1. For the analog transfer function

$$H(s) = \frac{2}{(s+1)(s+2)}$$

(a) determine $H(z)$ using impulse invariance method (assume T=1 sec).

(b) determine $H(z)$ using bilinear transformation (assume T=1 sec).

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{s - p_k} \quad \rightarrow \quad H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{p_k T} z^{-1}}$$

$$H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$C_1 = 2 \quad C_2 = -2$

$p_1 = -1 \quad p_2 = -2$

$$H(z) = \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}}$$

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$H(z) = \frac{2}{\left(2\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + 1\right)\left(2\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + 2\right)} = \frac{(1 + z^{-1})^2}{6\left(1 - \frac{1}{3} z^{-1}\right)}$$

2. The system function of a digital filter is

$$H(z) = \sum_{k=1}^{p} \frac{A_k}{1 - a_k z^{-1}}$$

a) If this filter was designed using impulse invariance with $T_s = 2$, find the system function, $H_a(s)$, of an analog filter that could have been the analog filter prototype. Is your answer unique?

b) Repeat part (a) assuming that the bilinear transformation was used with $T_s = 2$.

(*a*) Because $H(z)$ is expanded in a partial fraction expansion, the poles at $z = \alpha_k$ in $H(z)$ are mapped from poles in $H_a(s)$ according to the mapping

$$\alpha_k = e^{s_k T_s}$$

Therefore, if $T_s = 2$,

$$s_k = \tfrac{1}{2} \ln \alpha_k$$

and one possible analog filter prototype is

$$H_a(s) = \sum_{k=1}^{p} \frac{A_k}{s - \frac{1}{2} \ln \alpha_k}$$

Because the mapping from the s-plane to the z-plane is not one to one, this answer is not unique. Specifically, note that we may also write

$$\alpha_k = e^{s_k T_s + j2\pi}$$

Therefore, with $T_s = 2$, we may also have

$$s_k = \tfrac{1}{2} \ln \alpha_k + j\pi$$

and another possible analog filter prototype is

$$H_a(s) = \sum_{k=1}^{p} \frac{A_k}{s - \left(\frac{1}{2} \ln \alpha_k - j\pi\right)}$$

(*b*) With the bilinear transformation, because the mapping from the s-plane to the z-plane is a one-to-one mapping, with $T_s = 2$,

$$z = \frac{1+s}{1-s}$$

and the analog filter prototype that is mapped to $H(z)$ is unique and given by

$$H_a(s) = \sum_{k=1}^{p} \frac{A_k}{1 - \alpha_k \dfrac{1-s}{1+s}} = \sum_{k=1}^{p} \frac{A_k(1+s)}{(1-\alpha_k) + (1+\alpha_k)s}$$

3. With impulse invariance, a first-order pole in $_a(s)$ at $s = s_k$ is mapped to a pole in $H(z)$ at $z = e^{s_k T_s}$:

$$\frac{1}{s - s_k} \Rightarrow \frac{1}{1 - e^{s_k T_s} z^{-1}}$$

Determine how a second-order pole is mapped with impulse invariance.

With impulse invariance, a first-order pole in $H_a(s)$ at $s = s_k$ is mapped to a pole in $H(z)$ at $z = e^{s_k T_s}$:

$$\frac{1}{s - s_k} \Longrightarrow \frac{1}{1 - e^{s_k T_s} z^{-1}}$$

Determine how a second-order pole is mapped with impulse invariance.

If the system function of a continuous-time filter is

$$H_a(s) = \frac{1}{(s - s_k)^2}$$

the impulse response is

$$h_a(t) = t e^{s_k t} u(t)$$

where $u(t)$ is the unit step function. Sampling $h_a(t)$ with a sampling period T_s, we have

$$h(n) = h_a(nT_s) = nT_s e^{s_k n T_s} u(n)$$

Using the z-transform property

$$nx(n) \stackrel{z}{\longleftrightarrow} -z \frac{dX(z)}{dz}$$

and the z-transform pair

$$\alpha^n u(n) \stackrel{z}{\longleftrightarrow} \frac{1}{1 - \alpha z^{-1}}$$

it follows that the z-transform of $h(n)$ is

$$H(z) = -T_s z \frac{d}{dz} \left[\frac{1}{1 - e^{s_k T_s} z^{-1}} \right] = \frac{T_s e^{s_k T_s} z^{-1}}{(1 - e^{s_k T_s} z^{-1})^2}$$

Therefore, for a second-order pole, we have the mapping

$$\frac{1}{(s - s_k)^2} \Longrightarrow \frac{T_s e^{s_k T_s} z^{-1}}{(1 - e^{s_k T_s} z^{-1})^2}$$