Universal Bell Correlations Do Not Exist

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CS395T – Quantum Complexity Theory

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Quantum nonlocality

- Recall Bell's theorem: Entanglement allows interactions that can't be simulated using shared randomness / hidden variables

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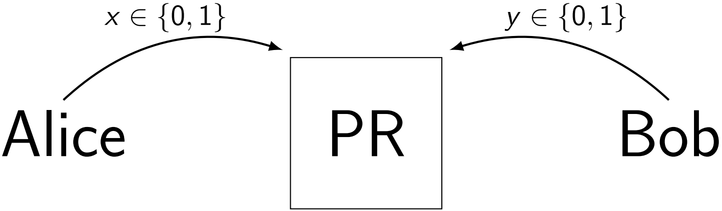
- Recall the no-communication theorem: Entanglement can't be used to send signals
- Contradictory?

PR box

Alice $\boxed{\text{PR}}$ Bob

PR box



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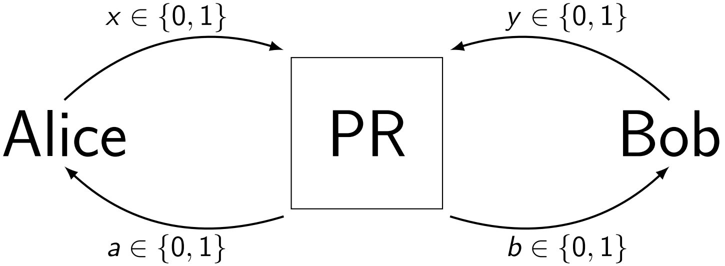

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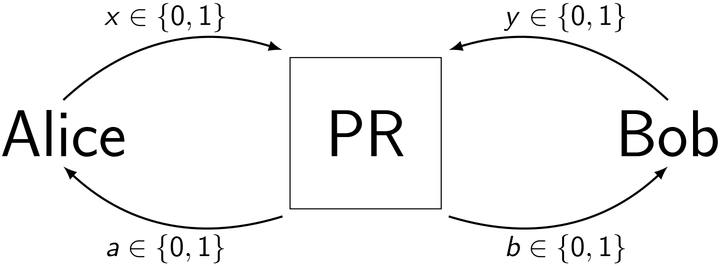

$$(a,b) = \begin{cases} (0, xy) & \text{with probability } 1/2 \\ (1, 1-xy) & \text{with probability } 1/2 \end{cases}$$

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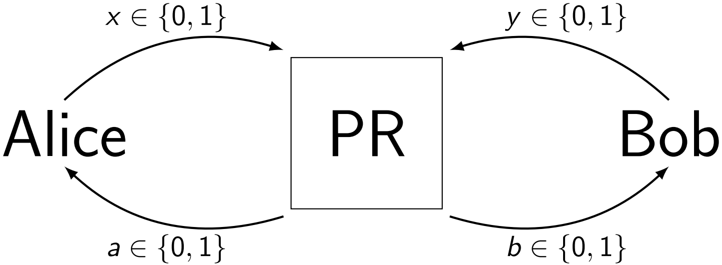

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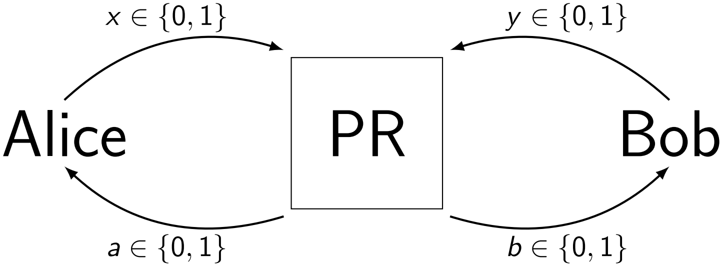

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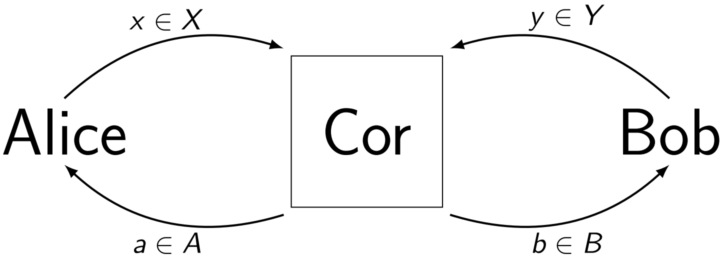
- Cannot be used to communicate
- But can be used to win CHSH game: $a + b = xy \pmod 2$

Correlation box



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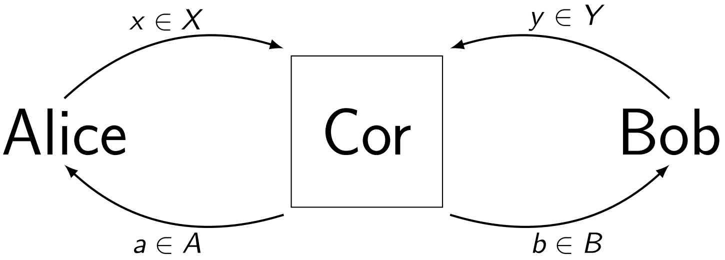

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  $$\mathrm{Cor} : X \times Y \to \{\mu : \mu \text{ is a probability distribution over } A \times B\}$$

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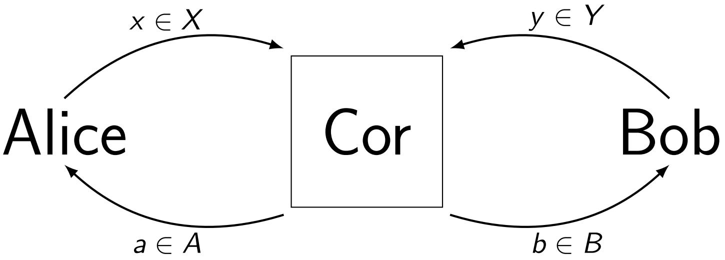

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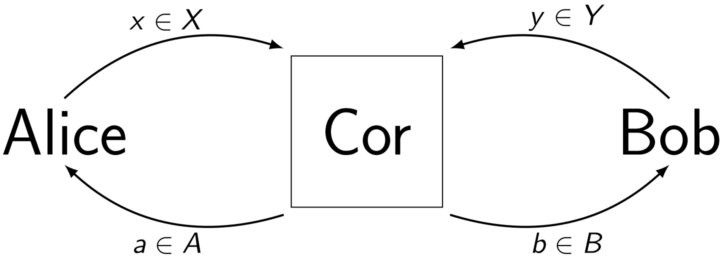

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  $$\mathrm{Cor} : X \times Y \to \{\mu : \mu \text{ is a probability distribution over } A \times B\}$$

- Assume X, Y, A, B are countable
- Abuse notation and write $\mathrm{Cor} : X \times Y \to A \times B$

Distributed sampling problems

$$
\begin{tikzcd}
 & \text{Referee} \arrow[dl, "x \in X"'] \arrow[dr, "y \in Y"] & \\
\text{Alice} \arrow[ur, "a \in A"'] & & \text{Bob} \arrow[ul, "b \in B"]
\end{tikzcd}
$$

- Can think of a correlation box as a *distributed sampling problem* – the problem of simulating the box

Distributed sampling complexity classes

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- Bell's theorem: $\mathbf{SR} \neq \mathbf{Q}$

Distributed sampling complexity classes (2)

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- Tsierelson bound: $\mathrm{PR} \notin \mathbf{Q}$, so $\mathbf{Q} \neq \mathbf{NS}$

Bell pair

- Goal: Understand $\mathbf{Q}$

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- **SR** $\subsetneq$ **BELL** $\subsetneq$ **Q**

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- Loose upper bound, since **BELL** $\subseteq$ **NS**

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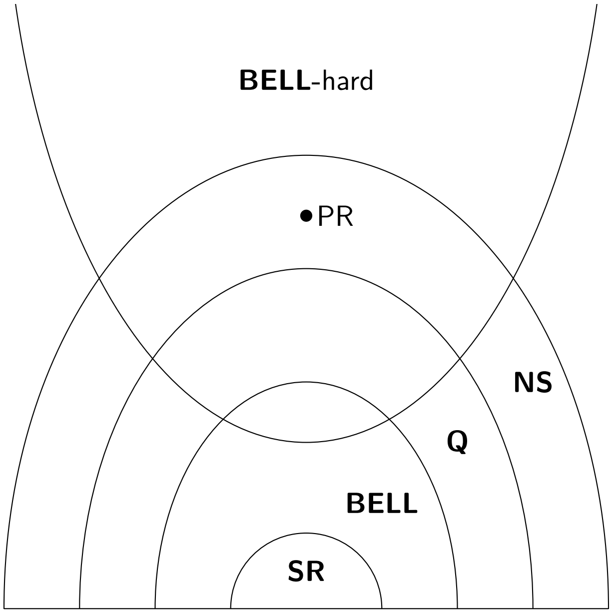
- In other words, PR is **BELL**-hard with respect to 1-query reductions

Distributed sampling complexity zoo



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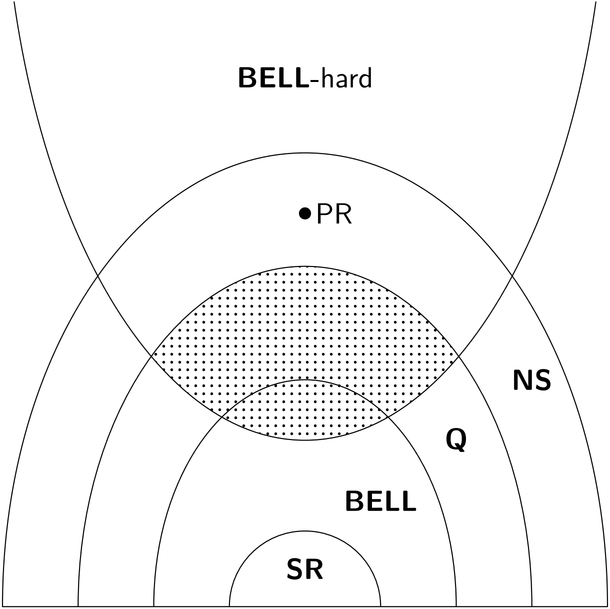
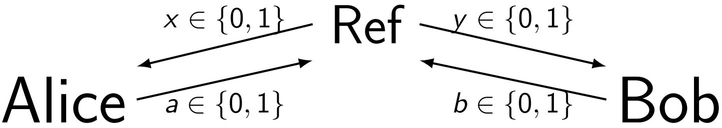


Biased CHSH game



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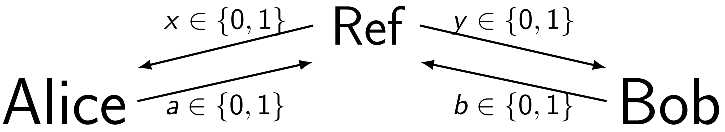

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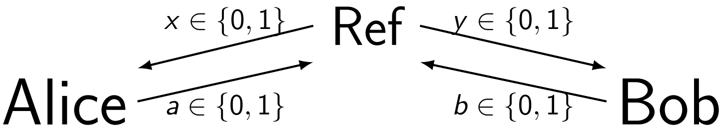

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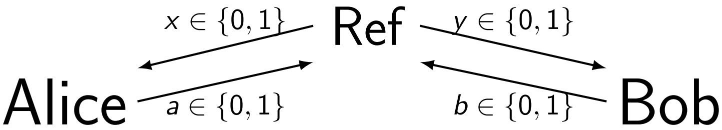

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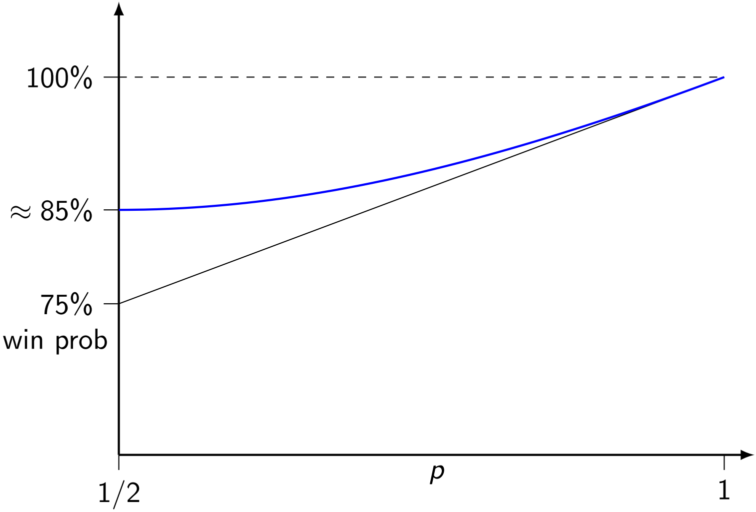
- Inputs x, y are chosen independently at random
- y is uniform, x is biased: $\Pr[x = 1] = p \in [1/2, 1]$
- Theorem (Lawson, Linden, Popescu '10): Optimal quantum strategy can be implemented in **BELL**, wins with probability

$$f(p) \stackrel{\text{def}}{=} \frac{1}{2} + \frac{1}{2}\sqrt{p^2 + (1-p)^2}$$

Quantum value of biased CHSH game



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- Affine function of p, for fixed reduction

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- Recall Cor : $X \times Y \to A \times B$ with X, Y countable, A, B finite
- Only countably many deterministic reductions!
- Countably many affine functions, so $\exists p$ where all the affine functions disagree with $f(p)$

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  - $\mathrm{Cor}_2 \in \mathbf{BELL}$
  - For every $\mathrm{Cor}_1 \in \mathbf{BELL}$, there is a 1-query ε-error reduction from Cor_1 to Cor_2

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- Thanks for listening! Questions?
- This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-1610403.
- Cole Graham gratefully acknowledges the support of the Fannie and John Hertz Foundation.