Kaynak: Oppenheim, Willsky, “Signals and Systems”

Ayrık-Zamanlı Fourier Serileri

Sürekli zamanlı işaretler için hatırlatma

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \\ a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt. \end{aligned}}$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt.$$

Benzer şekilde ayrık-zamanlı işaretler için, N temel periyod olmak üzere,

$$x[n] = x[n+N].$$

Periyodik işareti için

$$\boxed{\begin{aligned} x[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}, \\ a_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}. \end{aligned}}$$

$$a_k = a_{k+N}.$$

Örnek:

$$x[n] = \sin\omega_0 n,$$

şeklinde verilen işaretin Ayrık-Fourier Serisi katsayıları

$$\omega_0 = \frac{2\pi}{N},$$

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$$x[n] = \frac{1}{2j}e^{j(2\pi/N)n} - \frac{1}{2j}e^{-j(2\pi/N)n}.$$

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j},$$

N ile periyodik

Örnek:

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right).$$

verilsin.

$$x[n] = 1 + \frac{1}{2j}[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2}[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}] + \frac{1}{2}[e^{j(4\pi n/N+\pi/2)} + e^{-j(4\pi n/N+\pi/2)}].$$

$$x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j(2\pi/N)n} + \left(\frac{3}{2} - \frac{1}{2j}\right)e^{-j(2\pi/N)n} + \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2(2\pi/N)n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2(2\pi/N)n}.$$

yazılarak,

$$a_0 = 1,$$
$$a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j,$$
$$a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}j,$$
$$a_{-2} = -\frac{1}{2}j,$$

$$a_{-k} = a_k^*$$

N ile periyodik

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Ayrık-Zamanlı Fourier serilerinin özellikleri

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Çarpım:

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$x[n]y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} d_k = \sum_{l=\langle N \rangle} a_l b_{k-l}.$$

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Fark alma:

$$x[n] \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

$$x[n] - x[n-1] \stackrel{\mathfrak{F}S}{\longleftrightarrow} (1 - e^{-jk(2\pi/N)})a_k,$$

Parseval bağıntısı:

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2,$$

Fourier Serileri ve LZD sistemler

Hatırlatma

$$x(t) = e^{st}$$

sürekli -zamanlı girişi işareti için, sistem çıkışı

$$y(t) = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau,$$

$$s = j\omega,$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt.$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}.$$

şeklinde modellenen giriş işareti için, çıkış işareti doğrudan

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}.$$

yazılabilir.

Ayrık-zamanlı giriş işareti

$x[n] = z^n$

için, sistem çıkışı

$y[n] = H(z)z^n,$

elde edilir.

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k},$$

$z = e^{j\omega}$

alınarak

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}.$$

tanımlanır

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n},$$

şeklinde verilen giriş işareti için , çıkış

$$y[n] = \sum_{k=\langle N\rangle} a_k H(e^{j2\pi k/N})e^{jk(2\pi/N)n},$$

şeklinde doğrudan yazılabilir.

Örnek:

İmpuls cevabı

$$h[n] = \alpha^n u[n], -1 < \alpha < 1,$$

şeklinde tanımlanan sistemin girişine

$$x[n] = \cos\left(\frac{2\pi n}{N}\right).$$

işareti uygulanması durumunda sistem çıkışını yazınız.

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$$x[n] = \frac{1}{2}e^{j(2\pi/N)n} + \frac{1}{2}e^{-j(2\pi/N)n}.$$

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n.$$

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}.$$

Sistem çıkışı

$$\begin{aligned} y[n] &= \frac{1}{2}H\left(e^{j2\pi/N}\right)e^{j(2\pi/N)n} + \frac{1}{2}H\left(e^{-j2\pi/N}\right)e^{-j(2\pi/N)n} \\ &= \frac{1}{2}\left(\frac{1}{1 - \alpha e^{-j2\pi/N}}\right)e^{j(2\pi/N)n} + \frac{1}{2}\left(\frac{1}{1 - \alpha e^{j2\pi/N}}\right)e^{-j(2\pi/N)n}. \end{aligned}$$