Recitation 7W//

1) X and y are independent random veriables with pdf's.

$$f_X(x) = \begin{cases} \frac{1}{3} e^{-x/3} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2} e^{-y/2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a) $P(X>y)$

b) $E[Xy]$

c) $Cov(Xy)$

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soln

a) X and y are independent rvs. Therefore, to find $P(X>y)$, we need the joint pdf $f_{xy}(x,y)$

Since they are independent

$$f_{xy}(xy) = f_x(x) \cdot f_y(y)$$

$$P(X>y) = \iint_{x>y} f_x(x) f_y(y)\, dx\, dy$$

$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right\}$ for $P(X>y)$, $0 \leq y \leq x < \infty$

(1)

$$P(X>y) = \int_0^{\infty}\int_0^{x} \frac{1}{2} e^{-y/2} \cdot \frac{1}{3} e^{-x/3} \, dy \, dx$$

$$= \frac{1}{6}\int_0^{\infty}\int_0^{x} e^{-x/3} \qquad dx$$

$$= \frac{1}{6}\int_0^{\infty} e^{-x/3} \left[-2e^{-y/2} \Big|_0^x \right] dx$$

$$= \frac{1}{6}\int_0^{\infty} e^{-x/3} \left[-2e^{-x/2} + 2 \right] dx$$

$$= \frac{1}{3}\int_0^{\infty} e^{-5x/6} - e^{-x/3} \, dx = \frac{3}{5} \,//$$

b) Since X and y are exponential rvs.

$f_x(x) = \lambda_x e^{-\lambda x}$, $\lambda_x = 1/3$.

$E[X] = \frac{1}{\lambda_x} = 3$

$f_y(y) = \lambda_y e^{-\lambda y}$ $\lambda_y = 1/2$

$E[Y] = \frac{1}{\lambda_y} = 2.$

$E[XY] = E[X]E[Y] = 2 \cdot 3 = 6$

c) Since X and y are independent

$Cov(X, y) = 0$

2) The random vector $\underline{X}$ has pdf

$$f_{\underline{X}}(\underline{x}) = \begin{cases} e^{-x_3}, & 0 \le x_1 \le x_2 \le x_3 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal pdfs $f_{x_1}(x_1), f_{x_2}(x_2), f_{x_3}(x_3)$.

soln

$$f_{x_1}(x_1) = \int_{x_1}^{\infty}\int_{x_2}^{\infty} e^{-x_3}\, dx_3\, dx_2$$

$$= \int_{x_1}^{\infty} -e^{-x_3}\Big|_{x_2}^{\infty} dx_2$$

$$= \int_{x_1}^{\infty} e^{-x_2}\, dx_2 = e^{-x_1} \quad \text{for } x_1 \geq 0$$

$$f_{x_2}(x_2) = \int_{0}^{x_2}\int_{x_2}^{\infty} e^{-x_3}\, dx_3\, dx_1$$

$$= \int_{0}^{x_2} -e^{-x_3}\Big|_{x_2}^{\infty} dx_1$$

$$= \int_{0}^{x_2} e^{-x_2}\, dx_1 = x_1 e^{-x_2}\Big|_{0}^{x_2} = x_2 e^{-x_2} \quad \text{for } x_2 \geq 0$$

3

$$f_{X3}(x_3) = \int_0^{x_3}\int_{x_1}^{x_3} e^{-x_3}\,dx_2\,dx_1$$

$$= \int_0^{x_3} e^{-x_3}(x_3 - x_1)\,dx_1$$

$$= x_3 e^{-x_3} x_1 - e^{-x_3}\frac{x_1^2}{2}\Big|_0^{x_3} = \frac{1}{2} x_3^2 e^{-x_3} \quad // \text{ for } x_3 \geq 0$$

3) Random variables X_1 and X_2 have zero mean and $Var[X_1]=4$ and $Var[X_2]=9$. $Cov(X_1, X_2)=3$.

a) Find the covariance matrix $\underline{X} = [X_1 \ X_2]'$

b) = X_1 and X_2 are transformed to new variables y_1 and y_2.

$$Y_1 = X_1 - 2X_2$$

$$Y_2 = 3X_1 + 4X_2$$

Find the covariance matrix $\underline{Y} = [Y_1 \ Y_2]'$

a) $$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} [X_1 \ X_2] = \begin{bmatrix} Cov(X_1, X_1) & Cov(X_1, X_2) \\ Cov(X_2, X_1) & Cov(X_2, X_2) \end{bmatrix}$$

$$Cov(X_1, X_1) = E[(X_1 - \mu_{x1})(X_1 - \mu_{x1})]$$

$$= E[X_1^2] - \mu_{x1}^2 = Var(X_1) = 4$$

$$Cov(X_2, X_2) = Var(X_2) = 9$$

$$\underline{C}x = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_1, X_2) & Var(X_2) \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}.$$

→ off diagonal elements are same

b) $$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 - 2X_2 \\ 3X_1 + 4X_2 \end{bmatrix}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}_{\underline{X}} =$$

$$\underline{C}y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} = (\underline{A}\,\underline{X}) \cdot (AX)^T = AXX^TA^T$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \underline{C}x \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}$$