Functional Programming
Excercise Sheet 4

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Excercise 1

a) $f_{plus} : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp, f_{plus}(x, y) = \begin{cases} y & x = 0 \\ x & y = 0 \\ x + y & x, y \in \mathbb{Z} \\ \perp & otherwise \end{cases}$

b) We will show that f_{Plus} is strict, i.e. if $x_i = \perp$ for $1 \leq i \leq 2$ then $f_{Plus}(x_1, x_2) = \perp$.

Proof:
case 1: $x_1 = 0$ and $x_2 = \perp$. It follow: $f_{Plus}(x_1, x_2) = x_2 = \perp$ since $x_1 = 0$
case 1: $x_1 \neq 0$ and $x_2 = \perp$. It follow: $f_{Plus}(x_1, x_2) = \perp$ since $x_2 \notin \mathbb{Z}$ and $x_1 \neq 0$
case 1: $x_2 = 0$ and $x_1 = \perp$. It follow: $f_{Plus}(x_1, x_2) = x_1 = \perp$ since $x_2 = 0$
case 1: $x_2 \neq 0$ and $x_1 = \perp$. It follow: $f_{Plus}(x_1, x_2) = \perp$ since $x_1 \notin \mathbb{Z}$ and $x_2 \neq 0$

c) We will show that f_{Plus} is monotonic, i.e. if $d \sqsubseteq_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} d'$ then $f_{Plus}(d) \sqsubseteq_{\mathbb{Z}_\perp} f_{Plus}(d')$.

Proof:
Let $d \sqsubseteq_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} d'$ with $d, d' \in \mathbb{Z}_\perp \times \mathbb{Z}_\perp$. Then either $d = d'$ or d is less defined than d'.
If $d = d'$ then $f_{Plus}(d) = Plus(d')$, thus $f_{Plus}(d) \sqsubseteq Plus(d')$.
Otherwise $d \neq d'$. Let's say $d = (d_1, d_2)$ and $d' = (d'_1, d'_2)$ with $(d_1, d_2) \sqsubseteq (d'_1, d'_2)$. Then there exists an index $1 \leq i \leq 2$ with $d_i \neq \perp, d'_i \in \mathbb{Z}$ as well as $j \neq i$ with $d_j = d'_j$. Since f_{Plus} is strict $f_{Plus}(d) = \perp$ because $d_i = \perp$ for $1 \leq i \leq 2$.

Case 1: $d_j = d'_j = \perp$ then we know from the strictness of f_{Plus} that $f(d') = \perp$. $f_{Plus}(d) = \perp \sqsubseteq_{\mathbb{Z}_\perp} \perp = f_{Plus}(d')$ holds.

Case 2: $d_j = d'_j \neq \perp$ with $d_j, d'_j \in \mathbb{Z}$ then $d' \in \mathbb{Z}_\perp \times \mathbb{Z}_\perp$ and $f(d') = a \in \mathbb{Z}$. a is more defined than $\perp$, thus $f_{Plus}(d) = \perp \sqsubseteq_{\mathbb{Z}_\perp} a = f_{Plus}(d'), a \in \mathbb{Z}$ holds.

Excercise 2

a)
$-' : \mathbb{Z}_\perp \to \mathbb{Z}_\perp, -'(x) = \begin{cases} -x & x \in \mathbb{Z} \\ \perp & otherwise \end{cases}$

b)
$*' : \mathbb{N}_\perp \times \mathbb{N}_\perp \to \mathbb{N}_\perp, *'(x, y) = \begin{cases} x * y & x, y \in \mathbb{N} \\ \perp & otherwise \end{cases}$

$$max : \mathbb{N}_\perp \times \mathbb{N}_\perp \to \mathbb{N}_\perp, max'(x,y) = \begin{cases} x & x > y \\ y & y \geq x \\ \perp & otherwise \end{cases}$$

This is not monotonic is it? $max : \mathbb{N}_\perp \times \mathbb{N}_\perp \to \mathbb{N}_\perp, max'(x,y) = \begin{cases} x & x > y \\ y & otherwise \end{cases}$

Excercise 3

a) To show: If $\sqsubseteq_{D_1 \to D_2}$ is complete on $D_1 \to D_2$ then $\perp_{D_2}$ exists.

Proof:
Let $\sqsubseteq_{D_1 \to D_2}$ be complete on $D_1 \to D_2$, then $D_1 \to D_2$ has a smallest element with respect to $\sqsubseteq_{D_1 \to D_2}$. This element is denoted $\perp_{D_1 \to D_2}$. Since $\perp_{D_1 \to D_2}$ is the smallest element $\perp_{D_1 \to D_2} \sqsubseteq f$ holds for every $f : D_1 \to D_2$ i.e. $\perp_{D_1 \to D_2}(d) \sqsubseteq f(d)$ holds for every $d \in D_1$.
We say that $\perp_{D_1 \to D_2}(d) = a \in D_2$ and $\perp_{D_1 \to D_2}(d') = b \in D_2$. Then $a \sqsubseteq f(d)$ and $b \sqsubseteq f(d')$ with $d, d' \in D_1$. Now if $a \sqsubseteq b$ then a function g that maps every element of D_1 to a would exists with $g \sqsubseteq \perp_{D_1 \to D_2} \sqsubseteq f$ and $a \sqsubseteq f(d)$, $a \sqsubseteq b \sqsubseteq f(d')$ for every $f : D_1 \to D_2$. $\lightning$ $\perp_{D_1 \to D_2}$ is the smallest element! Therefore, $a \sqsubseteq b$ and similarily $b \sqsubseteq a$ only holds iff $a = b$.
With that we know that $\perp_{D_1 \to D_2}$ is a constant function.
It follows from $\perp_{D_1 \to D_2}(d) = a \sqsubseteq f(d)$ (for every $d \in D_1$) that $a \sqsubseteq x$ for every $x \in D_2$. Hence, D_2 must have a smallest element a with respect to $\sqsubseteq_{D_2}$ which we denote as $\perp_{D_2}$.

b) To show: If $\sqsubseteq_{D_1 \to D_2}$ is complete on $D_1 \to D_2$, then for all chains S on D_2 the least upper bound $\sqcup S$ of S exists in D_2.

Proof:
Let $\sqsubseteq_{D_1 \to D_2}$ be complete on $D_1 \to D_2$, then for every chain S of $D_1 \to D_2$ there exists a least upper bound $\sqcup S \in D_1 \to D_2$. For functions $f, f' \in S$ either $f \sqsubseteq f'$ or $f' \sqsubseteq f$ holds, i.e. $f(d) \sqsubseteq f'(d)$ or $f'(d) \sqsubseteq f(d)$ holds for every $d \in D_1$. It follows that for every $d \in D_1$ $S_d = \{f(d) | f \in S\}$ is a chain in D_2.
From Lemma 2.1.11 b) follows that every chain S_d has a least upper bound $\sqcup S_d$ since S has a least upper bound $\sqcup S$ ($\sqsubseteq_{D_1 \to D_2}$ is complete). Furthermore, $\sqcup S(d) = \sqcup S_d \in D_2$.

Excercise 4

a) To show: Let $a, b \in \mathbb{R}$. The standard ordering $\leq$ on the real numbers is complete on the closed interval $[a, b]$, i.e. $[a, b]$ has a smallest element w.r.t. $\leq$ and for every chain $S \subseteq [a, b]$ there exists a least upper bound $supS \in [a, b]$.

Proof:
The smallest element in $[a, b] = \{r | a \leq r \leq b\}$ with respect to $\leq$ is a, since $a \leq x$ for every $x \in [a, b]$ (this follows from the definition of a closed interval).
Since $[a, b]$ is bounded from above by b ($x \leq$ for every $x \in [a, b]$), every chain $S \subseteq [a, b]$ S is bounded from above as well:

Case 1. $S = [c, b] = \{r|c \leq r \leq b\}$ with $c \in [a, b]$. Then S is bounded from above by b.
Case 2. $S = [c, d] = \{r|c \leq r \leq d\}$ with $c, d \in [a, b]$ and $c \leq d$. Then S is bounded from above by d.
We know that $\mathbb{R}$ is complete, therefore all chains $S \subseteq [a, b]$ have a least upper bound with $supS = d$ for $S = [c, d] \subseteq [a, b]$.
Thus, the standard ordering $\leq$ on the real numbers is complete on the closed interval $[a, b]$.