EHB 315E – Digital Signal Processing

1. Find the discrete time Fourier transform of
   a) $x[n] = a^n u[n], \quad |a| < 1$
   b) $x[n] = \delta[n]$
   c) $x[n] = \delta[n-3]$
   d) $x[n] = \frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]$
   e) $x[n] = u[n+3] - u[n-4]$

a) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r|<1 \qquad X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

b) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{\delta[n]\, e^{-j\omega n}}_{\delta[n] e^{-j\omega 0}}$

$x[n] * \delta[n-n_0]$
$= x[n-n_0] \delta[n-n_0]$

$$= \sum_{n=-\infty}^{\infty} \delta[n] = 1$$

c) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n-3] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta[n-3] e^{-j3\omega} = e^{-j3\omega}$

d) $\delta[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0}$

$$x[n] = \frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]$$

$$X(e^{j\omega}) = \frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega} = 1 + \frac{1}{2}\left(e^{j\omega} + e^{-j\omega}\right)$$

$$= 1 + \cos(\omega)$$

e) $$x[n] = u[n+3] - u[n-4] = \begin{cases} 1, & -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \sum_{k=-3}^{3} \delta[n-k] = \delta[n+3] + \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$

$$X(e^{j\omega}) = e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega}$$

$$= 1 + 2\left(\cos 3\omega + \cos 2\omega + \cos\omega\right)$$

2. Find the DTFT of the two-sided sequence

$$x[n] = \left(\frac{1}{4}\right)^{|n|}$$

$$x(n) = \begin{cases} \left(\frac{1}{4}\right)^{n} & n \geq 0 \\ \left(\frac{1}{4}\right)^{-n} & n < 0 \end{cases}$$

$$= \left(\frac{1}{4}\right)^{n} u(n) + \left(\frac{1}{4}\right)^{-n} u(-n) - \delta(n)$$

$$\left(\frac{1}{4}\right)^{n} u(n) \overset{DTFT}{\longleftrightarrow} \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$\left(\frac{1}{4}\right)^{-n} u(-n) \overset{DTFT}{\longleftrightarrow} \frac{1}{1-\frac{1}{4}e^{j\omega}}$$

$$x(n) \overset{DTFT}{\longleftrightarrow} X(e^{j\omega})$$

$$x(-n) \overset{DTFT}{\longleftrightarrow} X(e^{-j\omega})$$

$$\delta(n) \overset{DTFT}{\longleftrightarrow} 1$$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}} + \frac{1}{1-\frac{1}{4}e^{j\omega}} + 1$$

$$= \frac{15/16}{\left(1-\frac{1}{4}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{j\omega}\right)} = \frac{15/16}{\frac{17}{16} - \frac{1}{2}\cos\omega}$$

3. Consider a causal LTI system described by the difference equation

$$y[n] + \frac{1}{2} y[n-1] = x[n]$$

a) Determine the frequency response $H(e^{j\omega})$ of this system.

b) Determine the impulse response $h[n]$ of this system.

c) Is this a stable system?

d) Find the output $y[n]$ for the input $x[n] = \left(\frac{1}{2}\right)^n u[n]$.

a) $$Y(e^{j\omega}) + \frac{1}{2} Y(e^{j\omega}) e^{-j\omega} = X(e^{j\omega})$$

$$Y(e^{j\omega}) \left(1 + \frac{1}{2} e^{-j\omega}\right) = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 + \frac{1}{2} e^{-j\omega}}$$

b) $$a^n u[n] \overset{F.T}{\longleftrightarrow} \frac{1}{1 - a e^{-j\omega}} \qquad |a| < 1$$

$$h[n] = \mathcal{F}^{-1} \left\{ \frac{1}{1 - \left(\frac{-1}{2}\right) e^{-j\omega}} \right\} = \left(\frac{-1}{2}\right)^n u[n]$$

c) $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2 < \infty$, so this system is stable.

d) $Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \frac{1}{1+\frac{1}{2}e^{-j\omega}} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1+\frac{1}{2}e^{-j\omega}}$$

$A = \frac{1}{2}$ $B = \frac{1}{2}$

$$Y(e^{j\omega}) = \frac{1/2}{1-\frac{1}{2}e^{-j\omega}} + \frac{1/2}{1+\frac{1}{2}e^{-j\omega}}$$

$$y[n] = \frac{1}{2}F^{-1}\left\{\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right\} + \frac{1}{2}F^{-1}\left\{\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right\}$$

$$y[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(-\frac{1}{2}\right)^n u[n]$$