

Functional Programming

Exercise Sheet 4

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Exercise 1

$$\text{a) } f_{plus} : \mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}, f_{plus}(x, y) = \begin{cases} y & x = 0 \\ x & y = 0 \\ x + y & x, y \in \mathbb{Z} \\ \perp & \text{otherwise} \end{cases}$$

b) We will show that f_{plus} is strict, i.e. if $x_i = \perp$ for $1 \leq i \leq 2$ then $f_{plus}(x_1, x_2) = \perp$.

Proof:

case 1: $x_1 = 0$ and $x_2 = \perp$. It follow: $f_{plus}(x_1, x_2) = x_2 = \perp$ since $x_1 = 0$

case 1: $x_1 \neq 0$ and $x_2 = \perp$. It follow: $f_{plus}(x_1, x_2) = \perp$ since $x_2 \notin \mathbb{Z}$ and $x_1 \neq 0$

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c) We will show that f_{plus} is monotonic, i.e. if $d \sqsubseteq_{\mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp}} d'$ then $f_{plus}(d) \sqsubseteq_{\mathbb{Z}_{\perp}} f_{plus}(d')$.

Proof:

Let $d \sqsubseteq_{\mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp}} d'$ with $d, d' \in \mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp}$. Then either $d = d'$ or d is less defined than d' .

If $d = d'$ then $f_{plus}(d) = plus(d')$, thus $f_{plus}(d) \sqsubseteq plus(d')$.

Otherwise $d \neq d'$. Let's say $d = (d_1, d_2)$ and $d' = (d'_1, d'_2)$ with $(d_1, d_2) \sqsubseteq (d'_1, d'_2)$. Then there exists an index $1 \leq i \leq 2$ with $d_i \neq \perp, d'_i \in \mathbb{Z}$ as well as $j \neq i$ with $d_j = d'_j$. Since f_{plus} is strict $f_{plus}(d) = \perp$ because $d_i = \perp$ for $1 \leq i \leq 2$.

Case 1: $d_j = d'_j = \perp$ then we know from the strictness of f_{plus} that $f(d') = \perp$. $f_{plus}(d) = \perp \sqsubseteq_{\mathbb{Z}_{\perp}} \perp = f_{plus}(d')$ holds.

Case 2: $d_j = d'_j \neq \perp$ with $d_j, d'_j \in \mathbb{Z}$ then $d' \in \mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp}$ and $f(d') = a \in \mathbb{Z}$. a is more defined than \perp , thus $f_{plus}(d) = \perp \sqsubseteq_{\mathbb{Z}_{\perp}} a = f_{plus}(d'), a \in \mathbb{Z}$ holds.

Exercise 2

Exercise 3

Exercise 4