Solutions of Transmission-Line Equations

Frequency-Domain Solution: (Steady-state solution)

Lossless-Line Case: For a lossless line, the equations in fr. domain are,

$$\frac{d^2V}{dz^2} = -\omega^2 LCV \quad \text{and} \quad \frac{d^2I}{dz^2} = -\omega^2 LCI$$

The general solution for the first eq. is:

$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z}$$

The first term satisfies the second order diff. eq., and the second term also satisfies. So the sum of two terms also satisfies the equation.

V_+ : complex voltage phasor in the $+z$ direction
V_- : " " " " " $-z$ "
$e^{-j\beta z}$: travelling wave " " $+z$ "
$e^{j\beta z}$: " " " " $-z$ "
$\beta = \omega\sqrt{LC}$ (rad/m) is the phase constant
βz : electrical length (rad)

According to Eq. $\frac{dV}{dz} = -j\omega LI$ (the first eq. on the top of this page) the current I is determined by,

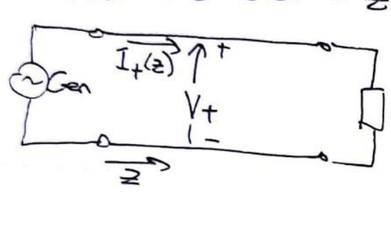
$$I = -\frac{1}{j\omega L}\frac{dV}{dz}$$

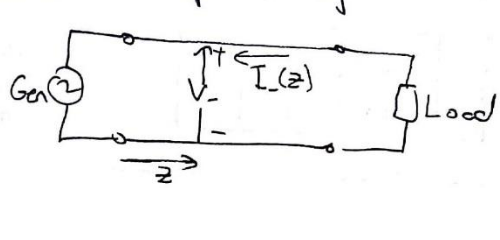
using the $V(z)$ solution in this eq. we obtain,

$$I = Y_0 V_+ e^{-j\beta z} - Y_0 V_- e^{+j\beta z} = I_+ e^{-j\beta z} - I_- e^{+j\beta z}$$

in which we define the characteristic impedance of the line for lossless case as $Z_0 = \frac{1}{Y_0} \equiv \sqrt{\frac{L}{C}}$

The factors I_+ and I_- represent complex currents travelling in the $+z$ and $-z$ direction respectively.





Line voltages and currents

Lossy-Line Case: $(R \neq 0, G \neq 0)$, for the lossy case, we have

$$\frac{d^2V}{dz^2} = \gamma^2 V \quad \text{and} \quad \frac{d^2I}{dz^2} = \gamma^2 I$$

$$\gamma = \alpha + j\beta = \sqrt{ZY} = \sqrt{(R+j\omega L)(G+j\omega C)}$$ is the propagation constant

α: attenuation constant (Np/m)
β: phase 〃 (rad/m)

We can rearrange the γ relation as,

$$\gamma = \sqrt{(j\omega)^2 LC}\sqrt{\left(1+\frac{R}{j\omega L}\right)\left(1+\frac{G}{j\omega C}\right)}$$

and at high frequencies (or with low losses), when $R \ll \omega L$ and $G \ll \omega C$ and by using the binomial expansion of $(1+b)^{\pm 1/2} = 1 \pm b/2$ for $b \ll 1$, we can express the γ as,

$$\gamma \simeq j\omega\sqrt{LC}\left[\left(1+\frac{1}{2}\frac{R}{j\omega L}\right)\left(1+\frac{1}{2}\frac{G}{j\omega C}\right)\right] \simeq j\omega\sqrt{LC}\left[1+\frac{1}{2}\left(\frac{R}{j\omega L}+\frac{G}{j\omega C}\right)\right]$$

$$= \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) + j\omega\sqrt{LC} = \alpha + j\beta$$

Thus, we obtain for α and β as,

$$\alpha = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) \quad \text{and} \quad \beta = \omega\sqrt{LC}$$

The general solution for the second order diff. voltage eq. in lossy case is,

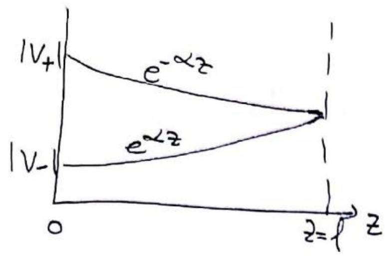
$$V = V_+ e^{-\gamma z} + V_- e^{\gamma z}$$

and using the relation $I = -\frac{1}{Z}\frac{dV}{dz}$, we can obtain,

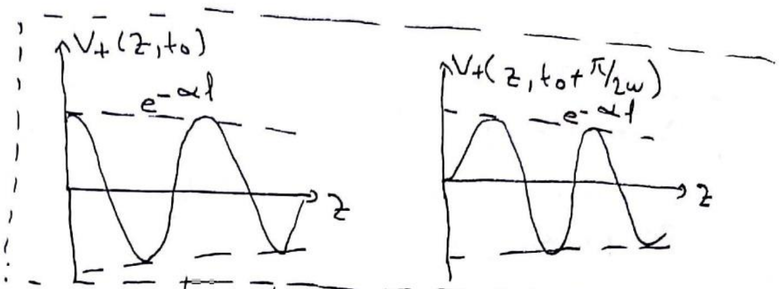
$$I = Y_0\left(V_+ e^{-\gamma z} - V_- e^{\gamma z}\right)$$

where the characteristic impedance is,

$$Z_0 = \frac{1}{Y_0} \equiv \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = R_0 \pm jX_0$$



Magnitude of voltage waves on a lossy line



Voltage wave at two instants in time.

We can express the voltage wave as,

$$v_+(z,t) = \mathrm{Re}\left[V_+(z)\, e^{j\omega t}\right] = \mathrm{Re}\left[V_+ e^{-\gamma z} e^{j\omega t}\right] = \mathrm{Re}\left[V_+ e^{-\alpha z} e^{j\omega t} e^{-j\beta z}\right]$$

$$= |V_+| e^{-\alpha z} \cos(\omega t - \beta z + \theta_+)$$

where $V_+ = |V_+| e^{j\theta_+}$

When the voltage wave moves from z_1 to z_2 position ($\ell = z_2 - z_1$), the ratios of the magnitudes of voltage V_+ and current I_+ of the two points are,

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{I_2}{I_1}\right| = e^{-\alpha\ell}$$

Taking the $\ln$ of both sides, we obtain the attenuation in Neper

$$\ln\left|\frac{V_2}{V_1}\right| = \ln e^{-\alpha\ell} = |\alpha\ell| \ (N_p)$$

The dB is defined as the logarithm (to the base 10) of a power ratio. ($1\ N_p = 8{,}686\ dB$)

$$(dB):\ 10 \log \frac{P_2}{P_1}$$

Since $P_1 = V_1^2/R_1$ and $P_2 = V_2^2/R_2$ and if $R_1 = R_2$, $dB = 20 \log\left|\frac{V_2}{V_1}\right|$

Since the voltage V_2 is smaller than V_1, the ratio in dB is negative. That means the traveling wave is attenuated by that number of dBs. The curve above represents a moving wave with an amplitude attenuated by a factor of $-\alpha\ell$.

Time-Domain Solution:

Lossless-Line Case: The wave equations for the time-domain solution are,

$$\frac{\partial^2 v}{\partial z^2} = LC \frac{\partial^2 v}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 i}{\partial z^2} = LC \frac{\partial^2 i}{\partial t^2}$$

Let, the following voltage function be a solution to the wave equations:

$$v_+ = f_+ (t - \sqrt{LC}\, z) \quad , \quad u_p = 1/\sqrt{LC} \text{ is the phase velocity}$$

defining the argument of the function as $A = (t - \sqrt{LC}\, z)$, then

$$v_+ = f_+(A)$$

and we can write the partial differentiation as,

$$\frac{\partial v_+}{\partial z} = \frac{df_+}{dA} \frac{\partial A}{\partial z} = -\sqrt{LC} \frac{df_+}{dA}$$

Differentiating this eq. with respect to z once again,

$$\frac{\partial^2 v_+}{\partial z^2} = LC \frac{d^2 f_+}{dA^2}$$

Similarly, $\frac{\partial v_+}{\partial t} = \frac{df_+}{dA} \overset{1}{\frac{\partial A}{\partial t}} = \frac{df_+}{dA}$ and $\frac{\partial^2 v_+}{\partial t^2} = \frac{d^2 f_+}{dA^2}$

and comparing the results, we can obtain: $\frac{\partial^2 v_+}{\partial z^2} = LC \frac{\partial^2 v_+}{\partial t^2}$

Therefore v_+ is one solution of voltage wave equation.

Similarly $v_- = f_- (t + \sqrt{LC}\, z)$ is also one solution of this equation.

Since the wave equation is linear, the sum of the solutions is also a solution:

$$v = f_+ (t - \sqrt{LC}\, z) + f_- (t + \sqrt{LC}\, z)$$

Similarly for the current wave eq.

$$i = g_+ (t - \sqrt{LC}\, z) + g_- (t + \sqrt{LC}\, z) \quad \text{is surely a solution.}$$

For example, we can replace the solution with the cosine function: $v_+ = V_+ \cos \omega(t - \sqrt{LC}\, z)$

If we use this solution in wave equation, we see (prove that) that the wave equation is satisfied. So, v_+ is a solution to the wave eq.

Lossy-Line Case: The wave eq.s for the time-domain on lossy line are,

$$\frac{\partial^2 v}{\partial z^2} = RGv + (RC+LG)\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2}$$ and similar eq. for current.

Let's assume the following solution,

$$v_+ = V_+ e^{-\alpha z}\cos(\omega t - \beta z) = \frac{V_+}{2}\left[e^{-(\alpha+j\beta)z}e^{j\omega t} + e^{-(\alpha-j\beta)z}e^{-j\omega t}\right]$$

Then,

$$\frac{\partial^2 v_+}{\partial z^2} = \frac{V_+}{2}\left[(\alpha+j\beta)^2 e^{-(\alpha+j\beta)z}e^{j\omega t} + (\alpha-j\beta)^2 e^{-(\alpha-j\beta)z}e^{-j\omega t}\right]$$

and

$$\frac{\partial v_+}{\partial t} = \frac{V_+}{2}\left[j\omega e^{-(\alpha+j\beta)z}e^{j\omega t} - j\omega e^{-(\alpha-j\beta)z}e^{-j\omega t}\right]$$

$$\frac{\partial^2 v_+}{\partial t^2} = \frac{V_+}{2}\left[-\omega^2 e^{-(\alpha+j\beta)z}e^{j\omega t} - \omega^2 e^{-(\alpha-j\beta)z}e^{-j\omega t}\right]$$

Substituting the results into wave eq., dividing both sides by $(V_+/2)$, and rearranging terms, we get

complex conjugate

$$\left[(\alpha+j\beta)^2 - RG - (RC+LG)j\omega - LC(j\omega)^2\right] + \left[(\alpha+j\beta)^2 - RG - (RC+LG)j\omega - LC(j\omega)^2\right]^* e^{-j(2\omega t - 2\beta z)} = 0$$

This eq. can be satisfied for independently chosen values of t and z if, and only if, the following two factors are zeros:

$(\alpha+j\beta)^2 - RG - (RC+LG)j\omega - LC(j\omega)^2 = 0$ and the second factor (taking the complex conjugate), $(\alpha-j\beta)^2 - RG - (RC+LG)(-j\omega) - LC(-j\omega)^2 = 0$

Let $\gamma = \alpha + j\beta$ defines the propagation constant and using 'γ' in the first eq. just above,

$$\gamma = \sqrt{RG + (RC+LG)j\omega + LC(j\omega)^2} = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{ZY}$$

as defined in the previous frequency domain result. Therefore our v_+ solution is surely one solution of the traveling wave equation. $v_- = V_- e^{\alpha z}\cos(\omega t + \beta z)$ is also a solution. The sum of the two solution

$$v = V_+ e^{-\alpha z}\cos(\omega t - \beta z) + V_- e^{\alpha z}\cos(\omega t + \beta z)$$ is also a solution.

Similarly

$$i = I_+ e^{-\alpha z}\cos(\omega t - \beta z) + I_- e^{\alpha z}\cos(\omega t + \beta z)$$ is the solution of the wave equation for current.

Characteristic Impedance and Line Impedance

Characteristic Impedance:

We define the characteristic impedance of a transmission line as,

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = R_0 \pm j X_0$$

This impedance is,
- independent of the length of the line
- " " " termination of " "
- not the impedance that a line itself possesses
- determined only by the parameters of the line per unit length.

At high frequencies or with low losses, since $R \ll \omega L$ and $G \ll \omega C$ from the binomial expansion, we can approximate,

$$Z_0 = \sqrt{\frac{L}{C}}\left(1+\frac{R}{j\omega L}\right)^{1/2}\left(1+\frac{G}{j\omega C}\right)^{-1/2} \simeq \sqrt{\frac{L}{C}}\left(1+\frac{1}{2}\frac{R}{j\omega L}\right)\left(1-\frac{1}{2}\frac{G}{j\omega C}\right)$$

$$\simeq \sqrt{\frac{L}{C}}\left[1+\frac{1}{2}\left(\frac{R}{j\omega L}-\frac{G}{j\omega C}\right)\right]$$

but $Z_0 = \sqrt{\frac{L}{C}}$ for very high frequency.

We define the characteristic admittance as $Y_0 = \frac{1}{Z_0} = G_0 \pm j B_0$

Example:

A coaxial line has the following parameters:

$R = 5\ \Omega/mi$, $L = 37.10^{-4}\ H/mi$, $G = 6{,}2 \cdot 10^{-3}\ \mho/mi$, $C = 0.0081 \cdot 10^{-6}\ F/mi$

The line operates at a frequency of 100 kHz. Determine its Z_0 and γ.

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{5+j2323}{(0{,}62+j0{,}51)\cdot 10^{-2}}} = \sqrt{\frac{2324\angle 90^\circ}{0{,}0080\angle 39{,}4^\circ}} = 539\angle 25{,}3^\circ$$

$$= 487 + j230\ \Omega$$

$$\gamma = \sqrt{ZY} = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{(2324\angle 90^\circ)(0.0080\angle 39{,}4^\circ)} = 4{,}31\angle 64{,}7^\circ$$

$$= 1{,}85 + j3{,}90$$

Since $\gamma = \alpha + j\beta$,

$\alpha = 1{,}85\ Np/mi$

and

$\beta = 3{,}90\ rad/mi$