Problem Set 4 – trying out numerical methods on Aiyagari

There are two goals for this problem set: Introduce production in the heterogeneous earnings model, and also to experiment with the solution techniques to compute the value function.

Let us consider a version of Aiyagari (1994) which extends Huggett's model by allowing for production within the model. In particular, there now are many competitive, profit-maximizing firms under constant returns to scale. From Macro I we remember that in such a case the firm sector can be modeled as a representative firm. For the production function we use the standard

$$F(K,N) = K^{\alpha}N^{1-\alpha}.$$

Firms take out loans for investment by borrowing on the asset market. An easy way to keep track of things is to think that the firm takes out a loan, buys all their capital stock for next period today, and after production tomorrow they sell the remaining capital on the goods market and repay the loan (since there is no uncertainty this timing convention will not cause any problems). The firm's accounting profits in any period are therefore $\pi(K,N;w_t,r_{t-1}) = F(K_t,N_t) - w_tN_t - r_tK_t + (1-\delta)K_t$. The firm maximization problem is then to maximize profits discounted with the interest rate:

$$\max_{\{K^d_{t+1},N^d_t\}} \sum_{t=0}^{\infty} \left(\frac{1}{\prod_{i=0}^{t} r_i} \right) \pi(K,N;w_t,r_t).$$

This looks more difficult than it is – it's really just a repeated static optimization problem.

The household side stays largely the same, except for the income side of the budget constraint. Instead of the exogenous variable being $s \in \{e,u\}$ the binary employment state of a worker as in Huggett, now we introduce the continuous random variable z which represents the worker's labor efficiency. We continue to assume that every worker supplies one unit of labor inelastically in the labor market, so $l_t = \bar{l} = 1$. A household's budget constraint is then

$$c_t + a_{t+1} = z_t w_t \bar{l} + r_t a_t.$$

Labor productivity follows an AR-1 in logs: $\ln z_{t+1} = \rho \ln z_t + \varepsilon_t$ where $\varepsilon \sim N(0,\sigma_\varepsilon^2)$. As in Huggett's model, households face a borrowing constraint prescribing $a_{t+1} > \underline{a}$. There is a unit mass of households so there is no aggregate uncertainty. They maximize expected utility, and have a CRRA utility function with risk aversion parameter σ.

Calibration We can use standard parameters by setting $\alpha = 1/3$,$\beta = 0.99$ and $\sigma = 2$, respectively. For now, let's say there is no borrowing at all and set $\underline{a} = 0$. The remaining two parameters we can take from looking at US data on

earnings. Aiyagari cites studies that find $\rho = 0.5$ and $\sigma_\varepsilon = 0.2$ by looking at households' labor incomes over time in the PSID.

We now want to solve the model for the steady-state equilibium in which all aggregate variables are constant over time. Proceed in the following way:

1. Write down the first-order conditions for the firm. They should look very familiar and tell you what factor prices are for given K and N!

2. Formulate the households' recursive problem.

3. Discretize the exogenous state variable into a grid with m points using ~~Tauchen's~~ Rouwenhorst's method (use the provided Matlab function), which will give you a discrete set of possible values for z as well as an $m \times m$ transition matrix Π. Let's use $m = 5$ for now. Find the invariant distribution of the implied Markov process over productivity states $\pi^{inv}(z)$. What is the aggregate *effective* labor supply $N^s = \int_Z z\bar{l}\pi^{inv}(dz)$? Given that households' labor supply is constant, we can use labor market clearing $N^s = N^d$ to figure out factor prices below.

4. Discretize the endogenous state variable into a grid with n points. Let's say $n = 500$. While the lowest gridpoint $a_{min} = \underline{a}$, we are not sure at this time how large to choose the upper bound for the grid. As discussed in class, we need to guess a_{max} and verify later that it does not constitute a binding constraint on how many assets households wish to hold.

5. Now start solving the model numerically. Liberally recycle your code from the previous homework!

   (a) Guess an equilibrium value for the aggregate capital stock K, and find the factor prices associated with this guess.

   (b) Given the factor prices w,r solve the households' recursive problem and find the policy function.

   (c) Given the policy function and the transition matrix, find the steady state distribution $\mu(z, a)$ over assets and labor productivity.

   (d) Check if the asset market clears, i.e. whether $\int_{Z,A} g(z, a; w, r)\,\mu(dz, da) = K$ (up to a small tolerance). If no, adjust the guess for the capital stock. If yes, you have found the steady state!

6. Analyze your results: What is the steady state interest rate, and how does it compare the the complete markets case in which $r^{CM} = 1/\beta$? Plot the policy functions for the m productivity states. Plot the Lorenz curve and compute the wealth gini (this time, we can take wealth to just be asset holdings, since there is now positive net supply of assets!). Describe how the wealth distribution compares to the Huggett model, and to the empirical wealth distribution.

7. Use the alternative ways to solve for the value function we discussed in class. In particular, try out

- solving the value function on a coarse grid, and using the result as a starting value (use linear interpolation)
- policy function iteration (use $k = 30$ policy iterations for each optimization step)
- linear interpolation
- extra credit: cubic spline interpolation. For both interpolation procedures, reduce the number of points for the endogenous variable n by a lot to, say, 12. In the very end, interpolate the policy function up to n points again.

For each method keep track of the runtime and compute the Euler equation error. How do the methods compare on accuracy and speed?