EHB 315E – Digital Signal Processing

1. Let us consider the design of a lowpass discrete-time filter by applying impulse invariance to an appropriate Butterworth continuous-time filter. The specifications for the discrete-time filter are ($T_d = 1$)

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$$

Since the parameter T_d cancels in the impulse invariance procedure, we can choose $T_d = 1$, so that $\omega = \Omega$. In Problem 7.2, this same example is considered, but with the parameter T_d explicitly included to illustrate how and where it cancels.

In designing the filter using impulse invariance on a continuous-time Butterworth filter, we must first transform the discrete-time specifications to specifications on the continuous-time filter. Recall that impulse invariance corresponds to a linear mapping between Ω and ω in the absence of aliasing. For this example, we will assume that the effect of aliasing is negligible. After the design is complete, we can evaluate the resulting frequency response against the specifications in Eqs. (7.13a) and (7.13b).

Because of the preceding considerations, we want to design a continuous-time Butterworth filter with magnitude function $|H_c(j\Omega)|$ for which

$$0.89125 \le |H_c(j\Omega)| \le 1, \qquad 0 \le |\Omega| \le 0.2\pi, \tag{7.14a}$$

$$|H_c(j\Omega)| \le 0.17783, \qquad 0.3\pi \le |\Omega| \le \pi. \tag{7.14b}$$

Since the magnitude response of an analog Butterworth filter is a monotonic function of frequency, Eqs. (7.14a) and (7.14b) will be satisfied if

$$|H_c(j0.2\pi)| \ge 0.89125 \tag{7.15a}$$

and

$$|H_c(j0.3\pi)| \le 0.17783. \tag{7.15b}$$

Specifically, the magnitude-squared function of a Butterworth filter is of the form

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}, \tag{7.16}$$

so that the filter design process consists of determining the parameters N and Ω_c to meet the desired specifications. Using Eq. (7.16) in Eqs. (7.15) with equality leads

to the equations

$$1 + \left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \tag{7.17a}$$

and

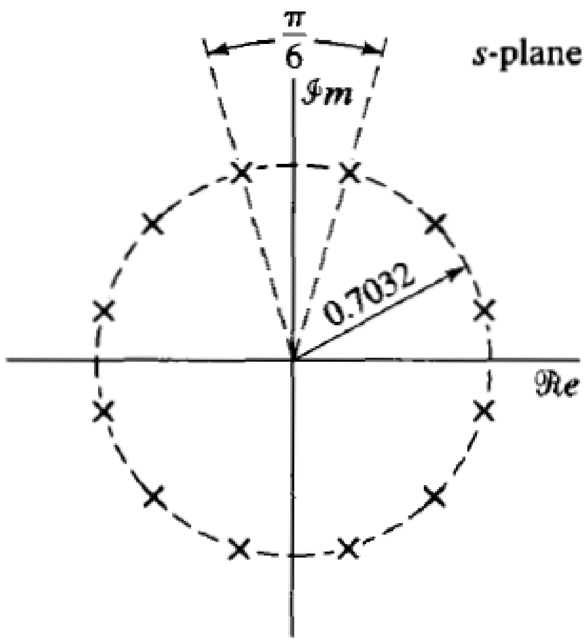
$$1 + \left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2. \tag{7.17b}$$

The solution of these two equations is $N = 5.8858$ and $\Omega_c = 0.70474$. The parameter N, however, must be an integer. Therefore, so that the specifications are met or exceeded, we must round N up to the nearest integer, $N = 6$. Because we have rounded N up to the next highest integer, the filter will not exactly satisfy both Eqs. (7.17a) and (7.17b) simultaneously. With $N = 6$, the filter parameter Ω_c can be chosen to exceed the specified requirements in either the passband, the stopband, or both. Specifically, as the value of Ω_c varies, there is a trade-off in the amount by which the stopband and passband specifications are exceeded. If we substitute $N = 6$ into Eq. (7.17a), we obtain $\Omega_c = 0.7032$. With this value, the passband specifications (of the continuous-time filter) will be met exactly, and the stopband specifications (of the continuous-time filter) will be exceeded. This allows some margin for aliasing in the discrete-time filter. With $\Omega_c = 0.7032$ and with $N = 6$, the 12 poles of the magnitude-squared function $H_c(s)H_c(-s) = 1/[1 + (s/j\Omega_c)^{2N}]$ are uniformly distributed in angle on a circle of radius $\Omega_c = 0.7032$, as indicated in Figure 7.4. Consequently, the poles of $H_c(s)$ are the three pole pairs in the left half of the s-plane with the following coordinates:

$$\text{Pole pair 1:} \quad -0.182 \pm j(0.679),$$

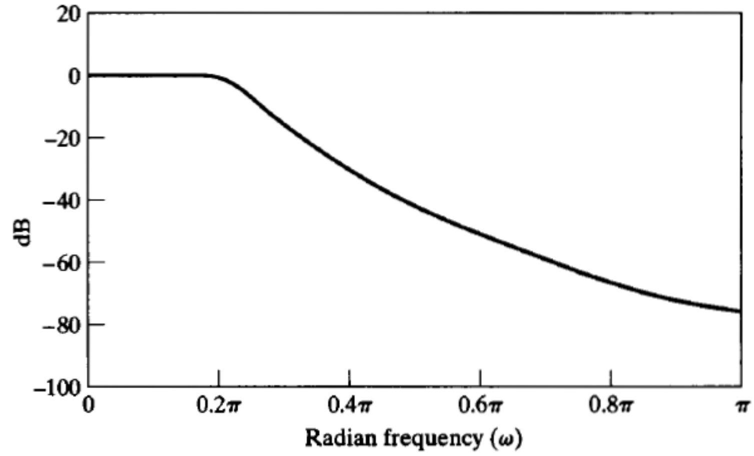
$$\text{Pole pair 2:} \quad -0.497 \pm j(0.497),$$

$$\text{Pole pair 3:} \quad -0.679 \pm j(0.182).$$

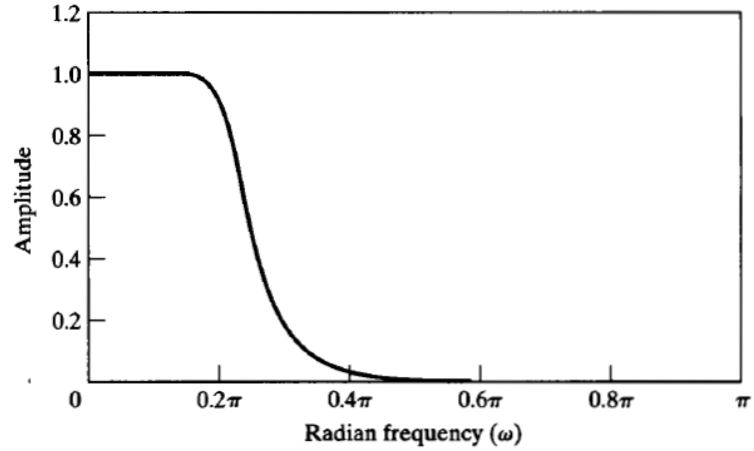


$$H_c(s) = \frac{0.12093}{(s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}.$$



(a)



(b)

2. Let us consider the design of a lowpass discrete-time filter by applying bilinear transfom to an appropriate Butterworth continuous-time filter. The specifications for the discrete-time filter are ($T_d = 1$)

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$$

Consider the discrete-time filter specifications of Example 7.2, in which we illustrated the impulse invariance technique for the design of a discrete-time filter. The specifications on the discrete-time filter are

$$0.89125 \le |H(e^{j\omega})| \le 1, \qquad 0 \le \omega \le 0.2\pi, \tag{7.30a}$$

$$|H(e^{j\omega})| \le 0.17783, \qquad 0.3\pi \le \omega \le \pi. \tag{7.30b}$$

In carrying out the design using the bilinear transformation, the critical frequencies of the discrete-time filter must be prewarped to the corresponding continuous-time frequencies using Eq. (7.28) so that the frequency distortion inherent in the bilinear transformation will map the continuous-time frequencies back to the correct discrete-time critical frequencies. For this specific filter, with $|H_c(j\Omega)|$ representing the magnitude-response function of the continuous-time filter, we require that

$$0.89125 \le |H_c(j\Omega)| \le 1, \qquad 0 \le \Omega \le \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right), \tag{7.31a}$$

$$|H_c(j\Omega)| \le 0.17783, \qquad \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right) \le \Omega \le \infty. \tag{7.31b}$$

For convenience, we choose $T_d = 1$. Also, as with Example 7.2, since a continuous-time Butterworth filter has a monotonic magnitude response, we can equivalently require that

$$|H_c(j2\tan(0.1\pi))| \ge 0.89125 \tag{7.32a}$$

and

$$|H_c(j2\tan(0.15\pi))| \le 0.17783. \tag{7.32b}$$

The form of the magnitude-squared function for the Butterworth filter is

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}. \tag{7.33}$$

Solving for N and Ω_c with the equality sign in Eqs. (7.32a) and (7.32b), we obtain

$$1 + \left(\frac{2\tan(0.1\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89}\right)^2 \tag{7.34a}$$

and

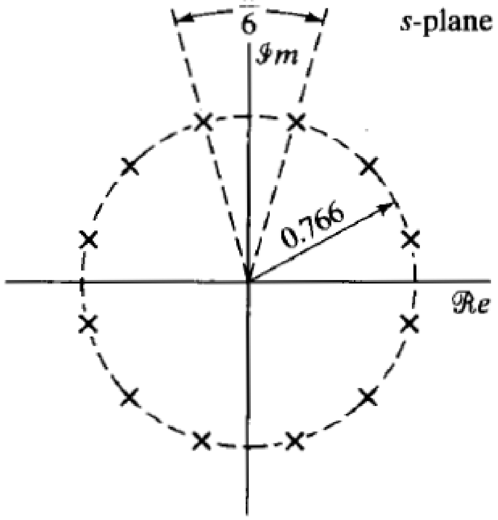
$$1 + \left(\frac{2\tan(0.15\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.178}\right)^2, \tag{7.34b}$$

and solving for N in Eqs. (7.34a) and (7.34b) gives

$$\begin{aligned} N &= \frac{\log\left[\left(\left(\frac{1}{0.178}\right)^2 - 1\right) \Big/ \left(\left(\frac{1}{0.89}\right)^2 - 1\right)\right]}{2\log[\tan(0.15\pi)/\tan(0.1\pi)]} \\ &= 5.305. \end{aligned} \tag{7.35}$$

Since N must be an integer, we choose $N = 6$. Substituting $N = 6$ into Eq. (7.34b), we obtain $\Omega_c = 0.766$. For this value of Ω_c, the passband specifications are exceeded and the stopband specifications are met exactly. This is reasonable for the bilinear transformation, since we do not have to be concerned with aliasing. That is, with proper prewarping, we can be certain that the resulting discrete-time filter will meet the specifications exactly at the desired stopband edge.

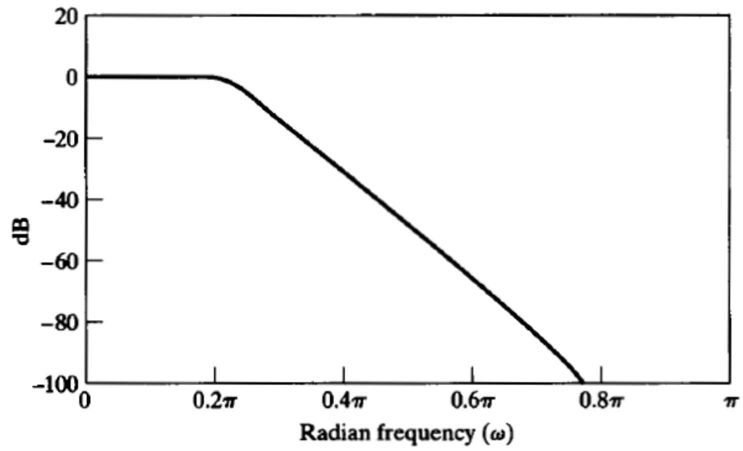
In the s-plane, the 12 poles of the magnitude-squared function are uniformly distributed in angle on a circle of radius 0.766, as shown in Figure 7.10. The system function of the continuous-time filter obtained by selecting the left half-plane



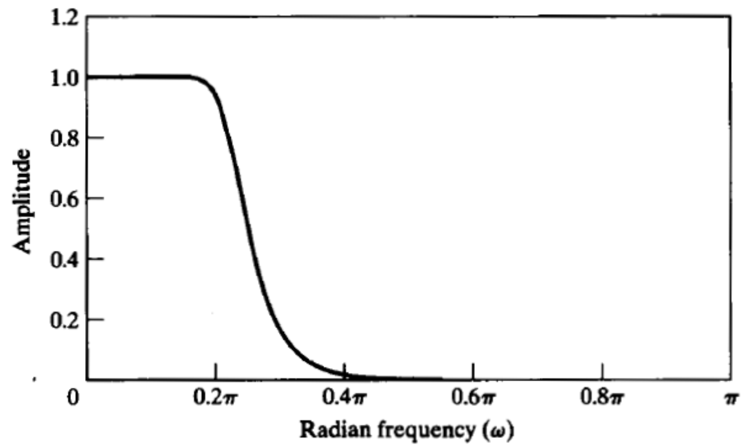
$$H_c(s) = \frac{0.20238}{(s^2 + 0.3996s + 0.5871)(s^2 + 1.0836s + 0.5871)(s^2 + 1.4802s + 0.5871)}.$$

$$H(z) = \frac{0.0007378(1 + z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})}$$

$$\times \frac{1}{(1 - 0.9044z^{-1} + 0.2155z^{-2})}.$$



(a)



3. Use the window design method to design a linear phase FIR filter of order N = 24 to approximate the following ideal frequency response magnitude:

$$|H_d(e^{j\omega})| = \begin{cases} 1 & |\omega| \le 0.2\pi \\ 0 & 0.2\pi \le |\omega| \le \pi \end{cases}$$

Use the window design method to design a linear phase FIR filter of order $N = 24$ to approximate the following ideal frequency response magnitude:

$$|H_d(e^{j\omega})| = \begin{cases} 1 & |\omega| \le 0.2\pi \\ 0 & 0.2\pi < |\omega| \le \pi \end{cases}$$

The ideal filter that we would like to approximate is a low-pass filter with a cutoff frequency $\omega_p = 0.2\pi$. With $N = 24$, the frequency response of the filter that is to be designed has the form

$$H(e^{j\omega}) = \sum_{n=0}^{24} h(n)e^{-jn\omega}$$

Therefore, the delay of $h(n)$ is $\alpha = N/2 = 12$, and the ideal unit sample response that is to be windowed is

$$h_d(n) = \frac{\sin[0.2\pi(n-12)]}{(n-12)\pi}$$

All that is left to do in the design is to select a window. With the length of the window fixed, there is a trade-off between the width of the transition band and the amplitude of the passband and stopband ripple. With a rectangular window, which provides the smallest transition band,

$$\Delta\omega = 2\pi \cdot \frac{0.9}{24} = 0.075\pi$$

and the filter is

$$h(n) = \begin{cases} \dfrac{\sin[0.2\pi(n-12)]}{(n-12)\pi} & 0 \le n \le 24 \\ 0 & \text{otherwise} \end{cases}$$

However, the stopband attenuation is only 21 dB, which is equivalent to a ripple of $\delta_s = 0.089$. With a Hamming window, on the other hand,

$$h(n) = \left[0.54 - 0.46\cos\left(\frac{2\pi n}{24}\right)\right] \cdot \frac{\sin[0.2\pi(n-12)]}{(n-12)\pi} \qquad 0 \le n \le 24$$

and the stopband attenuation is 53 dB, or $\delta_s = 0.0022$. However, the width of the transition band increases to

$$\Delta\omega = 2\pi \cdot \frac{3.3}{24} = 0.275\pi$$

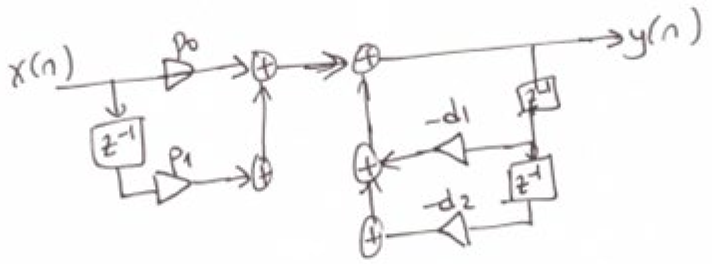
which, for most designs, would be too wide.

4. Obtain the direct form I and direct form II structures for the following system:

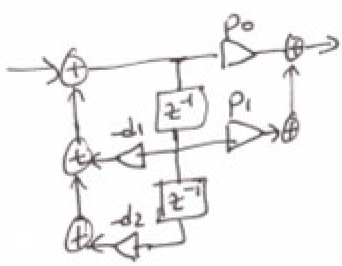
$$y[n] = \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + x[n] + \frac{1}{3}x[n-1]$$

$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = X(z)\left(1 + \frac{1}{3}z^{-1}\right)$$

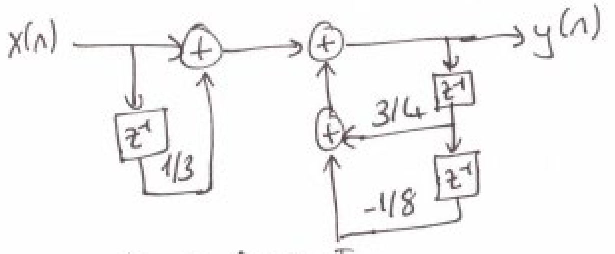
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \qquad H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1}}{1 + d_1 z^{-1} + d_2 z^{-2}}$$



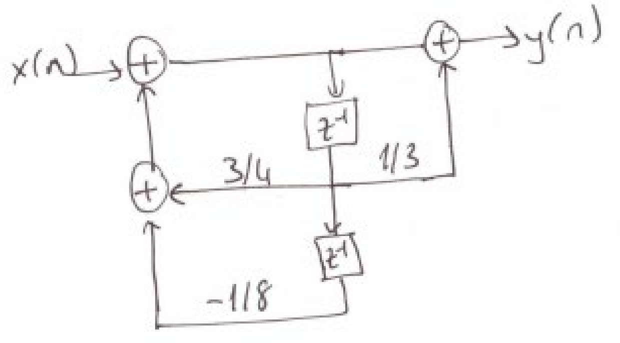
direct form I



direct form II



direct form I



direct form II