The input and the output of the LTI system given by

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$$

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n].$$

Find the transfer function, $H(z)$.

$$y[n] = x[n] * h[n] \overset{z}{\longleftrightarrow} X(z)\,H(z) = Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$X(z) = \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} - \underbrace{\frac{1}{1-2z^{-1}}}_{|z|<2} \qquad \text{ROC}: 2>|z|>\frac{1}{2}$$

$$X(z) = \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}$$

$$Y(z) = \underbrace{\frac{6}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} - \underbrace{\frac{6}{1-\frac{3}{4}z^{-1}}}_{|z|>\frac{3}{4}} \qquad \text{ROC}: |z|>\frac{3}{4}$$

$$Y(z) = \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$$

For the ROC of $H(z)$, we have two possibilities. Either $|z|>\frac{3}{4}$ or $|z|<\frac{3}{4}$. Because the ROC of $Y(z)$ is $|z|>\frac{3}{4}$ and includes the intersection of the region of $X(z)$ and $H(z)$, the ROC of $H(z)$ must be $|z|>\frac{3}{4}$.

For the following z-transform, determine the time-domain signal x[n]

$$X(z) = \frac{4 - \frac{13}{8}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{8}z^{-1}\right)}.$$

a) ROC: $|z| > \frac{1}{2}$
b) ROC: $|z| < \frac{1}{8}$
c) ROC: $\frac{1}{8} < |z| < \frac{1}{2}$

$$X(z) = \frac{4 - \frac{13}{8}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{8}z^{-1}}$$

$A = 1 \qquad B = 3$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{3}{1-\frac{1}{8}z^{-1}}$$

$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}}$, ROC: $|z| > \frac{1}{2}$

$-\left(\frac{1}{2}\right)^n u[-n-1] \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}}$ ROC: $|z| < \frac{1}{2}$

$\left(\frac{1}{8}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{8}z^{-1}}$, ROC: $|z| > \frac{1}{8}$

$-\left(\frac{1}{8}\right)^n u[-n-1] \xleftrightarrow{z} \frac{1}{1-\frac{1}{8}z^{-1}}$, ROC: $|z| < \frac{1}{8}$

a) $|z| > \frac{1}{2} > \frac{1}{8}$ $|z|$

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 3\left(\frac{1}{8}\right)^n u[n]$$

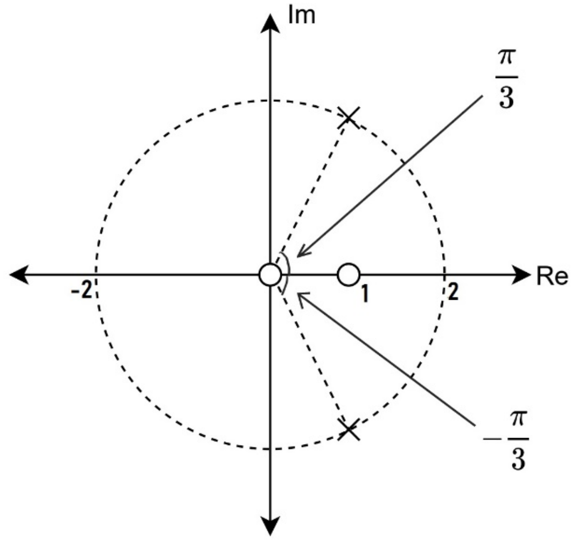
b) $|z| < \frac{1}{8} < \frac{1}{2}$

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - 3\left(\frac{1}{8}\right)^n u[-n-1]$$

c) $|z| > \frac{1}{8}$, $|z| < \frac{1}{2}$

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + 3\left(\frac{1}{8}\right)^n u[n]$$

The pole-zero diagram in figure below corresponds to the z-transform $H(z)$ of a causal sequence $h[n]$.



a) Find H(z), and indicate the ROC.
b) Find h[n] and determine if the system is stable.

a) $$H(z) = \frac{z(z-1)\ /z^2}{(z-2e^{-j\pi/3})(z-2e^{j\pi/3})\ /z^2} = \frac{(1-z^{-1})}{(1-2e^{-j\pi/3}z^{-1})(1-2e^{j\pi/3}z^{-1})} \qquad |z|>2$$

b) $$H(z) = \frac{A}{1-2e^{-j\pi/3}z^{-1}} + \frac{B}{1-2e^{j\pi/3}z^{-1}}$$

$A+B=1$

$2Be^{-j\pi/3} + 2Ae^{j\pi/3} = 1$

$\underbrace{B\cos\frac{\pi}{3}}_{1/2} - jB\sin\frac{\pi}{3} + A\cos\frac{\pi}{3} + jA\sin\frac{\pi}{3} = \frac{1}{2}$

$\left.\right\}$ $A+B=1$, $A-B=0$

$A=\frac{1}{2}\quad B=\frac{1}{2}$

$$h(n) = \frac{1}{2}2^n e^{-jn\pi/3}u(n) + \frac{1}{2}2^n e^{jn\pi/3}u(n)$$

$$= 2^n\left(\frac{e^{-jn\pi/3}+e^{jn\pi/3}}{2}\right)u(n)$$

$$= 2^n\cos\left(\frac{\pi}{3}n\right)u(n)$$

$$r^n\cos(w_0 n)u(n) \overset{z}{\longleftrightarrow} \frac{1-r\cos(w_0)z^{-1}}{1-2r\cos(w_0)z^{-1}+r^2z^{-2}}$$

as long as each of the terms $|x(n) z^{-n}|$ is finite.

For example;

If

$$x(n) = \delta(n) + \delta(n-5)$$

Then $$X(z) = 1 + z^{-5}, \quad |z| > 0.$$

Example 5: Finite-Length Truncated Exponential Squence

Let's consider the signal

$$x(n) = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (a z^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

where we have used the formula

$$\sum_{k=0}^{N-1} \alpha^k = \frac{1-\alpha^N}{1-\alpha} .$$

- The ROC is determined by the set of values of z for which

$$\sum_{n=0}^{N-1} |a z^{-1}|^n < \infty$$

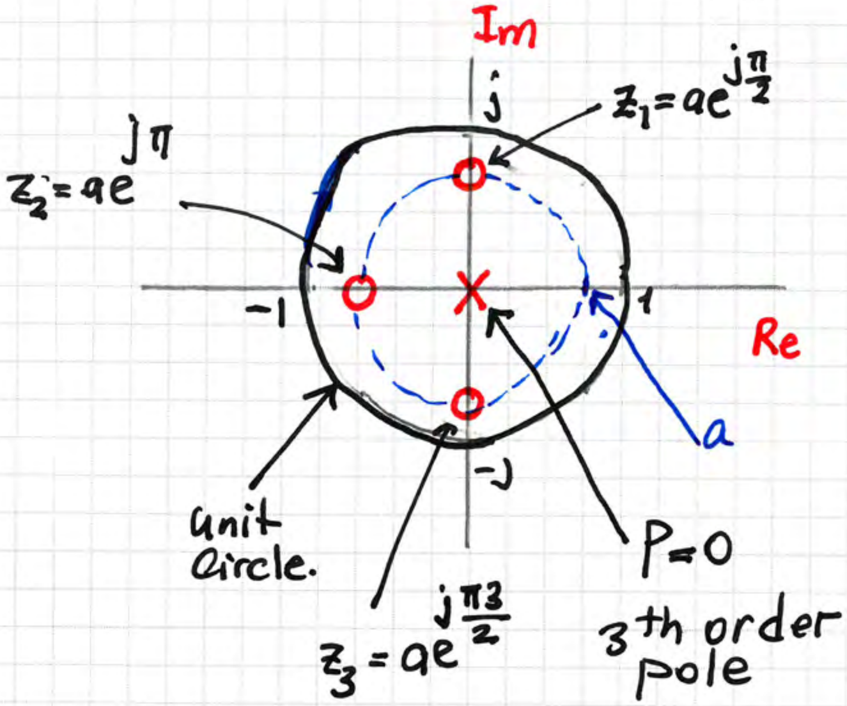
- This sum will be finite as long as az^{-1} is finite, because ther is only finite number of nonzero terms. Therefore, it requires only

$$|a| < \infty \quad \text{and} \quad z \neq 0 .$$

Assuming $|a|$ is finite, the ROC is the entire plane, with the exception of the origin, $z=0$.

- The pole-zero plot for this example, with $N=4$ and a real between zero and unity, shown in the Figure.

$$X(z) = \frac{1}{z^3}\,\frac{z^4 - a^4}{z - a}$$



- The numerator polynomial has n roots at z-plane locations

$$z_k = a\,e^{j\left(\frac{2\pi}{N}\right)k}, \quad k = 0, 1, \ldots, N-1$$

- The zero corresponding to $k=0$, namely

$$z_k = a$$

cancels the pole at $z=a$.
Consequently, there is no poles other than the $N-1$ poles at the origin.

Example:

Let $H(z)$ be

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}$$

with ROC $|z| > 0.9$.

Then $H_i(z)$ is

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}.$$

- Since $H_i(z)$ has only one pole, there are only two possibilities for its ROC.
- The only choice for the ROC of $H_i(z)$ that overlaps with $|z| > 0.9$ is $|z| > 0.5$.
- The impulse response of the system is

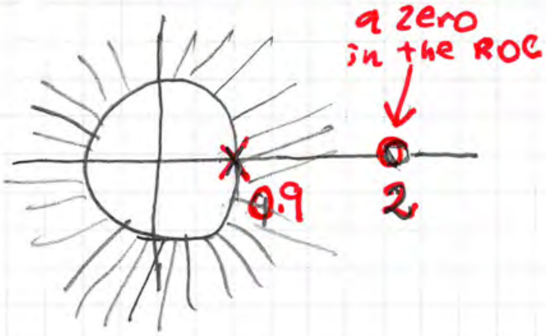
$$h_i(n) = 0.5^n u(n) - (0.9)(0.5)^{n-1} u(n-1).$$

- In this case, the inverse system is both causal and stable.

Example: Inverse system for System with a Zero in the ROC

Suppose that $H(z)$ is

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}}, \quad |z| > .9.$$



The inverse system function is

$$H_i(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5}$$

$$= \frac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}.$$

- $H_i(z)$ has two possible ROCs: $|z| < 2$ and $|z| > 2$.
- In this case, both regions overlap with the ROC of $H(z)$, namely, $|z| > 0.9$, so both are valid inverse systems.
- The corresponding impulse respons for an ROC $|z| < 2$ is

$$H_i(z) = \frac{-2}{1-2z^{-1}} + \frac{1.8\,z^{-1}}{1-2z^{-1}} \quad , \quad |z|<2$$

$$h_{i_1}(n) = 2\,(2)^n\,u(-n-1) - 1.8\;2^{n-1}\,u(-n)$$

and for an ROC $|z|>2$, is

$$h_{i_2}(n) = -2\,(2)^n\,u(n) + 1.8\,(2)^{n-1}\,u(n-1).$$

- We observe that

  $h_{i_1}(n)$ is stable and noncausal,

  while

  $h_{i_2}(n)$ is unstable and causal.

- $$H(z)\,H_{i_1}(z) = 1$$

  and

  $$H(z)\,H_{i_2}(z) = 1 .$$

Either system cascaded with $H(z)$ will result in the identity system.

TABLE

Some Common z-transform Pairs

				ROC
1.	$\delta(n)$	$\longleftrightarrow$	1	$\forall z$
2.	$u(n)$	$\longleftrightarrow$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3.	$-u(-n-1)$	$\longleftrightarrow$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4.	$\delta(n-m)$	$\longleftrightarrow$	z^{-m}	all z except $z=0$ (if $m>0$) or $z=\infty$ (if $m<0$).
5.	$a^n u(n)$	$\longleftrightarrow$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6.	$-a^n u(-n-1)$	$\longleftrightarrow$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7.	$n a^n u(n)$	$\longleftrightarrow$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8.	$\cos(w_0 n)\,u(n)$	$\longleftrightarrow$	$\dfrac{1-\cos(w_0)z^{-1}}{1-2\cos(w_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
9.	$\sin(w_0 n)\,u(n)$	$\longleftrightarrow$	$\dfrac{\sin(w_0)z^{-1}}{1-2\cos(w_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10.	$\begin{cases} a^n, & 0\le n\le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\longleftrightarrow$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$