Volume of a simplex

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1 Definition of the problem

It should be shown that the volume of a simplex tends to 0 if the dimension of simplex grows.

Say $v_n(a)$ is the volume of a simplex where n is the dimension. For convenience the volume will be determined for an arbitrary length a and later $a = 1$ will be set. Than

$$\lim_{n\to\infty} v(a = 1) = 0. \tag{1}$$

In geometry, a simplex (plural: simplexes or simplices) is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions (see Simplex in Wikipedia). For simplicity a canonical basis of unit vectors $\vec{e_i}$ is used. The simplex is determined by a set of points:

$$C = \left\{ c_1\vec{e_1} + \cdots + c_n\vec{e_n} \,|\, \forall i,\, c_i \geq 0,\, \sum_{i=1}^{n} c_i = 1 \right\}. \tag{2}$$

Borders of such a simplex are the unit vectors and straight lines between end points of the unit vectors respectively.

2 Determination of volume

- $n = 1$:
  In figure 1 is shown a 1-dimensional simplex (a straight line). The length (volume in one dimension) is determined by:

  $$v_1(a) = \int\limits_0^a \mathrm{d}x = \Big[x\Big]_0^a = a. \tag{3}$$

  Later the a can be substituted by 1.

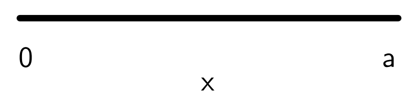

Figure 1: Volume of a 1-D simplex

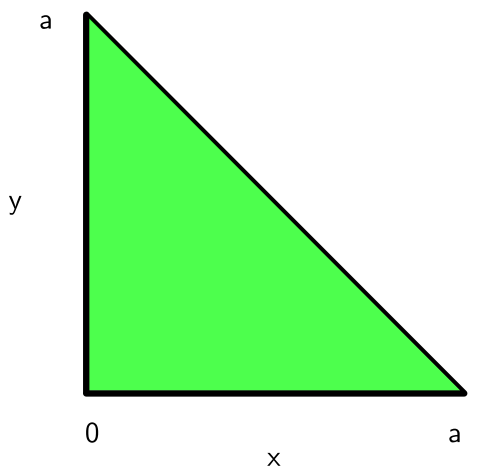

Figure 2: Volume of a 2-D simplex

- $n = 2$:
  In figure 2 is shown a 2-dimensional simplex (a triangle). The surface of the shaded triangle is determined by:
$$v_2(a) = \int_0^a \int_0^y \mathrm{d}x \, \mathrm{d}y = \int_0^a y \, \mathrm{d}y = \left[\frac{1}{2}\, y^2\right]_0^a = \frac{a^2}{2}. \tag{4}$$
- $n = 3$:
  The volume of the shaded area (see fig. 3) is determined by [1]:
$$v_3(a) = \int_0^a \int_0^z \int_0^y \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_0^a \int_0^z y \, \mathrm{d}y \, \mathrm{d}z = \int_0^a \frac{z^2}{2} \, \mathrm{d}z = \left[\frac{1}{2 \cdot 3}\, z^3\right]_0^a = \frac{a^3}{6}. \tag{5}$$

It seems that the volume $v_n(a)$ of dimension n for $a = 1$ is:
$$v_n(1) = \frac{1}{n!}. \tag{6}$$

3 Proof of (6)

The proof is made by means of induction. $v_n(a) = {}^1\!/_{n!}$ for $n = 1$ is shown above (3). Say the equation is valid for n. Next it has to be shown that the equation is valid for $n + 1$.

[1] The dashed triangle in fig. 3 is an arbitrary similar triangle of the 'base'. The size is direct related to the position z. At least all these triangles are 'added together'. This shows clear the recursive way to construct volumes of higher dimensions.

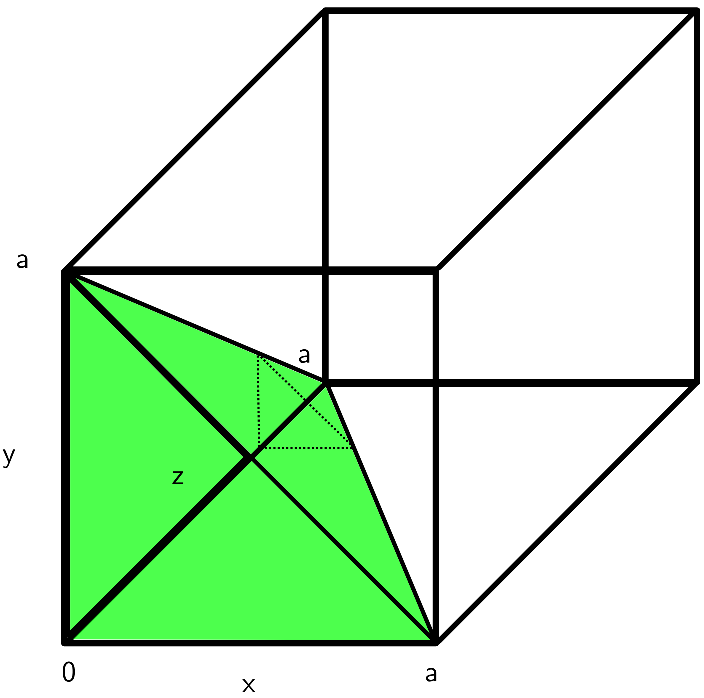

Figure 3: Volume of a 3-D simplex

Proof.

$$v_{n+1}(a) = \int_0^a \underbrace{\int_0^{x_{n+1}} \cdots \int_0^{x_2} \mathsf{d}x_1 \cdots \mathsf{d}x_n}_{= v_n(x_{n+1})} \, \mathsf{d}x_{n+1} \tag{7}$$

$$= \int_0^a \frac{1}{n!} x_{n+1}^n \, \mathsf{d}x_{n+1} \tag{8}$$

$$= \left[\frac{1}{n!} \frac{1}{n+1} x_{n+1}^{n+1} \right]_0^a = \frac{a^{n+1}}{(n+1)!} \qquad \square$$

In the formula above old x, y and z are replaced by x_1, x_2 and x_3. For $a = 1$ we get result for volume of a simplex.

At least we know that

$$\lim_{n \to \infty} v_n(1) = \lim_{n \to \infty} \frac{1}{n!} = 0. \tag{9}$$