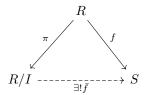
Proposition 0.1. (Universal property for quotients) Let R, S be a rings and $I \subseteq R$ an ideal. Suppose we have a function $f: R \to S$, which vanishes on I and is an additive group homomorphism when restricted to I. Then there exists a uniquely determined function

$$\bar{f}: R/I \to S$$

such that the following diagram commutes



In this case we say that f descends to the quotient, R/I. If f is a ring map then so is \bar{f} .

Proof. If there is a map \bar{f} so that the diagram commutes then what that means is that we have, for all $r \in R$

$$\bar{f}(\pi(r)) = f(r)$$

But π is surjective so this condition forces how \bar{f} is defined, and hence \bar{f} is unique if it exists. Suppose now that $x,y\in R$ so that $\bar{x}=\bar{y}$. Then $x-y\in I$ and so f(x-y)=0, so f(x)=f(y) as f is an additive homomorphism when restricted to I. Hence \bar{f} is well defined. If f is a ring map, then \bar{f} also be a ring map because the diagram commutes.

Suppose that v is a valuation as above. Let us briefly go through some important properties. Note first that property 2. above makes v into a homomorphism $v: K^* \to \mathbb{R}$. Thus, if $x \in K^*$ has finite order, then also v(x) has finite order. But then v(x) = 0 as 0 is the only element in \mathbb{R} that has finite order with respect to addition. In particular, v(-1) = 0 so v(-x) = v(-1) + v(x) = v(x) for all $x \in K$. It follows that v(x + y) = v(y) if v(x) > v(y), since

$$v(y) = v(x + y - x) \ge \min\{v(x + y), v(x)\} \ge \min\{v(x), v(y)\} = v(y)$$

There is another, perhaps more down-to-earth way of characterizing the completion of a field. But

Now for the alternative characterization.

Proposition 0.2. Let K be a valued field, \hat{K} a complete valued field and $\hat{\iota}: K \to \hat{K}$ a homomorphism preserving the absolute value. Then $(\hat{K}, \hat{\iota})$ is a completion of K if and only if K is dense in \hat{K} .

Proof. Assume first that the pair $(\hat{K}, \hat{\iota})$ is in fact the completion of K and let us show that K is dense in \hat{K} , by which we of course mean that the image $\hat{\iota}(K)$

is dense in \hat{K} . Now, as \hat{K} is complete also $\overline{K}(=\widehat{\iota(K)})$ is complete since it is closed and contained in \hat{K} . Also, as K is dense in \overline{K} the inclusion is a subfield of \hat{K} (SHOW THIS). Thus, we have the inclusion map $\psi: \overline{K} \to \hat{K}$. This shows that $(\overline{K}, \hat{\iota})$ satisfies the same universal property as $(\hat{K}, \hat{\iota})$ and hence $(\overline{K}, \hat{\iota})$ is the completion of K. (THIS PART IS UNFINISHED)

Let us now prove the converse. So suppose that $\hat{\iota}(K)$ is dense in \hat{K} and that (L, ι) is a pair as in (??).

Let us now look at some examples. We have already mentioned that \mathbb{R} is the completion of \mathbb{Q} . We have the inclusion $\mathbb{Q} \to \mathbb{R}$ which preserves absolute values, so this statement follows if we are willing to accept that \mathbb{R} is complete and that \mathbb{Q} is dense in \mathbb{R} . Here is another example. Suppose that K is a field and consider the formal power series K[[x]]. As we have mentioned already this is a local ring with maximal ideal $\mathfrak{p}=(x)$. Consider the valuation $v_m frakp$ on K[[x]] defined by

Theorem 0.1 (Ostrowski). Suppose that K is field which is complete with respect to an archimedian valuation. Then there is an isomorphism σ from K into \mathbb{R} or \mathbb{C} and a constant $s \in (0,1]$ so that

$$|x| = |\sigma(x)|^s$$

for all $x \in K$.

A nonarchimedian absolute value $|\cdot|$ on a field K extends to a nonarchimedian absolute value on K(t) by setting $|f| = \max |a_0|, ..., |a_n|$ where $f \in K[x]$ and $f(x) = a_n x^n + ... + a_0$. For an arbitrary element $\frac{g}{h} \in K(x)$ where $h \neq 0$ we then define $|\frac{g}{h}| = |g| - |h|$.