1. **(30 pts - 35 mins)** Suppose we have two discrete-time sequences $x[n]$ and $h[n]$ as follows:

$$x[n] = \cos\left(\frac{\pi n}{2}\right) \quad , n = 0, 1, 2, 3$$
$$h[n] = 2e^{j\frac{\pi}{2}n} \quad , n = 0, 1, 2, 3$$

a. **(8 pts)** Calculate $y[n] = x[n]$ ④ $h[n]$ by doing the circular convolution directly. (④: four-point circular convolution)

b. **(8 pts)** Calculate the four-point DFTs $X[k]$ and $H[k]$.

c. **(7 pts)** Calculate $Y[k]$ by multiplying $X[k]$ and $H[k]$, then calculate $y[n]$ by performing an inverse DFT.

d. **(7 pts)** Consider $w[n] = \{0,\ 7,\ 0,\ 5\}$. Determine if a sequence that satisfies $x[n]$ ④ $v[n] = w[n]$ can be found. If so, find $v[n]$. If not, prove it does not exist.

DFT Definition : $X[k] \triangleq \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \leq k \leq N-1$

Inverse DFT Definition : $x[n] \triangleq \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \leq n \leq N-1$

Circular Convolution : $x_3[n] = x_1[n]$ Ⓝ $x_2[n] = \sum_{m=0}^{N-1} x_1[m] x_2\left[((n-m))_N\right], \quad 0 \leq n \leq N-1$

2. **(30 pts - 35 mins)** For the causal discrete-time LTI system implemented using the difference equation

$$y[n] = -0.25y[n-1] + 0.125y[n-2] + 4x[n] + 0.25x[n-1]$$

where $x[n]$ is the input signal, and $y[n]$ is the output signal.

a. **(7 pts)** Draw the direct form-II block diagram of the system.

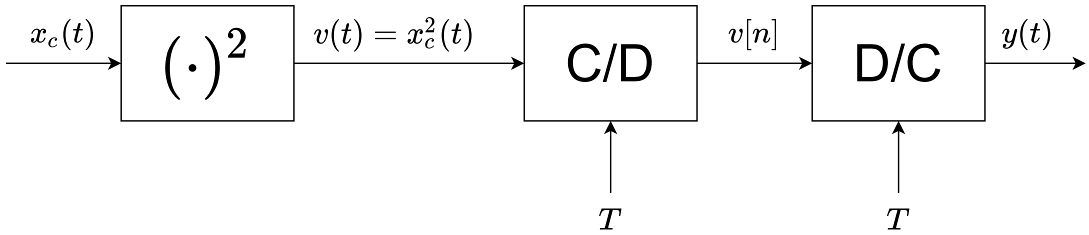
b. **(7 pts)** Find the transfer function of the system, $H(z)$.

c. **(8 pts)** Sketch the pole-zero diagram and indicate the ROC. Determine if the Fourier transform of the system $H(e^{j\omega})$ exists. If the Fourier transform exists, write $H(e^{j\omega})$.

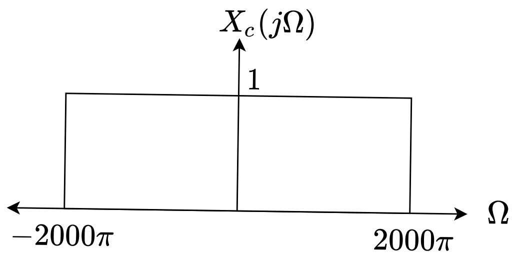
d. **(8 pts)** Find the z-transform of the output, $Y(z)$, when the input is $x[n] = u[-n-1]$. Specify the ROC for $Y(z)$.

Z-Transform Definition : $X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

3. **(20 pts - 30 mins)** Consider the system given in the figure below.



Suppose that the Fourier transform of $x_c(t)$ is shown in the figure below.



a. **(10 pts)** Find the largest value of T such that $y(t) = x_c^2(t)$. Sketch $V(j\Omega)$, $V(e^{j\omega})$, and $Y(j\Omega)$ that are the Fourier transform of $v(t)$, $v[n]$ and $y(t)$, respectively.

b. **(10 pts)** For the sampling period $T = \dfrac{1}{3000}$ second, sketch $V(e^{j\omega})$, and $Y(j\Omega)$.

4. **(20 pts - 30 mins)** Let $x[n]$ be the random process that is generated by filtering white noise $w[n] \sim \mathcal{N}(0, 2)$ with a discrete-time filter that is designed by applying the bilinear transform to a continuous-time filter. The system function of the continuous-time filter is given as

$$H(s) = \frac{2(s+1)}{3s+1}$$

a. **(6 pts)** For the sampling period $T_d = 2$, find the transfer function of the system $H(z)$.

b. **(6 pts)** Find the power spectrum $P_W(z)$ of $w[n]$ and $P_X(z)$ of $x[n]$.

c. **(8 pts)** Find the autocorrelation sequence $r_x(k)$ of $x[n]$.

Bilinear Transform Definiton : $H(z) = H(s)\Big|_{s=\frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}}}$