Universal Bell Correlations Do Not Exist

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CS395T – Quantum Complexity Theory

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Quantum nonlocality

- Recall Bell's theorem: Entanglement allows interactions that can't be simulated using shared randomness / hidden variables

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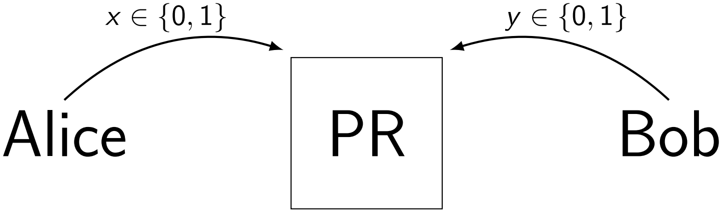
- Recall the no-communication theorem: Entanglement can't be used to send signals
- Contradictory?

PR box

Alice $\boxed{\text{PR}}$ Bob

PR box



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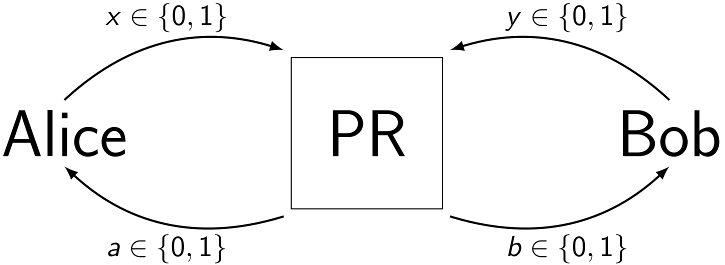

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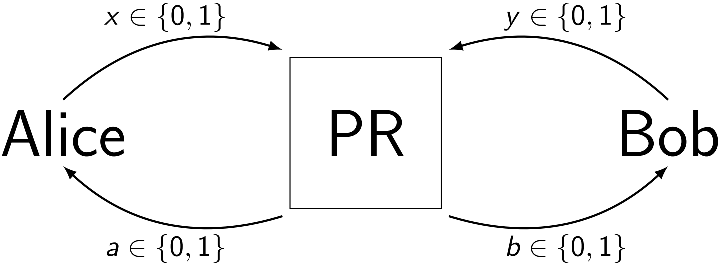

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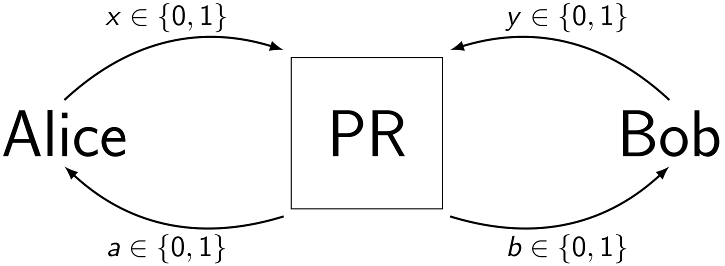

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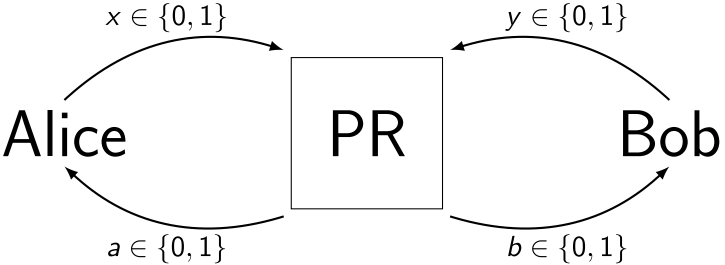

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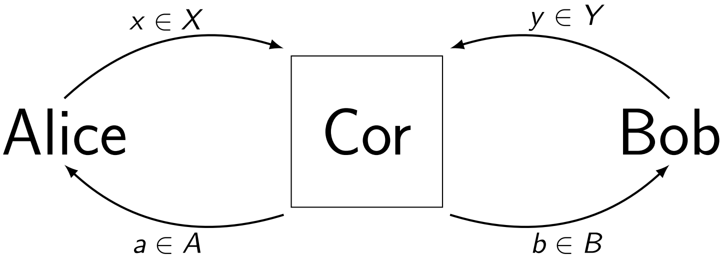
- Cannot be used to communicate
- But can be used to win CHSH game: $a + b = xy \pmod 2$

Correlation box



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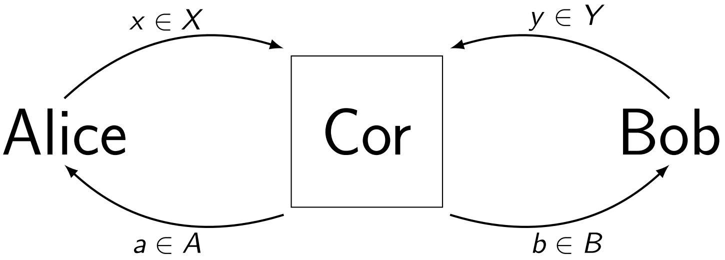

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  Cor : $X \times Y \to \{\mu : \mu$ is a probability distribution over $A \times B\}$

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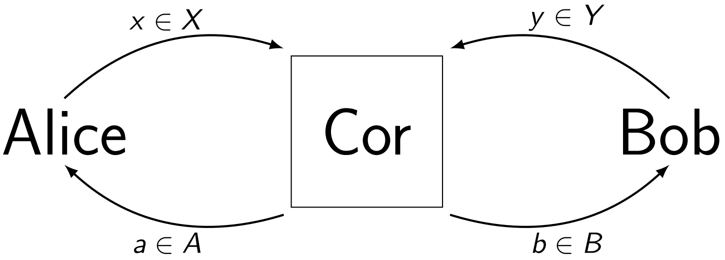

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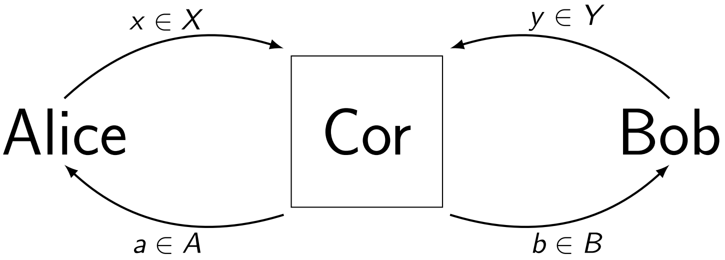

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  $$\text{Cor} : X \times Y \to \{\mu : \mu \text{ is a probability distribution over } A \times B\}$$

- Assume X, Y, A, B are countable
- Abuse notation and write $\text{Cor} : X \times Y \to A \times B$

Distributed sampling problems

- Can think of a correlation box as a *distributed sampling problem* – the problem of simulating the box

Distributed sampling complexity classes

- **SR**: class of correlation boxes that can be simulated using just shared randomness

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- **Q**: class of correlation boxes that can be simulated using shared randomness + arbitrary bipartite quantum state
- Obviously $\mathbf{SR} \subseteq \mathbf{Q}$
- Bell's theorem: $\mathbf{SR} \neq \mathbf{Q}$

Distributed sampling complexity classes (2)

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- Tsierelson bound: PR $\notin \mathbf{Q}$, so $\mathbf{Q} \neq \mathbf{NS}$

Bell pair

- Goal: Understand $\mathbf{Q}$

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- Baby step: Understand **BELL**: class of correlation boxes that can be simulated using shared randomness + $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ + projective measurements
- $\mathbf{SR} \subsetneq \mathbf{BELL} \subsetneq \mathbf{Q}$

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- Loose upper bound, since **BELL** $\subseteq$ **NS**

PR box is **BELL**-hard

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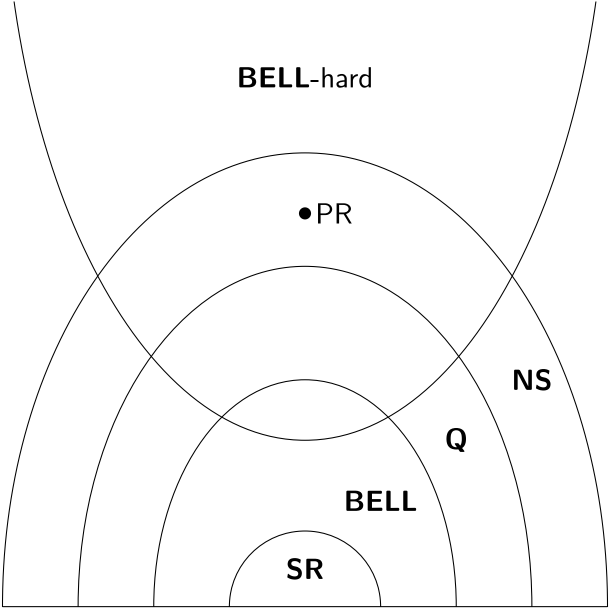
- Theorem (Cerf et al. '05): **BELL** can be simulated using shared randomness + 1 PR box
- In other words, PR is **BELL**-hard with respect to 1-query reductions

Distributed sampling complexity zoo



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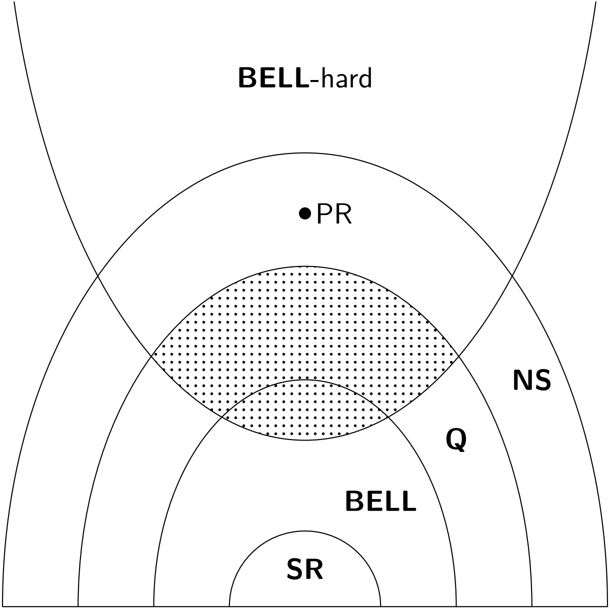
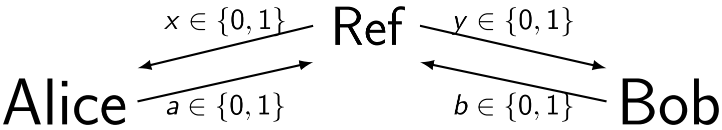


Biased CHSH game



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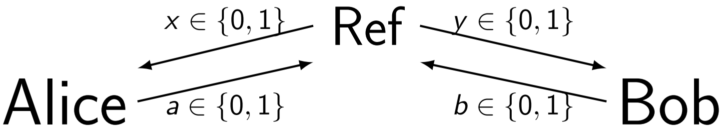

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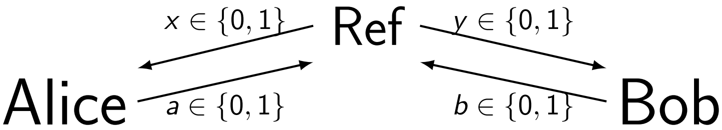

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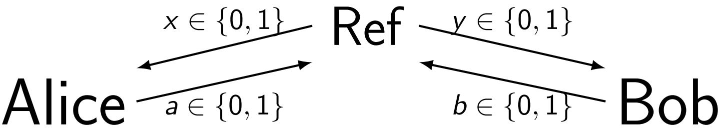

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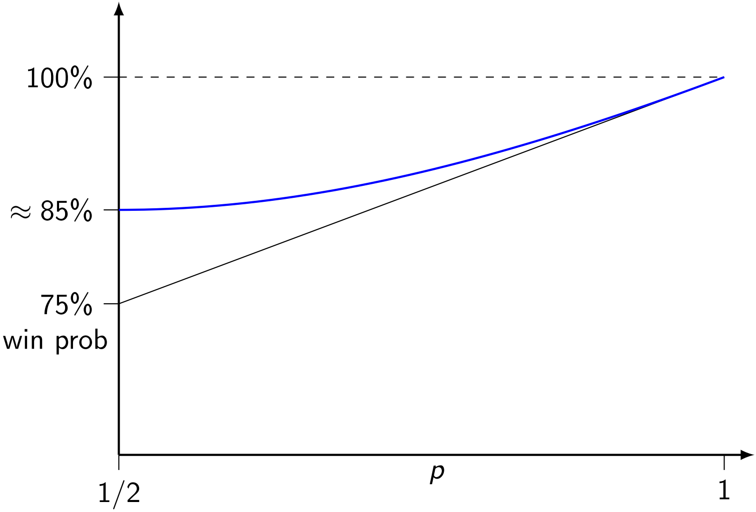
- Inputs x, y are chosen independently at random
- y is uniform, x is biased: $\Pr[x = 1] = p \in [1/2, 1]$
- Theorem (Lawson, Linden, Popescu '10): Optimal quantum strategy can be implemented in **BELL**, wins with probability

$$f(p) \stackrel{\text{def}}{=} \frac{1}{2} + \frac{1}{2}\sqrt{p^2 + (1-p)^2}$$

Quantum value of biased CHSH game



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- Affine function of p, for fixed reduction

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- Recall Cor : $X \times Y \to A \times B$ with X, Y countable, A, B finite
- Only countably many deterministic reductions!
- Countably many affine functions, so $\exists p$ where all the affine functions disagree with $f(p)$

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  - $\text{Cor}_2 \in \mathbf{BELL}$
  - For every $\text{Cor}_1 \in \mathbf{BELL}$, there is a 1-query ε-error reduction from Cor_1 to Cor_2

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- Thanks for listening! Questions?

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