EHB 315E Digital Signal Processing
Fall 2020
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HOMEWORK 5 - SOLUTIONS

1 [25 pts] Suppose we have two four-point sequences $x[n]$ and $h[n]$ as follows:

$$x(n) = \cos\left(\frac{\pi n}{2}\right), \quad n = 0,1,2,3$$
$$h(n) = 2^n. \quad n = 0,1,2,3$$

(a) Calculate the four-point DFT $X(k)$

(b) Calculate the four-point DFT $H(k)$

(c) Calculate $y(n) = x(n) \circledast_4 h(n)$ by doing the circular convolution directly.

(d) Calculate $y(n)$ of Part (c) by multiplying the DFTs of $x(n)$ and $h(n)$ and performing an inverse DFT.

S1 a. $X(k) = \sum_{n=0}^{3} x(n) W_4^{nk}$

$X(k) = x(0) W_4^{0k} + x(1) W_4^{1k} + x(2) W_4^{2k} + x(3) W_4^{3k}$

$= 1 + 0 - W_4^{2k} + 0$

$= 1 - W_4^{2k}$ ... $W_4^{2k} = e^{-j2\pi \cdot 2/4 \, k}$

X(k) = [0 2 0 2]

b. $H(k) = \sum_{n=0}^{3} h(n) W_4^{nk}$

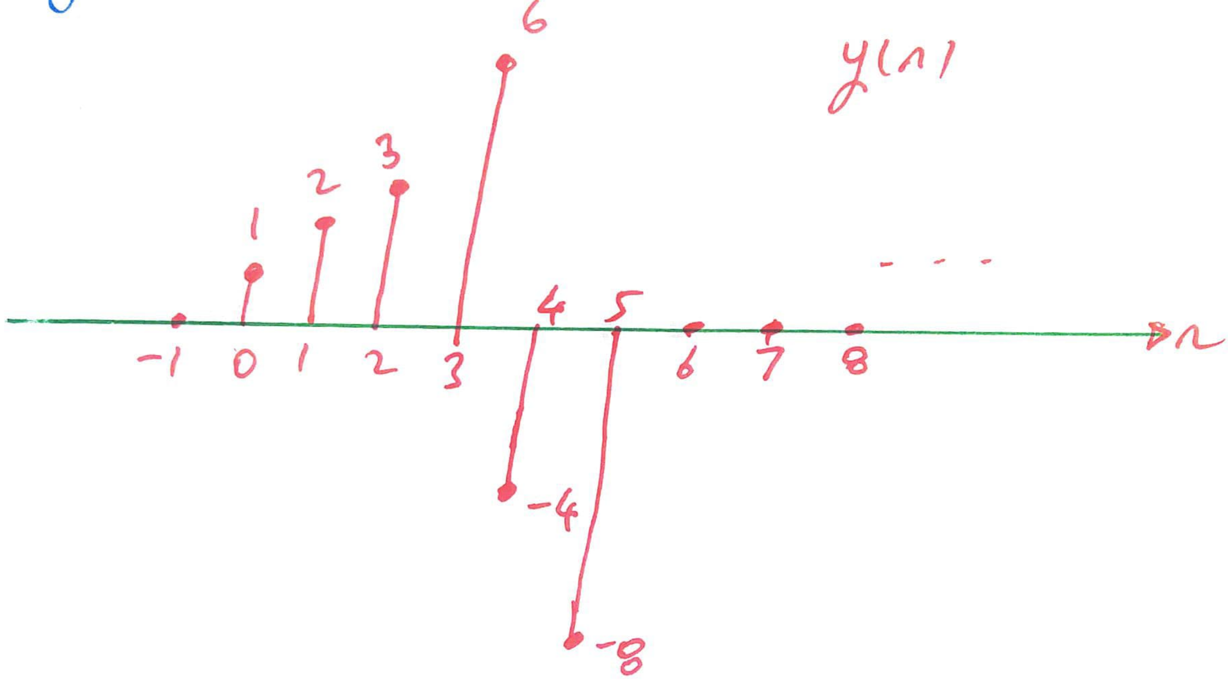
$= 2^0 W_4^{0k} + 2^1 W_4^{1k} + 2^2 W_4^{2k} + 2^3 W_4^{3k}$

$= 1 + 2W_4^{k} + 4W_4^{2k} + 8W_4^{3k}$

H(k) = [15 -3+6i -5 -3-6i]

c. Circular convolution = linear convolution
+
aliasing

$y(n) = x(n) * h(n)$



aliasing means the last 3 points $(n=4,5,6)$ will wrap-around on top of the first three points.

$y(n) = x(n) \; ④ \; h(n)$

$= -3\delta(n) - 6\delta(n-1) + 3\delta(n-2) + 6\delta(n-3)$

d. $Y(k) = H(k)\,X(k)$

$= (1 + 2W_4^{k} + 4W_4^{2k} + 8W_4^{3k})(1 - W_4^{2k})$

$= 1 + 2W_4^{k} + 4W_4^{2k} + 8W_4^{3k} - W_4^{2k} - 2W_4^{3k}$
$- 4W_4^{4k} - 8W_4^{5k}$

! $\cdot\; W_4^{4k} = W_4^{0k} = 1$

$\cdot\; W_4^{5k} = W_4^{k}$

$Y(k) = -3 - 6W_4^{k} + 3W_4^{2k} + 6W_4^{3k}$

$\text{IDFT}(Y(k)) = y(n)$

$= -3\delta(n) - 6\delta(n-1) + 3\delta(n-2)$
$+ 6\delta(n-3) \quad , \; 0 \le n \le 3$

2 **[25 pts]** You are given two signals $x_1(n)$ and $x_2(n)$

$$x_1(n) = \begin{cases} 1, & 0 \le n \le 99 \\ 0, & \text{otherwise} \end{cases}$$

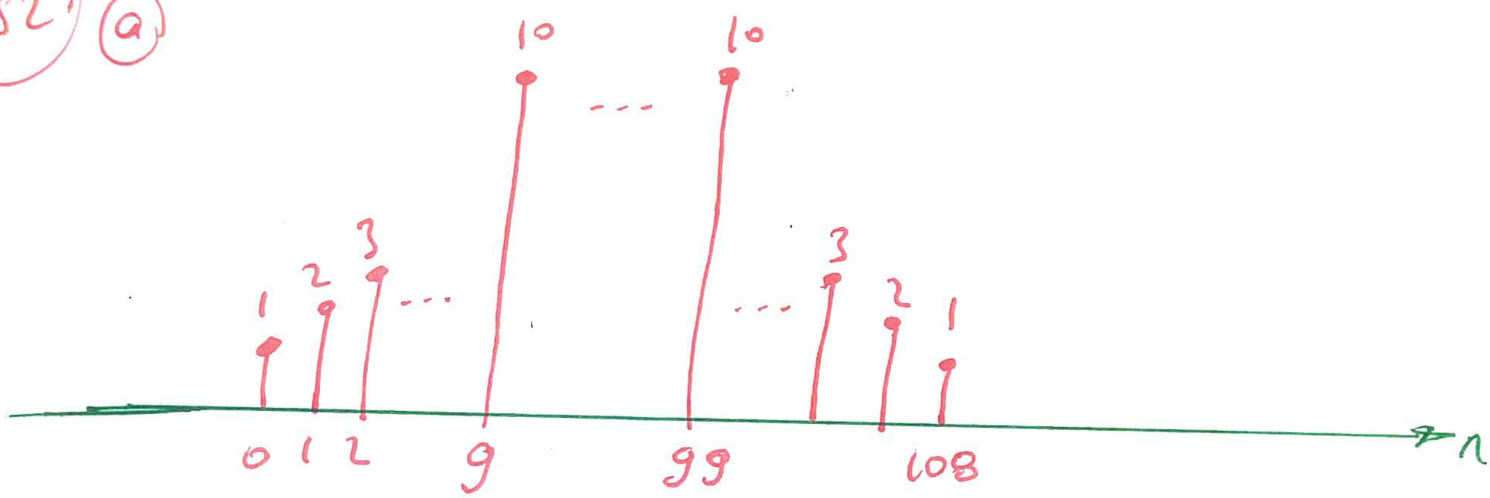
$$x_2(n) = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the linear convolution $x_1(n) * x_2(n)$

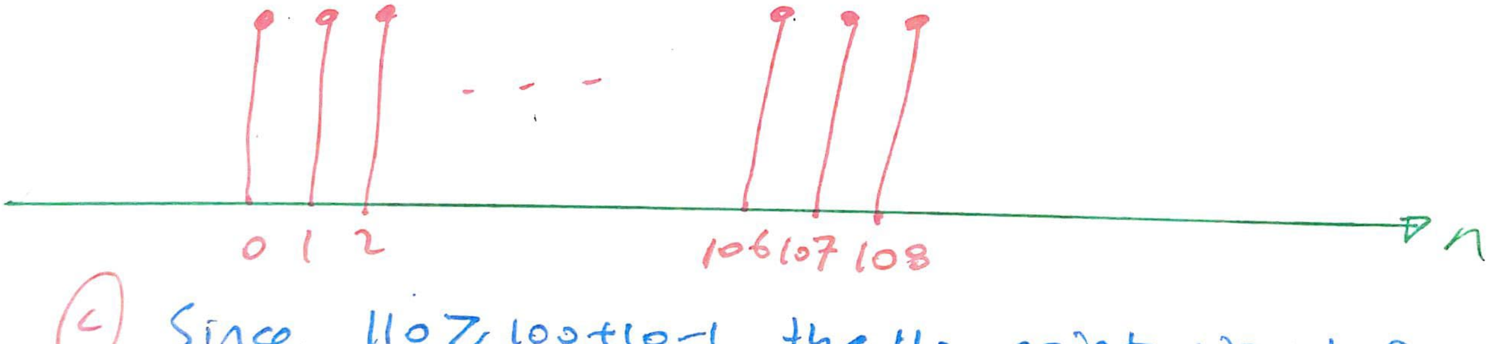
(b) Determine the 100-point circular convolution $x_1(n)$ (100) $x_2(n)$

(c) Determine the 110-point circular convolution $x_1(n)$ (110) $x_2(n)$

S2 (a)



(b) circular convolution (100-point) can be obtained by the first 9 points of the linear convolution above:



(c) Since $110 \geq 100 + 10 - 1$, the 110-point circular convolution will be equivalent to the linear convolution of part (a)

3 [25 pts] Consider the causal, linear shift-invariant filter with system (transfer) function

$$H(z) = \frac{1 - 0.5z^{-1}}{(1 - 3.5z^{-1} + 3z^{-2})(1 - 0.7z^{-1})}$$

(a) Write down the difference equation for this system. Is this system stable? Discuss.

(b) Draw a signal flowgraph (block diagram) for this system using

i. Direct form I

ii. Direct form II

iii. A parallel connection of first- and second-order systems realized in direct form II

iv. A cascade (serial connection) of first- and second-order systems realized in Direct Form II

S3

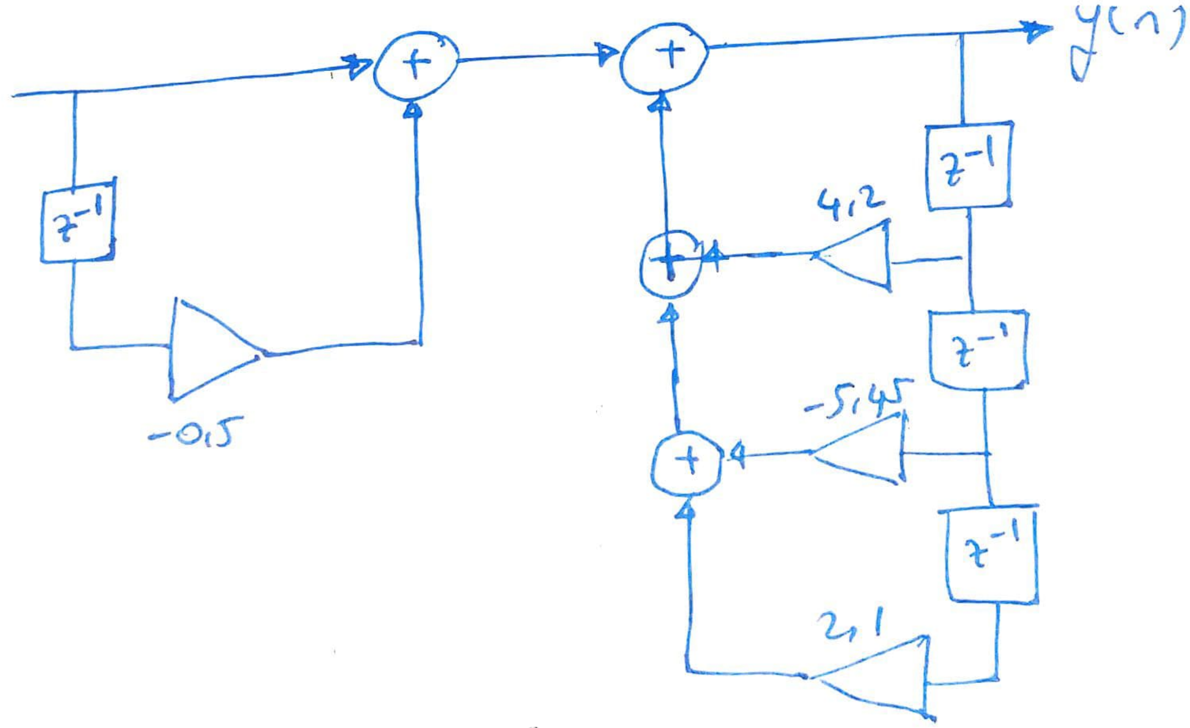
a

$$H(z) = \frac{(1 - 0,5z^{-1})}{-2,1z^{-3} + 5,45z^{-2} - 4,2z^{-1} + 1}$$

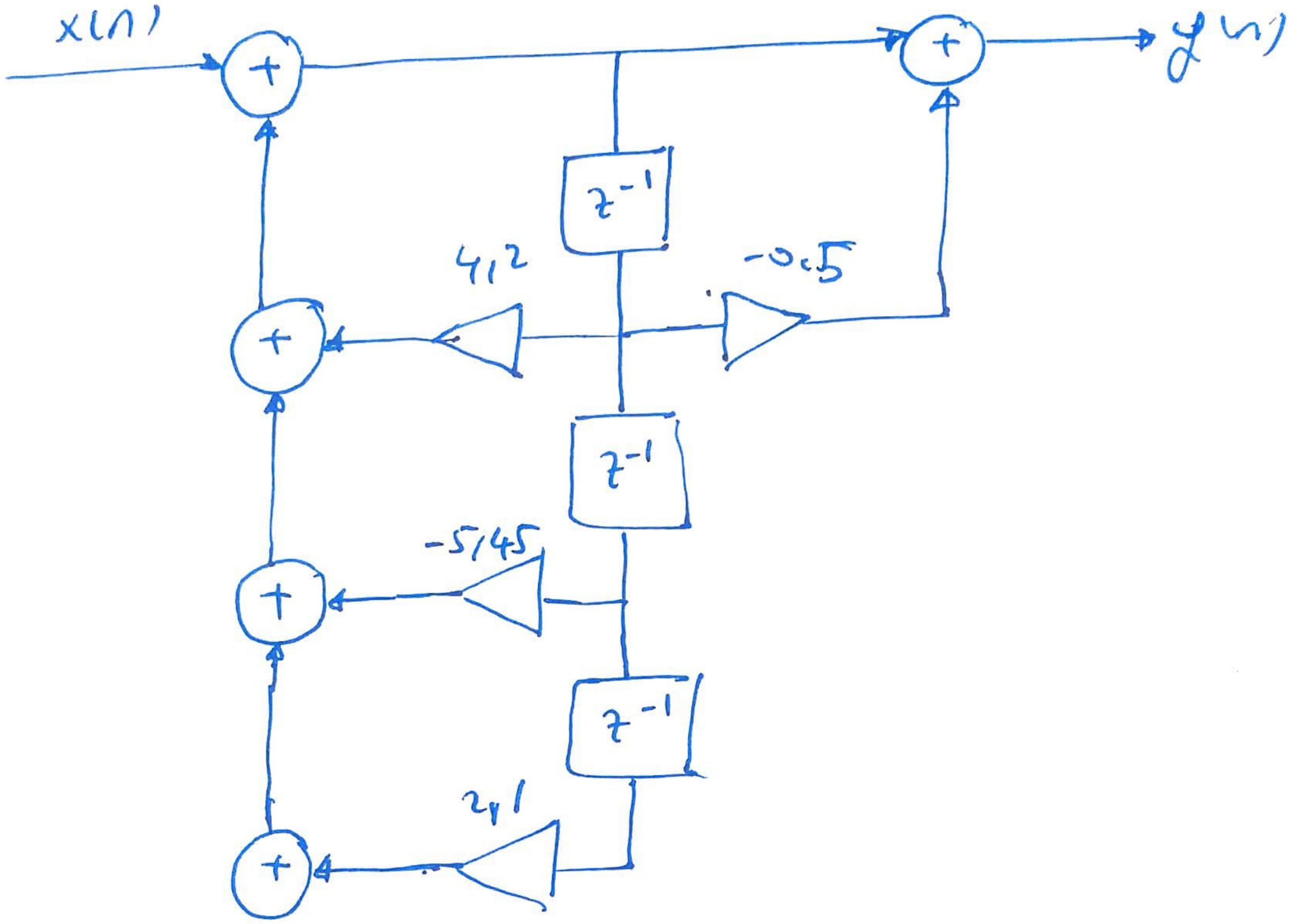
$$\frac{Y(z)}{X(z)} = H(z)$$

$$y(n) - 4,2\,y(n-1) + 5,45\,y(n-2) - 2,1\,y(n-3) = x(n) - 0,5\,x(n-1)$$

b i



ii)



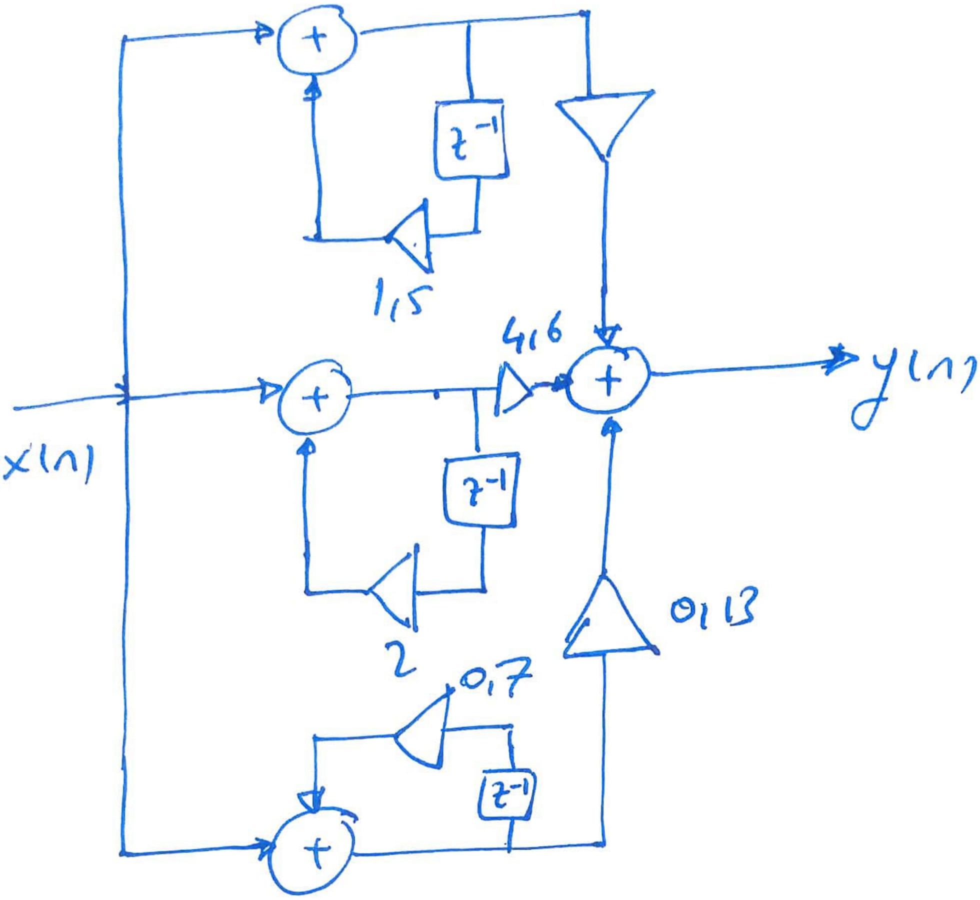
iii)

$$H(z) = \frac{1-0{,}5z^{-1}}{(1-3{,}5z^{-1}+3z^{-2})(1-0{,}7z^{-1})}$$

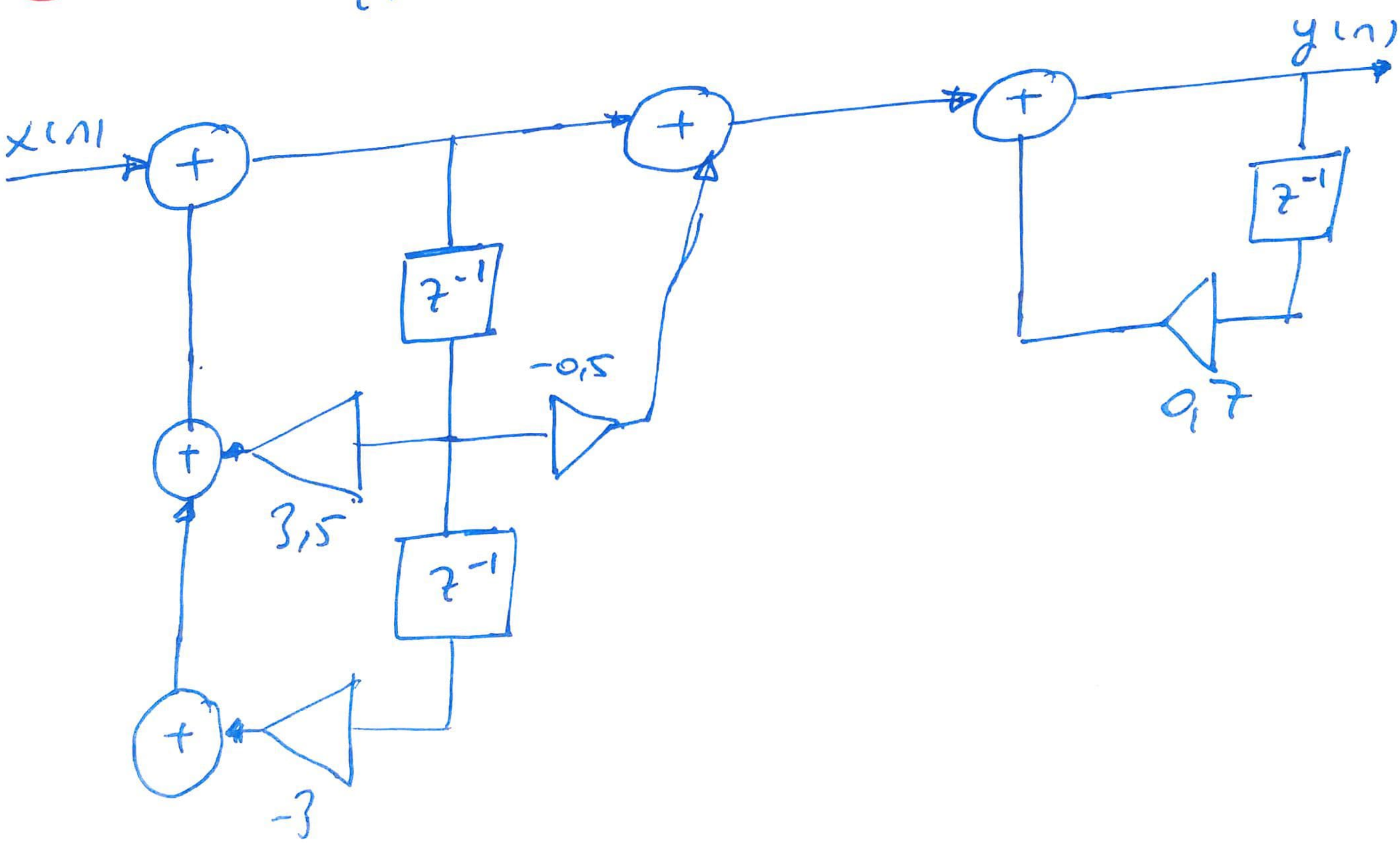
$$H(z) = \frac{(1-0{,}5z^{-1})}{(2-3z^{-1})(0{,}5-z^{-1})(1-0{,}7z^{-1})}$$

$$H(z) = \frac{-7{,}5}{2-3z^{-1}} + \frac{2{,}3}{0{,}5-z^{-1}} + \frac{0{,}13}{1-0{,}7z^{-1}}$$

$$= \frac{-3{,}75}{1-1{,}5z^{-1}} + \frac{4{,}6}{1-2z^{-1}} + \frac{0{,}13}{1-0{,}7z^{-1}}$$



iv $$H(z) = \frac{1-0{,}5z^{-1}}{(1-3{,}5z^{-1}+3z^{-2})(1-0{,}7z^{-1})}$$



4 [25 pts] Assume that a complex multiply takes 1 μs and that the amount of time to compute a DFT is determined by the amount of time it takes to perform all of the multiplications

(a) How much time does it take to compute a 1024-point DFT directly?

(b) How much time is required if an FFT is used?

(c) Repeat parts (a) and (b) for a 4096-point DFT

S4) Direct N-point DFT Computation $\longrightarrow$ N^2 multiplication

DFT Computation by FFT $\longrightarrow$ $\frac{N}{2}\log_2 N$ multp.

a. If it takes 1 μs per complex multiply, direct evaluation of a 1024-point DFT requires

$$t_{DFT} = (1024)^2 \cdot 10^{-6}\, s \approx \underline{1.05\, s}$$

b. $\frac{N}{2}\log_2 N = \frac{1024}{2}(\log_2 1024) = 5120$ complex multiplications

$$t_{FFT} = (5120)\cdot 10^{-6}\, s = \underline{5.12\, ms}$$

c. $t_{DFT} = (4096)^2 \cdot 10^{-6}\, s = \underline{16.78\, s}$

$$t_{FFT} = \left(\frac{4096}{2}\log_2 4096\right)\cdot 10^{-6}\, s = 24.576$$