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1 Introduction

Hopf algebras are a class of mathematical objects possessing a remarkably rich structure. Fundamentally, they are bialgebras which possesses certain properties analogous to the role of group inverses. In this paper we develop the basic machinery required to construct Hopf algebras, and in doing so we prove many of their fundamental properties. We begin with some definitions of algebras and coalgebras which are particularly conducive to the discussion of bialgebras, making use of commutative diagrams throughout.

In the third chapter we introduce the notion of a Lie algebra and its universal enveloping algebra, which we make use of in later sections. In chapter 4 we introduce the convolution product and its associated $\mathbb{K}$-algebra, and progress to a discussion of Hopf algebras in which we prove many of their fundamental properties for both finite-dimensional and infinite-dimensional cases. We provide several examples of Hopf algebras, including the classic construction introduced in (Sweedler, 1969). The final chapter makes use of the universal enveloping algebra constructed in earlier sections, to discuss a particular class of Hopf algebras sometimes called quantum groups. In the context of this paper, this refers to certain deformations of universal enveloping algebras which are infinite-dimensional, noncommutative and noncocommutative.

2 Algebras to Bialgebras (via Coalgebras)

2.1 $\mathbb{K}$-Algebras

Definition 2.1.1 (**K-Algebra**). A *unital* $\mathbb{K}$-*algebra* is a triple (A, μ, u) - a vector space A over a field $\mathbb{K}$ and two $\mathbb{K}$-linear maps,

$$\mu : A \otimes A \to A, \quad a \otimes b \mapsto ab, \qquad (\textit{product map})$$
$$u : \mathbb{K} \to A, \quad \lambda \mapsto \lambda 1_A. \qquad (\textit{unit map})$$

By convention we refer to the triple (A, μ, u) as A where there is no risk of confusion. If the following diagrams commute then A is a *unital associative algebra*:

$$\begin{tikzcd}
& A \otimes A \otimes A \arrow[dl, "\mu \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes \mu"] & \\
A \otimes A \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

(a) Associativity

$$\begin{tikzcd}
& A \otimes A \arrow[dd, "\mu"] & \\
K \otimes A \arrow[ur, "u \otimes \mathrm{id}"] \arrow[dr, "s_1"'] & & A \otimes K \arrow[ul, "\mathrm{id} \otimes u"'] \arrow[dl, "s_2"] \\
& A &
\end{tikzcd}$$

(b) Unitality

Figure 1: $\mathbb{K}$-Algebra structure maps.

The maps s_1 and s_2 are defined as $s_1 : \lambda \otimes a \mapsto \lambda a$ and $s_2 : a \otimes \lambda \mapsto \lambda a$, for $a \in A$ and $\lambda \in \mathbb{K}$. These encode scalar multiplication, and are in effect the canonical isomorphisms $\mathbb{K} \otimes A \simeq A \simeq A \otimes \mathbb{K}$.

Alternatively, we may express associativity and unitality via the following axioms:

$$\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu) \qquad (\textit{associativity})$$

$$\mu \circ (u \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes u). \qquad (\textit{unitality})$$

Proposition 2.1.2. The unit map u is uniquely determined for a given vector space A and product μ.[1]

Proof. Consider $u,\ u' : K \to A$ such that

$$\mu \circ (u \otimes \mathrm{id}_A)(\lambda \otimes a) = \lambda a = \mu \circ (\mathrm{id}_A \otimes u)(a \otimes \lambda),$$
$$\mu \circ (u' \otimes \mathrm{id}_A)(\lambda \otimes a) = \lambda a = \mu \circ (\mathrm{id}_A \otimes u')(a \otimes \lambda)$$

for $a \in A,\ \lambda \in \mathbb{K}$. Equating the LHS of each line we have

$$\mu(u(\lambda) \otimes a) = \lambda a = \mu(u'(\lambda) \otimes a),\ \text{i.e.}$$

$$a \otimes u'(\lambda) = a \otimes u(\lambda).$$

The tensor product is unique up to isomorphism, and our choice of a, λ was arbitrary. Hence, $u(\lambda) = u'(\lambda)\ \forall\ \lambda \in \mathbb{K}$ and so $u = u'$. ☐

Proposition 2.1.3. The identity element 1_A is given by $u(1_{\mathbb{K}})$.

Proof. Beginning with the LHS of the unitality condition, we have

$$\begin{aligned} \mu \circ (u \otimes \mathrm{id}_A)(\lambda \otimes a) &= \mu(u \otimes \mathrm{id}_A(\lambda(1_{\mathbb{K}} \otimes a)) \\ &= \mu(\lambda(u(1_{\mathbb{K}}) \otimes a)) = \lambda u(1_{\mathbb{K}})a. && \textit{(by linearity of } u,\ \mu) \end{aligned}$$

Similarly, the RHS of the unitality condition (using the map $\mathrm{id}_A \otimes u$) must also yield $\lambda u(1_{\mathbb{K}})a$. Hence, $u(\mathbb{K})$ is central in A (since our choice of a was arbitrary). Finally, setting $\lambda = 1_{\mathbb{K}}$ gives $au(1_{\mathbb{K}}) = u(1_{\mathbb{K}})a = a$, hence $u(1_{\mathbb{K}})$ is the identity element 1_A in A. ☐

Examples 2.1.4 ($\mathbb{K}$-*algebras*)**.**

- The simplest example of a $\mathbb{K}$-algebra is the ground field $\mathbb{K}$ considered as a 1-dimensional vector space over itself (and for which any non-zero element is a basis). We identify our product map with scalar multiplication in $\mathbb{K}$, and so naturally $1_A := 1_{\mathbb{K}}$, i.e. $u := \mathrm{id}_{\mathbb{K}}$.
- For any given $\mathbb{K}$-vector space V, the vector space $\mathrm{End}_{\mathbb{K}}(V)$ of $\mathbb{K}$-linear endomorphisms $f : V \to V$ is a $\mathbb{K}$-algebra, where the product map is given by function composition $\circ$ and the unit element is the identity map $\mathrm{id} :\ V \to V$.

Remarks 2.1.5. Given a $\mathbb{K}$-algebra A we may define the *opposite algebra*, A^{copp}. This is the $\mathbb{K}$-algebra $(A, \mu^{\mathrm{opp}}, u)$ where $\mu^{\mathrm{opp}} = \mu \circ \tau_{A,A}$ and $\tau_{A,A}$ is the *twist map*:

$$\tau_{A,A} : A \otimes A \to A \otimes A, \qquad \tau_{A,A} : a \otimes b \mapsto b \otimes a.$$

Definition 2.1.6. An algebra A is *commutative* if $\mu = \mu^{opp}$.

Definition 2.1.7 (Graded $\mathbb{K}$-Algebra). A *graded* $\mathbb{K}$-algebra is an associative $\mathbb{K}$-algebra such that there are $\mathbb{K}$-subspaces A^r for $r \geq 0$, where

1. $A = \bigoplus_{r \geq 0} A^r$, i.e. $A \oplus A \oplus \ldots \oplus A$ (r times)
2. $\forall\ p,\ q \geq 0$, if $x \in A^p$ and $y \in A^q$ then $xy \in A^{p+q}$, i.e. $A^p A^q \subseteq A^{p+q}$.

(Rotman, 2010, p.704)

[1] Proofs of propositions 2.1.2 and 2.1.3 adapted from exercises in Brown, Ken A. (2014), Chapter 1.

We say that elements $x, y \in A^r$ are *homogeneous of grade* (or *degree*) r.

Examples 2.1.8 (Tensor algebra). For a $\mathbb{K}$-vector space V, define $T^{(r)}(V) := V \otimes V \otimes \ldots \otimes V$ (r times). Hence $V = T^{(1)}(V)$, and by convention we identify $T^{(0)}(V)$ with the ground field $\mathbb{K}$.

The *tensor algebra* $T(V)$ is the unital associative $\mathbb{K}$-algebra:

$$T(V) = \bigoplus_{r \geq 0} V^{\otimes r}$$

where $\bigoplus$ is the direct sum and the product is the concatenation operation $\bigotimes$:

$$(v_1 \otimes v_2 \otimes \ldots \otimes v_p) \bigotimes (w_1 \otimes \ldots \otimes w_q) := (v_1 \otimes v_2 \otimes \ldots \otimes v_p \otimes w_1 \otimes \ldots \otimes w_q) \in T^{(p+q)}(V).$$

$T(V)$ is $\mathbb{Z}_+$-graded, and the homogeneous elements of subspace $T^{(r)}(V)$ are tensors of rank r (r-tensors). That is to say, $T(V) = \bigoplus_{r=0}^{\infty} T^{(r)}(V)$, and a general element of $T(V)$ is a finite linear combination of tensors of varying rank. The unit element is $1_{\mathbb{K}}$ (again, using the canonical isomorphism $1_{\mathbb{K}} \otimes_{\mathbb{K}} A \cong A \cong A \otimes_{\mathbb{K}} 1_{\mathbb{K}}$).
Note that $T(V)$ is infinite-dimensional regardless of provided $\neq \{0\}$, and in particular

$$\dim T^{(r)}(V) = \dim V^{\otimes r} = (\dim V)^r.$$

If V is of finite dimension $n > 1$, we can identify $T(V)$ with the algebra of polynomials over $\mathbb{K}$ in n *non-commuting* indeterminates[2], also known as the *free algebra on* V(Kassel, 1995, p.34).

Examples 2.1.9. Recall that a two-sided ideal in an algebra A is a linear subspace $I \subseteq A$ such that $\mu(a \otimes x) \in I$ and $\mu(x \otimes a) \in I$ for any $x \in I,\ a \in A$. Quotienting $T(V)$ by the two-sided ideal generated by $\{x \otimes y - y \otimes x | x, y \in V\}$, we obtain the *symmetric algebra, Sym*(V), whose equivalence classes force commutativity in our product operation; that is, we obtain the commutative unital $\mathbb{K}$-algebra of polynomials over $\mathbb{K}$ in n indeterminates (Kassel, 1995).

Definition 2.1.10 (Algebra morphisms). Given $\mathbb{K}$-algebras (A, μ_A, u_B) and (B, μ_B, u_B), a $\mathbb{K}$-linear map $\theta : A \to B$ is a $\mathbb{K}$-*algebra morphism* if

$$\theta \circ \mu_A = \mu_B \circ (\theta \otimes \theta) \qquad \text{and} \qquad \theta \circ u_A = u_B,$$

which is to say that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \arrow[r, "\theta \otimes \theta"] \arrow[d, "\mu_A"'] & B \otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, "\theta"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\theta"] & & B \\ & K \arrow[ul, "u_A"] \arrow[ur, "u_B"'] & \end{tikzcd}$$

Figure 2: Algebra Morphism.

The second condition entails that θ preserves units, that is $\theta(1_A) = 1_B$. If an injective algebra morphism $\iota : A \to B$ exists then A is a *subalgebra* of B (Kassel, 1995, p.3). Note that any ideal of a $\mathbb{K}$-algebra is a subalgebra.

Remarks 2.1.11. Let I be a two-sided ideal of a $\mathbb{K}$-algebra A. Then the quotient space A/I inherits a $\mathbb{K}$-algebra structure:

$$\mu_{A/I} : A/I \otimes A/I \to A/I, \quad (a + I) \otimes (b + I) \mapsto \mu_A(a \otimes b) + I,$$
$$u_{A/I} : K \to A/I, \quad \lambda \mapsto u_A(\lambda) + I.$$

This is to say that the canonical projection $\pi : A \to A/I$ mapping elements of A onto their respective cosets is a morphism of algebras.

[2] The case where $\dim(V) = 1$ is of course commutative, there being only one indeterminate.

2.2 $\mathbb{K}$-Coalgebras

Definition 2.2.1. A *counital* $\mathbb{K}$-*coalgebra* is a triple (C, Δ, ϵ) - a $\mathbb{K}$-vector space C and two $\mathbb{K}$-linear maps,

$$\Delta : C \to C \otimes C \qquad (\textit{coproduct map})$$
$$\epsilon : C \to K. \qquad (\textit{counit map})$$

The properties of *coassociativity* and *counitality* are encoded in what is essentially a reversal of the directions of the arrows in Definition 2.1.1. We require that the following diagrams commute:

$$\begin{tikzcd}
& C \otimes C \otimes C & \\
C \otimes C \arrow[ur, "\mathrm{id} \otimes \Delta"] & & C \otimes C \arrow[ul, "\Delta \otimes \mathrm{id}"'] \\
& C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

(a) Coassociativity

$$\begin{tikzcd}
& C \otimes C \arrow[dl, "\epsilon \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes \epsilon"] & \\
K \otimes C & & C \otimes K \\
& C \arrow[ul, "1_{\mathbb{K}} \otimes _"] \arrow[ur, "_ \otimes 1_{\mathbb{K}}"'] \arrow[uu, "\Delta"] &
\end{tikzcd}$$

(b) Counitality

Figure 3: Coalgebra structure maps

where $1_{\mathbb{K}} \otimes \_ : c \mapsto 1_{\mathbb{K}} \otimes c$ expresses the canonical isomorphism $\mathbb{K} \otimes C \simeq C$, and similar for $\_ \otimes 1_{\mathbb{K}}$. If we choose to identify these vector spaces we may encode counitality as the requirement that $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$.

Remarks 2.2.2. For readability's sake we will adopt what is known as *Sweedler notation*, as introduced in (Sweedler, 1969).

First note that for finite-dimensional $\mathbb{K}$-vector spaces V and W, with bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ respectively, the general form of an element of $V \otimes_{\mathbb{K}} W$ is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} (v_i \otimes w_j)$$

where $\lambda_{ij} \in \mathbb{K}$. Given a coalgebra (C, Δ, ϵ) and $c \in C$ we may write $\Delta(c)$ as

$$\Delta(c) = \sum_{i \in I} c_i^{(1)} \otimes c_i^{(2)} \in C \otimes C$$

where I is some finite indexing set. Note that $\Delta(c)$ must be a finite linear sum of pure tensors given that $\mathbb{K} \otimes C \simeq C \simeq C \otimes \mathbb{K}$. Hence, in Sweedler notation we may remove the subscripts and generalize to summation over c:

$$\Delta(c) = \sum_{c} c_1 \otimes c_2.$$

We may omit the index c and even the summation sign if the context allows it. Using Sweedler notation, coassociativity can be succinctly expressed in terms of elements:

$$
\begin{array}{rrcl}
 & (\mathrm{id}_C \otimes \Delta) \circ \Delta(c) & = & (\Delta \otimes \mathrm{id}_C) \circ \Delta(c) \\
\iff & (\mathrm{id}_C \otimes \Delta) \, (\sum_c c_1 \otimes c_2) & = & (\Delta \otimes \mathrm{id}_C) \, (\sum_c c_1 \otimes c_2) \\
\iff & \sum_c (c_1 \otimes \Delta(c_2)) & = & \sum_c (\Delta(c_1) \otimes (c_2)) \\
\iff & \sum_c \Big(c_1 \otimes \sum_{c_2} (c_{2(1)} \otimes c_{2(2)}) \Big) & = & \sum_c \Big(\sum_{c_1} (c_{1(1)} \otimes c_{1(2)}) \otimes c_2 \Big) \\
 & & := & c_1 \otimes c_2 \otimes c_3,
\end{array}
$$

while the counit axiom asserts that

$$
\begin{array}{rc}
 & (\epsilon \otimes \mathrm{id}_C) \circ \Delta(c) = (\mathrm{id}_C \otimes \epsilon) \circ \Delta(c) = c, \\
\iff & \epsilon(c_1) \otimes c_2 = c_1 \otimes \epsilon(c_2) = c
\end{array}
$$

where the summation symbols in the second line are implicit.

Proposition 2.2.3. The counit map ϵ as defined above is unique in C.

Proof. If ϵ, ϵ' are maps fulfilling the counit axiom then

$$
\begin{aligned}
\epsilon(c) &= \epsilon\big(\epsilon'(c_1) \otimes c_2)\big) \\
&= \epsilon'\big(c_1 \otimes \epsilon(c_2)\big) \\
&= \epsilon'(c)
\end{aligned}
$$

□

Remarks 2.2.4. A coalgebra (C, Δ, ϵ) is *cocommutative* if $\Delta = \tau_{C,C} \circ \Delta$. This is to say that as $\mathbb{K}$-coalgebras, $C = C^{\mathrm{cop}} = (C, \tau_{C,C} \circ \Delta, \epsilon)$. C^{cop} is called the *opposite coalgebra* of C.

(Brown, Ken A., 2014, p.7)

Examples 2.2.5 ($\mathbb{K}$-*coalgebras*)**.**

- The simplest example of a $\mathbb{K}$-coalgebra is the ground field $\mathbb{K}$ taken as a 1-dimensional vector space over itself, as in Example 2.1.4. The coproduct map is defined as $\Delta : \mathbb{K} \to \mathbb{K} \otimes \mathbb{K}$, $\Delta : \lambda \mapsto \lambda \otimes 1_{\mathbb{K}}$, and the counit map is given by $\epsilon : \mathbb{K} \to \mathbb{K}$, $\epsilon : \lambda \mapsto \lambda$. This is the *trivial coalgebra.*
- Denote by $\mathbb{K}[X]$ the infinite-dimensional vector space of polynomials in a single indeterminate X with coefficients in a field $\mathbb{K}$. Define $\Delta : K[X] \to K[X] \otimes K[X]$ on the basis $\{1, X, X^2, \ldots\}$ of $\mathbb{K}[x]$ by:

  $$\Delta(X^n) = \sum_{k=0}^{n} \binom{n}{k} X^k \otimes X^{n-k},$$

  with a counit given by

  $$\epsilon(X^n) = \delta_{0,n}$$

  i.e. $\epsilon(X^n) = 1$ for $n = 0$ and 0 otherwise. This is the *divided power coalgebra,* $(\mathbb{K}[X], \Delta, \epsilon)$ (Underwood, 2015, p.16). Note that since Δ and ϵ are linear maps, it was sufficient to define the structure maps over a basis of $\mathbb{K}[X]$.

Definition 2.2.6. Let I be a linear subspace of a coalgebra C:

- If $\Delta I \subseteq I \otimes I$ then I is a *subcoalgebra*;
- If $\Delta I \subseteq C \otimes I$ then I is a *left coideal*;
- If $\Delta I \subseteq I \otimes C$ then I is a *right coideal*;
- I is a *two-sided coideal* (or simply a *coideal*) if

$$\Delta I \subseteq I \otimes C + C \otimes I \qquad \text{and} \qquad \epsilon(I) = 0.$$

(Radford, 2012)

Definition 2.2.7 (Coalgebra morphisms). Given $\mathbb{K}$-coalgebras $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$, a $\mathbb{K}$-linear map $\psi : C \to D$ is a $\mathbb{K}$-*coalgebra morphism* if

$$(\psi \otimes \psi) \circ \Delta_C = \Delta_D \circ \psi, \qquad \text{and} \qquad \epsilon_C = \epsilon_D \circ \psi,$$

which is to say that the following diagrams commute:

$$\begin{tikzcd} C \otimes C \arrow[r, "\psi \otimes \psi"] & D \otimes D \\ C \arrow[u, "\Delta_C"] \arrow[r, "\psi"] & D \arrow[u, "\Delta_D"'] \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\psi"] \arrow[dr, "\epsilon_C"'] & & D \arrow[dl, "\epsilon_D"] \\ & K & \end{tikzcd}$$

Figure 4: Coalgebra Morphism.

Remarks 2.2.8.

- Let $I \subseteq C$ be a two-sided coideal of a $\mathbb{K}$-coalgebra C. Then the quotient space C/I is a $\mathbb{K}$-coalgebra, inheriting a structure given by

$$\Delta_{C/I} : C/I \to C/I \otimes C/I, \quad c + I \mapsto \Delta_C(c) + I = \sum_{(c)} (c_1 + I) \otimes (c_2 + I)$$
$$\epsilon_{C/I} : C/I \to K, \quad c + I \mapsto \epsilon_C(c).$$

This is to say that the canonical projection map $\pi : C \to C/I$ is a morphism of coalgebras. For a proof that C/I is indeed a $\mathbb{K}$-coalgebra (as well the analogous case for $\mathbb{K}$-algebras), making use of the the universal mapping property of the kernel, see (Underwood, 2015).

- If $\psi : C \to D$ is a morphism of coalgebras then $\ker(\psi)$ is a two-sided coideal of C. To show this, we first note that by definition $c \in \ker(\psi)$ then $\psi(c) = 0$. Therefore,

$$0 = \Delta_D(\psi(c)) = (\psi \otimes \psi)(\Delta_C(c))$$

in accordance with the first morphism diagram and linearity of Δ_D. Hence we have

$$\begin{aligned} \Delta_C(c) &\in \ker(\psi \otimes \psi) \\ &= \ker(\psi) \otimes C + C \otimes \ker(\psi). \end{aligned}$$

By $\mathbb{K}$-linearity of the counit the further requirement that $\epsilon_D(\ker(\psi)) = \epsilon_D(0) = 0$ is met.

From these remarks one might surmise the following.

Theorem 2.2.1 (Isomorphism Theorem for Coalgebras)**.** *For a morphism $\psi : C \to D$ of coalgebras, there exists a canonical isomorphism $C/\ker(\psi) \cong \mathrm{Im}(\psi)$ where $\cong$ is an isomorphism of coalgebras* (Dascalescu et al., 2001).

Proof. By the first isomorphism theorem for vector spaces, there exists a linear map $\overline{\psi} : C/\ker(\psi) \to D$. We need only check that this injective map is a morphism of coalgebras as follows:

$$\epsilon_D(\overline{\psi}(c + \ker(\psi)) = \epsilon_D(\psi(c)) = \epsilon_C(c),$$

$$\begin{aligned}
\Delta_D(\overline{\psi}(c + \ker(\psi)) &= \Delta_D(\psi(c)) \\
&= (\psi \otimes \psi)(\Delta_C(c)) \\
&= \sum_{(c)} \psi(c_1) \otimes \psi(c_2) \\
&= \sum_{(c)} \overline{\psi}(c_1 + \ker(\psi)) \otimes \overline{\psi}(c_2 \otimes \ker(\psi))
\end{aligned}$$

for any $c \in C$. □

2.3 Bialgebras

Definition 2.3.1. Let H be a $\mathbb{K}$-vector space. The 5-tuple $(H, \mu, u, \Delta, \epsilon)$ is a $\mathbb{K}$-*bialgebra* if (H, μ, u) is a unital associative $\mathbb{K}$-algebra, (H, Δ, ϵ) is a coassociative and counital $\mathbb{K}$-coalgebra, and Δ and ϵ are morphisms of algebras.

Proposition 2.3.2. The following statements are equivalent:

(i) Δ and ϵ are algebra morphisms.

(ii) μ and u are coalgebra morphisms.

To prove this equivalence we make use of the following definition.

Definition 2.3.3. Given $\mathbb{K}$-algebras A and B, there exists a *tensor product* $\mathbb{K}$-*algebra* on $A \otimes B$, defined by the multiplication map

$$\mu_{A\otimes B} := (\mu_A \otimes \mu_B) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B)$$

and unit map

$$u_{A\otimes B} := u_A \otimes u_B.$$

More perspicuously, we have

$$\mu_{A\otimes B}(a_1 \otimes b_1) \otimes (a_2 \otimes b_2) = \mu_A(a_1 \otimes a_2) \otimes \mu_B(b_1 \otimes b_2).$$

We can similarly define a tensor product coalgebra $C \otimes D$ of coalgebras C, D, by the maps

$$\Delta_{C\otimes D} := (\mathrm{id}_C \otimes \tau \otimes \mathrm{id}_D) \circ (\Delta_C \otimes \Delta_D),$$
$$\epsilon_{C\otimes D} := \epsilon_C \otimes \epsilon_D.$$

Proof of Proposition 2.3.2. Our proof follows that of (Dascalescu et al., 2001, p.147).

First note that μ is a coalgebra morphism if and only if the following diagrams commute:

$$\begin{tikzcd}
H \otimes H \otimes H \otimes H \arrow[r, "\mu_H \otimes \mu_H"] & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[u, "\mathrm{id}_H \otimes \tau \otimes \mathrm{id}_H"] & \\
H \otimes H \arrow[u, "\Delta_H \otimes \Delta_H"] \arrow[r, "\mu_H"] & H \arrow[uu, "\Delta_H"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes H \arrow[r, "\mu_H"] \arrow[d, "\epsilon_H \otimes \epsilon_H"'] & H \arrow[dd, "\epsilon_H"] \\
\mathbb{K} \otimes \mathbb{K} \arrow[d, "\phi"'] & \\
\mathbb{K} \arrow[r, "\mathrm{id}"] & \mathbb{K}
\end{tikzcd}$$

Figure 5: μ as coalgebra morphism

ϕ is the canonical isomorphism $\phi : \mathbb{K} \otimes \mathbb{K} \to \mathbb{K},\ 1 \otimes 1 \mapsto 1 \cdot 1$, and we have made use of the tensor product coalgebra $H \otimes H$. Note that ϕ possesses an inverse ϕ^{-1} by virtue of the fact that $\mathbb{K}$ is our ground field, i.e. $a \otimes b = 1 \otimes ab = ab \otimes 1$.

Similarly, u is a coalgebra morphism if and only if the following commute:

$$\begin{tikzcd}
\mathbb{K} \otimes \mathbb{K} \arrow[r, "u \otimes u"] & H \otimes H \\
\mathbb{K} \arrow[u, "\phi^{-1}"] \arrow[r, "u"] & H \arrow[u, "\Delta_H"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} \arrow[rr, "u"] \arrow[dr, "\mathrm{id}"'] & & H \arrow[dl, "\epsilon_H"] \\
& \mathbb{K} &
\end{tikzcd}$$

Figure 6: ϵ as coalgebra morphism

The first and third diagrams are identical to those which would determine that Δ is a morphism of $\mathbb{K}$-algebras (see definition 2.1.10). Similarly, ϵ is an algebra morphism according to the conditions expressed by the second and fourth diagrams. Therefore, the equivalence of (i) and (ii) is evident. $\square$

Examples 2.3.4 (*Bialgebras*).

- Let $\mathbb{K}$ be a field and let G be any group. The *group algebra* $\mathbb{K}G$ is the $\mathbb{K}$-algebra over the vector space whose basis comprises the elements of G. The product μ in $\mathbb{K}G$ is identified with the group operation $\cdot$ in G, extended linearly (Jahn, Astrid, 2015, p.14). That is to say,

$$\mathbb{K}G = \{\sum_{i=1}^{n} \lambda_i g_i : \lambda_i \in \mathbb{K}, g_i \in G, n = |G|\},$$

$$\mu(\sum_{g \in G} \lambda_g g \otimes \sum_{h \in G} \nu_h h) = \sum_{(g,h) \in G \times G} \lambda_g \nu_h \mu(g \otimes h) = \sum_{(g,h) \in G \times G} \lambda_g \nu_h g \cdot h.$$

  The unit element $1_{\mathbb{K}G}$ is naturally identified with 1_G. We may further define a coalgebra structure on $\mathbb{K}G$ by:

$$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1_{\mathbb{K}}$$

  for all $g \in G$ (Majid, 1995, p.13).

  The quintuple $(\mathbb{K}G, \mu, u, \Delta, \epsilon)$ is a bialgebra.

  Proof.

  (i) Clearly $(\mathbb{K}G, \mu, u)$ is an algebra; the $\mathbb{K}$-algebra axioms are fulfilled by the group axioms. Associativity of μ is inherited from the associativity of the group operation, and unitality holds because $u(1_{\mathbb{K}}) = 1_G$ and $\forall g \in G,\ 1_G \cdot g = g \cdot 1_G = g$.

(ii) To show that (KG, Δ, ϵ) is a coalgebra, following (Laugwitz, 2010, p.8) we note that for $g \in G$

$$\begin{aligned}(\mathrm{id}_{\mathbb{K}G} \otimes \Delta) \circ \Delta(g) = (\mathrm{id}_{\mathbb{K}G} \otimes \Delta)(g \otimes g) = g \otimes (g \otimes g) \\ = (g \otimes g) \otimes g = (\Delta \otimes \mathrm{id}_{KG}) \circ \Delta(g)\end{aligned}$$

where we have made use of the associativity of $\otimes$. Note that because G is a basis for $\mathbb{K}G$, we only needed to consider $g \in G$.

(iii) For counitality, note

$$(\epsilon \otimes \mathrm{id}_{\mathbb{K}G}) \circ \Delta(g) = (\epsilon \otimes \mathrm{id}_{KG})(g \otimes g) = \epsilon(g) \otimes g = 1_{\mathbb{K}} \otimes g = g$$

where we have again made use of the canonical isomorphism $\mathbb{K} \otimes C \simeq C$ as implicit scalar multiplication. It is clear that $(\mathrm{id}_{\mathbb{K}G} \otimes \epsilon) \circ \Delta(g)$ is also equal to g, as is required.

(iv) Δ is an algebra (homo)morphism because

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = (\Delta g)(\Delta h)$$

where we have made use of the tensor product multiplication $\mu_{\mathbb{K}G \otimes \mathbb{K}G}$.

(v) Finally, ϵ is an algebra (homo)morphism because

$$\epsilon(gh) = 1_{\mathbb{K}} = 1_{\mathbb{K}} \cdot 1_{\mathbb{K}} = \epsilon(g) \cdot \epsilon(h).$$

Therefore the group algebra $\mathbb{K}G$ is a bialgebra. □

Note that group-multiplicative inverses were not required at any stage in the above proof. Thus, monoid algebras are also bialgebras.

- The tensor algebra $T(V)$ as defined in Example 2.1.8 possesses a bialgebra structure when equipped with coproduct $\Delta(v) = 1 \otimes v + v \otimes 1$ and counit $\epsilon(v) = 0 \;\; \forall\, v \in V$, $\epsilon(1_{\mathbb{K}}) = 1_{\mathbb{K}}$ (recall that $T^{(0)}(V)$ is identified with $\mathbb{K}$). It is a cocommutative structure, and $\forall\, v_1, \ldots, v_n \in V$:

$$\epsilon(v_1 \otimes \ldots \otimes v_n) = \epsilon(v_1) \otimes \ldots \otimes \epsilon(v_n) = 0 \otimes \ldots \otimes 0 = 0,$$

$$\Delta(v_1 \otimes \ldots \otimes v_n) = 1 \otimes v_1 \otimes \ldots \otimes v_n + \sum_{p=1}^{n-1} \sum_{\sigma} \big(v_{\sigma(1)} \otimes \ldots \otimes s v_{\sigma(p)}\big) \otimes \big(v_{\sigma(p+1)} \otimes \ldots \otimes v_{\sigma(n)}\big) + v_1 \otimes \ldots \otimes v_n \otimes 1.$$

where σ ranges over all members of S_n such that

$$\sigma(1) < \sigma(2) < \ldots < \sigma(p) \qquad \text{and} \qquad \sigma(p+1) < \sigma(p+2) < \ldots < \sigma(n).$$

Such a permutation σ is called a *$(p, n-p)$-shuffle* (named for the interleaving 'riffle shuffle' of a split deck of n cards). Standard proofs of counitality, coassociativity and cocommutativity can be found in (Kassel, 1995, p.48); we omit them here for reasons of space.

Definition 2.3.5. For bialgebras H_1, H_2 over a common field $\mathbb{K}$, a linear map $f : H_1 \to H_2$ is a *morphism of bialgebras* if it is simultaneously a $\mathbb{K}$-algebra morphism and a $\mathbb{K}$-coalgebra morphism.

Remarks 2.3.6.

- A *sub-bialgebra* of a $\mathbb{K}$-bialgebra H is a linear subspace $L \subseteq H$ which is simultaneously a subalgebra of H and a subcoalgebra of H.
- A linear subspace of a bialgebra H is a *biideal* if it is both an ideal in H and a coideal in H. Note that any biideal is necessarily a sub-bialgebra.
- As in the case of $\mathbb{K}$-algebras and $\mathbb{K}$-coalgebras, if I is a biideal in H then H/I inherits a $\mathbb{K}$-bialgebra structure from H via the canonical surjection. If H is cocommutative (resp. commutative) then H/I is cocommutative (resp. commutative).

3 Interlude - Lie Algebras

3.1 Lie Algebras

All of the algebras we have considered up to this point are associative. We now consider a particular class of non-associative algebras.

Definition 3.1.1. Let $\mathbb{K}$ be a field. A *Lie algebra* over $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{g}$ equipped with a bilinear map, the *Lie bracket*

$$[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \qquad [\ ,\ ] : x \otimes y \mapsto [x, y],$$

such that the following axioms hold:

$$[x, x] = 0 \ \forall x \in \mathfrak{g} \tag{L1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \ \forall x, y, z, \in \mathfrak{g}. \tag{L2}$$

(Erdmann and Wildon, 2006)

Remarks 3.1.2.

- Axiom L1 is the *antisymmetry property.* L2 is known as the *Jacobi identity.* It is an immediate consequence of L1 and the bilinearity of $[\ ,\ ]$ that

$$[x, y] = -[y, x] \ \forall\ x, y \in \mathfrak{g}. \qquad (\textit{anticommutativity})$$

- Given a Lie algebra $\mathfrak{g}$, we define the opposite Lie algebra $\mathfrak{g}^{opp}$ as having the same underlying vector space, and Lie bracket $[\ ,\ ]_{opp} := -[x, y] = [y, x]$ (Schweigert, 2015, p.13).

- Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over a field $\mathbb{K}$. A $\mathbb{K}$-linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism of Lie algebras if it preserves the respective Lie brackets:

$$\phi([x, y]_{\mathfrak{g}_1}) = [\phi(x), \phi(y)]_{\mathfrak{g}_2} \qquad \forall\ x,\ y \in \mathfrak{g}_1.$$

- The anticommutativity of the Lie bracket ensures that any left ideal of $\mathfrak{g}$ - that is, a linear subspace $I \subseteq \mathfrak{g}$ such that $[x, a] \in I$ for all $x \in \mathfrak{g}$, $a \in I$ - is also a right ideal.

Examples 3.1.3 (*Lie algebras*)**.**

- Any vector space V is a Lie algebra when equipped with the Lie bracket $[x, y] = 0$ for all $x, y \in V$. Such a Lie algebra is called *abelian.*

- Let V be a finite-dimensional vector space over a field $\mathbb{K}$. $\mathrm{End}_{\mathbb{K}}(V)$ is the $\mathbb{K}$-vector space of all endomorphisms of V, i.e. linear transforms $V \to V$. It becomes a Lie algebra when equipped with the Lie bracket

$$[f, g] := f \circ g - g \circ f \ \ \forall\ f,\ g \in \mathrm{End}_{\mathbb{K}}(V)$$

where $\circ$ is composition of maps. We denote this Lie algebra by $\mathfrak{gl}(V)$.

- For V of finite dimension n over a field $\mathbb{K}$ there exists a matrix analogue to $\mathfrak{gl}(V)$, denoted $\mathfrak{gl}(n, \mathbb{K})$. The underlying vector space is given by the set $\{e_{ij}\}$ of $n \times n$ matrices having 1 in the ijth entry and 0 everywhere else for $1 \leq i, j \leq n$, and the Lie bracket is defined as the commutator of matrices.

- Consider $\mathrm{End}_{\mathbb{K}}(A)$, the $\mathbb{K}$-algebra of $\mathbb{K}$-linear endomorphisms on a $\mathbb{K}$-algebra A. A *derivation* is an endomorphism $\phi : A \to A$ which obeys the *Leibniz rule* (effectively a generalized product rule):
$$\phi(\mu(a \otimes b)) = \mu(\phi(a) \otimes b) + \mu(a \otimes \phi(b)) \quad \forall a,\ b \in A.$$
Let $\mathrm{Der}(A)$ denote the set of all derivations on A. $\mathrm{Der}(A)$ is Lie subalgebra of $\mathrm{End}_{\mathbb{K}}(A)$, i.e. a vector subspace of $\mathrm{End}_{\mathbb{K}}(A)$ that is closed under the Lie bracket.

- The *specal linear Lie algebra* is the Lie subalgebra of $\mathfrak{gl}(n, \mathbb{K})$ consisting of all $n \times n$ matrices having vanishing trace. It is denoted by $\mathfrak{sl}(n, \mathbb{K})$. The subalgebra structure is clear, given that $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + Tr(B)$ and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any $n \times n$ matrices.

  A basis for the vector space of $\mathfrak{sl}(n, \mathbb{K})$ is given by $\{e_{ij}\}$ as defined above with the further condition $i \neq j$, and the addition of the matrix $e_{ii} - e_{i+1,i+1}$ for $1 \leq i < n$. For example, the standard basis for $\mathfrak{sl}(2, \mathbb{K})$ is:
$$\left\{ e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

3.2 Universal Enveloping Algebras

Definition 3.2.1. To any Lie algebra $\mathfrak{g}$ there corresponds a unital associative algebra called the *universal enveloping algebra of* $\mathfrak{g}$, denoted $U(\mathfrak{g})$.
We construct $U(\mathfrak{g})$ by quotienting the tensor algebra $T(\mathfrak{g})$ by the ideal I generated by elements of the form
$$f \otimes g - g \otimes f - [f, g] \text{ for } f, g \in \mathfrak{g}.$$ [3]
i.e. finite strings of elements of $\mathfrak{g}$ such that $[f, g] = fg - gf$.

Examples 3.2.2.

- When $\mathfrak{g}$ is abelian the construction of $U(\mathfrak{g})$ yields the symmetric algebra $Sym(\mathfrak{g})$ as defined in Example 2.1.9.

  In this instance the generators for I are homogeneous modulo the tensor algebra grading, and so the universal enveloping algebra inherits a grading from $T(\mathfrak{g})$.

- We can explicitly construct the universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{K})$. By direct calculation of the commutators we find that
$$[e, f] = ef - fe = h, \quad [h, e] = he - eh = 2e, \quad hf - fh = -2f.$$
Hence,
$$U(\mathfrak{sl}(2, \mathbb{K})) = T(\mathfrak{sl}(2, \mathbb{K})) \,/\, I$$
$$\text{where } I = \langle e \otimes f - f \otimes e - h, h \otimes e - e \otimes h - 2e, h \otimes f - f \otimes h + 2f \rangle.$$
Unlike $Sym(\mathfrak{g})$, the enveloping algebra of $\mathfrak{sl}(2, \mathbb{K})$ is non-commutative.

Proposition 3.2.3 (PBW Basis)**.** The set of monomials
$$\{f^i h^j e^k : i, j, k \geq 0\}$$
is a basis for $U(\mathfrak{sl}(2, \mathbb{K}))$.

[3] Recall that $\mathfrak{g} = T^1(\mathfrak{g}) \subset T(\mathfrak{g})$

Proposition 3.2.3 is an instance of the Poincaré-Birkhoff-Witt (PBW) theorem, which is a fundamental result in Lie theory. Proofs can be found in (Mazorchuk, 2010) and (Humphreys, 1974); here we will only state the result.

Theorem 3.2.1 (Poincaré-Birkhoff-Witt Theorem)**.** *Let* $(x_1, x_2, x_3, \ldots)$ *be an ordered basis of* $\mathfrak{g}$*. Then the elements* $x_{i(1)} \ldots x_{i(m)} = \pi(x_{i(1)} \otimes \ldots \otimes x_{i(m)}), m \in \mathbb{Z}^+$*,* $i(1) \leq i(2) \leq \ldots \leq i(m)$*, along with* 1*, form a basis of* $U(\mathfrak{g})$ *.* (Humphreys, 1974, p.92)

Proposition 3.2.4. $U(\mathfrak{g})$ is a bialgebra for any Lie algebra $\mathfrak{g}$.

Proof. We define a map $i_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ as the composite of the inclusion $\iota : \mathfrak{g} \to T(\mathfrak{g})$ and the projection $\pi : T(\mathfrak{g}) \to T(\mathfrak{g})/I = U(\mathfrak{g})$. Because $U(\mathfrak{g})$ is generated as a $\mathbb{K}$-algebra by $\mathfrak{g}$ (or rather, the image of $\mathfrak{g}$ under $i_{\mathfrak{g}}$), we need only define our bialgebra structure on the basis elements of $\mathfrak{g}$.

Since $I \subset \bigoplus_{i>0} T^i(\mathfrak{g})$, π embeds our ground field $\mathbb{K} = T^0(\mathfrak{g})$ isomorphically into $U(\mathfrak{g})$; and in fact, it is a direct consequence of the PBW theorem that $i_{\mathfrak{g}}$ is an injective embedding $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$.

The Lie algebra structure on $U(\mathfrak{g})$ is given by the commutator:

$$[\ ,\ ]_{U(\mathfrak{g})} := xy - yx \ \ \forall\, x, y \in \mathfrak{g}$$

where multiplicative products $xy,\ yx$ in $U(\mathfrak{g})$ are inherited from the tensor algebra. We may therefore identify $\mathfrak{g}$ with its image in $U(\mathfrak{g})$. It is clear that $i_{\mathfrak{g}}$ is a morphism of Lie algebras, i.e. that $i_{\mathfrak{g}}([x, y]) = [i_{\mathfrak{g}}(x), i_{\mathfrak{g}}(y)]_{U(\mathfrak{g})}$ - the respective brackets are identical, $\mathfrak{g}$ being a Lie subalgebra of $U(\mathfrak{g})$.

The bialgebra structure on $U(\mathfrak{g})$ uses the same definitions of coproduct and counit as the tensor algebra in 2.3.4:

$$\Delta : \mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad x \mapsto x \otimes 1 + 1 \otimes x,$$
$$\epsilon : U(\mathfrak{g}) \to K, \quad \epsilon(1_{\mathbb{K}}) = 1_{\mathbb{K}}, \quad \epsilon(x) = 0\ \forall x \in \mathfrak{g}.$$

In accordance with Proposition 2.3.2, it remains only to show that Δ, ϵ are algebra morphisms. Let m denote the multiplication of the tensor product algebra on $\mathfrak{g} \otimes \mathfrak{g}$. Following (Hazewinkel et al., 2010), we proceed by calculation:

$$\begin{aligned} m((\Delta \otimes \Delta)(x \otimes y)) &= m((1 \otimes x + x \otimes 1) \otimes (1 \otimes y + y \otimes 1)) \\ &= 1 \otimes xy + y \otimes x + x \otimes y + xy \otimes 1. \end{aligned}$$

We multiply in reverse order and subtract to calculate the Lie product in $U(\mathfrak{g})$:

$$\begin{aligned} [(1 \otimes x + x \otimes 1), (1 \otimes y + y \otimes 1)]_{U(\mathfrak{g})} &= 1 \otimes xy + y \otimes x + x \otimes y + xy \otimes 1 \\ &\quad - 1 \otimes yx - x \otimes y - y \otimes x - yx \otimes 1 \\ &= 1 \otimes (xy - yx) + (xy - yx) \otimes 1 \\ &= \Delta([x, y]) \end{aligned}$$

and so Δ is a morphism of algebras. By linearity of ϵ, we see that

$$\epsilon(xy - yx) = \epsilon(x)\epsilon(y) - \epsilon(y)\epsilon(x) = 0 = \epsilon([x, y]),$$

so ϵ is an algebra morphism. We conclude that $U(\mathfrak{g})$ is a bialgebra. □

Remarks 3.2.5.

- An element x such that $\Delta(x) = x \otimes x$ is called *grouplike*. If $\Delta(x) = 1 \otimes x + x \otimes 1$, x is called *primitive*. Note that $\mathfrak{g}$ is the set of primitive elements in $U(\mathfrak{g})$, and that $\mathfrak{g}$ contains no nonzero grouplike elements.

- $U(\mathfrak{g})$ is cocommutative, inheriting this property from $T(\mathfrak{g})$.

4 Hopf Algebras

4.1 The Convolution Algebra

Let (A, μ, u) be a unital $\mathbb{K}$-algebra and (C, Δ, ϵ) a counital coalgebra over $\mathbb{K}$. The set of $\mathbb{K}$-linear maps from C to A, denoted $\mathrm{Hom}_{\mathbb{K}}(C, A)$, is a vector space when equipped with operations

$$\begin{aligned}(f+g)(c) &= f(c) + f(c)\\ (\lambda f)(c) &= \lambda f(c)\end{aligned}$$

for $f, g \in \mathrm{Hom}_{\mathbb{K}}(C, A)$, $c \in C$ and $\lambda \in \mathbb{K}$.

Definition 4.1.1. We define a bilinear product $\star$ on $\mathrm{Hom}_{\mathbb{K}}(C, A)$, which we call *convolution*:

$$f \star g : C \to A, \quad (f \star g)(c) := \mu(f \otimes g)\Delta(c) = \sum_{(c)} f(c_1) \cdot g(c_2)$$

where $\cdot$ represents products via μ. In general we will write products $x \cdot y$ simply as xy.

Proposition 4.1.2. $(\mathrm{Hom}_{\mathbb{K}}(C, A), \star, u \circ \epsilon)$ is a unital associative $\mathbb{K}$-algebra.

Proof. We prove associativity first. Consider $f, g, h \in \mathrm{Hom}_{\mathbb{K}}(C, A)$, $c \in C$:

$$\begin{aligned}(f \star (g \star h))(c) &= \mu(f \otimes (g \star h))\Delta(c)\\ &= \sum_{(c)} f(c_1)(g \star h)(c_2)\\ &= \sum_{(c)} f(c_1) \sum_{(c_2)} g(c_{2(1)})h(c_{2(2)})\\ &= \sum_{(c)} f(c_1)g(c_2)h(c_3)\\ &= \sum_{(c)} \sum_{(c_1)} f(c_{1(1)})g(c_{1(2)})h(c_2) && (\textit{by coassociativity})\\ &= \sum_{(c)} (f \star g)(c_1 h(c_2))\\ &= \mu((f \star g) \otimes h)\Delta(c)\\ &= ((f \star g) \star h)(c),\end{aligned}$$

as required. We now show that $u \circ \epsilon$ is a left identity element. For arbitrary $f \in \mathrm{Hom}_{\mathbb{K}}(C, A)$, $c \in C$:

$$\begin{aligned}((u \circ \epsilon) \star f)(c) &= \mu((u \circ \epsilon) \otimes f)\Delta(c)\\ &= \mu\Big(\sum_{(c)} u(\epsilon(c_1)) \otimes f(c_2)\Big)\\ &= \sum_{(c)} u(\epsilon(c_1))f(c_2)\\ &= \sum_{(c)} \epsilon(c_1)u(1_{\mathbb{K}})f(c_2)\\ &= \sum_{(c)} \epsilon(c_1)1_A f(c_2)\\ &= f(c), && (\textit{by counitality})\end{aligned}$$

hence $(u \circ \epsilon) \star f = f$. Mutatis mutandis, we see that $f \star (u \circ \epsilon) = f$. $\square$

Remarks 4.1.3.

- It follows that for a $\mathbb{K}$-bialgebra $(H, \mu, u, \Delta, \epsilon)$ there exists a convolution $\mathbb{K}$-algebra $(\mathrm{End}_{\mathbb{K}}(H), \star, u \circ \epsilon)$. Note that while id_H is an element of this algebra, it is *not* the unit element.

- It is clear from the definition of $\star$ that the algebra $\mathrm{Hom}_{\mathbb{K}}(C, A)$ is commutative if the algebra A is commutative or the coalgebra C is cocommutative.

Proposition 4.1.4. For any $\mathbb{K}$-coalgebra $(C, \Delta_C, \epsilon_C)$, we can use the convolution product to construct a $\mathbb{K}$-algebra on the dual space $C^* = \mathrm{Hom}_{\mathbb{K}}(C, \mathbb{K})$ (the space of linear functionals on C) where $\mathbb{K}$ is taken as a one-dimensional vector space. We call this the *dual algebra* of the coalgebra C.

We make use of the following lemma, taken from (Sweedler, 1969, Appendix I).

Lemma 4.1.5. For arbitrary vector spaces V, W, the map $\zeta : V^* \otimes W^* \to (V \otimes W)^*$ defined by $\zeta(v^* \otimes w^*)(v \otimes w) = v^*(v)w^*(w)$ is injective.[4] Hence we have *a fortiori* a containment $V^* \otimes W^* \subseteq (V \otimes W)^*$.

Proof. Assume $0 \neq \sum v_i^* \otimes w_i^* \in V^* \otimes W^*$. We may assume without loss of generality that the set $\{v_i^*\} \subset V^*$ is linearly independent, and that $w_1^* \neq 0$. Note that we do not specify a basis for V^*, which at any rate is *not* the dual set of any basis of V in the infinite-dimensional case. Let X be the linear subspace of V^* spanned by $\{v_i^*\}$. Note that X is necessarily finite-dimensional because any element of $V^* \otimes W^*$ is a finite linear sum of pure tensors by definition of $\otimes$.

A map $R : V \to X^*$ is defined by the relation $R(v)(x) = x(v)$ for $x \in X \subseteq V^*, v \in V$. For any non-zero $x \in X$ (i.e. as long as x is not the zero map $0_{V^*} : V \to 0_{\mathbb{K}}$) there must exist at least one $v \in V$ such that $x(v) = R(v)(x) \neq 0$. Therefore by the finite-dimensionality of X, the map $R : V \to X^*$ is surjective and hence there is some $v \in V$ such that

$$\delta_{i,1} = R(v)(v_i^*) = v_i^*(v)$$

where $\delta_{i,1}$ is the Kronecker-delta. Let $w \in W$ be such that $w_1^*(w) \neq 0$. Then,

$$\zeta(\sum_{(i)} v_i^* \otimes w_i^*)(v \otimes w) = \sum_{(i)} (v_i^*(v))(w_i^*(w)) = w_1^*(w) \neq 0.$$

This implies that $\zeta(\sum_{(i)} v_i^* \otimes w_i^*) \neq 0_{(V \otimes W)^*}$ and we are done. □

Remarks 4.1.6. It can be shown that ζ is an isomorphism if the vector space C is finite-dimensional (see e.g. (Kassel, 1995, p.27)). In general, however, ζ is merely injective.

Also of interest is the fact that $\mathrm{Im}(\zeta)$ is dense in $(V \otimes W)^*$ (Dascalescu et al., 2001, p.17).

Proof of proposition 4.1.4. We require a multiplication $m : C^* \otimes C^* \to C^*$ and a unit $\eta : \mathbb{K} \to C^*$. Taking the linear transpose of Δ_C gives a $\mathbb{K}$-linear map $\Delta_C^* : (C \otimes C)^* \to C^*$. We define m as the composite $(\Delta_C^* \circ \zeta) = \Delta_C^*|_{C^* \otimes C^*}$, where ζ is defined as in Lemma 4.1.5. Specifically, we have

$$\begin{aligned} m(f \otimes g)(c) &= \Delta_C^*(f \otimes g)(c) \\ &= (f \otimes g)(\Delta_C(c)) \\ &= (f \otimes g)(\Delta_c(c)) \\ &= \sum_{(c)} f(c_1)g(c_2) \end{aligned}$$

[4] It should be stressed that $v^* \in V^*, w^* \in W^*$ are arbitrary elements - they bear no particular relation to $v \in V, w \in W$, by which we mean that v^* is not necessarily the "dual of v" corresponding to some choice of basis.

for $f, g \in C^*$ and $c \in C$. Note that multiplication in the field is implied in the last line, and in fact all this work has amounted to reconstructing the convolution product on $\mathrm{Hom}_{\mathbb{K}}(C, \mathbb{K})$! Henceforth we can denote products in C^* using $\star$. We let ϕ denote the canonical isomorphism $\mathbb{K} \to \mathbb{K}^*$ and define a unit map $\eta := \epsilon_C^* \circ \phi$ by

$$\eta(\lambda)(c) = \lambda(\epsilon_C(c)) = \lambda\epsilon_C(c)$$

for $\lambda \in \mathbb{K}, c \in C$. To check associativity and unitality (the latter being the condition that $\eta(1_{\mathbb{K}}) \star f = f \star \eta(1_{\mathbb{K}}) = f \ \forall \ f \in C^*$) would be to repeat the method used in proving Proposition 4.1.2; instead we simply state that the associativity of m is due to the coassociativity of Δ_C, and that η is a unit map due to the counitality of ϵ. $\square$

Remarks 4.1.7. In light of the above one might hope - for reasons of aesthetics at least - that any $\mathbb{K}$-algebra (A, μ_A, u_A) will admit a $\mathbb{K}$-coalgebra structure on its dual space A^*. Unfortunately, this is not always the case. To construct a dual coalgebra we require a coproduct map $\Delta_{A^*} : A^* \to A^* \otimes A^*$. Note that the linear transpose of μ_A is a map $\mu_A^* : A^* \to (A \otimes A)^*$; but as stated in remark 4.1.6, if A is infinite-dimensional then there is a strict containment $A^* \otimes A^* \subsetneq (A \otimes A)^*$ and so μ_A^* may not descend to a map $A^* \to A^* \otimes A^*$.[5] There are two possible remedies for such a case; to substitute the tensor product for its Hausdorff completion in some appropriate topology (Pittner, 1996, p.72) or to replace the dual space A^* with its *finite dual*, A°, defined as

$$A^\circ := \{f \in A^* : \exists I \subset A \text{ such that} \dim_{\mathbb{K}}(A/I) < \infty \text{ and } I \subset \ker(f)\}$$

where I is a two-sided ideal in A (Hazewinkel et al., 2010, p.113).

Proposition 4.1.8. If (A, μ_A, u_A) is a finite-dimensional algebra then we may define coalgebra structure maps $\Delta_{A^*} : A^* \to A^* \otimes A^*$ and $\epsilon_{A^*} : A^* \to K$ by $\Delta_{A^*} := \zeta^{-1} \circ \mu_A^*$ and $\epsilon_{A^*} := \psi \circ u_A^*$ respectively, where ζ^{-1} is the inverse of the isomorphism ζ as defined in 4.1.5, ψ is the canonical isomorphism $\psi : K^* \to K$, $\psi(f) = f(1_K)$ and * denotes linear transposes of maps.

If A is infinite-dimensional then we may replace A^* in the above with A° to define a $\mathbb{K}$-coalgebra on A°. A° is the largest subspace of A^* such that Δ_{A^*} defines a comultiplication.

Remarks 4.1.9. For reasons of space we omit a proof of proposition 4.1.8, which again would be similar to those already encountered. The second assertion requires a proof that ζ may be regarded as an isomorphism between $A^\circ \otimes A^\circ$ and $(A \otimes A)^\circ$ - see for example (Dascalescu et al., 2001, p.34). It should be noted that in the case where A is finite-dimensional, $A^* = A^\circ$.

Definition 4.1.10. For a bialgebra $(H, \mu, u, \Delta, \epsilon)$ over a field $\mathbb{K}$, we define the *dual bialgebra* as the dual coalgebra A° regarded as a subalgebra of the dual algebra on H^* (Radford, 2012, p.177). Explicitly stated, this is the coalgebra structure on H° together with the restriction to H° of $m : H^* \otimes H^* \to H^*$, where $m = \Delta^* \circ \zeta$ as before.

4.2 The Antipode

Definition 4.2.1 (Antipode)**.** Let H be a bialgebra. An *antipode* for H is a $\mathbb{K}$-linear map $S : H \to H$ which is a two-sided inverse for the identity $\mathrm{id}_H : H \to H$ with respect to the convolution product on $\mathrm{End}_{\mathbb{K}}(H)$. Symbolically, this is to say that

$$\mu \circ (S \otimes \mathrm{id}_H) \circ \Delta = (u \circ \epsilon) = \mu \circ (\mathrm{id}_H \otimes S) \circ \Delta,$$
$$\sum_{(c)} S(c_1)c_2 = \epsilon(c)1_H = \sum_{(c)} c_1 S(c_2)$$

[5] See (Underwood, 2015, p.25) for such an example, in which $A = \mathbb{K}X$ the $\mathbb{K}$-algebra of polynomials in one indeterminate.

for any $c \in H$ (Brown and Goodearl, 2002, p.83).

Expressed as a commutative diagram we have

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S \otimes \mathrm{id}_H"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & K \arrow[rr, "u"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{id}_H \otimes S"] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Figure 7: The antipode property.

By the standard result that two-sided inverses in an associative structure are unique, a bialgebra possesses *at most* one antipode; in general, a bialgebra need not possess an antipode at all.

Definition 4.2.2. A bialgebra $(H, \mu, u, \Delta, \epsilon)$ with an antipode S is a ***Hopf algebra***. A morphism ψ of Hopf algebras H, K is a bialgebra morphism $H \to K$ which respects antipodes. This is to say that $\psi(S_H(c)) = S_K(\psi(c)) \; \forall \; c \in H$ (Dascalescu et al., 2001, p.152).

Remarks 4.2.3. Let H be a Hopf algebra with antipode S.

- A *sub-Hopf algebra* of H is a sub-bialgebra $K \subseteq H$ which is closed under the antipode, i.e. $S(K) \subseteq K$. K is itself a Hopf algebra with antipode $S|_K$.
- A *Hopf ideal* in H is a biideal $I \subseteq H$ with the further condition that $S(I) \subseteq I$.
- If $\psi : H \to K$ is a morphism of Hopf algebras then $\ker(\psi)$ is a Hopf ideal in H (reminiscent of the first isomorphism theorem).

Proposition 4.2.4. For I a Hopf ideal in a Hopf algebra H, $\pi : H \to H/I$ is a morphism of Hopf algebras, i.e. H/I is itself a Hopf algebra (one should already suspect this to be the case).

Proof. From the fact that H is a bialgebra and I is a biideal, we know that H/I has a bialgebra structure. Because I is furthermore a Hopf ideal, S induces a $\mathbb{K}$-linear map $S_{H/I} : H/I \to H/I$ given by $c + I \mapsto S(c) + S(I) = S(c) + I$. Following (Underwood, 2015, p.76), we have

$$\begin{aligned}
\mu_{H/I}(\mathrm{id}_{H/I} \otimes S_{H/I})\Delta_{H/I}(c + I) &= \mu_{H/I}(\mathrm{id}_{H/I} \otimes S_{H/I})(\sum_{(c)} c_1 + I \otimes c_2 + I) \\
&= \sum_{(c)} (c_1 S(c_2) + I) \\
&= \epsilon(c)1_H + I \\
&= \epsilon_{H/I}(c + I)1_{H/I}.
\end{aligned}$$

Similarly, we have

$$\mu_{H/I}(S_{H/I} \otimes \mathrm{id}_{H/I})\Delta_{H/I}(c + I) = \epsilon_{H/I}(c + I)1_{H/I}.$$

Hence $S_{H/I}$ is the antipode in our bialgebra H/I, and so H/I is a Hopf algebra. □

Proposition 4.2.5. The antipode possesses a number of interesting properties; here we consider only a few.

1) $S(xy) = S(y)S(x)$ - i.e. S is an algebra *anti*-homomorphism[6].

2) $S(\Delta(c)) = \sum_{(c)} c_2 \otimes c_1$ - i.e. S is a coalgebra *anti*-homomorphism.

3) $S(1_H) = 1_H$ and $\epsilon(S(c)) = \epsilon(c)$.

4) Given a Hopf algebra with antipode S over a field $\mathbb{K}$, the following are equivalent[7]:

   a. H^{op} is a Hopf algebra.
   b. H^{cop} is a Hopf algebra.
   c. S is a linear automorphism of H.

Furthermore, if the conditions of 4) are met then S^{-1} is the antipode in H^{op} and H^{cop} (Lambe and Radford, 1997, p.44).

Proof of 1). Our proof follows that of (Dascalescu et al., 2001). For a given Hopf algebra, let $H \otimes H$ denote the tensor product of the coalgebra structure and let H denote the algebra structure. This allows us to discuss the convolution algebra $\mathrm{Hom}_{\mathbb{K}}(H \otimes H, H)$. Let x, y be elements in H and define maps ν, ρ, χ in this algebra by:

$$\nu(x \otimes y) = S(y)S(x), \quad \rho(x \otimes y) = S(xy), \quad \chi(x \otimes y) = xy.$$

We show that χ is a left inverse for ν and a right inverse for ρ in the convolution algebra as follows:

$$\begin{aligned}
(\chi \star \nu)(x \otimes y) &= \sum \chi((x \otimes y)_1)\nu((x \otimes y)_2) \\
&= \sum \chi(x_1 \otimes y_1)\nu(x_2 \otimes y_2) \\
&= \sum x_1 y_1 S(y_2) S(x_2) \\
&= \sum x_1 \epsilon(y) 1_H S(x_2) && \textit{(by definition of } S \textit{ for } y) \\
&= \epsilon(x)\epsilon(y) 1_H && \textit{(by definition of } S \textit{ for } x) \\
&= \epsilon_{H\otimes H}(x \otimes y) 1_H \\
&= u_H \epsilon_{H\otimes H}(x \otimes y),
\end{aligned}$$

i.e. $\chi \star \nu = u_H \epsilon_{H\otimes H}$, which is our identity element in $\mathrm{Hom}_{\mathbb{K}}(H \otimes H, H)$ - so χ is a left inverse for ν. Note that $\epsilon(y) \in \mathbb{K}$ so the move from line 4 to 5 does not presume algebra commutativity in H. Similarly,

$$\begin{aligned}
(\rho \star \chi)(x \otimes y) &= \sum \rho((x \otimes y)_1)\chi((x \otimes y)_2) \\
&= \sum \rho(x_1 \otimes y_1)\chi(x_2 \otimes y_2) \\
&= \sum S(x_1 y_1) x_2 y_2 \\
&= \sum S((xy)_1)(xy)_2 \\
&= \epsilon(xy) 1_H && \textit{(by definition of } S \textit{ for } xy) \\
&= u_H \epsilon_{H\otimes H}(x \otimes y),
\end{aligned}$$

[6] Compare this with the 'reversal rule' for inverses of products in a group - there is a close relationship between antipodes and inverse elements.

[7] Refer back to Remarks 2.1.5 and 2.2.4 for definitions of H^{op} and H^{cop}.

and so $\rho \star \chi = u_H \epsilon_{H \otimes H}$, i.e. χ is a right inverse for ρ in our associative algebra. By uniqueness of inverses, we have shown that $\nu = \rho$, i.e. $S(xy) = S(y)S(x)$ as required. $\square$

We omit a proof for 2), but this would employ a similar strategy. Note that as a pair, 1) and 2) in proposition 4.2.5 are equivalent to the assertion that S is a bialgebra morphism from a Hopf algebra $H = (H, \mu, u, \Delta, \epsilon)$ with antipode S to a Hopf algebra $H^{\text{opcop}} = (H, (\mu \circ \tau), u, (\tau \circ \Delta), \epsilon)$ with the same antipode; i.e. S is a morphism of Hopf algebras. This morphism is both unital and counital, as implied by 3). Furthermore, as suggested by 4), this mapping is not necessarily bijective. We expand on this point later.

Claim 3) follows almost immediately from the properties of the antipode (Definition 4.2.1):

Proof of 3).

$$\begin{aligned} S(1_H) &= \mu(\mathrm{id}_H \otimes S)(1_H \otimes 1_H) \\ &= \mu(\mathrm{id}_H \otimes S)\Delta(1_H) \\ &= \epsilon(1_H)1_H \\ &= 1_{\mathbb{K}}1_H = 1_H, \end{aligned}$$

$$\begin{aligned} \epsilon(c) &= \epsilon(c)\epsilon(1_H) \\ &= \epsilon\big(\epsilon(c)1_H\big) \\ &= \epsilon\Big(\sum_{(c)} c_1 S(c_2)\Big) \\ &= \sum_{(c)} \epsilon(c_1)\epsilon(S(c_2)) \\ &= \sum_{(c)} \epsilon\big(S(\epsilon(c_1)c_2)\big) \\ &= \epsilon(S(c)). \end{aligned}$$

$\square$

Proof of 4). We have observed that for any Hopf algebra H, S_H is always a Hopf algebra morphism $H \to H^{\text{opcop}}$. Hence, the equivalence of a. and b. is immediate from the fact that $H^{\text{cop}} = (H^{\text{op}})^{\text{opcop}}$ and $H^{\text{op}} = (H^{\text{cop}})^{\text{opcop}}$ (Radford, 2012, p.214).

We now show that a. $\implies$ c.. Let H^{op} be a Hopf algebra with antipode T. For any $c \in H$ we have an equivalence $c_2 T(c_1) = \epsilon(c)1_H = T(c_2)c_1$ by defining properties of the antipode. Composing S with T in the left side of this equation we have

$$(S \circ T)(c_1)S(c_2) = \epsilon(c)1_H,$$

where we have used the antihomomorphism property (1) and the fact that $S(1_H) = 1_H$ (3). Using the second part of (3), we can replace c with $S(c)$ in the right hand equation to give

$$(T \circ S)(c_2)S(c_1) = \epsilon(c)1_H.$$

Hence $(S \circ T) * S = (T \circ S) * S = (u \circ \epsilon)$, which is to say that $(S \circ T)$ is a left inverse for S in the convolution algebra. By the definition of S, this is to say that $(S \circ T) = (T \circ S) = \mathrm{id}_H$. Hence, S is invertible and so we have established the bijectivity required for (c.).

To prove c. $\implies$ a., let S be a linear automorphism on H with inverse S^{-1}. We know that S is an algebra morphism $H \to H^{\text{op}}$ - this is the just the algebra antihomomorphism property. Therefore by definition S^{-1} is an algebra morphism $H^{\text{op}} \to H$. By definition of S in H we have the equivalence $S(c_1)c_2 = \epsilon(c)1_H = c_1 S(c_2)$. Applying S^{-1} gives

$$S^{-1}(c_2)c_1 = \epsilon(c)1_H = c_2 S^{-1}(c_1)$$

using the fact that $\epsilon(S(c)) = \epsilon(c)$. This is precisely to say that S^{-1} is the antipode in H^{op}. $\square$

Proposition 4.2.6. If a Hopf algebra H is either commutative or cocommutative then the antipode S has order 2, i.e. $S^2 = S \circ S = \mathrm{id}_H$ (we call such a Hopf algebra *involutory*).

Proof. This follows immediately from 4.2.5 part 4), the antihomomorphism properties, and the definitions of commutativity and cocommutativity in terms of opposite algebras and coalgebras respectively. $\square$

Remarks 4.2.7. As noted above, in the general case S is *not* a linear automorphism. In fact, in the case of an infinite-dimensional Hopf algebra the antipode may fail even injectivity (see (Takeuchi, 1971)). Note that such a Hopf algebra must fail commutativity and cocommutativity by Proposition 4.2.6, which holds in general. For the finite-dimensional case we can prove the following.

Proposition 4.2.8. If H is a finite-dimensional Hopf algebra with antipode S and over a field $\mathbb{K}$, then $S : H \to H$ is a linear automorphism.

Proof. Our proof is adapted from that of (Lambe and Radford, 1997).

The case where $\dim(H) = 1$ is the trivial Hopf algebra - this is the field $\mathbb{K}$ itself with antipode $S_{\mathbb{K}} = 1_{\mathbb{K}}$. We proceed by induction on $\dim(H)$.

Let $\dim(H) > 1$. By hypothesis, the antipodes of finite-dimensional Hopf algebras with dimension less than $\dim(H)$ are linear automorphisms. Because S is a morphism of Hopf algebras $H \to H^{(\mathrm{opcop})}$ and $H^{(\mathrm{opcop})} \to H$, $K = S(H)$ is a sub-Hopf algebra of H and $\ker(S)$ is a Hopf ideal in H. Note that we are considering the image K of H under $S : H \to H$. If $K = H$ then $\ker(S) = 0$, i.e. S is both surjective and injective and we are done. If $K \neq H$ then by our initial assumption the restriction $S|_K$ is a linear automorphism on the sub-Hopf algebra $K \subset H$. Hence, $H = \ker(S) \bigoplus K$, and $S|_K$ is injective in any case.

Consider the projection $\pi : H \to K$. Because $\ker(S)$ is a Hopf ideal in H, and by injectivity $\ker(\pi) = \ker(S)$, we have both $\epsilon(\ker(S)) = 0$ and $\Delta(\ker(S)) \subseteq \ker(S) \otimes H + H \otimes \ker(H) = \ker(\pi) \otimes H + H \otimes \ker(S)$. Hence $(\pi * S)(c) = 0 = \epsilon(c)1 \;\forall\; c \in \ker(S)$. For $c \in K$, we compute $(\pi * S)(c) = \pi(c_1 S(c_2)) = c_1 S(c_2) = c_1 S(c_2) = \epsilon(c)1_H$. This is to say that $\pi * S = u \circ \epsilon$, i.e. π is the inverse of S in the convolution algebra $\mathrm{End}_{\mathbb{K}}(H)$. By the definition of the antipode, this means that $\pi = 1_H$. Hence, $K = H$ and $S = S|_K$, and so S is a linear automorphism of H. $\square$

Definition 4.2.9. Given a Hopf algebra H over a field $\mathbb{K}$ with antipode S, the *dual Hopf algebra of* H is the dual bialgebra H° as defined in 4.1.10 with antipode $S^{\circ} = S^*_{H^{\circ}}$, i.e. the restriction of the linear tranpose of S to H°.

Remarks 4.2.10. From the discussion above, it should be clear that any finite-dimensional Hopf algebra H formally gives rise to eight Hopf algebras (including itself) - viz. H, H^{opcop}, H^{op}, H^{cop} and the dual structures H^*, $(H^*)^{\mathrm{opcop}} = (H^{\mathrm{copop}})^*$, $(H^*)^{\mathrm{op}} = (H^{\mathrm{cop}})^*$ and $(H^*)^{\mathrm{cop}} = (H^{\mathrm{op}})^*$.

4.3 Examples of Hopf Algebras

Examples 4.3.1 (The group Hopf algebra)**.** Apart from the trivial Hopf algebra, the simplest example is probably the group algebra $\mathbb{K}G$. We have already proven that $\mathbb{K}G$ is a bialgebra (see examples 2.3.4), so all we have to do is find an antipode. Recall that G is a basis for $\mathbb{K}G$ and that we may identify $1_{\mathbb{K}G}$ with 1_G. Let S be the $\mathbb{K}$-linear map defined by

$$S : \mathbb{K}G \to \mathbb{K}G, \quad g \mapsto g^{-1}$$

for all $g \in G$. Recalling that: μ is the group multiplication extended linearly; the comultiplication is $\Delta : g \mapsto g \otimes g \;\forall\; g \in G$; and that the counit is $\epsilon(g) = 1_G \;\forall\; g \in G$, we have

$$(\mathrm{id}_{\mathbb{K}G} \star S)(g) = \mu(\mathrm{id}_{\mathbb{K}G} \otimes S)\Delta(g) = \mu(\mathrm{id}_{\mathbb{K}G} \otimes S)(g \otimes g) = \mu(g \otimes g^{-1}) = gg^{-1} = 1_{\mathbb{K}G} = \epsilon(g)1_{\mathbb{K}G}.$$

The linearity of S ensures that the proof extends from the basis G to all $x \in \mathbb{K}G$, and so S is a right inverse to $\mathrm{id}_{\mathbb{K}G}$ under $\star$. The proof that S is also a left inverse under $\star$ is similar. This is precisely to say that S is the antipode in $\mathbb{K}G$. We note that the Hopf algebra $\mathbb{K}G$ is cocommutative, has dimension equal to $|G|$ (which could be ∞), and is commutative if and only if G is an abelian group.

Examples 4.3.2 (Sweedler's 4-dimensional Hopf algebra)**.** The following example was first described in (Sweedler, 1969). Let $\mathbb{K}$ be a field of characteristic $\neq 2$.[8] Let H_4 be given as a $\mathbb{K}$-algebra by generators c, x such that $c^2 = 1, x^2 = 0$ and $cx = -xc$. A basis for H_4 is therefore given by $\{1, c, x, cx\}$. We define a comultiplication $\Delta : H_4 \to H_4 \otimes H_4$ by

$$c \mapsto c \otimes c, \quad x \mapsto c \otimes x + x \otimes 1.$$

and a counit by $\epsilon(c) = 1, \epsilon(x) = 0$. To show that H_4 is a bialgebra, it is sufficient to check that the definitions of Δ and ϵ on the generators of H_4 extend to an algebra homomorphism $H_4 \to (H_4 \otimes H_4)$, where $(H_4 \otimes H_4)$ is the tensor product algebra. Using linearity of Δ we have

$$\begin{aligned}\Delta(cx) &= \Delta(c)\Delta(x) \\ &= (c \otimes c)(c \otimes x + x \otimes 1) \\ &= (cc \otimes cx) + (cx \otimes c) = (1 \otimes cx) + (cx \otimes c).\end{aligned}$$

Letting m denote the multiplication in the tensor product algebra on $H_4 \otimes H_4$, we have

$$\begin{aligned}m\big((\Delta \otimes \Delta)(c \otimes x)\big) &= m(\Delta(c) \otimes \Delta(x)) \\ &= m((c \otimes c) \otimes (c \otimes x + x \otimes 1)) \\ &= (1 \otimes cx) + (cx \otimes c).\end{aligned}$$

Similarly, $\Delta(xx) = \Delta(0) = 0$ by linearity of Δ and we may compute

$$\begin{aligned}m\big((\Delta \otimes \Delta)(x \otimes x)\big) &= m((c \otimes x + x \otimes 1) \otimes (c \otimes x + x \otimes 1)) \\ &= (1 \otimes 0) + (cx \otimes c) + (xc \otimes c) + (0 \otimes 1) \\ &= 0 + (cx \otimes c) - (cx \otimes c) + 0 = 0.\end{aligned}$$

Lastly, the equivalence of $\Delta(cc) = \Delta(1) = 1$ and $m(\Delta(c) \otimes \Delta(c)) = 1 \otimes 1$ is obvious. Hence we have shown that Δ is an algebra morphism. For ϵ we note that $\epsilon(cc) = 1 = 1 \cdot 1 = \epsilon(c) \cdot \epsilon(c)$, $\epsilon(xx) = 0 = 0 \cdot 0 = \epsilon(x) \cdot \epsilon(x)$, and $\epsilon(cx) = 0 = 0 \cdot 1 = \epsilon(c) \cdot \epsilon(c)$. Hence, H is a bialgebra.

Define a linear map $S : H_4 \to H_4$ by

$$S(c) = c, \quad S(x) = -cx$$

and let μ denote the product operation in H_4. We check that S defines an antipode on the generators as follows:

$$\begin{aligned}(\mathrm{id}_{H_4} \star S)(c) &= \mu(\mathrm{id}_{H_4} \otimes S)\Delta(c) \\ &= \mu(\mathrm{id}_{H_4} \otimes S)(c \otimes c) \\ &= \mu(c \otimes c) = cc = 1 = \epsilon(c)1_{H_4},\end{aligned}$$

$$\begin{aligned}(\mathrm{id}_{H_4} \star S)(x) &= \mu(\mathrm{id}_{H_4} \otimes S)\Delta(x) \\ &= \mu(\mathrm{id}_{H_4} \otimes S)(c \otimes x + x \otimes 1) \\ &= \mu(c \otimes -cx) + \mu(x) \\ &= -x + x = 0 \cdot 1_{H_4} = \epsilon(x)1_{H_4}.\end{aligned}$$

[8] As an aside, we note that fields of characteristic 2 often produce "pathological cases" because they dictate that $x = -x$. One wonders how far the prevalence with which one encounters these cases is attributable to properties of the even prime, and how much is a kind of mathematical anthropocentrism. Dually, one might ask whether structures such as H_4 are as "artificial" as they might appear.

The calculations above show that $\mu \otimes (id_{H_4} \otimes S) = (u \circ \epsilon)$, i.e. S is a right inverse for id_{H_4} under the convolution product. Mutatis mutandis, we see that S is also a left inverse, and so S is the antipode in H_4.

Sweedler's 4-dimensional example was the first Hopf algebra known to be neither commutative nor cocommutative; moreover, H_4 it is the smallest Hopf algebra for which this can be the case. We note that the antipode has order 4 - indeed, $S^4(x) = S^2(S^2(x)) = S^2(-x) = x$, and $S^2(c) = c$.

Remarks 4.3.3. Sweedler's example H_4 is a specific case of the *Taft algebras*, which were introduced in (Taft, 1972). These are K-algebras $H_{n^2}(\lambda)$, where $n \in \mathbb{Z}, n \geq 2$ and λ is a primitive n-th root of unity. They are defined by the generators c, x with relations

$$c^n = 1, \quad x^n = 0, \quad xc = \lambda cx.$$

The coalgebra structure is induced by the definitions of coproduct and counit used in examples 4.3.2. This produces a bialgebra structure of dimension n^2 with basis $\{c^i x^j : 0 \leq i, j \leq n-1\}$, for which an antipode is defined by $S(c) = c^-1$ and $S(x) = -c^{-1}x$. Note that setting parameters to $n = 2$ and $\lambda = -1$ produces H_4.

Examples 4.3.4 (Universal enveloping algebra of a Lie algebra)**.** In example 3.2.4 we showed that the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ admits a bialgebra structure when equipped with the maps

$$\Delta(x) = x \otimes 1 + 1 \otimes x \ \ \forall \ \in \mathfrak{g},$$
$$\epsilon(1_{\mathbb{K}}) = 1_{\mathbb{K}}, \quad \epsilon(x) = 0 \, \forall \, x \in \mathfrak{g}.$$

Recall that $U(\mathfrak{g})$ is generated by $\mathfrak{g}$ and that there exists a canonical morphism $i_g : \mathfrak{g} \to U(\mathfrak{g})$. We define a linear map $S : \mathfrak{g} \to \mathfrak{g}$ by

$$S(x) = -x \quad \forall \, x \in \mathfrak{g}.$$

For $x \in \mathfrak{g}$ we have

$$\mu \circ (S \otimes \mathrm{id}) \otimes \Delta(x) = \mu(-x \otimes 1 + 1 \otimes x) = -x + x = 0 = 1\epsilon(x),$$

so S is a left inverse for id under convolution. Clearly, S is also a right inverse. To demonstrate that the definition of S extends to all of $U(\mathfrak{g})$, we will consider a more general case. Let H be a bialgebra that is generated as a $\mathbb{K}$-algebra by some $X \subset H$. Assume that the definition of the antipode applies for all $x \in X$, i.e. we have

$$\mathrm{id}_H \star S(x) = S(x) \star \mathrm{id}_H = u\epsilon(x).$$

If S is an antipode in H, then this relation must hold for any $h \in H$. Note that $h = xy$ for some $x, y \in X$. Therefore we have

$$\begin{aligned}
(xy)_1 S(xy)_2 &= x_1 y_1 S(x_2 y_2) \\
&= x_1 y_1 S(y_2) S(x_2) && (\textit{anti-morphism property}) \\
&= x_1 \epsilon(y) S(x_2) && (\textit{by definition of } S) \\
&= \epsilon(x)\epsilon(y) = \epsilon(xy) 1_H.
\end{aligned}$$

Hence, S extends to a right inverse for all $h \in H$. Similarly, S extends to a left inverse. This is to say that S is the antipode in H.

We note that $U(\mathfrak{g})$ is an infinite-dimensional cocommutative Hopf algebra. This is precisely because the generating set $\mathfrak{g}$ is the set of primitive elements in $U(\mathfrak{g})$, and hence $\tau \circ \Delta = \Delta$. It is a commutative Hopf algebra if and only if $\mathfrak{g}$ is abelian.

5 Quantum Groups

5.1 Defining a Quantum Group

Our final chapter is devoted to a discussion of a particular class of Hopf algebras sometimes referred to as *quantum groups.* More precisely, we will be discussing the *quantized enveloping algebras* $U_q(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$, with a particular focus on $U_q(\mathfrak{sl}(2,\mathbb{K}))$. We restrict our discussion to cases where the deformation parameter q is *not* a root of unity.

Definition 5.1.1. Let $\mathbb{K}$ be a field and let $q \in \mathbb{K}$ with $q \neq 0$ and $q^2 \neq 1$. Then $U_q(\mathfrak{sl}(2,\mathbb{K}))$ is defined as the unital associative $\mathbb{K}$-algebra with generators E, F, K, K^{-1} subject to the relations

$$KK^{-1} = 1 = K^{-1}K, \tag{R1}$$

$$KEK^{-1} = q^2E, \tag{R2}$$

$$KFK^{-1} = q^{-2}F, \tag{R3}$$

$$EF - FE = [E,F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{R4}$$

(Brown and Goodearl, 2002, p.25)

We consider first some elementary properties of $U_q(\mathfrak{sl}(2,\mathbb{K}))$, which we hereafter abbreviate to U_q. This section draws on (Jantzen, 1996).

Proposition 5.1.2. There exists a unique algebra automorphism ω on U_q such that $\omega(E) = F$, $\omega(F) = E$ and $\omega(K) = K^{-1}$.

Proof. This is elementary, requiring only that we verify that the relations R1 - R4 are satisfied in $\mathrm{Im}(\omega)$. For example, R2 becomes $K^{-1}FK = q^2F$ under ω. Using R3 and the fact that $K = K^{-1}$ we have

$$\begin{aligned} & KFK^{-1} = q^{-2}F \\ \iff & K^{-1}KFK^{-1} = K^{-1}q^{-2}F \\ \iff & FK^{-1} = q^{-2}K^{-1}F \\ \iff & FK^{-1} = q^2KF \\ \iff & K^{-1}FK = q^2F. \end{aligned}$$

The other relations are checked in a similar way. The uniqueness of ω is evident from the fact that ω maps the set of generators back onto itself. ☐

It is also clear that $\omega^2 = \mathrm{id}$. The mapping ω is sometimes called the *Cartan automorphism* (Kassel, 1995, p.123).

Proposition 5.1.3. There exists a unique algebra antiautomorphism $\tau : U_q \to U_q^{opp}$ such that $\tau(E) = E, \tau(F) = F$ and $\tau(K) = K^{-1}$.

Proof. In this case we verify that R1 - R4 are satisfied in the opposite algebra U_q^{opp}, wherein the product $a \cdot b$ is equal to ba in U_q. For example, R4 is satisfied by the fact that $E \cdot F - F \cdot E = FE - EF = (K^{-1} - K)/(q - q^{-1})$. The other relations are proved in the same way. ☐

We note that $\tau^{-1} = \mathrm{id}$, and that τ is unique for the same reasons that ω is unique.

Theorem 5.1.1 (PBW basis for $U_q(\mathfrak{sl}(2,\mathbb{K}))$)**.** *The set of monomials*

$$\{F^sK^nE^r : r, s, n \in \mathbb{Z}, r, s \geq 0\}$$

is a basis for $U_q(\mathfrak{sl}(2,\mathbb{K}))$.

Remarks 5.1.4. This is an application of the PBW theorem from 3.2.1. For a detailed proof in the context of U_q, see (Jantzen, 1996).

5.2 A Hopf Algebra Structure on $U_q(\mathfrak{sl}(2,\mathbb{K}))$

We define a comultiplication $\Delta : U_q \to U_q \otimes U_q$ and a counit $\epsilon : U_q \to \mathbb{K}$ as follows:

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$
$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = \epsilon(K^{-1}) = 1.$$

We also define an antiautomorphism $S : U_q \to U_q$ which we take as a putative antipode:

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K.$$

Proof that U_q is a Hopf algebra. To prove that U_q is a Hopf algebra, we must first check that Δ is an algebra morphism $U_q \to U_q \otimes U_q$ (i.e. a morphism onto the tensor product algebra) and that ϵ is an algebra morphism $U_q \to \mathbb{K}$. This is to say that the relations R1 - R4 are respected under these linear maps. The relation R1 requires that

$$\Delta(K)\Delta(K^{-1}) = \Delta(K^{-1})\Delta(K) = 1$$

which is self-evident. Relation R2 requires that

$$\Delta(K)\Delta(E)\Delta(K^{-1}) = q^2\Delta(E).$$

Following (Kassel, 1995, p.141), we proceed by calculation:

$$\begin{aligned}\Delta(K)\Delta(E)\Delta(K^{-1}) &= (K \otimes K)(1 \otimes E + E \otimes KK^{-1}) \otimes K^{-1} \\ &= 1 \otimes KEK^{-1} + KEK^{-1} \otimes K \\ &= q^2(1 \otimes E + E \otimes K) \\ &= q^2\Delta(E)\end{aligned}$$

as required. The proof that R3 is respected is analogous. For R4 we compute that

$$\begin{aligned}&[\Delta(E), \Delta(F)] \\ &\quad = (1 \otimes E + E \otimes K)(K^{-1} \otimes F + F \otimes 1) \\ &\qquad\qquad - (K^{-1} \otimes F + F \otimes 1)(1 \otimes E + E \otimes K) \\ &\quad = K^{-1} \otimes EF + F \otimes E + EK^{-1} \otimes KF + EF \otimes K \\ &\qquad\qquad - K^{-1} \otimes FE - K^{-1}E \otimes FK - F \otimes E - FE \otimes K \\ &\quad = K^{-1} \otimes [E,F] + [E,F] \otimes K \\ &\quad = \frac{K^{-1} \otimes (K - K^{-1}) + (K - K^{-1}) \otimes K}{q - q^{-1}} \\ &\quad = \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}\end{aligned}$$

as required. Hence, Δ defines an algebra morphism. That ϵ defines an algebra morphism is clear from the fact that the image of the generators under ϵ is the set $\{0, 0, 1, 1\}$ - for example, R3 is verified by computing

$$\epsilon(K)\epsilon(F)\epsilon(K^{-1}) = 1 \cdot 0 \cdot 1 = 0 = q^{-1} \cdot 0 = q^{-1}\epsilon(F).$$

All that remains is to check that S is indeed the antipode on U_q. First we show that S is an anti-homomorphism $H \to H$, by showing that it is a $\mathbb{K}$-algebra morphism $U_q \to U_q^{\mathrm{opp}}$. Again, this is straightforward computation. Considering first R1 we have:

$$S(K^{-1})S(K) = S(K)S(K^{-1}) = K^{-1}K = 1$$

as required. For R2 we have

$$S(K^{-1})S(E)S(K) = -K(EK^{-1} = -q^2 EK^{-1} = q^2 S(E)$$

as required. Checking R3 is analogous. For R4 we have

$$\begin{aligned}[S(F), S(E)] &= KFEK^{-1} - EK^{-1}KF \\ &= [F, E] \\ &= \frac{K^{-1} - K}{q - q^{-1}} = \frac{S(K) - S(K^{-1})}{q - q^{-1}}\end{aligned}$$

as required. Finally, we check that the antipode condition $\mathrm{id} \star S(h) = S(h) \star \mathrm{id} = u\epsilon(h)$ holds for any $h \in H$. Letting f denote the map $m \circ (1 \otimes S) \circ \Delta$, where m is the product operation, we compute that f maps our generators as follows:

$$\begin{aligned}&E \mapsto E \otimes 1 + K \otimes E \mapsto E \otimes 1 + K \otimes (-K^{-1}E) \mapsto E + K(-K^{-1}E) = 0 = \epsilon(E)1 \\ &F \mapsto F \otimes K^{-1} + 1 \otimes F \mapsto F \otimes K + 1 \otimes (-FK) \mapsto FK - FK = 0 = \epsilon(F)1 \\ &K \mapsto K \otimes K \mapsto K \otimes K^{-1} \mapsto KK^{-1} = 1 = \epsilon(K)1,\end{aligned}$$

and it is clear that $f(K^{-1}) = \epsilon(K^{-1})1$ will also hold. This shows that S is the right inverse to the identity of U_q in the convolution algebra. The proof that it is also a right inverse is, as usual, analogous. Hence we conclude that S is an antipode and U_q is indeed a Hopf algebra. □

Remarks 5.2.1. It is clear from the actions on the generators that U_q is an infinite-dimensional Hopf algebra that is neither commutative nor cocommutative. By 4.2.6, we know that $S^2 \neq \mathrm{id}$.

Finally, we would like to say something about how the universal enveloping algebra $U(\mathfrak{sl}(2,\mathbb{K}))$ (which we now call U) relates to its quantized analogue $U(\mathfrak{sl}(2,\mathbb{K})) = U_q$. By considering the generating relations of in definition 5.1.1, one would expect that the structure of U_q approaches something like U as $q \to 1$. Note, however, that U_q fails to be well-defined for $q = 1$. To discuss the relation requires us to reformulate the definition of U_q.

We begin by replacing q with e^h and $K = e^{hH}$ in 5.1.1, which allows us to consider the limit as $q \to 1$ in terms of $h \to 0$. For example, by an application of L'Hôpital's rule for the case of a limit of form $\frac{0}{0}$ we have:

$$\begin{aligned}\lim_{h\to 0}[E, F] &= \lim_{h\to 0} \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}} \\ &= \lim_{h\to 0} \frac{He^{hH} + He^{hH}}{e^h + e^h} \\ &= \frac{H + H}{1 + 1} = H.\end{aligned}$$

By similar calculation we have $[H, F] = -2F$ and $[E, F] = H$, which are precisely the relations that can be used to define $U(\mathfrak{sl}(2,\mathbb{K})$. In order to properly compare the constructions in the general case we require a slight reformulation of the definition of U_q, taken from (Klimyk and Schmudgen, 1997, p.57).

First, we note that the elements E, F, K, K^{-1} as defined in 5.1.1, with the addition of $G := (q - q^{-1})^{-1}(K - K^{-1})$, satisfy the following relations:

$$[G, E] = E(qK + q^{-1}K^{-1}), \quad [G, F] = -(qK + q^{-1}K^{-1})F, \tag{Q1}$$

$$[E, F] = G, \quad (q - q^{-1})G = K - K^{-1} \tag{Q2}$$

.

We let $\tilde{U}_q$ denote the algebra generated by E, F, K, K^{-1} and G subject to the relations R1, R2, R3, Q1 and Q2. Then the algebras $\tilde{U}_q$ and U_q are isomorphic via a mapping $E \to E, F \to F, K \to K$ and $G \to (q - q^{-1})(K - K^{-1})$. Therefore, a Hopf algebra structure is induced on $\tilde{U}_q$ by the Hopf algebra structure on U_q subject to the further relations that $\Delta(G) = G \otimes K + K^{-1} \otimes G$, $\epsilon(G) = 0$ and $S(G) = -G$. We note that this endows $\tilde{U}_q$ with a Hopf algebra structure that applies even in the cases $q = 1$ or $q = -1$, which were excluded in U_q. In this sense, we can consider $\tilde{U}_1(\mathfrak{sl}(2, \mathbb{K}))$ to be the classical limit of the Hopf algebra $\tilde{U}_q(\mathfrak{sl}(2, \mathbb{K}))$.

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