Announcement: Seminar on characteristic classes

This is the announcement for the upcoming (online) seminar on **characteristic classes of manifolds and singular spaces** organized by Nivaldo de Góes Gruhla Júnior, Matthias Zach and Timo Essig.

Time	Tuesday, 2pm Brazilian time (= 6pm European time)
Platform	BBB/Jisti/Zoom
Link	*will be added later*

Idea of the seminar

Since the participants are scattered all over the world, we plan to have it online. This is a research oriented seminar and some kind of mix of a classical seminar, a research discussion and a reading course. One session should consist of a concise presentation of the day's topic, lasting approximately half an hour, and then mostly well prepared and guided examples and exercises. Thus, it is the speaker's responsibility to gather and organize the material in such a way that the other participants obtain the possibility to learn the topic "hands on".

Organization of the seminar

Information will be distributed via the Github Wiki created for the seminar. Files like the one you are looking at are saved on that repository as well. If you have any questions concerning the wiki or git in general, feel free to email Timo.

It would be nice if everyone who is considering participation could send an email to Matthias, in which they briefly describe their particular topics of interest, further suggestions and, if applicable, connections to their own research.

Overview on the topics of the seminar

Disclaimer: To not get lost into too many technical details, we focus on the complex setting and mainly work with Chern classes.

1. **Introduction** by *M. Zach, T. Essig* on 03/29/2022
   Examples for applications of characteristic classes on smooth manifolds are the starting point for the seminar. We present some relations to fiber bundle theory and to bordism theory. Via the Poincaré-Hopf-Theorem we get to Grothendieck's conjecture for the existence of Chern classes on singular spaces.

2. **CCs on manifolds via obstruction theory 1** by *Edmundo Martins* on 04/05/2022

3. **CCs on manifolds via obstruction theory 2** by *Edmundo Martins* on 04/05/2022
   In these two talks we learn about the obstruction theory apporach - an algebraic topology approach to characteristic classes.

4. **CCs on manifolds via differential forms** by *Ivan* on 04/19/2022
   An alternative approach to Chern classes on smooth manifolds following Bott-Tu (?)

5. **Functorial approach to CCs on manifolds** by *M. Zach* on 04/26/2022 This talk contains a more axiomatic approach to CCs on manifolds, the axioms for them and the question for a general framework that covers the different approaches. We examine the Riemann-Roch-Theorem in its different forms.

The topics for the other talks are not fully determined yet. After these first talks, we want to focus on characteristic classes on singular spaces. We cover MacPherson's solution to the Grothendieck conjecture first and understand his construction of Chern classes using the local Euler obstruction ([9, 11]). After that, we examine the Fulton class and the Milnor class, that measures the deviation of the two aforementioned classes.

Unordered collection of ideas and topics

In the following, you can find a rather unordered list of topics that will be or might be covered in the seminar. We will pick the interesting and reachable topics during the seminar, so we do not chain ourselves to the list.

Characteristic classes of manifolds

- Introduce the Chern class calculus: What are the axioms. How does it work? Why is it useful?
- Present some explicit approaches to Chern classes: Chern-Weyl theory for smooth de Rham cohomology, algebraic Chern classes taking values in the duals of Chow groups, Chern classes via topological obstruction theory, and Chern classes of holomorphic vector bundles taking values in algebraic de Rham cohomology.

  Can we see examples for these? Can we explain in which sense these theories are "all the same", i.e. can we describe the functoriality?
- How can other characteristic classes, such as Pontrjagin and L-classes be defined? Present the alternative approach to L-classes via maps to spheres and explain again, why theses theories are "the same".
- What is bordism and surgery, why are these concepts important and what is the relation to characteristic classes?
- K-theory: What is the K-group $K^0(X)$ of a topological space X? What is the Grothendieck group of coherent sheaves $K_0(X)$? What can it be used for? Spoiler: These K-groups should sit at the top of all the theory of characteristic classes that were discussed in the previous paragraph in the sense that all characteristic classes are functors factoring through these K-groups. Can we make these functors explicit?
- What is a Riemann-Roch-type theorem?
- What does Motivic integration have to do with this?

Characteristic classes for singular varieties

- What does the "Grothendieck-Deligne conjecture" say? Explain this in explicit examples.
- Explain the original work of MacPherson [9]: What is the local Euler obstruction and what is it used for?
- What is a "characteristic class" in general? There are others, such as, for instance, the L-class. How is the L-class defined for sing. varieties in contrast to manifolds?

Characteristic classes in singularity theory

- What is the "Buchsbaum-Rim multiplicity"?
- What are the Fulton-Johnson and the Milnor classes?
- What is the Euler obstruction of a 1-form?

References

Characteristic classes on manifolds

- [10]: Rather broad book on characteristic classes of real and complex vector bundles.
- [8]: Touches upon obstruction theory.
- [4]: Treats, for instance, Characteristic classes with values in smooth Cech-de-Rham cohomology in an elementary way.
- [6]: The reference for Chern classes and intersection theory of algebraic schemes; mostly using Chow groups.
- [7]: Monograph on the proof of the Riemann-Roch-Theorem.
- [13]: Short (and steep) introduction to CCs in Chapter 5.
- [1]: Covers the case of complex analytic vector bundles on projective manifolds using Hodge theory.

Characteristic classes on singular spaces

The first four references are survey like. All of them have different goals and come from different time frames.

- [15] is a survey that is "an appendix to MacPherson's" survey, which I had a hard time finding in the internet. A newer survey is [14].
- [12] is a rather general survey with a nice exposition, which later focusses on motivic CCs. Those are probably beyond the scope of our seminar.
- [5] surveys CCs on manifolds via obstruction theory and Schubert cycles and the generalizations of these ideas to singular spaces. It is a good overview and does not cover many technical details.

- [11] is a survey on Milnor classes, which also covers the basics on the MacPherson and Fulton classes. The approach is via characteristic cycles.
- [9] is MacPhersons original article on his approach to the Chern class for singular spaces. He makes use of the local Euler obstruction in the construction. We will focus on this article in the second part of the seminar.
- [2] and [3]: discuss the Fulton class and Riemann-Roch-type theorems for both algebraic and topological K-theories.

References

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