EHB 315E Digital Signal Processing
Fall 2020
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İTÜ

HOMEWORK 2 - SOLUTIONS

1 [20 pts] Indicate which of the following discrete-time signals are eigenfunctions of stable, linear time-invariant discrete-time systems:

(a) $e^{j2\pi n/3}$

(b) 3^n

(c) $2^n u[-n-1]$

(d) $\cos(\omega_0 n)$

(e) $(1/4)^n$

(f) $(1/4)^n u[n] + 4^n u[-n-1]$

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a)

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k} h(k)\, e^{j2\pi(n-k)/3}$$

$$= e^{j2\pi n} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-j2\pi k/3}}_{H(e^{j2\pi/3})} \longrightarrow \text{YES, EIGENFUNCTION}$$

b)

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, 3^{n-k} = \sum_{k=-\infty}^{\infty} h(k)\, 3^{n} 3^{-k}$$

$$= 3^n \sum_{k=-\infty}^{\infty} h(k)\, 3^{-k}$$

YES, EIGENFUNCTION

(c) $$y[n] = \sum_{k=-\infty}^{\infty} h(k)\, 2^{n-k} u(-n+k-1)$$

$$= \sum_{k=n+1}^{\infty} h(k)\, 2^{n} 2^{-k}$$

$$= 2^{n} \sum_{k=n+1}^{\infty} h(k)\, 2^{-k}$$

Since the summation depends on n, $x[n]$ is NOT AN EIGENFUNCTION

(d) $$x[n] = \cos(w_0 n) = \frac{1}{2} e^{jw_0 n} + \frac{1}{2} e^{-jw_0 n}$$

$$y[n] = \sum_{k=-\infty}^{\infty} h(k) \left[\frac{1}{2} e^{jw_0(n-k)} + \frac{1}{2} e^{-jw_0(n-k)} \right]$$

$$= \frac{1}{2} \sum_{k=-\infty}^{\infty} h(k)\, e^{jw_0 n} e^{-jw_0 k} + \frac{1}{2} \sum_{k=-\infty}^{\infty} h(k)\, e^{-jw_0 n} e^{jw_0 k}$$

$$= \frac{1}{2} e^{jw_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-jw_0 k}}_{H(e^{jw_0})} + \frac{1}{2} e^{-jw_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{jw_0 k}}_{H(e^{-jw_0})}$$

The sum of complex exponentials is NOT AN EIGENFUNCTION. $x[n] = \cos(w_0 n)$ is NOT AN EIGENFUNCTION

(e) $y(n) = \sum_{k=-\infty}^{\infty} h(k) \left(\frac{1}{4}\right)^{n-k}$

$$= \left(\frac{1}{4}\right)^n \underbrace{\sum_{k=-\infty}^{\infty} h(k) \left(\frac{1}{4}\right)^{-k}}_{H(1/4)} \rightarrow \text{YES, EIGENFUNCTION}$$

(f) $y(n) = \sum_{k=-\infty}^{\infty} h(k) \left[\left(\frac{1}{4}\right)^{n-k} u(n-k) + (4)^{n-k} u(-n+k-1)\right]$

$$= \left(\frac{1}{4}\right)^n \underbrace{\sum_{k=-\infty}^{n} h(k) \left(\frac{1}{4}\right)^{-k}}_{\text{the sum depends on } n} + 4^n \underbrace{\sum_{k=n+1}^{\infty} h(k)\, 4^{-k}}_{\text{the sum depends on } n}$$

$\oplus$

$y(n)$ is NOT AN EIGENFUNCTION

2 [20 pts] Consider an LTI system with frequency response

$$H\left(e^{j\omega}\right) = e^{-j\left(\omega-\frac{\pi}{4}\right)}\left(\frac{1+e^{-j2\omega}+4e^{-j4\omega}}{1+\frac{1}{2}e^{-j2\omega}}\right), \qquad -\pi < \omega \leq \pi$$

Determine the output $y[n]$ for all n if the input for all n is

$$x[n] = \cos\left(\frac{\pi n}{2}\right)$$

Let's rewrite $x(n)$ as a sum of complex exponentials

$$x(n) = \cos\left(\frac{\pi}{2} n\right) = \frac{1}{2} e^{j\pi/2\, n} + \frac{1}{2} e^{-j\pi/2\, n}$$

Since complex exponentials are eigenfunctions of LTI systems,

$$y(n) = \frac{1}{2} e^{j\pi/2\, n} H(e^{j\pi/2}) + \frac{1}{2} e^{-j\pi/2\, n} H(e^{-j\pi/2})$$

We need to calculate the frequency response at $w = \mp \pi/2$:

$$H(e^{j\pi/2}) = e^{-j\pi/4}\left(\frac{1+e^{-j\pi}+4e^{-j2\pi}}{1+\frac{1}{2}e^{-j\pi}}\right) = e^{-j\pi/4}\,\frac{1-1+4}{1+\frac{1}{2}(-1)}$$

$$H(e^{-j\pi/2}) = e^{+j3\pi/4}\left(\frac{1+e^{j\pi}+4e^{j2\pi}}{1+\frac{1}{2}e^{j\pi}}\right) = e^{j3\pi/4}\,\frac{1-1+4}{1+\frac{1}{2}(-1)}$$

$$H(e^{j\pi/2}) = e^{-j\pi/4}.8$$

$$H(e^{j\pi/2}) = e^{j3\pi/4}.8$$

$$y(n) = \frac{1}{2} e^{j\pi/2 n} H(e^{j\pi/2}) + \frac{1}{2} e^{-j\pi/2 n} H(e^{-j\pi/2})$$

$$= \frac{1}{2} e^{j\pi/2 n} (8 \cdot e^{-j\pi/4}) + \frac{1}{2} e^{-j\pi/2 n} (8 \cdot e^{j3\pi/4})$$

$$= 4 e^{j\pi/2 n} (e^{-j\pi/4}) + 4 e^{-j\pi/2 n} (e^{j3\pi/4})$$

$$= 4 e^{j\pi/2 n} \left(\frac{\sqrt{2}}{2} (1-j)\right) + 4 e^{-j\pi/2 n} \left(\frac{\sqrt{2}}{2} (-1+j)\right)$$

$$y(n) = 2\sqrt{2} (1-j) e^{j\pi/2 n} + 2\sqrt{2} (-1+j) e^{-j\pi/2 n}$$

3 [20 pts] The frequency response $H^f(\omega)$ of a discrete-time LTI system is

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega} & -0.4\pi < \omega < 0.4\pi \\ 0 & 0.4\pi < |\omega| < \pi \end{cases}$$

Find the output $y(n)$ when the input $x(n)$ is

$$x(n) = 1.2\cos(0.3\pi n) + 1.5\cos(0.5\pi n)$$

Put $y(n)$ in simplest real form (your answer should not contain j).

Let's decompose $x(n)$ into its complex exponentials,

$$x(n) = 0.6\left[e^{j0.3\pi n} + e^{-j0.3\pi n}\right] + 0.75\left[e^{j0.5\pi n} + e^{-j0.5\pi n}\right]$$

Recall that complex exponential sequences are eigenfunctions of LTI systems

$$\begin{aligned} y(n) = {} & 0.6\, e^{j0.3\pi n} H(e^{j0.3\pi}) \\ & + 0.6\, e^{-j0.3\pi n} H(e^{j0.3\pi}) \\ & + 0.75\, e^{j0.5\pi n} H(e^{j0.5\pi}) \\ & + 0.75\, e^{-j0.5\pi n} H(e^{j0.5\pi}) \end{aligned}$$

Since $H(e^{j0.5\pi})$ and $H(e^{-j0.5\pi})$ are equal to 0, they are cancelled.

$$y(n) = 0.6\, e^{j0.3\pi n} H(e^{j0.3\pi}) + 0.6\, e^{-j0.3\pi n} H(e^{-j0.3\pi})$$

Evaluating the frequency response at $\omega = \mp 0.3\pi$

$$\left.\begin{aligned} H(e^{j0.3\pi}) &= e^{-j0.3\pi} \\ H(e^{-j0.3\pi}) &= e^{j0.3\pi} \end{aligned}\right\} \quad \begin{aligned} y(n) &= 0.6\left(e^{j0.3\pi(n-1)} + e^{-j0.3\pi(n-1)}\right) \\ y(n) &= 1.2\left(\frac{e^{j0.3\pi(n-1)} + e^{-j0.3\pi(n-1)}}{2}\right) \end{aligned}$$

$$\boxed{y(n) = 1.2\cos(0.3\pi(n-1))}$$

4 **[20 pts]** The frequency response of a discrete-time LTI system is given by

$$H(e^{j\omega}) = \begin{cases} 0, & |\omega| \le 0.25\pi \\ e^{-j2.5\omega}, & 0.25\pi < |\omega| \le 0.5\pi \\ 0, & 0.5\pi < |\omega| \le \pi \end{cases}$$

(a) Sketch the frequency response magnitude $|H(e^{j\omega})|$ for $|\omega| \le \pi$

(b) Sketch the frequency response phase $\angle H(e^{j\omega})$ for $|\omega| \le \pi$

(c) Find the output signal produced by the input signal

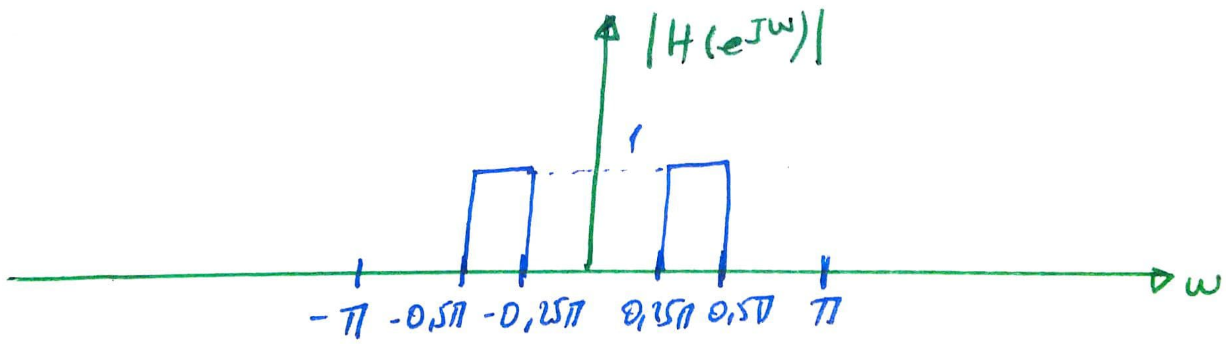
$$x(n) = 3 + 2\cos(0.3\pi n) + 2\cos(0.7\pi n) + (-1)^n$$

Simplify your answer so that it does not contain j.

(d) Classify the system as a low-pass filter, high-pass filter, band-pass filter, band-stop filter, or none of these.

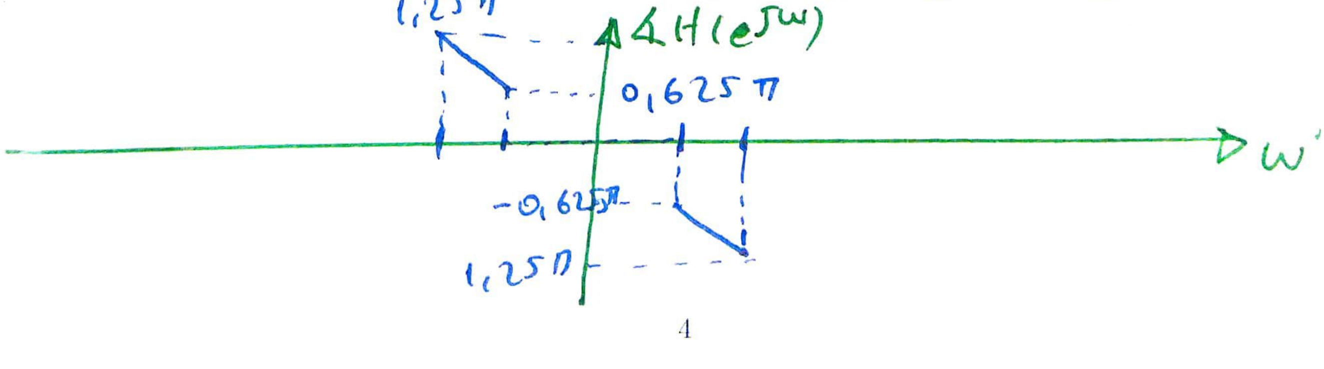
(a)

$$|H(e^{jw})| = \begin{cases} 0, & |w| \le 0{,}25\pi \\ 1, & 0{,}25\pi < |w| \le 0{,}5\pi \\ 0, & 0{,}5\pi < |w| \le \pi \end{cases}$$



(b)

$$\measuredangle H(e^{Jw}) = \begin{cases} 0, & |w| \le 0{,}25\pi \\ -2{,}5w, & 0{,}25\pi \le |w| \le 0{,}5\pi \\ 0, & 0{,}5\pi \le |w| \le \pi \end{cases}$$



(c) $x[n] = 3e^{j(0)} + e^{j(0,3\pi n)} + e^{-j(0,3\pi n)} + e^{j(0,7\pi n)} + e^{-j(0,7\pi n)}$
$+ (e^{j\pi})^n$

$$y[n] = 3e^{j(0)} H(e^{j(0)}) + e^{j0,3\pi n} H(e^{j0,3\pi}) + e^{-j0,3\pi n} H(e^{-j0,3\pi}) + e^{j0,7\pi n} H(e^{j0,7\pi}) + e^{-j0,7\pi n} H(e^{-j0,7\pi}) + e^{j\pi n} H(e^{j\pi})$$

Notice that $H(e^{jw})$ has nonzero values at the interval $0,25\pi < |w| \leq 0,5\pi$. It means that except for the interval, $H(e^{jw})$ has zero values. Hence,

$$y[n] = 0 + e^{j0,3\pi n} e^{-j\frac{3\pi}{4}} + e^{-j0,3\pi n} e^{j3\pi/4} + 0 + 0 + 0$$

$$y[n] = e^{j(0,3\pi n - 3\pi/4)} + e^{-j(0,3\pi n + 3\pi/4)}$$

$$y[n] = 2\cos(0,3\pi n - 3\pi/4)$$

(d) BANDPASS FILTER

5 [20 pts] Define three discrete-time signals:

$$a(n) = u(n) - u(n-4)$$
$$b(n) = \delta(n) + 2\delta(n-3)$$
$$c(n) = \delta(n) - \delta(n-1)$$

Define three new Z-transforms:

$$D(z) = A(-z), \quad E(z) = A(1/z), \quad F(z) = A(-1/z)$$

(a) Sketch $a(n), b(n), c(n)$

(b) Write the Z -transforms $A(z), B(z), C(z)$

(c) Write the Z -transforms $D(z), E(z), F(z)$

(d) Sketch $d(n), e(n), f(n)$

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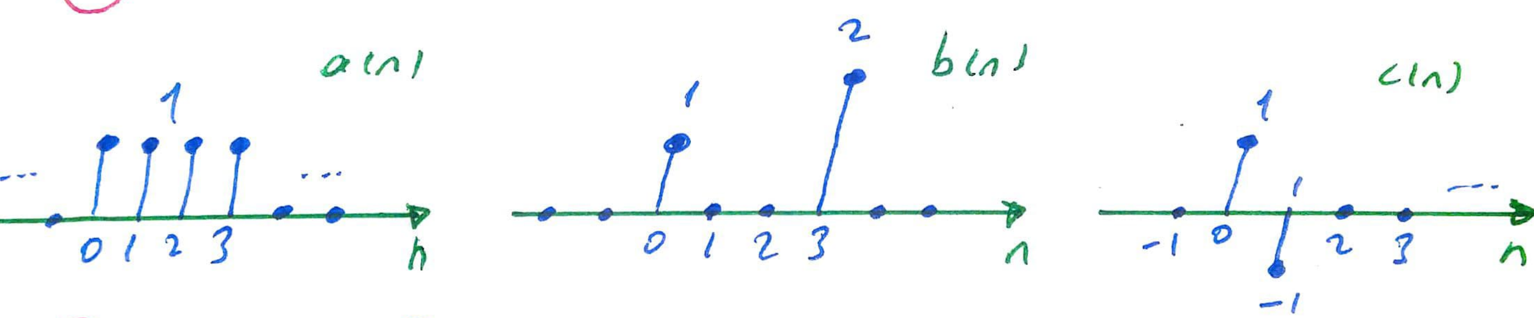
a



b $\quad A(z) = \sum_{n=-\infty}^{\infty} a(n) z^{-n}$

! $\quad \delta(n) \xleftrightarrow{Z} 1$

$\delta(n-n_0) \xleftrightarrow{Z} z^{-n_0}$

$A(z) = 1 + z^{-1} + z^{-2} + z^{-3}$

$B(z) = 1 + 2z^{-3}$

$C(z) = 1 - z^{-1}$

c $\quad$ For $D(z)$, we replace $(-z)$ in $A(z)$

$D(z) = 1 + (-z)^{-1} + (-z)^{-2} + (-z)^{-3}$

$= 1 - z^{-1} + z^{-2} - z^{-3}$

$E(z) = 1 + (z^{-1})^{-1} + (z^{-1})^{-2} + (z^{-1})^{-3}$

$E(z) = 1 + z + z^2 + z^3$

$F(z) = 1 + (-z^{-1})^{-1} + (-z^{-1})^{-2} + (-z^{-1})^{-3}$

$= 1 - z + z^2 - z^3$

d) $D(z) = 1 - z^{-1} + z^{-2} - z^{-3}$

$d(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$

$E(z) = 1 + z + z^2 + z^3$

$e(n) = \delta(n) + \delta(n+1) + \delta(n+2) + \delta(n+3)$

$F(z) = 1 - z + z^2 - z^3$

$f(n) = \delta(n) - \delta(n+1) + \delta(n+2) - \delta(n+3)$