Chebyshev Inequality:

$$P[|X-\mu| \geq a] \leq \frac{\sigma_X^2}{a^2}, \quad a>0$$

→ proof

Quiz 1) X: Head → -1, Tail → 1 $P[H] = P[T] = \frac{1}{2}$

$$E[X] = \sum_{x \in R_X} P[X=x] . x$$

$$= \frac{1}{2} . -1 + \frac{1}{2} . 1 = 0$$

unfair coin: $P[H] = p$ $P[T] = 1-p$

Quiz 2)

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$Y = g(x) = \frac{X-(-2)}{\sqrt{2}} \qquad E[Y] = \frac{E[X]-(-2)}{\sqrt{2}} = 0$$

$Y \to (0,1)$: μ, σ^2

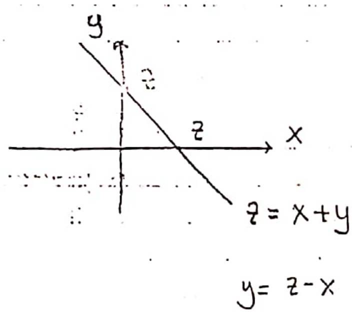
$g(x) = Y \to$ CDF of X

Function of Two Random Variables

$Z = g(X,Y)$, $X+Y$, $\frac{X}{Y}$, $\min(X,Y)$, $\max(X,Y)$, $\frac{\max(X,Y)}{\min(X,Y)}$

Ex. $Z = X+Y$

$$F_Z(z) = P[Z \leq z] = P[X+Y \leq z]$$



$$F_Z(z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} \underbrace{f_{XY}(x,y)}_{\text{Joint pdf of x and y}} \, dy \, dx$$

$$= \int_{y=-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x,y) \, dx \, dy$$

Independency

Given random vars X and Y are independent,

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y) \Rightarrow \text{for indepent events}$$

joint pdf — marginal pdf of X — marginal pdf of Y

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy} \qquad \boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$

general formulas

X → para atımı
Y → zar "

	1	2	3	4	5	6	
H	1/12	1/12					1/2
T	1/12				1/12	1/12	1/2
	1/6	1/6				1/6	

$$F_Z(z) = \int_{x=-\infty}^{x=z-y} \underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}_{f_X(x)}\,dx = \int_{x=-\infty}^{x=z-y} f_X(x)\,dx$$

$$= \int_{y=-\infty}^{z-x} f_Y(y)\,dy$$

they have the same dist

Let X, Y are i.i.d. (independent identically distributed),
(2 fair zar atılması buna örnek olabilir)

then, $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$ Given X, Y are independent.

$$F_Z(z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_{XY}(x,y)\,dy\,dx$$

If they are also identical, then $f_X(x_1)$ $f_X(x_2)$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_X(x)\, f_Y(y)\,dy\,dx$$

$$= \underbrace{\int_{x=-\infty}^{\infty} f_X(x)\,dx}_{=1} \int_{y=-\infty}^{z-x} f_Y(y)\,dy = F_Y(z-x) = P[Y \leq z-x]$$

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Leibnitz's Rule

$$f_z(z) = \frac{\partial F_z(z)}{\partial z} = \frac{\partial}{\partial z}\left[\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{y=z-x} f_x(x)\, f_y(y)\, dx\, dy\right]$$

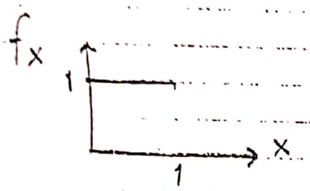
independent olmasaydı $f_{xy}(x, z-x)$ yazacaktık çarpım şeklinde ayıramadığımız için. Leibnitz's Rule hala geçerli ama indep. olmasalar da.

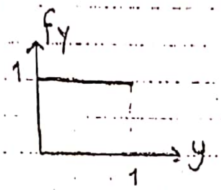
Leibnitz's Rule

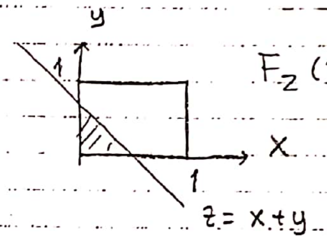
$$= \frac{\partial(z-x)}{\partial z}\int_{x=-\infty}^{\infty} f_x(x)\, f_y(z-x)\, dx$$

1

ex. $Z = X + Y$, where X and Y are i.i.d uniform between 0 and 1. Find pdf of z.





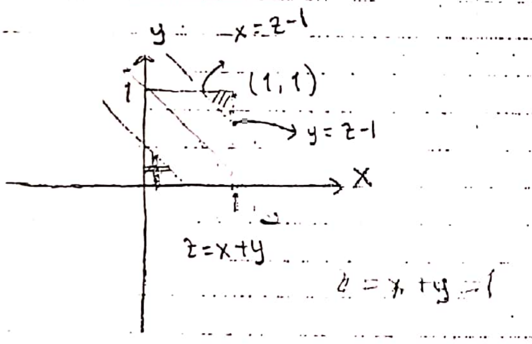


$F_z(z) = P[Z \leq z]$

$$F_z = \begin{cases} 0, & z < 0 \\ ?, & 0 \leq z \leq 2 \\ 1, & z > 2 \end{cases}$$

$\nearrow$ $0 \leq z \leq 1$

$\searrow$ $1 \leq z \leq 2$



For $0 \leq z \leq 1$;

$$F_z(z) = P[Z \leq z]$$

$$= P[X + Y \leq z]$$

$$= \int_{y=0}^{z}\int_{x=0}^{z-y} 1\, dx\, dy$$

$$= \int_{x=0}^{z}\int_{y=0}^{z-x} 1\, dy\, dx$$

$$= \left. zy - \frac{y^2}{2}\right|_0^z = \frac{z^2}{2}$$

$$f_{xy}(x,y) = f_x(x)\, f_y(y)$$

$$= 1 \cdot 1 = 1$$

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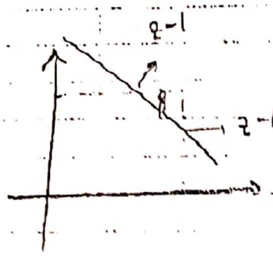
For $1 \leq z \leq 2$; $F_Z(z) = P[Z \leq z]$

$$= P[X + Y \leq z]$$



$$= 1 - \int_{y=z-1}^{1} \int_{x=z-y}^{1} 1 \, dx \, dy = 1 - \frac{(2-z)^2}{2}$$

$$= 1 - \int_{x=z-1}^{1} \int_{y=z-x}^{1} 1 \, dy \, dx$$

$$F_Z(z) = \begin{cases} 0 & ; \ z < 0 \\ \frac{z^2}{2} & ; \ 0 \leq z \leq 1 \\ 1 - \frac{(2-z)^2}{2} & ; \ 1 \leq z \leq 2 \\ 1 & ; \ z \geq 2 \end{cases} \qquad f_Z(z) = \begin{cases} 0 & ; \ z < 0 \\ z & ; \ 0 \leq z \leq 1 \\ 2-z & ; \ 1 \leq z \leq 2 \\ 0 & ; \ z \geq 2 \end{cases}$$

X, Y are two random variables

$$F_{XY}(x,y) = P[X \leq x, \ Y \leq y] = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY} \, dx \, dy$$

$$F_X(x) = F_X(x, \infty) = P[X \leq x, \ Y \leq \infty] = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY} \, dy \, dx$$

$$F_Y(y) = F_Y(\infty, y)$$

$Var[X] = E[(X - \mu_X)^2]$ (covariance of X and X)

$C(X,Y) =$ Covariance $(X,Y) = E[(X - \mu_X)(Y - \mu_Y)] = R(X,Y) - \mu_X \mu_Y$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} \qquad -1 \leq \rho_{XY} \leq 1$$

correlation coefficient = normalized covariance

$R(X,Y) = E[XY]$

correlation

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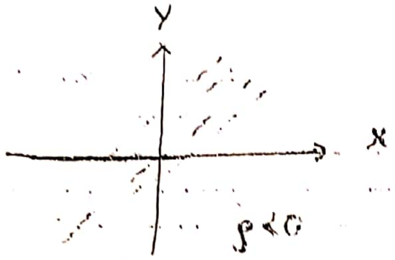
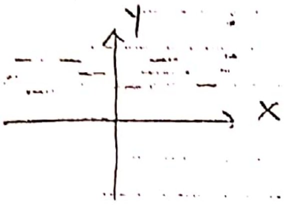


$\rho > 0$ (x azalırken y azalır, x artarken y artar!)

exercise! $Y = aX$

$\rho_{X,Y} = ?$

↳ 1 ya da -1 mi?



$\rho = 0$

* Joint pdf of two Gaussian RVs

$Cov(X,Y) = \Sigma_{XY}$

$$f_{XY}(x,y) = \frac{1}{2\pi |\Sigma_{XY}|^{1/2}} e^{-\frac{1}{2}\cdot(x-\mu_x)\Sigma_{XY}^{-1}(y-\mu_x)}$$

determinant

$$X \rightarrow F_X(x) \rightarrow f_X(x) \rightarrow \underbrace{E[X]}_{\text{mean}}, \ \underbrace{E[(X-\mu_x)^2]}_{\text{variance}}$$

$$X, Y \rightarrow F_{XY}(x,y) \rightarrow f_{XY}(x,y) \rightarrow E[XY] : R$$

$$C(X,Y) = E[(X-\mu_x)(Y-\mu_y)] = \Sigma_{XY}$$

↳ covariance

$$\rho_{XY} = \frac{C(X,Y)}{\sigma_X \sigma_Y}, \quad [-1, 1]$$

correlation coefficient : normalized covariance

$$F_{XY}(x,y) = P[X \leq x, Y \leq y]$$

$$F_{XY}(\infty, \infty) = 1 \qquad F_{XY}(-\infty, -\infty) = 0$$

$$F_X(x) = F_{XY}(x, \infty) \qquad f_X(x) = \int_{-\infty}^{\infty} f_{XY}\, dy$$

$$F_Y(y) = F_{XY}(\infty, y) \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}\, dx$$

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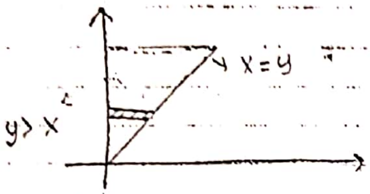
✓ continuous (CDF'ye bakarak)

ex. $f_{XY}(x,y) = Kxy \quad ; \quad 0 \leq x \leq y \leq 1$ (klasik sınav sorusu)

$Z = X + Y$, Find pdf of Z ? $\quad R_Z : [0,2]$



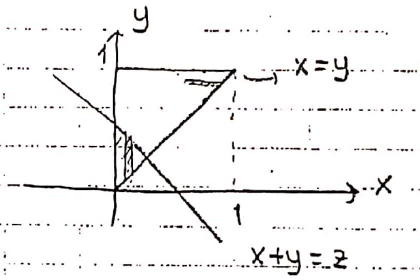
$$\int_{y=0}^{y=1}\int_{x=0}^{y} f_{XY}\, dx\, dy = 1$$

$$\int_0^1\int_0^y Kxy\, dx\, dy = \int_0^1 K\frac{x^2}{2}y\Big|_0^y dy$$

$$= \int_0^1 Ky\frac{y^2}{2}\, dy = K\frac{y^4}{8}\Big|_0^1 = \frac{K}{8} = 1 \qquad K=8$$

$F_Z(z) = P[Z \leq z]$

$= P[X+Y \leq z]$



$$F_Z(z) = \begin{cases} 0 & , z<0 \\ & , 0<z<1 \\ & , 1<z<2 \\ 1 & , z>2 \end{cases}$$

$$1 - \int_{y=\frac{z}{2}}^{1}\int_{x=z-y}^{y} 8xy\, dx\, dy$$

(yatay çubuk aldık sample)

$$\int_{x=0}^{x=\frac{z}{2}}\int_{y=x}^{z-x} 8xy\, dy\, dx$$

(dik çubuklar alırken sample olarak)

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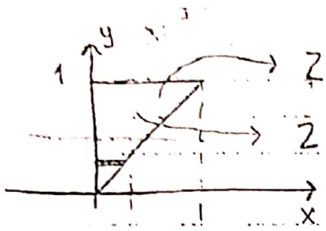
ex $Z = \max(X, Y)$ or $Z = \min(X, Y)$

$f_{XY}(x,y) = 8xy$, $0 \le x \le y \le 1$ Find $f_Z(z) = ?$



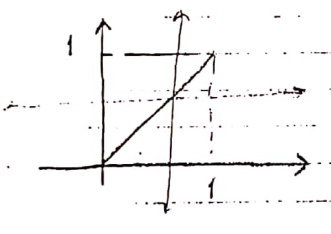
$\rightarrow Z = \max(X, Y) = Y$ $R_Z : [0, 1]$

$\rightarrow Z = \min(X, Y) = X$

$$F_Z(z) = P[Z \le z] = P[Y \le z] = \int_{y=0}^{z}\int_{x=0}^{y} 8xy\,dx\,dy$$

$$F_Z(z) = P[Z \le z] = P[X \le z] = \int_{x=0}^{z}\int_{y=x}^{1} 8xy\,dy\,dx$$

ex $Z = \max(X, Y)$ $f_{XY}(x,y) = 4xy$ $0 < x < 1$, $0 < y < 1$ $f_Z(z) = ?$



$z = y$

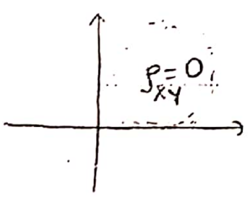
$z = x$

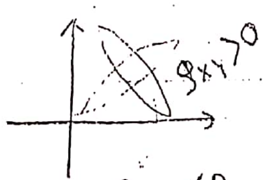
$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$\Sigma_{XY} = \begin{array}{c|cc} & X & Y \\ \hline X & \sigma_X^2 & C_{XY} \\ Y & C_{XY} & \sigma_Y^2 \end{array}$$

$\downarrow$

$E[(X - \mu_X)(Y - \mu_Y)]$





$\rho_{XY} < 0$

$y = ax + b$, $\mu_Y = E[Y] = E[aX + b] = aE[X] + b$

$= a\mu_X + b$

$\sigma_Y^2 = a^2 \sigma_X^2$

$\sigma_Y = |a| \sigma_X$

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$$\rho_{XY} = \frac{\operatorname{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$$

$$= \frac{E[(X-\mu_X)(aX+\cancel{b}-a\mu_X-\cancel{b})]}{\sigma_X \, |a| \, \sigma_X}$$

$$= \frac{E[aX^2 - aX\mu_X - a\mu_X X + a\mu_X^2]}{|a| \, \sigma_X^2}$$

$$= \frac{aE[X^2] - 2a\mu_X^2 + a\mu_X^2}{|a| \, \sigma_X^2} = \frac{aE[X^2] - a\mu_X^2}{|a| \, \sigma_X^2}$$

$$= \frac{a(\sigma_X^2 + \mu_X^2) - a\mu_X^2}{|a| \, \sigma_X^2} = \frac{a\sigma_X^2}{|a| \, \sigma_X^2} = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases}$$

EX The joint pdf of RVs X and Y are given by

$$f_{XY}(x,y) = \begin{cases} e^{x+y} & x>0 \cap y>0 \cap x+y<k \\ 0 & \text{otherwise} \end{cases}$$

a) Find $k = ?$

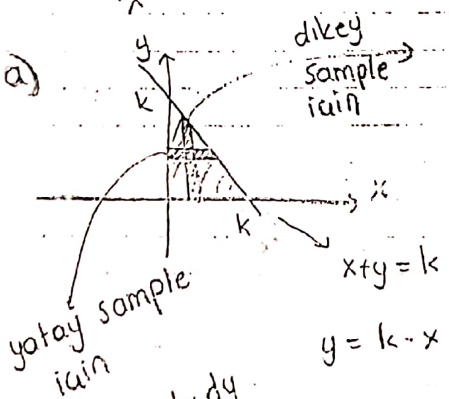
b) $P[Y > X]$ e) $E[X / X > Y]$

c) f_X, f_Y

d) $f_X(x / X > Y)$

sınavda var $\begin{cases} \text{a funct of RV} \\ \text{a funct of two RV} \\ \text{probability theory} \end{cases}$

a)



$$\int_{y}\int_{x} \ldots \, dx\, dy$$

joint pdf yazıyoruz içeri

$$\int_{x=0}^{k} \int_{y=0}^{y=k-x} e^{x+y} \, dy\, dx = 1$$

$$= \int_0^k e^{x+y} \Big|_0^{k-x} = \int_0^k (e^k - e^x)\, dx = 1$$

$$= xe^k - e^x \Big|_0^k = 1$$

$$(ke^k - e^k) - (-1) = 1$$

$$ke^k - e^k + 1 = 1$$

$$ke^k = e^k \Rightarrow k = 1$$

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