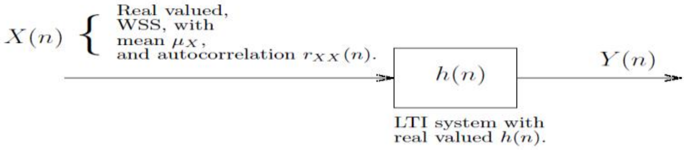
**EHB 315E – Digital Signal Processing**

1. What will happen to the mean and autocorrelation of a random process when it goes through an LTI system? Given $\mu_X$, $r_{XX}(n)$ and h(n), can we find $E[Y(n)]$ and $r_{YY}(n)$?




$$E[Y(n)] = E\left[\sum_{k=-\infty}^{\infty} h(k)\, x(n-k)\right] = \sum_{k=-\infty}^{\infty} h(k) \underbrace{E[x(n-k)]}_{\mu_x}$$

$$= \mu_x \sum_{k=-\infty}^{\infty} h(k) e^{-j0} = \mu_x H(e^{j0})$$

The mean of $Y(n)$ is constant and it is related to the mean of $X(n)$ by a scale factor that is frequency response of the filter at $w=0$.

$$r_y(n+k, n) = E[Y(n+k)\, Y(n)] = E\left[Y(n+k) \sum_{l=-\infty}^{\infty} x(l)\, h(n-l)\right]$$

$$= \sum_{l=-\infty}^{\infty} h(n-l) \underbrace{E[Y(n+k) X(l)]}_{r_{yx}(n+k-l)} = \sum_{l=-\infty}^{\infty} h(n-l)\, r_{yx}(n+k-l)$$

$$\overset{m=n-l}{=} \sum_{m=-\infty}^{\infty} h(m)\, r_{yx}(m+k) = r_{yx}(k) * h(-k)$$

$$r_{yx}(n+k, n) = E[Y(n+k) X(n)] = E\left[\sum_{l=-\infty}^{\infty} h(l)\, x(n+k-l)\, X(n)\right]$$

$$= \sum_{l=-\infty}^{\infty} h(l) \underbrace{E[X(n+k-l)\, X(n)]}_{r_x(k-l)} = \sum h(l)\, r_x(k-l)$$

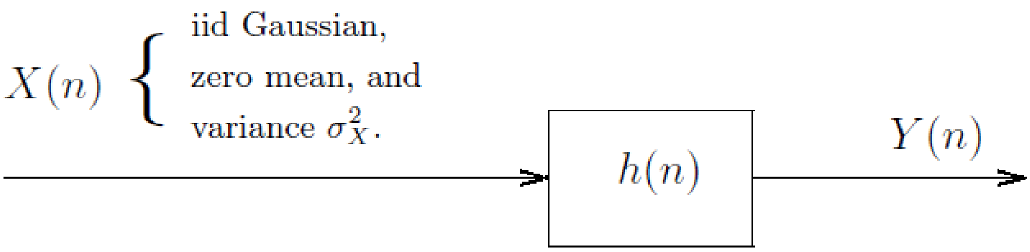
$$r_{yx}(k) = r_x(k) * h(k)$$

$$r_y(k) = r_{yx}(k) * h(-k) = r_x * h(k) * h(-k)$$

$$\mathcal{F}\{r_y(n)\} = P_y(e^{jw}) = P_x(e^{jw})\, |H(e^{jw})|^2$$

$$\mathcal{Z}\{r_y(n)\} = P_y(z) = P_x(z)\, H(z)\, H(z^{-1})$$

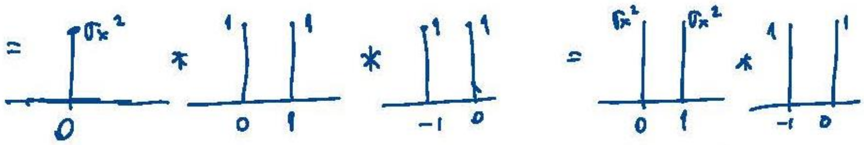
2. Let Y (n) = X(n) + X(n − 1) in the system illustrated in the figure below. Find $r_{YY}(n)$. I.e., how correlated is the output with itself shifted by some lag value n?

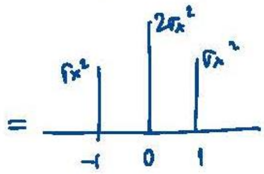



$$r_X(m) = E[X(n+m)\,X(n)]$$

$$= \begin{cases} E[X^2(n)] = \sigma_X^2 & , m = 0 \\ E[X(n+m)\,X(n)] = \underbrace{E[X(n+m)]\,E[X(n)]}_{\text{since } X \text{ is independent}} = 0 & , m \neq 0 \end{cases} = \sigma_X^2 \delta(m)$$

$$r_Y(k) = r_X(m) * h(m) * h(-m)$$




$$r_X(m) = \sigma_X^2\,(\delta(m+1) + 2\delta(m) + \delta(m-1))$$

3. Let x[n] be the random process that is generated by filtering white noise w[n] with a first order linear time invariant filter having a system function

$$H(z) = \frac{1}{1 - 0.25z^{-1}}$$

and $w[n] \sim N(0, \sigma_w^2), \sigma_w^2 = 1$.

a) Find the power spectrum of x[n], $P_X(z)$.
b) Find the autocorrelation of x[n].

a) The autocorrelation of white noise w(n) : $r_w(k) = \sigma_w^2 \delta(k)$

$$P_w(z) = Z\{r_w(k)\} = Z\{\sigma_w^2 \delta(k)\} = \sigma_w^2$$

$$P_x(z) = P_w(z)\, H(z)\, H(z^{-1}) = \underbrace{\sigma_w^2}_{1} \frac{1}{1-0.25z^{-1}} \frac{1}{(1-0.25z)} = \frac{z^{-1}}{(1-0.25z^{-1})(z^{-1}-0.25)}$$

ROC : $|z| > 0.25$ $\quad |z| < 4$ $\quad 0.25 < |z| < 4$

b) $r_x(k) = Z^{-1}\{P_x(z)\}$

$$P_x(z) = \frac{z^{-1}}{(1-0.25z^{-1})(z^{-1}-0.25)} = \frac{16/15}{(1-0.25z^{-1})} + \frac{4/15}{z^{-1}-0.25}$$

$$P_x(z) = \frac{16/15}{1-0.25z^{-1}} - \frac{16/15}{(1-4z^{-1})}$$

$$r_x(k) = \frac{16}{15}\left(\frac{1}{4}\right)^k u(k) + \left(\frac{16}{15}\right) 4^k u(-k-1) = \frac{16}{15}\left(\frac{1}{4}\right)^{|k|}$$

4. Suppose that we would like to generate a random process having a power spectrum of the form

$$P_x\left(e^{j\omega}\right)=\frac{5+4\cos 2\omega}{10+6\cos\omega}$$

by filtering unit variance white noise with a linear shift-invariant filter. Writing $P_x\left(e^{j\omega}\right)$ in terms of complex exponentials we have

$$P_x\left(e^{j\omega}\right)=\frac{5+2e^{j2\omega}+2e^{-j2\omega}}{10+3e^{j\omega}+3e^{-j\omega}}$$

Replacing $e^{j\omega}$ by z gives

$$P_x(z)=\frac{5+2\left(z^2+z^{-2}\right)}{10+3\left(z+z^{-1}\right)}=\frac{\left(2z^2+1\right)\left(2z^{-2}+1\right)}{(3z+1)\left(3z^{-1}+1\right)}$$

Performing the factorization

$$P_x(z)=H(z)H\left(z^{-1}\right)$$

where

$$H(z)=\frac{2z^2+1}{3z+1}=z\frac{2}{3}\frac{1+\frac{1}{2}z^{-2}}{1+\frac{1}{3}z^{-1}}$$

we see that $H(z)$ is a stable filter. Since introducing a delay into $H(z)$ will not alter the power spectrum of the filtered process, we may equivalently use the filter

$$H(z)=\frac{2}{3}\frac{1+\frac{1}{2}z^{-2}}{1+\frac{1}{3}z^{-1}}$$

which is causal and has a unit sample response given by

$$h(n)=\frac{2}{3}\left(-\frac{1}{3}\right)^n u(n)+\frac{1}{3}\left(-\frac{1}{3}\right)^{n-2} u(n-2)$$