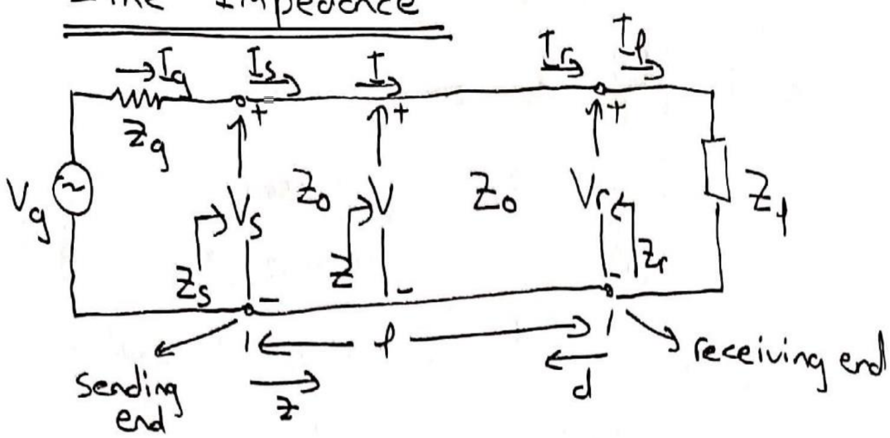
Line Impedance



The line impedance of a transmission line is the ratio of the voltage to the current at any z point:

$$Z \equiv \frac{V(z)}{I(z)}$$

where, $V = V_{inc} + V_{ref} = V_+ e^{-\gamma z} + V_- e^{\gamma z}$

and $I = I_{inc} + I_{ref} = (V_+ e^{-\gamma z} - V_- e^{\gamma z}) Y_0$

Impedance Computed from the Sending End (or input port)

At the sending end, $z=0$,

$$V_s = I_s Z_s \quad \text{and} \quad I_s = I_s$$

and $V(z=0) = V_s = V_+ + V_- = I_s Z_s$, $I(z=0)\cdot Z_0 = V_+ - V_- = I_s Z_0$

Solving these two equations for V_+ and V_-, we have

$$V_+ = \frac{I_s}{2}(Z_s + Z_0) \quad \text{and} \quad V_- = \frac{I_s}{2}(Z_s - Z_0)$$

Substituting V_+ and V_- into $V(z)$ and $I(z)$ yields,

$$V = \frac{I_s}{2}\left[(Z_s+Z_0)e^{-\gamma z} + (Z_s - Z_0)e^{\gamma z}\right]$$

and $$I = \frac{I_s}{2Z_0}\left[(Z_s+Z_0)e^{-\gamma z} - (Z_s - Z_0)e^{\gamma z}\right]$$

Then, the line impedance at a point z (from the sending end) is

$$Z = Z_0 \frac{(Z_s+Z_0)e^{-\gamma z} + (Z_s-Z_0)e^{\gamma z}}{(Z_s+Z_0)e^{-\gamma z} - (Z_s-Z_0)e^{\gamma z}}$$

For $z=l$, we express the line impedance at the receiving end Z_r in terms of Z_s and Z_0.

If $Z_s = Z_0$ (Z_s matches Z_0), Z_r will equal to Z_0 and the line impedance Z at any z point will also equal to Z_0.

We can find the line impedance at any point from the source voltage V_g, current I_g, and impedance Z_g by using these relations,

$$I_s = I_g \quad \text{and} \quad \frac{V_g}{I_g} = Z_g + Z_s, \quad Z_s = \frac{V_g}{I_g} - Z_g$$

Impedance Computed from the Receiving End (or Output port)

At the receiving end, $z = \ell$ and $V_r = I_\ell Z_\ell$. We can express the line impedance in terms of Z_ℓ and Z_0:

$$I_\ell Z_\ell = V_+ e^{-\gamma \ell} + V_- e^{\gamma \ell} \quad \text{and} \quad I_\ell Z_0 = V_+ e^{-\gamma \ell} - V_- e^{\gamma \ell}$$

We can obtain V_+ and V_- by solving the equations above:

$$V_+ = \frac{I_\ell}{2}(Z_\ell + Z_0)e^{\gamma \ell} \quad \text{and} \quad V_- = \frac{I_\ell}{2}(Z_\ell - Z_0)e^{-\gamma \ell}$$

Then, substituting these results into $V(z)$ and $I(z)$ and letting $\ell - z = d$, we have

$$V = \frac{I_\ell}{2}\left[(Z_\ell + Z_0)e^{\gamma d} + (Z_\ell - Z_0)e^{-\gamma d}\right] \quad \text{and} \quad I = \frac{I_\ell}{2Z_0}\left[(Z_\ell + Z_0)e^{\gamma d} - (Z_\ell - Z_0)e^{-\gamma d}\right]$$

Next, we find the line impedance at any point from the receiving end in terms of Z_ℓ and Z_0:

$$Z = Z_0 \frac{(Z_\ell + Z_0)e^{\gamma d} + (Z_\ell - Z_0)e^{-\gamma d}}{(Z_\ell + Z_0)e^{\gamma d} - (Z_\ell - Z_0)e^{-\gamma d}}$$

We get the line impedance at the sending end (Z_s) by setting $d = \ell$ into this equation.
If $Z_\ell = Z_0$ (Z_ℓ matches Z_0), Z_s will equal to Z_0.

Transfer Impedance

Transfer impedance, Z_{tr}, is defined as the ratio of V_s / I_r. We get Z_{tr} by using V and I definitions (which are given in terms of Z_ℓ, Z_0 and d):

$$Z_{tr} = \frac{V_s}{I_r} = \frac{1}{2}\left[(Z_\ell + Z_0)e^{\gamma \ell} + (Z_\ell - Z_0)e^{-\gamma \ell}\right]$$

This eq. is useful for finding the V_s from the known quantities of the receiving end.

Impedance in Terms of Hyperbolic or Circular Functions:

The hyperbolic functions are $e^{\pm\gamma z} = \cosh(\gamma z) \pm \sinh(\gamma z)$

Substituting these functions into the $Z(Z_s)$ eq. yields the line impedance at any point from the sending end in terms of the hyperbolic functions:

$$Z = Z_0 \frac{Z_s \cosh(\gamma z) - Z_0 \sinh(\gamma z)}{Z_0 \cosh(\gamma z) - Z_s \sinh(\gamma z)} = Z_0 \frac{Z_s - Z_0 \tanh(\gamma z)}{Z_0 - Z_s \tanh(\gamma z)}$$

Similarly, for $Z(Z_\ell)$ eq., we obtain Z from the receiving end;

$$Z = Z_0 \frac{Z_\ell \cosh(\gamma d) + Z_0 \sinh(\gamma d)}{Z_0 \cosh(\gamma d + Z_\ell \sinh(\gamma d)} = Z_0 \frac{Z_\ell + Z_0 \tanh(\gamma d)}{Z_0 + Z_\ell \tanh(\gamma d)}$$

For a lossless line, $\gamma = j\beta$, $(Z_0 = R_0)$ and using the following relationships,

$\sinh(j\beta z) = j\sin(\beta z)$ and $\cosh(j\beta z) = \cos(\beta z)$

we can express the Z impedance in terms of the circular functions,

$$Z = R_0 \frac{Z_s - jR_0 \tan(\beta z)}{R_0 - jZ_s \tan(\beta z)} \quad \text{(from the sending end)}$$

or

$$\boxed{Z = R_0 \frac{Z_\ell + jR_0 \tan(\beta d)}{R_0 + jZ_\ell \tan(\beta d)}} \quad \text{(from the receiving end)}$$

For a lossless line, the line impedance or admittance repeats at intervals of $\lambda/2$

Two special cases are,

- Short circuit: if $Z_\ell = 0$, the input impedance becomes,

$$Z_{in} = jR_0 \tan(\beta d) \quad \text{(inductor)}$$

- Open circuit: if the load is open, that is, $Z_\ell = \infty$,

$$Z_{in} = -jR_0 \cot(\beta d) \quad \text{(capacitor)}$$

For the general lossy case and for the $Z_\ell = 0$ we obtain the sending-end impedance as, $(d = \ell)$

$$Z_{sc} = Z_0 \tanh(\gamma \ell)$$

For the $Z_\ell = \infty$, we obtain, (letting $d = \ell$)

$$Z_{oc} = Z_0 \coth(\gamma \ell) = Z_0 / \tanh(\gamma \ell)$$

Now, we can calculate the characteristic impedance from

$$Z_0 = \sqrt{Z_{oc} Z_{sc}}$$

and the propagation constant from

$$\gamma \ell = \alpha \ell + j\beta \ell = \tanh^{-1} \sqrt{Z_{sc}/Z_{oc}}$$

- if we let $d = n\lambda/2$ in eq. $Z(Z_\ell)$ for lossless case, $n = 1, 2, 3, \ldots$ then

$$Z_s = Z_\ell$$

- If we let $d = \lambda/4$, then $Z_s = Z_0^2 / Z_\ell$ or $Z_s \cdot Z_\ell = Z_0^2$

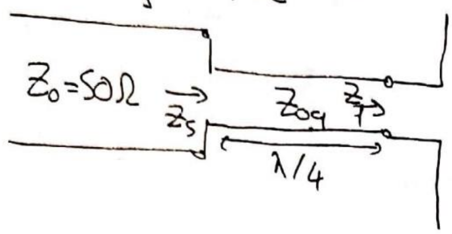
If we define normalized impedances $z_s = \frac{Z_s}{Z_0}$ and $z_\ell = \frac{Z_\ell}{Z_0}$ then the above relationship becoms $z_s = 1/z_\ell$

The line which has the length of $\lambda/4$, is referred to as quarter-wave transformer. If a given <u>resistive</u> load $Z_\ell \neq Z_0$, a quarter-wave transformer can be used to convert Z_ℓ to Z_0 of the main transmission line. To obtain this result, the characteristic impedance of the transformer will be,

$$Z_{0q} = \sqrt{Z_\ell Z_0} \ \Omega \qquad \text{(Note that } Z_{0q}, Z_\ell \text{ an } Z_0 \text{ are pure resistances)}$$

Example:

A half-wave, center-fed antenna has a driving-point impedance of $73\ \Omega$. But the transmission line connected to the antenna has $Z_0 = 50\ \Omega$. Determine the characteristic impedance of a quarter wavelength line to be used for matching the dipole antenna to the line.



For a proper match, $Z_0 = Z_s = 50\ \Omega$

then, $Z_{oq} = \sqrt{Z_p \cdot Z_s} = \sqrt{73 \cdot 50} = 60\ \Omega.$

Example:

A certain open-wire telephone line has the following parameters at a frequency of 2 kHz:

$R = 6{,}75\ \Omega/mi$, $L = 0{,}00340\ H/mi$, $G = 0{,}400\ \mu\mho/mi$, $C = 0{,}00862\ \mu F/mi$

$D = 100\ mi$ and $Z_p = 200 - j200\ \Omega$

Calculate the a) Z_0, b) γ of the line and c) line impedance Z_{line} at a distance D from the receiving load Z_p.

a) $Z = R + j\omega L = 6{,}75 + j42{,}7$, $Y = G + j\omega C = (0{,}4 + j108{,}26) \cdot 10^{-6}$

$Z_0 = \sqrt{Z/Y} = 632\angle{-4{,}4^\circ} = 630 - j48{,}3\ \Omega$

b) The propagation constant is $\gamma = \sqrt{ZY} = 0{,}0685\angle{85{,}4^\circ} = 0{,}00545 + j0{,}068$

$\alpha = 0{,}00545\ Np/mi$ and $\beta = 0{,}0680\ rad/mi$

c) $Z_{line} = Z_0 \dfrac{Z_p \cosh(\gamma d) + Z_0 \sinh(\gamma d)}{Z_0 \cosh(\gamma d) + Z_p \sinh(\gamma d)}$

$\cosh(\gamma d) = \cosh(\alpha + j\beta)d = \cosh(\alpha d)\cos(\beta d) + j\sinh(\alpha d)\sin(\beta d)$
$= 0{,}990 + j0{,}296$

$\sinh(\gamma d) = \sinh(\alpha + j\beta)d = \sinh(\alpha d)\cos(\beta d) + j\cosh(\alpha d)\sin(\beta d)$
$= 0{,}494 + j0{,}593$ $j9{,}7$ $Z_{line} = 460{,}1\angle{4{,}5} = 458{,}6 + j36{,}80\ \Omega$