Recitation W6

1) Toss a fair coin 3 times.

X: the number of heads on the first toss

Y: the total number of heads on the last two tosses, and

F: the number of heads on the first two tosses.

a) Give the joint probability table for X and Y.

b) Compute $Cov(X, Y)$.

c) Give the joint probability table for X and F.

d) Compute $Cov(X, F)$.

soln.

a) $p_{XY}(x, y) = P_{XY}(X = x, Y = y)$

$$p_{XY}(0,0) = \underset{X}{\frac{1}{2}} \cdot \underset{Y}{\frac{1}{4}} = \frac{1}{8}$$

$$p_{XY}(0,1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$p_{XY}(0,2) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

$$p_{XY}(1,0) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

$$p_{XY}(1,1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$p_{XY}(1,2) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

y \ X	0	1	P_Y
0	1/8	1/8	1/4
1	1/4	1/4	1/2
2	1/8	1/8	1/4
P_X	1/2	1/2	1

b) Since, the tosses are independent from each other, rvs X and Y are independent.

Therefore, $Cov(X, Y) = 0$ //

c) $P_{X,F}(x,f) = P_{XF}(X=x, F=f)$

$$\quad X \quad F$$

$$P_{XF}(0,0) = P\{\underbrace{TT}H, \underbrace{TT}T\} = \frac{1}{4}$$

$$P_{XF}(0,1) = P\{THH, THT\} = \frac{1}{4}$$

$$P_{XF}(0,2) = 0.$$

$$P_{XF}(1,0) = 0.$$

$$P_{XF}(1,1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P_{XF}(1,2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

F \ X	0	1	P_F
0	1/4	0	1/4
1	1/4	1/4	1/2
2	0	1/4	1/4
P_X	1/2	1/2	1

d) $Cov(X,F) = E[(X - E[X])(F - E[F])]$

$$= E[XF] - E[X]E[F] - \cancel{E[X]E[F]} + \cancel{E[X]E[F]}$$

$$= E[XF] - E[X]E[F]$$

$$E[X] = \sum_{i=0}^{1} x_i p(x_i) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}. ///$$

$$E[F] = \sum_{j=0}^{2} f_j p(f_j) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1 //$$

$$E[XF] = \sum x_i f_j p(x_i, f_j) = 1 \cdot 1 \cdot \frac{1}{4} + 2 \cdot 1 \cdot \frac{1}{4} = 3/4 //$$

$$Cov(X,F) = E[XF] - E[X]E[F]$$

$$= \frac{3}{4} - \frac{1}{2} \cdot 1 = \frac{1}{4} //$$

2) Let X and Y be continous rvs with joint pdf

$$f_{x,y}(x,y) = \frac{3}{2}(x^2+y^2), \quad 0 \le x \le 1, \; 0 \le y \le 1$$

a) Find marjinal pdfs of X and Y.

b) Are X and Y independent?

$$f_x(x) = \int f_{xy}(x,y)\,dy$$

$$= \int_0^1 \frac{3}{2}(x^2+y^2)\,dy$$

$$= \frac{3}{2}\left[x^2 y + \frac{y^3}{3}\right]\Bigg|_0^1 = \frac{3}{2}\left[x^2 + \frac{1}{3}\right] = \frac{3}{2}x^2 + \frac{1}{2} \;//$$

Therefore,

$$f_x(x) = \begin{cases} \frac{1}{2} + \frac{3}{2}x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$f_y(y) = \int f_{xy}(x,y)\,dx$$

$$= \int_0^1 \frac{3}{2}(x^2+y^2)\,dx = \frac{3}{2}y^2 + \frac{1}{2} \;//$$

$$f_y(y) = \begin{cases} \frac{1}{2} + \frac{3}{2}x^2, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

2b) X and Y <u>are not independent</u>

$f_{XY}(x,y) \neq f_X(x) f_Y(y)$

3) Let the continuous rvs X, Y have joint distribution

$$f_{X,Y}(x,y) = \begin{cases} 1/x, & 0<y<x<1 \\ 0, & \text{otherwise} \end{cases}$$

a) Compute $E[X]$ and $E[Y]$.
b) Compute the conditional pdf of Y given $X=x$, for all $0<x<1$.
c) Compute $E[Y|X=x]$ for all $0<x<1$.
d) Compute $Cov(X,Y)$.

Soln.

$$E[X] = \iint x \cdot f_{XY}(x,y)\, dy\, dx$$

$$= \int_0^1\int_0^x x \cdot \frac{1}{x}\, dy \cdot dx = \int_0^1 \left[y \Big|_0^x \right] dx$$

$$= \int_0^1 x\, dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2} \;//$$

$$E[Y] = \iint y \cdot f_{XY}(x,y)\, dy\, dx$$

$$= \int_0^1\int_0^x y \cdot \frac{1}{x}\, dy\, dx = \int_0^1 \frac{1}{x}\left[\frac{y^2}{2} \Big|_0^x \right] dx$$

$$= \int_0^1 \frac{1}{2} x\, dx = \frac{1}{4} \;//$$

b) $f_{Y|X}(y|x) = ?$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

f(y\x) $) = \int f_{XY}(x,y)\,dy$

$$= \int_0^x \frac{1}{x}\,dy = \frac{y}{x}\Big|_0^x = 1 \quad // \quad \text{for } 0<x<1.$$

Therefore,

f(y\x) $= \dfrac{1/x}{1} = \dfrac{1}{x}, \quad \text{for } 0<y<x.$

c) $E[Y|X=x] = \int y \cdot f_{Y|X}(y|x)\,dy$

$$= \int_0^x y \cdot \frac{1}{x}\,dy = \frac{1}{2}x \quad //$$

d) $Cov(X,Y) = E[XY] - E[X]E[Y]$

$$E[XY] = \iint xy\, f_{XY}(x,y)\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]\Big|_0^x dx$$

$$= \int_0^1\int_0^x xy\,\frac{1}{x}\,dy\,dx$$

$$= \int_0^1 \frac{x^2}{2}\,dx = \frac{x^3}{6}\Big|_0^1 = \frac{1}{6} \quad //$$

We calculated $E[X] = \frac{1}{2}$ and $E[Y] = \frac{1}{4}$ before.

So, $Cov(X,Y) = E[XY] - E[X]E[Y]$

$$= \frac{1}{6} - \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{24} \quad //$$

5

4) Suppose X and Y have joint pdf $f_{XY}(x,y) = cx^2y(1+y)$ for $0 \le x \le 3$ and $0 \le y \le 3$.

a) Find c.

b) Find the probability $P(1 \le X \le 2, 0 \le Y \le 1)$.

c) Determine the joint cdf of X and Y, $F_{XY}(x,y)$ for $0 \le x \le 3$ and $0 \le y \le 3$.

d) Find marjinal cdf $F_X(x)$ for $0 \le x \le 3$.

e) Are X and Y independent?

soln.

a) $\iint f(x,y)\,dx\,dy = 1$

$$1 = \int_0^3\int_0^3 c\,x^2y(1+y)\,dy\,dx = c\int_0^3 x^2\left[\frac{y^2}{2}+\frac{y^3}{3}\right]\Big|_0^3 dx$$

$$= c \cdot \frac{243}{2} = 1 \Rightarrow c = \frac{2}{243} \;//$$

b) $$P(1 \le X \le 2, 0 \le Y \le 1) = \int_1^2\int_0^1 f_{XY}(x,y)\,dy\,dx$$

$$= \int_1^2\int_0^1 \frac{2}{243}(x^2y + x^2y^2)\,dy\,dx$$

$$= \frac{70}{4374} \;//$$

c) $F_{xy}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{xy}(x,y)\,dx\,dy$

since, $0 \le x \le 3$ and $0 \le y \le 3$.

$F_{xy}(x,y) = \int_{0}^{x}\int_{0}^{y} f_{xy}(x,y)\,dy\,dx$

$= c\cdot\left(\frac{x^3y^2}{6} + \frac{x^3y^3}{9}\right)$

d) $F_x(x) = \int_{0}^{x} \underbrace{f_x(x)}_{\Downarrow}\,dx$

$= \int_{0}^{x}\overbrace{\int_{0}^{3} f_{xy}(x,y)\,dy}\,dx$

$= \int_{0}^{x}\underbrace{\int_{0}^{3} c(x^2y + x^2y^2)\,dy}_{f_x(x)}\,dx$

$= \int_{0}^{x} \underbrace{\frac{1}{9}x^2}_{f_x(x)}\,dx = \frac{x^3}{27}$ //

2nd way.

From c) $F_{xy}(x,y) = c\cdot\left(\frac{x^3y^2}{6} + \frac{x^3y^3}{9}\right)$

$F_x(x) = F_{xy}(x,3)$ → since $y=3$ is the max value for Y.

$F_x(x) = c\cdot\left(x^3\frac{9}{6} + 3x^3\right) = \frac{x^3}{27}$ //

4e) For independency,

$$f_{xy}(x,y) = f_x(x) f_y(y)$$

$$f_y(y) = \int_0^3 c(x^2y + x^2y^2)\,dx = (y + y^2)\frac{2}{27}.$$

we found $f_x(x)$ in d) as $\frac{1}{9}x^2$.

Therefore, X and Y are independent.

5) X and Y are jointly cont. with joint pdf

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & , \quad 0 < x \leq 1 \text{ and } 0 \leq y \leq 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

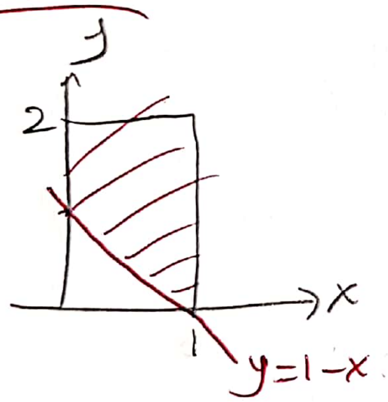
a) Find $P(X+Y \geq 1)$

b) Find $f_X(x)$ and $f_Y(y)$.

c) Are X and Y independent.

soln

a)



$P(X+Y \geq 1)$

$$P(Y \geq 1-X) = \int\int f_{XY}(x,y)\, dy\, dx$$

$$= \int_0^1 \int_{1-x}^2 x^2 + \frac{xy}{3}\, dy\, dx$$

$$= \int_0^1 \left[x^2 y + \frac{xy^2}{6} \right]_{1-x}^2 dx$$

$$= \frac{65}{72} \; //$$

5 b)

5 $f_X(x) = \int f_{XY}(x,y)\,dy$

$$= \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy = 2x^2 + \frac{2x}{3}, \quad \text{if } 0 \le x \le 1$$

$$f_Y(y) = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx = \frac{1}{3} + \frac{y}{6}, \quad \text{if } 0 \le y \le 2$$

5 c) since joint pdf $f_{XY}(x,y) \neq f_X(x) \cdot f_Y(y)$.

They are not independent.