Functional Programming

Excercise Sheet 3

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Excercise 1

a)

```
collatz :: Int -> [Int]
collatz x = iterate (\y -> if mod y 2 == 0 then div y 2 else 3*y+1) x


total_stopping_time :: Int -> Int
total_stopping_time 1 = 3
total_stopping_time x = length (takeWhile (\y -> y /= 1) (collatz x))
```

b)

```
check_collatz :: Int -> Bool
check_collatz 1 = False
check_collatz n = elem 1 (take n (collatz n))
```

Excercise 2

a)

```
drop_mult :: Int -> [Int] -> [Int]
drop_mult x xs = [y | y <- xs , mod y x /= 0]

dropall :: [Int]-> [Int]
dropall (x:xs) = x : dropall (drop_mult x xs)

primes :: [Int]
primes = dropall [2 ..]

goldbach :: Int -> [(Int,Int)]
goldbach n = [(x,y)| mod n 2 == 0, x<-takeWhile (<n) primes, y<-filter
   (>=x) (takeWhile (<n) primes), x+y == n]
```

b)

```
range :: [a] -> Int -> Int -> [a]
range xs a b = [xs !! i| i <- [a .. b]]
```

Excercise 3

Excercise 4

a)

$$\begin{tikzcd}
 & ((\text{-}1,\text{False}),0) = y_1 & \\
((\perp,\text{False}),0) \arrow[ur] & ((\text{-}1,\perp),0) \arrow[u] & ((\text{-}1,\text{False}),\perp) \arrow[ul] \\
((\text{-}1,\perp),\perp) \arrow[u] \arrow[ur] & ((\perp,\text{False}),\perp) \arrow[ul] \arrow[ur] & ((\perp,\perp),0) \arrow[ul] \arrow[u] \\
 & ((\perp,\perp),\perp) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

All elements $x \in ((\mathbb{Z}_\perp \times \mathbb{B}_\perp) \times \mathbb{Z}_\perp$ that are less defined than y_1 are given by $((-1,\perp),0),((\perp,False),0),((-1,False),\perp),((\perp,False),\perp),((-1,\perp),\perp),((\perp,\perp),0)$ and the smallest Element $((\perp,\perp),\perp)$.

$$\begin{tikzcd}
 & ((\text{-}1,\perp),2) = y_2 & \\
((\text{-}1,\perp),\perp) \arrow[ur] & & ((\perp,\perp),2) \arrow[ul] \\
 & ((\perp,\perp),\perp) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

All elements $x \in ((\mathbb{Z}_\perp \times \mathbb{B}_\perp) \times \mathbb{Z}_\perp$ that are less defined than y_2 are given by $((\perp,\perp),2),((-1,\perp),\perp)$ and the smallest Element $((\perp,\perp),\perp)$.

b)
Given a domain $D = \underbrace{\mathbb{Z}_\perp \times ... \times \mathbb{Z}_\perp}_{n \text{ times}}$ for $0 < n \in \mathbb{N}$ and a chain $S \subseteq D$ then $\sup\{|S| \mid S \subseteq D, \text{S is a chain}\} = n + 1$.

We use induction on n to show that this holds for all $n \in \mathbb{N}, n > 0$.

Base case. $n = 1$
Let $D = \mathbb{Z}_\perp$, then $\perp \in D$ and for all $x \in \mathbb{Z} : x \in \mathbb{Z}_\perp$. For $x, y \in S$ with $x, y \in \mathbb{Z}$ it follows from the definition of S that either $x \sqsubseteq y$ or $y \sqsubseteq x$. By the definition of $\sqsubseteq$ and x and y, this is only true iff $x = y$. As a result, $|\{x \in \mathbb{Z} \mid x \in S \text{ with } S \subseteq D, \text{S is a chain}\}| \leq 1$.
However, $\perp \sqsubseteq x$ for all $x \in \mathbb{Z}$ hence $\sup\{|S| \mid S \subseteq D, \text{S is a chain}\} = 1 + 1 = 2 = n + 1$.

Induction hypothesis
$\sup\{|S| \mid S \subseteq D, \text{S is a chain}\} = n+1$ holds for $n \in \mathbb{N}, 0 < n$ with $D = \underbrace{\mathbb{Z}_\perp \times ... \times \mathbb{Z}_\perp}_{n \text{ times}}$.

Induction step $n \to n + 1$
Let $D = \underbrace{\mathbb{Z}_\perp \times ... \times \mathbb{Z}_\perp}_{n+1 \text{ times}}$.
It follows from the hypothesis that if $D_n = \underbrace{\mathbb{Z}_\perp \times ... \times \mathbb{Z}_\perp}_{n \text{ times}}$ then $\sup\{|S_n| \mid S_n \subseteq D_n, S_n \text{ is a chain}\} = n + 1$.

Now we extend each n-tuple $(n_1, n_2, ..., n_n)$ in the chain S_n $(n_1, n_2, ..., n_n \in \mathbb{Z}_\perp)$ so that we get (n+1)-tuples $(n_1, n_2, ..., n_n, x) \in D, x \in \mathbb{Z}_\perp$.

Case 1: $x \in \mathbb{Z}$
For two tuples $t_i = (n_{i_1}, n_{i_2}, ..., n_{i_n}), t_j = (n_{j_1}, n_{j_2}, ..., n_{j_n}) \in S_n$ either $t_i \sqsubseteq t_j$ or $t_j \sqsubseteq t_i$ applies. Then also $(n_{i_1}, n_{i_2}, ..., n_{i_n}, x) \sqsubseteq (n_{j_1}, n_{j_2}, ..., n_{j_n}, x)$ or $(n_{j_1}, n_{j_2}, ..., n_{j_n}, x) \sqsubseteq (n_{i_1}, n_{i_2}, ..., n_{i_n}, x)$ since in particular $x \sqsubseteq x$ and consequently $(n_{j_1}, n_{j_2}, ..., n_{j_n}, x)$, $(n_{i_1}, n_{i_2}, ..., n_{i_n}, x) \in S$ with $\sup\{|S| \mid S \subseteq D, \text{S is a chain}\} \geq \sup\{|S_n| \mid S_n \subseteq D_n, S_n \text{ is a chain}\} = n + 1$.
If $(n_1, n_2, ..., n_n, x) \in S$ and $(n_1, n_2, ..., n_n, y) \in S$ then $x = y$ because neither $x \sqsubseteq y$ nor $y \sqsubseteq x$ applies. Similarily, If $t_i = (n_1, n_2, ..., n_n, x) \in S$ and $t_j = (n_1, n_2, ..., x, ..., n_n) \in S$ then neither $t_i \sqsubseteq t_j$ nor $t_j \sqsubseteq t_i$ applies, because $t_i \neq t_j$ and both are equally defined. This leads on to our second case.

Case 2: $(n_1, n_2, ..., n_n, x), n_1 = n_2 = ... = n_n = x = \perp$
Then clearly $(n_1, n_2, ..., n_n, x) \in S$ since it is the smallest element of D and therefore $(n_1, n_2, ..., n_n, x) \sqsubseteq y$ for all $y \in D$. Resulting from this, $\sup\{|S| \mid S \subseteq D, \text{S is a chain}\} = (n + 1) + 1 = n + 2$. The equation holds. $\square$