Recitation W5

1) Suppose that X takes values between 0 and 1 and has probability density function $2x$.

a) Compute $Var(X)$

b) Compute $Var(X^2)$

Soln.

a) $E[X^n] = \int x^n f_X(x)\, dx \rightarrow$ n^{th} order moment.

$$Var(X) = E[(X - E[X])^2]$$
$$= E[X^2] - 2E[X]E[X] + E[X]^2$$
$$= E[X^2] - E[X]^2.$$

First we compute, n-th order moments

$$E[X] = \int_0^1 x \cdot 2x\, dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$E[X^2] = \int_0^1 x^2 \cdot 2x\, dx = \frac{2x^4}{4}\Big|_0^1 = \frac{1}{2}$$

Therefore,

$$Var(X) = E[X^2] - (E[X])^2$$
$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

b) $Var(X^2) = ?$

$Y = X^2$

$Var(Y) = E[Y^2] - (E[Y])^2 = E[X^4] - (E[X^2])^2$

$E[Y] = E[X^2] = \int_0^1 x^2 f_X(x)\,dx$

$= \int_0^1 x^2 \cdot 2x \cdot dx = \frac{1}{2}$

$E[Y^2] = E[X^4] = \int x^4 f_X(x)\,dx$

$= \int_0^1 x^4 \cdot 2x = \frac{2x^6}{6}\Big|_0^1 = \frac{1}{3}$

$Var(Y) = Var(X^2) = E[X^4] - (E[X^2])^2$

$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$

2) Let X and Y be independent random variables.

Random variable X has mean μ_X and variance σ_X^2

and, random variable Y has mean μ_Y and variance σ_Y^2.

Let $Z = 2X - 3Y$. Find mean and variance of Z

in terms of the means and variances of X and Y.

Soln.

$$E[aX + bY + c] = a\,E[X] + b\,E[Y] + c$$

$$E[Z] = E[2X - 3Y]$$
$$= 2E[X] - 3E[Y]$$

$$E[Z] = \mu_Z = \underline{2\mu_X - 3\mu_Y}.$$

$\mathrm{Cov}(K, L) = E[(K - E[K])(L - E[L])]$

Covariance of rvs K and L.

Therefore,

$$\mathrm{Cov}(Z, Z) = E[(Z - \mu_Z)(Z - \mu_Z)]$$
$$= \mathrm{Var}(Z)$$

$$\mathrm{Cov}(Z, Z) = \mathrm{Cov}(2X - 3Y, 2X - 3Y)$$

1 2 3 4

$$= 4\mathrm{Cov}(X, X) + 9\,\mathrm{Cov}(Y, Y) - 12\,\mathrm{Cov}(X, Y)$$

$$= 4\,\mathrm{Var}(X) + 9\,\mathrm{Var}(Y)$$

$$\sigma_Z^2 = \underline{4\sigma_X^2 + 9\sigma_Y^2}.$$

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since X and Y are independent $\mathrm{Cov}(X, Y) = 0$

3) Probability mass function of discrete rv X is given in the following table.

x	-2	-1	0	1	2
p(x)	1/15	2/15	3/15	4/15	5/15

Compute $E[X]$ and $Var(X)$.

soln.

for discrete rvs expected value $E[X]$

$$E[X] = \sum_{j=1}^{n} p(x_j)\, x_j$$

$$= \sum_{j=1}^{5} p(x_j)\, x_j$$

$$= -2 \cdot \frac{1}{15} - 1 \cdot \frac{2}{15} + 0 \cdot \frac{1}{15} + 1 \cdot \frac{4}{15} + 2 \cdot \frac{5}{15}$$

$$= \frac{-2}{15} - \frac{2}{15} + \frac{4}{15} + \frac{10}{15} = \frac{2}{3}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$E[X^2] = \sum_{j=1}^{n} p(x_j)\, x_j^2$$

$$= 4 \cdot \frac{1}{15} + 1 \cdot \frac{2}{15} + 0 + \frac{4}{15} + \frac{20}{15} = \frac{30}{15} = 2.$$

$$Var(X) = 2 - \left(\frac{2}{3}\right)^2 = \frac{14}{9}$$

2nd way

for discrete rvs.

$$Var(X) = \sum_{j=1}^{5} (x_j - \mu_X)^2 \, p(x_j)$$

$$= \left(-2-\frac{2}{3}\right)^2 \frac{1}{15} + \left(-1-\frac{2}{3}\right)^2 \frac{2}{15} + \left(0-\frac{2}{3}\right)^2 \frac{3}{15} + \left(1-\frac{2}{3}\right)^2 \frac{4}{15} + \left(2-\frac{2}{3}\right)^2 \frac{5}{15}$$

$$= \frac{14}{9}.$$

4) The pdf of X is given by $f_X(x) = \frac{1}{2} e^{-|x-1|}$, $x \in \mathbb{R}$

a) Find the MGF of X.

b) Use the MGF to find $E[X]$ and $Var(X)$.

soln

$$f_X(x) = \begin{cases} \frac{1}{2} e^{x-1} & \text{if } x \leq 1 \\ \frac{1}{2} e^{1-x} & \text{if } x > 1 \end{cases}$$

$$m_X(t) = E\left[e^{tX}\right]$$

$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

$$= \int_{-\infty}^{1} e^{tx} \frac{1}{2} e^{x-1}\, dx + \int_{1}^{\infty} e^{tx} \frac{1}{2} e^{1-x}\, dx$$

$$= \frac{1}{2}\left[\left.\frac{e^{x(t+1)-1}}{t+1}\right|_{-\infty}^{1} + \left.\frac{e^{x(t-1)-1}}{t-1}\right|_{1}^{\infty} \right] = \frac{e^t}{1-t^2}$$

$E[X^n]$: n^{th} order moment

$$E[X^n] = \frac{d^n}{dt^n} m_X(t) \Big|_{t=0}$$

Therefore,

$$E[X] = \frac{d}{dt} m_X(t) \Big|_{t=0} = \frac{d}{dt}\left(\frac{e^t}{1-t^2}\right) \Big|_{t=0}$$

$$= \frac{e^t(1-t^2) + 2t e^t}{(1-t^2)^2} \Big|_{t=0} = 1$$

$$E[X^2] = \frac{d^2}{dt^2} m_X(t) \Big|_{t=0} = 3$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= 3 - 1^2 = 2$$