Standing Wave and Standing Wave Ratio

The general solutions of the transmission-line voltage and current equations are:

$$V = V_+ e^{-\gamma z} + V_- e^{\gamma z} \quad \text{and} \quad I = Y_0 (V_+ e^{-\gamma z} - V_- e^{\gamma z})$$

We can rewrite the voltage solution as,

$$V = V_+ e^{-\alpha z} e^{-j\beta z} + V_- e^{\alpha z} e^{j\beta z}$$

$$= V_+ e^{-\alpha z}[\cos(\beta z) - j\sin(\beta z)] + V_- e^{\alpha z}[\cos(\beta z) + j\sin(\beta z)]$$

$$= (V_+ e^{-\alpha z} + V_- e^{\alpha z})\cos(\beta z) - j(V_+ e^{-\alpha z} - V_- e^{\alpha z})\sin(\beta z)$$

We can assume that $V_+ e^{-\alpha z}$ and $V_- e^{\alpha z}$ are real; we can then obtain the voltage standing wave equation by expessing the above equation as,

$$V_s = V_0 e^{-j\phi_0}$$

where V_0 is the standing-wave pattern, or the amplitude of the standing wave;

$$V_0 = [(V_+ e^{-\alpha z} + V_- e^{\alpha z})^2 \cos^2(\beta z) + (V_+ e^{-\alpha z} - V_- e^{\alpha z})^2 \sin^2(\beta z)]^{1/2}$$

and ϕ_0 is the phase pattern of the standing wave,

$$\phi_0 = \arctan\left[\frac{V_+ e^{-\alpha z} - V_- e^{\alpha z}}{V_+ e^{-\alpha z} + V_- e^{\alpha z}} \tan(\beta z)\right]$$

We can find the max. and min. values of V_0 by diffentiating the V_0 with respect to βz and equating the result to zero. When we do so and substitute the proper values of βz into V_0:

the max. amplitude is : $V_{max} = V_+ e^{-\alpha z} + V_- e^{\alpha z}$

which occurs at $\beta z = n\pi$, $n = 0, \pm 1, \pm 2, \dots$

the min. amplitude is : $V_{min} = V_+ e^{-\alpha z} - V_- e^{\alpha z}$

which occurs at $\beta z = (2n-1)(\pi/2)$, $n = 0, \pm 1, \pm 2, \dots$

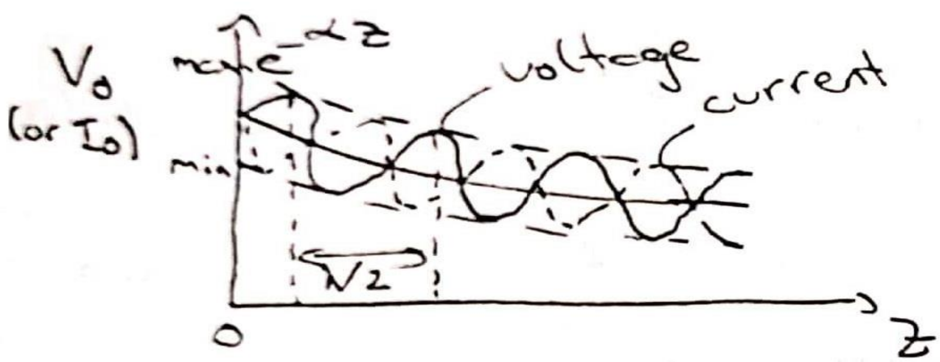
The distance between any two maxima or minima is one-half wavelength, since $\beta z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, so

$$z = \frac{n\pi}{\beta} = \frac{n\pi}{(2\pi)/\lambda} = n\frac{\lambda}{2}$$

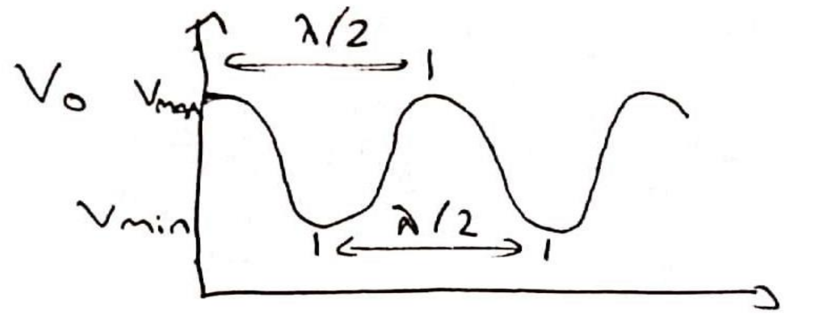
Similarly, the maximum and minimum currets are,

$$I_{max} = I_+ e^{-\alpha z} + I_- e^{\alpha z} \quad \text{and} \quad I_{min} = I_+ e^{-\alpha z} - I_- e^{\alpha z}$$

The figures below show the standing wave patterns in a lossy and lossless line :



lossy line



lossless line

In the lossy-line case, the maxima and minima are functions of position z and reflection coefficient Γ ($\Gamma = \frac{V_- e^{\gamma z}}{V_+ e^{-\gamma z}}$, $|\Gamma| = \frac{V_- e^{\alpha z}}{V_+ e^{-\alpha z}}$)

$$V_{max} = V_+ e^{-\alpha z}(1+|\Gamma|) \quad \text{and} \quad V_{min} = V_+ e^{-\alpha z}(1-|\Gamma|)$$

$$I_{max} = I_+ e^{-\alpha z}(1+|\Gamma|) \quad \text{and} \quad I_{min} = I_- e^{-\alpha z}(1-|\Gamma|)$$

In the lossless-line, max. and min. amplitudes remain constant,

$$V_{max} = V_+ (1+|\Gamma|) \quad \text{and} \quad V_{min} = V_+ (1-|\Gamma|)$$

$$I_{max} = I_+ (1+|\Gamma|) \quad \text{and} \quad I_{min} = I_+ (1-|\Gamma|)$$

when $V_- = 0$ ($\Gamma = 0$), V_0 becomes $V_0 = V_+ e^{-\alpha z}$ and $V_s = V_+ e^{-\gamma z}$ which is a pure traveling wave.

when $|V_+ e^{-\alpha z}| = |V_- e^{\alpha z}|$, ($|\Gamma| = 1$), $V_s = 2V_+ e^{-\alpha z}\cos(\beta z)$ which is a pure standing wave.

Similarly, for the current, the equation of a pure standing wave is

$$I_s = -j2Y_0 V_+ e^{-\alpha z} \sin(\beta z)$$

We can express the voltage or real function of time and space:

$$v_s(z,t) = Re[V_s(z) e^{j\omega t}] = 2V_+ e^{-\alpha z} \cos(\beta z)\cos(\omega t)$$

Similarly for current, $i_s(z,t) = Re[I_s(z) e^{j\omega t}] = 2Y_0 V_+ e^{-\alpha z} \sin(\beta z)\sin(\omega t)$

The sum of the two time-average densities is constant on the line. The complex power in the lossless line is,

$$P = \frac{1}{2} V_s(z) I_s^*(z) = \frac{1}{2}[2V_+ \cos(\beta z)][j2Y_0 V_+ \sin(\beta z)] = j|V_+|^2 Y_0 \sin(2\beta z)$$

This power has only imaginary part. Thus a pure standing wave of a lossless line transmits no time-average power.

Standing Wave Ratio

Standing-wave ratio is defined as, (designated by s)

$$S = \frac{\text{max. voltage or current}}{\text{min. voltage or current}} = \frac{|V_{max}|}{|V_{min}|} = \frac{|I_{max}|}{|I_{min}|}$$

For the pure traveling wave ($V_- = 0$), $S = 1$, and for the pure standing wave, we have $S \to \infty$. The standing-wave ratio (SWR) cannot be defined on a lossy line because the standing-wave pattern changes from one position to another. On a lossless line, SWR stays the same throughout the line.

SWR is related to the reflection coefficient Γ as follows:

$$S = \frac{1+|\Gamma|}{1-|\Gamma|} \quad \text{or} \quad |\Gamma| = \frac{S-1}{S+1} \qquad \left(\text{Since } |\Gamma| \leq 1,\ S \geq 1 \right)$$

Since $|V_+| = |I_+| Z_0$ and $|V_-| = |I_-| Z_0$, we can express $|V_{max}|$ and $|V_{min}|$ as,

$$|V_{max}| = |I_{max}| Z_0 \quad \text{and} \quad |V_{min}| = |I_{min}| Z_0$$

When the voltage is max. and the current is min., the impedance at that point must be max and purely resistive.

$$Z_{max} = \frac{|V_{max}|}{|I_{min}|} \qquad Z_{min} = \frac{|V_{min}|}{|I_{max}|}$$

The power transmitted by a line:

$$P = |V_{max}||I_{min}| = |V_{min}||I_{max}| \qquad \text{also,}$$

$$P = \frac{|V_{max}||V_{min}|}{Z_0} = \frac{|V_{max}|^2}{Z_{max}} = \frac{|V_{min}|^2}{Z_{min}}$$

in terms of current,

$$P = |I_{max}||I_{min}| Z_0 = |I_{max}|^2 Z_{min} = |I_{min}|^2 Z_{max}$$

Since $|V_{max}| = |V_+| + |V_-|$ and $|V_{min}| = |V_+| - |V_-|$

$$P = \frac{|V_{max}||V_{min}|}{Z_0} = \frac{|V_+|^2}{Z_0} - \frac{|V_-|^2}{Z_0}$$

also in terms of current, $P = |I_+|^2 Z_0 - |I_-|^2 Z_0$

The result says that if the Z_0 is a pure resistance the power is related (or calculated) with incident and reflected waves.

Example:

For a transmission line, $Z_0 = 50 + j0{,}01\Omega$ and $Z_L = 73 - j12{,}5\ \Omega$, $S = ?$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = 0{,}377 \angle -42{,}7^0$$

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0{,}377}{1 - 0{,}377} = \underline{\underline{2{,}21}}$$