Volume of a simplex

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1 Definition of the problem

It should be shown that the volume of a simplex tends to 0 if the dimension of simplex grows.

Say n is the dimension and v the volume of a simplex than

$$\lim_{n\to\infty} v(n) = 0. \tag{1}$$

In geometry, a simplex (plural: simplexes or simplices) is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions (see Simplex in Wikipedia). For simplicity a canonical basis of unit vectors $\vec{e_i}$ is used. The simplex is determined by the set of points:

$$C = \left\{ c_1\vec{e_1} + \cdots + c_n\vec{e_n} \,|\, c_i \geq 0, 0 \leq i \leq n, \sum_{i=1}^{n} c_i = 1 \right\}. \tag{2}$$

The borders of such a simplex are the unit vectors and straight lines between the end points of the unit vectors respectively.

2 Determination of volume

$n = 1$: Bild Linie

The length (volume in one dimension) is determined by:

$$v(1) = \int_0^a \mathrm{d}x = \Big[x\Big]_0^a = a. \tag{3}$$

Later the a can be substitute by 1.

$n = 2$: Bild Dreieck

The surface of the shaded triangle is determined by:

$$v(2) = \int_0^a \int_0^y \mathrm{d}x\,\mathrm{d}y = \int_0^a y\,\mathrm{d}y = \left[\frac{1}{2}y^2\right]_0^a = \frac{a^2}{2}.$$

$n = 3$: Bild Wuerfel

The volume of the shaded area is determined by:

$$v(3) = \int_0^a \int_0^z \int_0^y \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_0^a \int_0^z y \, \mathrm{d}y \, \mathrm{d}z = \int_0^a \frac{z^2}{2} \, \mathrm{d}z = \left[\frac{1}{2 \cdot 3} z^3 \right]_0^a = \frac{a^3}{6}.$$

It seems that the volume $v(n)$ of dimension n for $a = 1$ is:

$$v(n) = \frac{1}{n!}. \tag{4}$$

3 Proof of the volume

The proof is made by induction. $v(n) = {}^1\!/_{n!}$ for $n = 1$ is shown above. Say the equation is valid for n. Next it has to be shown that the equation is valid for $n + 1$.