# Modal Analysis

## VU 325.100

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# Modal Analysis Identification Methods

# Eigenvalue, Natural Frequency and Damping Ratio

For sub-critical damping ($0 \leq \zeta < 1$) we have a complex-valued eigenvalue pair

$$s = -\omega_n \zeta \pm j\omega_n \sqrt{1-\zeta^2} = -\delta \pm j\omega_d, \tag{4}$$

where the damped natural frequency $\omega_d = \Im\{s\}$ corresponds to the imaginary part, and the real part contains the damping-information. The damping ratio is obtained from

$$\zeta = \sqrt{\frac{r^2}{1+r^2}}, \quad \text{with} \quad r = \frac{\Re\{s\}}{\Im\{s\}} = \frac{-\delta}{\omega_d}, \tag{5}$$

which can be used to compute the un-damped natural frequency

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}. \tag{6}$$

# Receptance for Proportionally Damped System

For a proportionally damped system the receptance can be expressed via the modal parameters by

$$H_{ab}(\omega) = \sum_{k=1}^{n} \frac{v_{ak} v_{bk}}{\omega_k^2 - \omega^2 + 2j\zeta_k \omega_k \omega}. \tag{7}$$

Thus, the modal parameters consisting of natural frequencies $\omega_k$, damping ratios $\zeta_k$ and corresponding mode shapes $\boldsymbol{v}_k$ completely describe the dynamical system.

# Identification Task

We want to identify the dynamic behaviour of a system from measurement data.

Identification is a *fitting process* of the measurement data to a *system model*.

# Classification of Identification Methods

in terms of system description

## Indirect Methods

Rely on a modal description to identify the FRFs, i.e. the description of the system to be identified is done based on modal parameters: natural frequencies, damping ratios, mode shapes.

## Direct Methods

Try to direly determine the parametrisation of the spacial model, e.g. the entries of the system matrices. Modal parameters may be determined from them afterwards.

Since the course is on *modal analysis*, we will only consider indirect methods.

# Classification of Identification Methods

in terms of simultaneously considered inputs and outputs

**SISO** single-input single-output methods process one FRF at a time.

**SIMO** single-input multiple-output methods analyse multiple FRFs generated with a single input simultaneously.

**MIMO** multiple-input multiple-output, also called *polyreference* methods can handle all FRFs from experiments with several simultaneous excitations and output channels.

# Classification of Identification Methods

in terms of domain

## Time Domain Methods

- use measured time histories directly to estimate system parameters,
- work best when data contains contributions from a wide frequency range

## Frequency Domain Methods

- operate on FRFs and, thus, typically rely on FFTs of the time domain data
- may suffer from the deficiencies of the Fourier transform (e.g. leakage)
- work best on a limited number of frequencies (modes)

In Single degree of freedom methods the FRF data in a certain frequency range is fit to a single degree of freedom oscillator, whereas multi degree of freedom methods, fit the whole FRF data simultaneously.

Tuned Sinosiodal Methods are a special class, and partly rely on other methods.

# Time Domain Methods

*Indirect* methods in time domain include

Complex Exponential (CE) method is based on Prony's method and operates on a single FRF (from which an impulse response function is computed). It is a SISO method, but can be extended to the

Least-Squares Complex Exponential (LSCE) method, which is a SIMO method, and can be extended to the,

Polyreference Complex Exponential (PRCE) method, which is MIMO.

Eigensystem Realisation Algorithm (ERA) is another MIMO method.

Ibrahim Time Domain (ITD) method operates on free oscillation data considering several simultaneous outputs.

Some selected *direct methods* in the time domain are the

Autoregressive Moving-Average (ARMA) method which exists in several versions like the *autoregressive moving-average with exogenous variables* (ARMAX) or *autoregressive moving-average vector* (ARMAV) which are all SISO, and the

Direct Parameter Identification method which is a MIMO method.

# Ibrahim Time Domain (ITD) Method - Derivation I

The *free oscillation response* of a linear second order ODE system, i.e. the solution to

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0} \tag{8}$$

can be written in terms of the $m$ participating modes in the fundamental solution from

$$\boldsymbol{x}(t) = \sum_{k=1}^{2m} \boldsymbol{v}_k e^{\lambda_k t}, \tag{9}$$

where $\lambda_k$ are the $2m$ eigenvalues of the characteristic equation of Eq. (8) and $\boldsymbol{v}_k$ are the corresponding complex eigenvectors.

# Ibrahim Time Domain (ITD) Method - Derivation II

The response of a freely oscillating structure $\boldsymbol{x}(t)$ can be measured at $n$ positions at constantly sampled points in time. Writing Eq. (9) for $p$ times $t_i$ we obtain

$$\begin{bmatrix} x_1(t_1) & x_1(t_2) & \cdots & x_1(t_p) \\ x_2(t_1) & x_2(t_2) & \cdots & x_2(t_p) \\ \vdots & \vdots & & \vdots \\ x_n(t_1) & x_n(t_2) & \cdots & x_n(t_p) \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1,2m} \\ v_{21} & v_{22} & \cdots & v_{2,2m} \\ \vdots & \vdots & & \vdots \\ v_{n,1} & v_{n,2} & \cdots & v_{n,2m} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t_1} & \cdots & e^{\lambda_1 t_p} \\ e^{\lambda_2 t_1} & \cdots & e^{\lambda_2 t_p} \\ \vdots & & \vdots \\ e^{\lambda_{2m} t_1} & \cdots & e^{\lambda_{2m} t_p} \end{bmatrix}, \text{ or,}$$

$$\boldsymbol{X} = \boldsymbol{V}\boldsymbol{\Lambda}. \tag{10}$$

Writing the above relation for response data that occurs $\Delta t_1$ later in time yields

$$x_i(t_j + \Delta t_1) = \sum_{k=1}^{2m} v_{ik}\, e^{\lambda_k (t_j + \Delta t_1)} = \sum_{k=1}^{2m} \left[ v_{ik}\, e^{\lambda_k \Delta t_1} \right] e^{\lambda_k t_j} = \sum_{k=1}^{2m} \hat{v}_{ik}\, e^{\lambda_k t_j}, \tag{11}$$

or again in matrix form

$$\hat{\boldsymbol{X}} = \hat{\boldsymbol{V}}\boldsymbol{\Lambda}. \tag{12}$$

# Ibrahim Time Domain (ITD) Method - Derivation III

We now introduce the so-called, square $(n \times n)$ *system matrix* $\boldsymbol{A}$ such that

$$\boldsymbol{A}\boldsymbol{V} = \hat{\boldsymbol{V}}. \tag{13}$$

Since $\hat{\boldsymbol{v}}_k = \boldsymbol{v}_k e^{\lambda_k \Delta t_1}$ we see that solving the standard EVP

$$\boldsymbol{A}\boldsymbol{v} = s\boldsymbol{v}, \qquad \text{or,} \qquad (\boldsymbol{A} - s\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}, \tag{14}$$

for the $s_k = e^{\lambda_k \Delta t_1}$ eigenvalues, yields the originally sought eigenvalues $\lambda_k$ of Eq. (8) and the corresponding mode shapes $\boldsymbol{v}_k$.

# Ibrahim Time Domain (ITD) Method - Derivation IV

The *system matrix* can be computed from the known data matrix $\boldsymbol{X}$ and from its time shifted companion $\hat{\boldsymbol{X}}$: Use to the expression for the system response Eq. (10), pre-multiply $\boldsymbol{A}$, use its definition Eq. (13), and the expression for the time-shifted system response Eq. (12), to obtain

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{A}\boldsymbol{V}\Lambda = \hat{\boldsymbol{V}}\Lambda = \hat{\boldsymbol{X}}. \tag{15}$$

Now apply a suitable technique to solve for $\boldsymbol{A}$, e.g. the *pseudo-inverse* method. Post-multiply by $\boldsymbol{X}^T$ or $\hat{\boldsymbol{X}}^T$, then solve for $\boldsymbol{A}$ to obtain

$$\boldsymbol{A}_1 = \hat{\boldsymbol{X}}\boldsymbol{X}^T\left(\boldsymbol{X}\boldsymbol{X}^T\right)^{-1}, \qquad \text{or,} \qquad \boldsymbol{A}_2 = \hat{\boldsymbol{X}}\hat{\boldsymbol{X}}^T\left(\boldsymbol{X}\hat{\boldsymbol{X}}^T\right)^{-1}. \tag{16}$$

One typically uses the average of the two options, i.e. $\boldsymbol{A} = (\boldsymbol{A}_1 + \boldsymbol{A}_2)/2$, known as the *double least squares* method.

# Properties of the ITD Method

- Operates on data from free oscillation response,
- and does not need information of the excitation force, thus cannot deliver mass-normalised modes.
- It is well suited to identify closely spaced modes.
- Extracts $n$ modes since $\boldsymbol{A}$ is $n \times n$ if $n$ measurement locations are used in the data matrices $\boldsymbol{X}$.
- If $n < 2m$ there are not sufficient measurement stations to account for all modes present in the system, and the identification quality is bad.
- If $n > 2m$, i.e. the size of $\boldsymbol{A}$ is larger then twice the number of modes contributing to the response one obtains so called *computational* modes.

Several methods exist to identify computational modes, the simplest is visual inspection since they look un-physical.

# Artificial Measurement Positions

To improve the identification quality one can add artificial measurements (artificial sensors) by using time shifting actual measurements $k$ to $l$ by $\Delta t_2$.

$$\boldsymbol{X} = \begin{bmatrix} x_1(t_1) & \cdots & x_1(t_p) \\ \vdots & & \vdots \\ x_n(t_1) & \cdots & x_n(t_p) \\ x_k(t_1 + \Delta t_2) & \cdots & x_k(t_p + \Delta t_2) \\ \vdots & & \vdots \\ x_l(t_1 + \Delta t_2) & \cdots & x_l(t_p + \Delta t_2) \end{bmatrix}$$

The measurement matrix now has more rows, only the first part of the eigenvector (first $n$ elements) constitute the actually measured mode.

# Identifying Computational Modes

Computational modes can be identified by doubling the data matrices using another time shift $\Delta t_3$. The identified mode can then split into an upper and lower part

$$\boldsymbol{v}_k = \begin{bmatrix} {}^{u}\boldsymbol{v}_k \\ {}^{l}\boldsymbol{v}_k \end{bmatrix},$$

which must stratify the fundamental time shift relationship

$${}^{l}\boldsymbol{v}_k = {}^{u}\boldsymbol{v}_k e^{\lambda_k \Delta t_3}.$$

For perfect identification the *mode shape coherence and confidence factor*

$$\mathrm{MSCCF}_k = \frac{{}^{l}\boldsymbol{v}_k^{*}\,{}^{u}\boldsymbol{v}_k e^{\lambda_k \Delta t_3}}{{}^{l}\boldsymbol{v}_k^{*}\,{}^{l}\boldsymbol{v}_k}, \tag{17}$$

must be $\approx 1$, for computational modes it's significantly lower.

## Data matrices for the ITD method

$$\boldsymbol{X} = \begin{bmatrix} x_1(t_1) & \cdots & x_1(t_p) \\ \vdots & & \vdots \\ x_n(t_1) & \cdots & x_n(t_p) \\ x_k(t_1 + \Delta t_2) & \cdots & x_k(t_p + \Delta t_2) \\ \vdots & & \vdots \\ x_l(t_1 + \Delta t_2) & \cdots & x_l(t_p + \Delta t_2) \\ x_1(t_1 + \Delta t_3) & \cdots & x_1(t_p + \Delta t_3) \\ \vdots & & \vdots \\ x_l(t_1 + \Delta t_2 + \Delta t_3) & \cdots & x_l(t_p + \Delta t_2 + \Delta t_3) \end{bmatrix} \tag{18}$$

$$\hat{\boldsymbol{X}} = \begin{bmatrix} x_1(t_1 + \Delta t_1) & \cdots & x_1(t_p + \Delta t_1) \\ \vdots & & \vdots \\ x_l(t_1 + \Delta t_1 + \Delta t_2 + \Delta t_3) & \cdots & x_l(t_p + \Delta t_1 + \Delta t_2 + \Delta t_3) \end{bmatrix} \tag{19}$$

# Single-DoF Methods in Frequency Domain

The basic assumption is that only a single DoF, i.e. mode, contributes to the FRF at a single frequency point. Selecting a natural frequency $\omega_k$ we obtain (from Eq. (7))

$$H_{ab}(\omega_k) \approx \frac{v_{ak} v_{bk}}{2j\zeta_k\omega_k^2} = \frac{C_{ab,k}}{2j\zeta_k\omega_k^2} \tag{20}$$

## Steps of the identification procedure

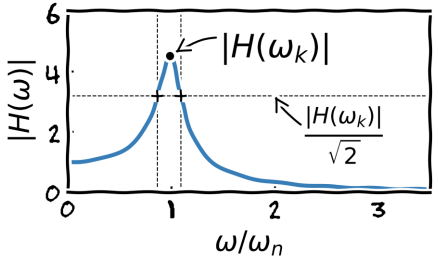
1. Select *frequencies* $\omega_k$ from the FRF data, then
2. determine damping constants $\zeta_k$ at these frequencies, and
3. use them to compute the modal constants $C_{ab,k}$ from the FRF data.
4. Doing this for a complete row or column of the FRF matrix allows to determine all mode shapes: Start with the drive-point FRF ($b = a$) to determine $v_{ak} = \sqrt{C_{aa,k}} = \sqrt{2j\zeta_k\omega_k^2 H_{aa}(\omega_k)}$, then determine the other $v_{bk} = C_{ab,k}/v_{ak}$.

Methods essentially differ only in the way the $\omega_k$ and $\zeta_k$ are determined.

# Peak-Amplitude Method (Peak-Picking)

Take the $k$ peaks of FRF-amplitude curve as the damped natural frequencies $\omega_k$, and estimate the damping ratios from the sharpness of the peaks.

## Half-power method

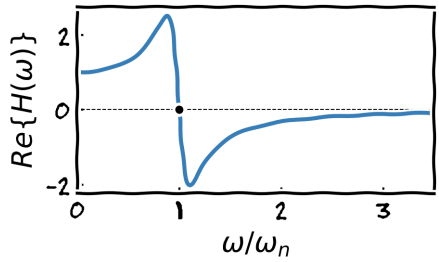



1. Find the half-power width by determining the frequencies $\omega_{2,k} > \omega_k > \omega_{1,k}$ at which the response amplitude is $|H(\omega_k)|/\sqrt{2}$.
2. Estimate the damping ratio from

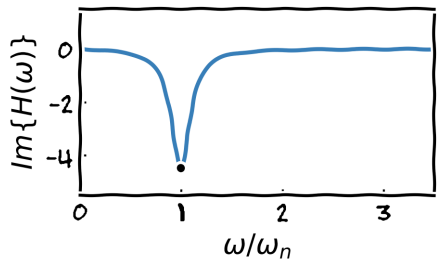
$$\zeta_k = \frac{\omega_{2,k} - \omega_{1,k}}{2\omega_k} \tag{21}$$

# Quadrature Response methods

**Quadrature Response** method locates the $k$ points where the in-phase part of the response is zero, i.e. the real part of the FRF is zero.




**Maximum Quadrature Component** method selects natural frequencies at $k$ points where the quadrature component of the response, i.e. the imaginary part of the FRF, is extremal, i.e. a maximum or minimum.



# Circle Fitting I

We assume the points of the FRF around a natural frequency $\omega_k$ are described by

$$H(\omega) = \frac{C_k}{\omega_k^2 - \omega^2 + 2j\zeta_k\omega_k\omega} + D_k, \tag{22}$$

where the constant $D_k$ is used to take account for the effect of all other modes on mode $k$.

FRF values at $\omega_i$ close to the natural frequency form a circle in the complex plane.

The task is to select $L$ data points $H(\omega_i)$ and fit a circle to them, i.e. we need to minimize

$$e = \sum_{j=1}^{L} \left( R_k - \left| M_k - H(\omega_j) \right| \right)^2, \tag{23}$$

which is a non-linear fit for the unknown circle radius $R_k \in \mathbb{R}$ and center position $M_k \in \mathbb{C}$.

# Circle Fitting II

A smart modification of the error function to

$$e' = \sum_{i=1}^{L} \left( R_k^2 - \left| M_k - H(\omega_i) \right|^2 \right)^2 = \sum_{i=1}^{L} \left( R_k^2 - (x_k - x_i)^2 - (y_k - y_i)^2 \right)^2$$
$$= \sum_{i=1}^{L} \left( c - x_i^2 - a x_i - b y_i - y_i^2 \right)^2, \tag{24}$$

and the suitable new variables

$$a = -2x_k, \qquad b = -2y_k, \qquad c = R_k^2 - x_k^2 - y_k^2, \tag{25}$$

allow to directly minimize $e'$ with respect to $a$, $b$, and $c$ by solving the linear system

$$\frac{\partial e'}{\partial a} = 0, \qquad \frac{\partial e'}{\partial b} = 0, \qquad \frac{\partial e'}{\partial c} = 0. \tag{26}$$

# Circle Fitting III

The linear system to solve is

$$\begin{bmatrix} \sum x_i^2 & \sum x_i y_i & -\sum x_i \\ \sum x_i y_i & \sum y_i^2 & -\sum y_i \\ -\sum x_i & -\sum y_i & L \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\sum x_i^3 - \sum x_i y_i^2 \\ -\sum y_i^3 - \sum x_i^2 y_i \\ \sum x_i^2 + \sum y_i^2 \end{bmatrix}, \tag{27}$$

where the system matrix and right hand side are computed from the $L$ data points

$$x_i = \Re\{H(\omega_i)\}, \qquad y_i = \Im\{H(\omega_i)\}. \tag{28}$$

Solving Eq. (27) and using Eq. (26) allows to compute the circle radius $R_k$ and circle centre $M_k = x_k + jy_k$, respectively.

# Circle Fitting IV

Once the center is determined the natural frequency is found based on a *frequency spacing* technique.

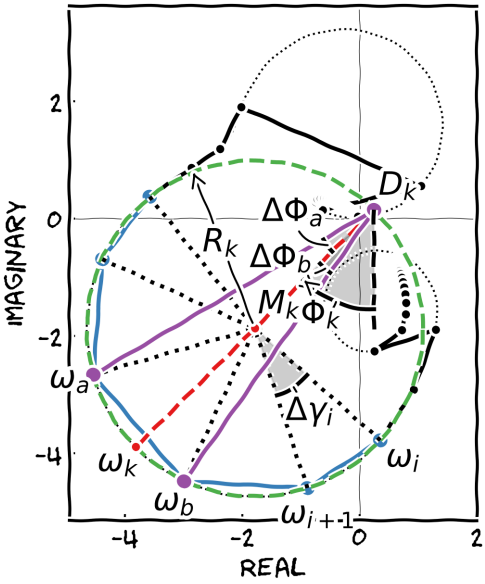
The natural frequency is located where the change in angle $\gamma$ is maximum, i.e. we need to find $\frac{d^2\omega}{d\gamma^2} = 0$.

With the natural frequency one can find the circle shift $D_k$, and the *damping ratio* $\zeta_k$ can be computed by a modified *half power* method

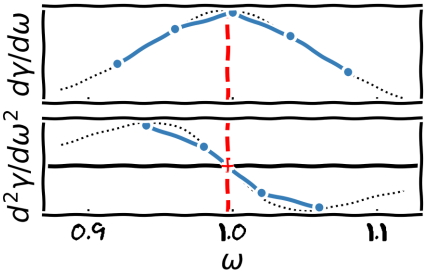
$$\zeta_k = \frac{\omega_a^2 - \omega_b^2}{2\omega_k^2} \frac{1}{\tan(\Delta\phi_a) + \tan(\Delta\phi_b)}. \tag{29}$$

Finally one can compute the *modal constant*, $C_k$ which is related to the circle radius $R_k$ and the circle angle $\phi_k$

$$C_k = 4R_k\omega_k^2\zeta_k e^{j\phi_k}. \tag{30}$$




1. Select points for mode $k$
2. Fit a circle: center $M_k$, radius $R_k$
3. Compute *frequency spacing* $\gamma$
4. Determine natural frequency $\omega_k$
5. Compute damping ratio
6. Compute modal constant



# Multi-DoF Methods in Frequency Domain

Multi-DoF methods typically try to fit the FRF data using a pre-defined number of modes. Examples are the Ewins-Gleeson, and Complex Exponential Frequency Domain methods which are SISO.

The Rational Fraction Polynomial (RFP) method assumes the FRF as

$$H_{ab}(\omega) = \sum_{k=1}^{N} \frac{A_r + j\omega B_r}{\omega_r^2 - \omega^2 + j2\zeta_r\omega_r\omega} = \frac{\sum_{k=0}^{N-1} a_k (j\omega)^k}{\sum_{k=0}^{N-1} b_k (j\omega)^k}. \tag{31}$$

Since the system poles defined by $b_k$ are global, i.e. must be the same for all FRFs, the method can be generalised to a SIMO version, the Global Rational Fraction Polynomial (GRFP). It can also be extended to a MIMO version.

The RFP method is one of the most popular frequency domain methods, and is also known as *Rational Fraction Orthogonal Polynomial* method.

# Comparing Mode Shapes

The *mode scale factor* (MSF) and the *modal assurance criterion* (MAC) describe how well two mode shapes $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ correspond. They are defined by

$$\mathrm{MSF}(\boldsymbol{v}_i, \boldsymbol{v}_j) = \frac{\boldsymbol{v}_i^* \boldsymbol{v}_j}{\boldsymbol{v}_i^* \boldsymbol{v}_i} \neq \mathrm{MSF}(\boldsymbol{v}_j, \boldsymbol{v}_i) \tag{32}$$

$$\mathrm{MAC}(\boldsymbol{v}_i, \boldsymbol{v}_j) = \frac{\left| \boldsymbol{v}_i^* \boldsymbol{v}_j \right|^2}{(\boldsymbol{v}_i^* \boldsymbol{v}_i)(\boldsymbol{v}_j^* \boldsymbol{v}_j)} = \mathrm{MAC}(\boldsymbol{v}_j, \boldsymbol{v}_i) \tag{33}$$

They are insensitive to constant phase shifts and assume an absolute value of 1 for perfect shape correlation.

The MAC is sometimes also called *mode shape correlation coefficient* (MSCC).

# Comparing Sets of Modes

Often one needs to compare two sets of modes *a* and *b*, e.g. the experimentally obtained modes and the analytical modes. To compare

- make sure they use corresponding spacial discretisations,
- compute MAC or MSF for all combinations, forming a matrix.
- The highest entry in a row or column shows which mode of set *a* corresponds best to which modes of set *b*.