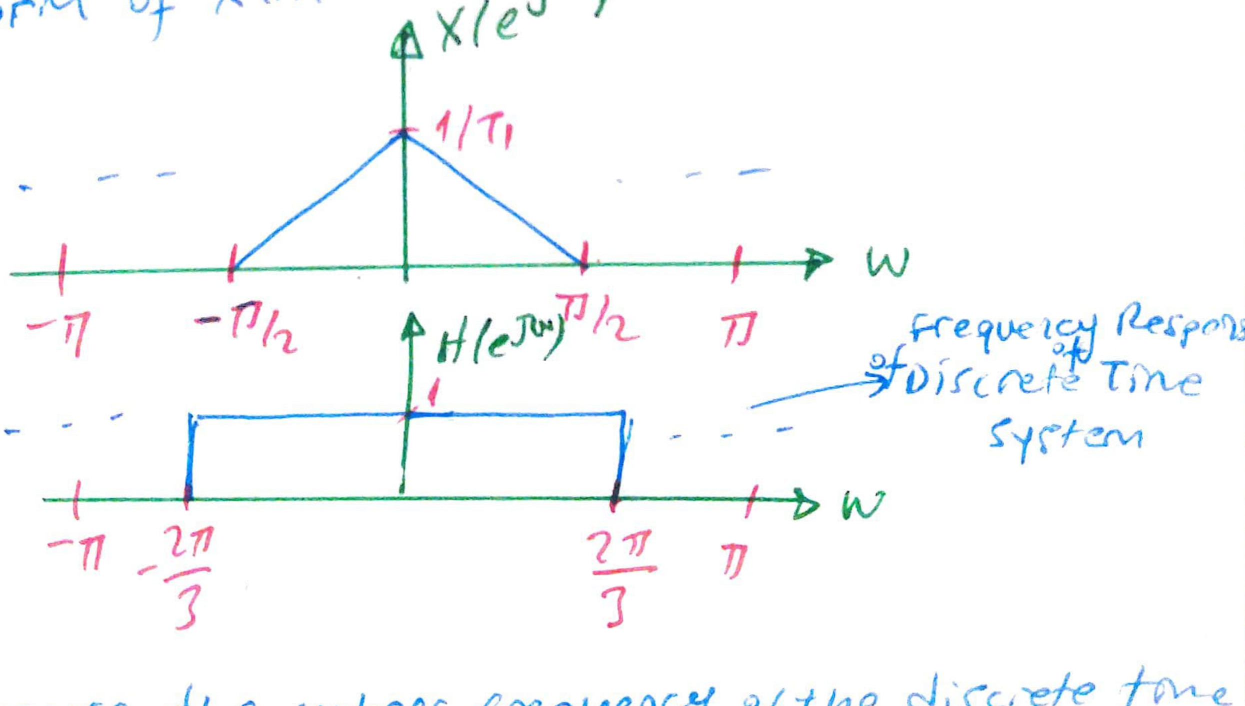
Therefore, if $H(e^{j\omega})$ is a LPF with a cut-off frequency $2\pi/3$, the cutoff frequency of $H_c(j\Omega)$, denoted by Ω_0, is given by

$$\Omega_0 T_1 = \frac{2\pi}{3}$$

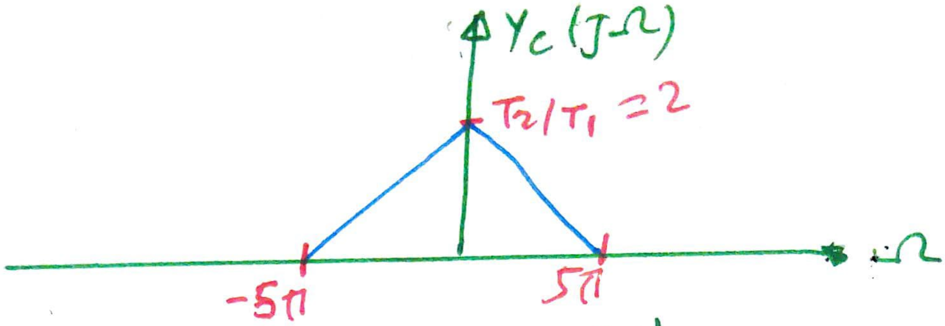
$$\Omega_0/10 = \frac{2\pi}{3}$$

$$\Omega_0 = \frac{20\pi}{3} \text{ rad/s}$$

(b) When the sampling frequencies of the C/D and D/C are different, it is best to plot the spectrum of the signals as they progress through the system. With $X_c(j\Omega)$ as shown above, the discrete-time Fourier Transform of $x[n]$ is

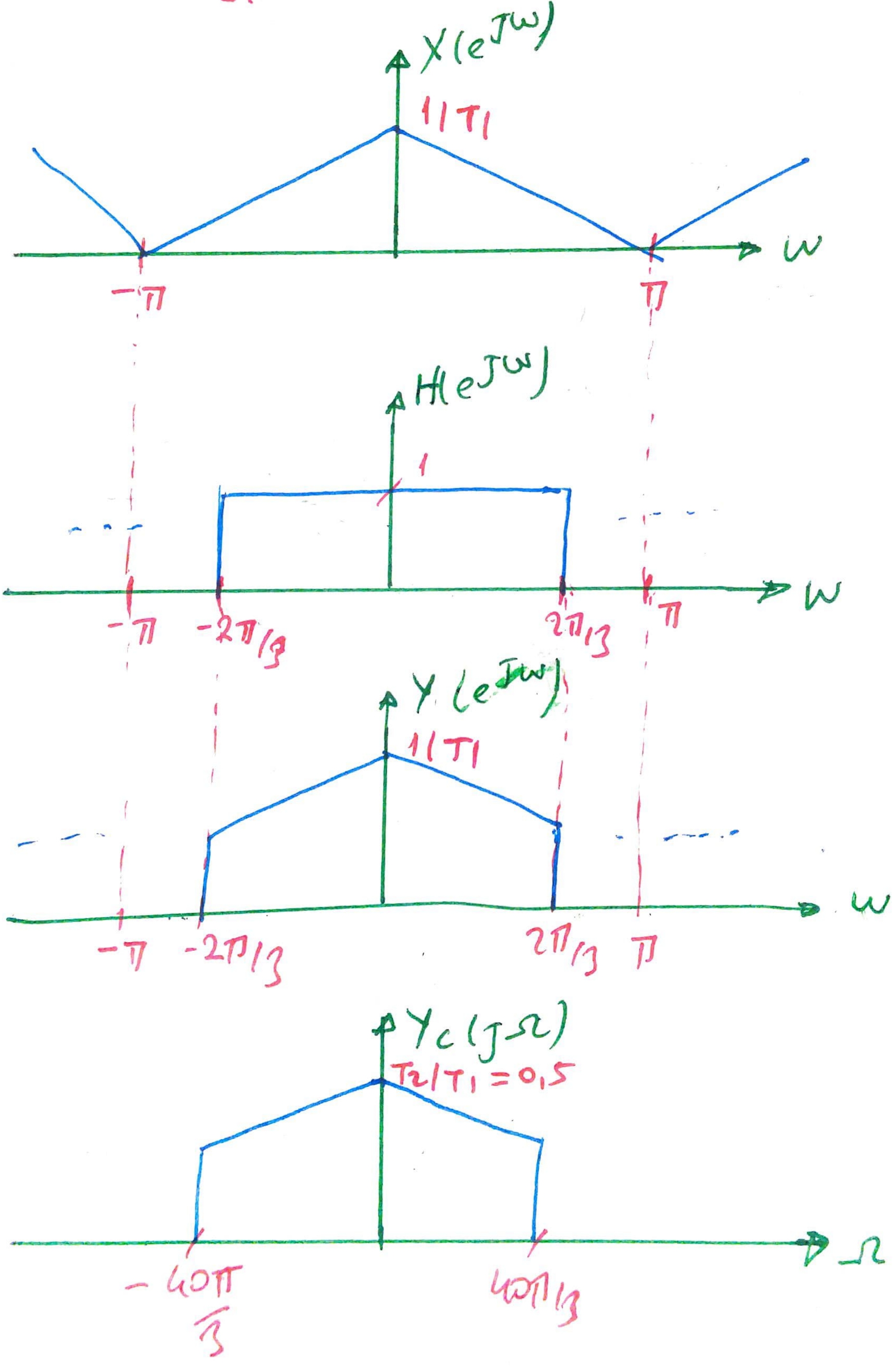


Because the cutoff frequency of the discrete time LPF is $\frac{2\pi}{3}$, $y[n] = x[n]$, and the output of the DC converter is drawn as follows:



$Y_c(j\Omega)$
$T_2/T_1 = 2$
-5π
5π
Ω

(c)



$X(e^{j\omega})$
$1/T_1$
$-\pi$
π
ω
$H(e^{j\omega})$
1
$-\pi$
$-2\pi/3$
$2\pi/3$
π
ω
$Y(e^{j\omega})$
$1/T_1$
$-\pi$
$-2\pi/3$
$2\pi/3$
π
ω
$Y_c(j\Omega)$
$T_2/T_1 = 0{,}5$
$-\frac{40\pi}{3}$
$40\pi/3$
Ω

5 **[20 pts]** Determine the minimum sampling frequency for each of the following signals

(a) $x_c(t)$ is real with $X_c(j\Omega)$ nonzero only for 6 krad/s $< |\Omega| <$ 9 krad/s

(b) $x_c(t)$ is real with $X_c(j\Omega)$ nonzero only for 16 krad/s $< |\Omega| <$ 27 krad/s

(c) $x_c(t)$ is complex with $X_c(j\Omega)$ nonzero only for 17 krad/s $< |\Omega| <$ 73 krad/s

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(a) signal bandwith B is $B = 9-6 = 3$ krad/s and $\Omega_2 = 9$ krad/s. $\Omega_2 = 9 = 3B$ is an integer multiple of B. Hence,

$\Omega_s = 2B = 6$ krad/s

(b) $B = 27-16 = 11$ krad/s $\}$ which is not an integer multiple of B.

$\Omega_2 = 27$ krad/s

$\lfloor \Omega_2/B \rfloor = \lfloor 27/11 \rfloor = 2$ → (!) $\lfloor \cdot \rfloor$: rounding operator

$B' = \Omega_2 / \lfloor \Omega_2/B \rfloor = 27/2 = 13{,}5$ krad/s

$\Omega_s = 2B' = 2(13{,}5) = 27$ krad/s

(c) For a complex bandpass signal with a spectrum that is non-zero for $\Omega_1 < \Omega < \Omega_2$, the minimum sampling frequency $\Omega_s = \Omega_2 - \Omega_1$. Thus,

$\Omega_s = 73-17 = 56$ krad/s