EHB 315E – Digital Signal Processing

1. The input-output relationship of a discrete-time LTI system is given by

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

a) Determine the frequency response, $H(e^{j\omega})$, and impulse response, $h[n]$, of the system.

b) Find the Fourier series coefficient of the output for the input $x[n] = cos\left(\frac{3\pi}{4}n\right)$

a)

$y[n] \xleftrightarrow{F} Y(e^{j\omega})$ $\quad y[n-1] \xleftrightarrow{F} e^{-j\omega}Y(e^{j\omega})$

$x[n] \xleftrightarrow{F} X(e^{j\omega})$

$y[n] - \frac{1}{4}y[n-1] = x[n] \xleftrightarrow{F} Y(e^{j\omega}) - \frac{1}{4}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega})$

$Y(e^{j\omega})\left(1 - \frac{1}{4}e^{-j\omega}\right) = X(e^{j\omega})$

$y[n] = x[n] * h[n] \xleftrightarrow{F} X(e^{j\omega})H(e^{j\omega}) = Y(e^{j\omega})$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

$a^n u[n], \; |a| < 1 \xleftrightarrow{F} \frac{1}{1 - ae^{-j\omega}}$

$h[n] = \left(\frac{1}{4}\right)^n u[n]$

b)

Becouse of the system is time-invariant, y(n) is also periodic with same fundamental period of x[n].

$y[n] = x[n] * h[n]$ $\quad Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$

$y[n] = \sum_{k=\langle N\rangle} b_k e^{jk\omega_0 n}$ $\quad x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$

$Y(e^{j\omega}) = \mathcal{F}\{\sum_{k=\langle N\rangle} b_k e^{jk\omega_0 n}\} = \sum_{k=\langle N\rangle} b_k \underbrace{\mathcal{F}\{e^{jk\omega_0 n}\}}_{2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi l)}$

$Y(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} b_k \delta(\omega - k\omega_0)$

$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$

$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$

$2\pi \sum b_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{\infty} a_k \underbrace{\delta(\omega - k\omega_0) H(e^{j\omega})}_{H(e^{jk\omega_0})\delta(\omega - k\omega_0)}$

$b_k = a_k H(e^{jk\omega_0})$

$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{8}n} = \dfrac{e^{j3\frac{2\pi}{8}n} + e^{j(-3)\frac{2\pi}{8}n}}{2}$

$a_3 = \frac{1}{2} \quad a_{-3} = \frac{1}{2}$

$b_3 = a_3 H(e^{j3\frac{2\pi}{8}}) = \frac{1}{2} \dfrac{1}{1 - \frac{1}{4}e^{-j\frac{3\pi}{4}}}$

$b_{-3} = a_{-3} H(e^{-j3\frac{2\pi}{8}}) = \frac{1}{2} \dfrac{1}{1 - \frac{1}{4}e^{j\frac{3\pi}{4}}}$

2. Find the z-transform of of
   a) $x[n] = a^n$u[n]
   b) $x[n] = -a^n$u[-n-1]

a) $x[n] = a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |az^{-1}|<1 \, , \; |z|>|a|$$

b) $-a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z}$$

$$X(z) = \frac{-a^{-1}z \,/\, -a^{-1}z}{1-a^{-1}z \,/\, a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z|<|a|$$