Q1. An elliptically polarized plane wave and its phasor domain expression

Time-dependent expression of an EM wave propagating in air has been given below.

$$\boldsymbol{E}(z,t) = 5\cos(-\omega t - 10z)\,\boldsymbol{e}_x - 10\sin(-\omega t - 10z)\,\boldsymbol{e}_y$$

Then,

a) Find phasor expression of the electric field vector.
b) Find frqeuency and wavelength of the wave.
c) Find polarization of the wave.

A:

a) Remember that cosine is an even, sine is an odd function. Then,

$$\boldsymbol{E}(z,t) = 5\cos(\omega t + 10z)\,\boldsymbol{e}_x + 10\sin(\omega t + 10z)\,\boldsymbol{e}_y$$

For phasor expression, cosine fuction is selected as reference function. Then, we must write *sine* expression in terms of *cosine*.

$$\Rightarrow \quad \boldsymbol{E}(z,t) = 5\cos(\omega t + 10z)\,\boldsymbol{e}_x - 10\cos(\omega t + 10z + \pi/2)\,\boldsymbol{e}_y$$

$$\Rightarrow \quad \boldsymbol{E}(z) = 5e^{-j10z}\boldsymbol{e}_x - 10e^{-j10z}e^{-\frac{j\pi}{2}}\boldsymbol{e}_y = 5e^{-j10z}\boldsymbol{e}_x - 10e^{-j\left(10z+\frac{\pi}{2}\right)}\boldsymbol{e}_y$$

b) For air,

$$\beta = \frac{\omega}{c} = \frac{2\pi}{\lambda} \quad \Rightarrow \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{10} = \frac{\pi}{5} \quad \Rightarrow \frac{c}{\lambda} = f = \frac{3\times 10^8}{\pi/5} = 4.77\times 10^8\,Hz = 477\,MHz$$

c) Phase difference between the components of electric field vector is 90 degree, and amplitudes are different. Then, the polarization is elliptic.

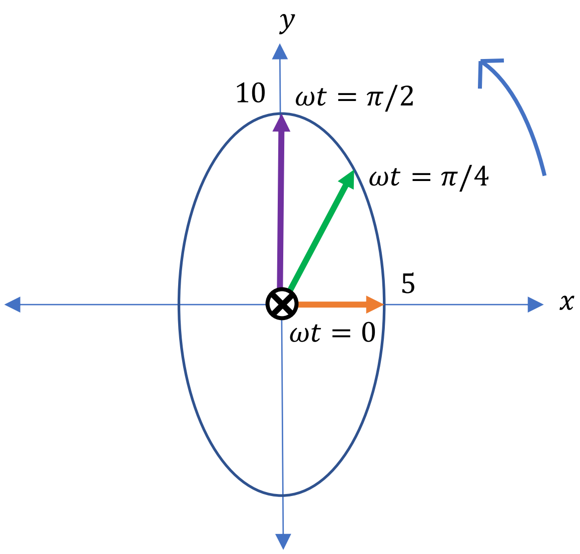
For the right-hand left-hand examination, we can consider $z = 0$ plane. We must check the orientation of the electric field vector with respect to increasing t moments.

$$\boldsymbol{E}(0,t) = \boldsymbol{e}_x 5\cos(\omega t) - \boldsymbol{e}_y 10\cos\left(\omega t + \frac{\pi}{2}\right)$$

$$\omega t = 0 \quad \Rightarrow \quad \boldsymbol{E} = 5\boldsymbol{e}_x$$

$$\omega t = \frac{\pi}{4} \quad \Rightarrow \quad \boldsymbol{E} = 3.54\boldsymbol{e}_x + 7.08\boldsymbol{e}_y$$

$$\omega t = \frac{\pi}{2} \quad \Rightarrow \quad E = 10\boldsymbol{e}_y$$



As it is seen from the figure, electric field vector is oriented from positive x to positive y axis. This is direction of four finger when we use left-hand and when direction of propagation is $-z$, which shows direction of thumb. Therefore, the polarization is called as left-hand elliptical polarization (LHEP).

Q2. A plane wave in lossy media

Electric field vector of an EM wave propagating in a non-magnetic lossy media is given below.

$$\boldsymbol{E}(x, y, t) = e^{-\frac{x}{3}} e^{\frac{-\sqrt{2}y}{3}} \cos\left(-x - \sqrt{2}y + 10^8 t\right) \boldsymbol{e}_z$$

Determine,

a) Frequency.
b) Wavenumber, attenuation constant, and phase constant.
c) Direction of propagation, direction of E and H vector.
d) Dielectric constant and conductivity.
e) Phase velocity.

A:

General expression for a z-polarized plane wave in phasor domain can be written as follows.

$$\boldsymbol{E}(r) = E_0 e^{j\boldsymbol{kr}} \boldsymbol{e}_z = E_0 e^{jk\boldsymbol{nr}} \boldsymbol{e}_z$$

In this expression, k is the propagation constant, $\boldsymbol{n}$ is unit vector in direction of the propagation, and r is position vector defined as,

$$\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$$

If the expression is rearranged in this way,

$$\boldsymbol{E}(r) = \boldsymbol{e}_z e^{-\frac{x}{3}} e^{\frac{-\sqrt{2}y}{3}} e^{j(x+\sqrt{2}y)} = \boldsymbol{e}_z e^{-\left(\frac{x}{3}+\frac{\sqrt{2}y}{3}\right)} e^{j3\left(\frac{x}{3}+\frac{\sqrt{2}y}{3}\right)}$$

$$= \boldsymbol{e}_z e^{j\left(3-\frac{1}{j}\right)\left(\frac{x}{3}+\frac{\sqrt{2}y}{3}\right)} = \boldsymbol{e}_z e^{j\left(\sqrt{3}-\frac{\sqrt{3}}{3j}\right)\left(\frac{x}{\sqrt{3}}+\frac{\sqrt{2}y}{\sqrt{3}}\right)} = \boldsymbol{e}_z E_0 e^{jk\boldsymbol{nr}}$$

Then,

$$E_0 = 1; \qquad k = \beta + j\alpha = \sqrt{3} + j\frac{\sqrt{3}}{3}; \qquad \boldsymbol{n}\cdot\boldsymbol{r} = \frac{x}{\sqrt{3}} + \frac{\sqrt{2}y}{\sqrt{3}}$$

As we see, magnitude of $\boldsymbol{n}$ vector is 1. This is required because we are searching the true k value. For this purpose, the phase expression has been arranged to give magnitude of direction vector as 1.

a) Easily,

$$\omega = 2\pi f = 10^8 \;\Rightarrow f = 7.9\ MHz$$

b) We can write,

$$k = \beta + j\alpha = \sqrt{3} + j\frac{\sqrt{3}}{3} \quad \Rightarrow \quad \beta = \sqrt{3} \;\; ; \;\; \alpha = \frac{\sqrt{3}}{3}$$

c) And,

$$\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y \quad \Rightarrow \quad \boldsymbol{n}\cdot\boldsymbol{r} = \frac{x}{\sqrt{3}} + \frac{\sqrt{2}y}{\sqrt{3}} \quad \Rightarrow \quad \boldsymbol{n} = \boldsymbol{e}_x\frac{1}{\sqrt{3}} + \boldsymbol{e}_y\frac{\sqrt{2}}{\sqrt{3}}$$

Direction of electric field vector can be seen directly from the expression given in the question, and it is $+z$.

Direction of the magnetic field vector can be found by considering following expression,

$$\boldsymbol{H} = \frac{1}{\eta}\boldsymbol{n}\times\boldsymbol{E}$$

Then,

$$\left(\boldsymbol{e}_x\frac{1}{\sqrt{3}} + \boldsymbol{e}_y\frac{\sqrt{2}}{\sqrt{3}}\right)\times\boldsymbol{e}_z = \frac{1}{\sqrt{3}}\boldsymbol{e}_x\times\boldsymbol{e}_z + \frac{\sqrt{2}}{\sqrt{3}}\boldsymbol{e}_y\times\boldsymbol{e}_z = -\boldsymbol{e}_y\frac{1}{\sqrt{3}} + \boldsymbol{e}_x\frac{\sqrt{2}}{\sqrt{3}}$$

Cross product of the vectors has been written by right-hand rule.

d) If the expression of the propagation constant is considered,

$$k^2 = \omega^2\epsilon\mu + j\omega\sigma\mu = \left(\sqrt{3} + j\frac{\sqrt{3}}{3}\right)^2 = 2.66 + 2j$$

$$\Rightarrow \quad \epsilon_r = \frac{2.66}{\omega^2\mu_0\epsilon_0} = \frac{2.66\times(3.10^8)^2}{10^{16}} = 23.94$$

$$\Rightarrow \quad \sigma = \frac{2}{\omega\mu_0} = \frac{2}{10^8\times4\pi\times10^{-7}} = 0.0159 \quad [S/m]$$

e) Finally,

$$v = \frac{\omega}{Re(k)} = \frac{\omega}{\beta} = \frac{10^8}{\sqrt{3}} = 5.77\times10^7 \;\; m/s$$

Q3. Poynting vector and power passing through a surface

Electric field vector for a wave propagating in free space is given below. Then, find the total average power passing through a circular region with radius 2.5 m on plane defined as $x = z$.

$$\boldsymbol{E}(z,t) = 50\cos(\omega t - \beta z)\,\boldsymbol{e}_x \quad [V/m]$$

A:

Complex Poynting vector can be written as,

$$\boldsymbol{P_c} = \frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^*$$

Then, average power density vector is given,

$$\boldsymbol{P_{av}} = Re\{\boldsymbol{P_c}\}$$

And,

$$\boldsymbol{E}(z) = 50e^{j\beta z}\boldsymbol{e_x}$$

We see that direction of propagation is +z. Then,

$$\boldsymbol{H}(z) = \frac{1}{\eta}\boldsymbol{n} \times \boldsymbol{E} = \frac{50}{\eta_0}e^{j\beta z}\boldsymbol{e}_y = 0.13e^{j\beta z}\boldsymbol{e}_y$$

$$\boldsymbol{P_{av}} = Re\left\{\frac{1}{2}\left(50e^{j\beta z}\boldsymbol{e}_x\right) \times \left(0.13e^{-j\beta z}\boldsymbol{e}_y\right)\right\} = 3.25\boldsymbol{e_z} \quad [W/m^2]$$

$$P_{tot} = \int_S \boldsymbol{P}_{av} d\boldsymbol{s}$$

$$d\boldsymbol{s} = ds.\boldsymbol{n}$$

Here, $\boldsymbol{n}$ denotes the unit vector, which is perpendicular to the plane. This vector can be found by,

$$\boldsymbol{n} = \frac{\boldsymbol{e}_x - \boldsymbol{e}_z}{\sqrt{1^2 + 0^2 + (-1)^2}} = \frac{1}{\sqrt{2}}\boldsymbol{e}_x - \frac{1}{\sqrt{2}}\boldsymbol{e}_z$$

where $\boldsymbol{e}_x - \boldsymbol{e}_z$ vector indicates grandient of the left side of the equation, which defines the surface. The denominator is length of the vector $\boldsymbol{e}_x - \boldsymbol{e}_z$.

Then,

$$P_{tot} = \int_S \boldsymbol{P}_c d\boldsymbol{s} = \int_S (3.25\boldsymbol{e}_z)\left(\frac{1}{\sqrt{2}}\boldsymbol{e}_x - \frac{1}{\sqrt{2}}\boldsymbol{e}_z\right) ds = (3.25\boldsymbol{e}_z)\left(\frac{1}{\sqrt{2}}\boldsymbol{e}_x - \frac{1}{\sqrt{2}}\boldsymbol{e}_z\right)\int_S ds$$

$$= -\frac{3.25}{\sqrt{2}}S = -\frac{3.25}{\sqrt{2}}(\pi 2.5^2) \quad \Rightarrow \quad |P_{tot}| = 45.12\,W$$