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1 Introduction

Diophantine equations, named after the greek mathematician Diophantus, are polynomial equations in multiple variables where only integer solutions are of interest. These have been studied for a very long time and are known for being notoriously hard to solve. Therefore we often ask simpler questions like: Does this equation even have a solution? If it has finitely many solutions, can we then obtain an upper bound on the number of these? These are the kinds of questions we will be asking in this thesis. To make life simpler we will restrict our attention to a certain type of diophantine equations, namely those of the type

$$N_{K/\mathbb{Q}}(x_1\mu_1 + ... + x_m\mu_m) = c,$$

where K is an algebraic number field, $\mu_1, ..., \mu_m \in K$ and $x_1, ..., x_m \in \mathbb{Z}$. It can be shown that any polynomial equation, $f(x, y) = c$, with coefficients in $\mathbb{Q}$ is an example of a form of this type, and adding a suitable number of additional assumption on f, we get theorem 12.2 which says that this equation only has finitely many solutions. This theorem was proven in 1909 by the Norwegian mathematician Axel Thue, and further advancements have been made since. Notably, Alan Baker showed in 1969 that the solutions can be bounded and he even managed to weed out some of the assumptions in the original theorem

by Thue. This allows for an implementation of an algorithm that, in principle, could find all solutions to equations like this.

2 Dual basis

Proposition 2.1. *Suppose that K/k is a field extension of degree n. Given any basis $\mu_1, ..., \mu_n$ for K over k there exists a basis, $\mu_1^*, ..., \mu_n^*$ called the dual basis of $\mu_1, ..., \mu_n$ so that*

$$\mathrm{Tr}_{K/k}(\mu_i^* \mu_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

Proof. For ease of notation we will write Tr instead of $\mathrm{Tr}_{K/k}$. Choose n elements, $c_1, ..., c_n$, in k. We know that the $n \times n$ matrix, $(\mathrm{Tr}(\mu_i \mu_j))$, is non-singular since

$$0 \neq \mathrm{disc}(\mu_1, ..., \mu_n) = |\mathrm{Tr}(\mu_i \mu_j)|^2$$

This means that there is a unique solution, $x_1, ..., x_n \in k$, to the n equations

$$\sum_{j=1}^{n} \mathrm{Tr}(\mu_i \mu_j) x_j = \frac{c_i}{n} \quad (i = 1, ..., n)$$

Let $\alpha = \sum_{j=1}^{n} x_j \mu_j$. Using rules of the trace, we get for any i that

$$c_i = \sum_{j=1}^{n} \mathrm{Tr}(x_j \mu_i \mu_j) = \mathrm{Tr}(\sum_{j=1}^{n} x_j \mu_i \mu_j) = \mathrm{Tr}(\alpha \mu_i)$$

Thus, we have demonstrated that for any choice of $c_1, ..., c_n \in k$, there is a unique $\alpha \in K$ such that $\mathrm{Tr}(\alpha \mu_i) = c_i$. For $(i, j) \in \{1, ..., n\}^2$ we now define

$$c_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

For every $i \in \{1, ..., n\}$ we get a unique $\mu_i^* \in K$ such that $\mathrm{Tr}(\mu_i^* \mu_j) = c_{ij}$ for $j \in \{1, ..., n\}$. To show that $\mu_1^*, ..., \mu_n^*$ is a basis, it suffices to show that they are linearly independent over k. So suppose that

$$\sum_{i=1}^{n} x_i \mu_i^* = 0.$$

Multiplying by μ_j and taking the trace, we get

$$0 = \mathrm{Tr}(\sum_{i=1}^{n} x_i \mu_i^* \mu_j) = \sum_{i=1}^{n} x_i \mathrm{Tr}(\mu_i^* \mu_j) = x_j,$$

which shows that $x_j = 0$ is zero. Doing this for all $j \in \{1, ..., n\}$ we get $x_1 = ... = x_n = 0$. □

3 Formal power series

Let K be a field. Recall the ring of formal power series, $K[[x]]$. Addition of two elements $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{i=0}^{\infty} b_i x^i$ is defined in the obvious way and multiplication is defined as

$$f(x)g(x) = \sum_{i=0}^{\infty} c_i x^i$$

where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$. Thus, multiplication is really just the process of expanding out brackets and collecting like terms. In some cases it is also possible to compose power series. Let $\omega : K[[x]] \to \mathbb{Z}$ be defined by sending a power series $\sum_{i=m}^{\infty} a_i x^i$ to m. Note that if $f(x) = \sum_{i=1}^{\infty} a_i x^i$ then the sequence $\omega(f^0), \omega(f^1), ...$ is strictly increasing. Hence for any $n \in \mathbb{N}_0$ it is only the first n power series in the sequence $f^0, f^1, f^2, ...$ that can contain a term of degree n. Hence we can in this case define, for any $g(x) = \sum_{i=0} b_i x^i$ the composite

$$g(f(x)) = \sum_{i=0}^{\infty} b_i f(x)^i$$

The coefficients, c_i, of this power series are given by

$$c_i = \sum_{k \in \mathbb{N}} \sum_{\substack{j \in \mathbb{N}^k, \\ j_1 + ... j_k = k}} b_k a_{j_1} ... a_{j_k}$$

Formal differentiation can be defined on $K[[x]]$, simply by formal differentiation of each term, that is,

$$f'(x) = \sum_{i=0}^{\infty} i a_i x^{i-1}$$

This operation is clearly linear and the well known product and chain rules actually hold in $K[[x]]$. Using the product rule and an induction argument it follows that $(f(x)^n)' = nf(x)^{n-1} f'(x)$. Hence if $f(x) = \sum_{i=1} a_i x^i$ and $g(x) = \sum_{i=0} b_i x^i$ then

$$(g(f(x)))' = \sum_{i=0}^{\infty} i b_i f(x)^{i-1} f'(x) = (\sum_{i=0}^{\infty} i b_i f(x)^{i-1}) f'(x) = g'(f(x)) f'(x).$$

We can also consider the fraction field of $K[[x]]$ and we will denote this field as $K((x))$. This is called the field of Laurant series and each element in it can be written as $\sum_{i=m}^{\infty} a_i x^i$, for some $m \in \mathbb{Z}$. As such we can extend ω defined above in an obvious way to $K((x))$.

Lemma 3.1. *Suppose K is a field and $f \in K((t))$. If $f \notin K$ then f is transcendental over K. That is, there exists no non-zero polynomial with coefficients in K having f as a root.*

Proof. Suppose that $f \in K((t)) \setminus K$. First, let us assume that $\omega(f) > 0$. Then the numbers $\omega(f^0), \omega(f^1), \omega(f^2)...$ constitute a strictly increasing sequence in $\mathbb{Z}$. In particular, these are all different numbers. So if we have a K-linear combination

$$a_0 + a_1 f_1 + ... + a_n f^n = 0,$$

then we must have $0 = a_0 = ... = a_n$. A similar argument can be made when $\omega(f) < 0$ and thus f cannot be algebraic when $\omega(f) \neq 0$. Lastly, suppose that $\omega(f) = 0$. This means that f has a non-zero constant term, say $a_0 \in K$. For contradiction, assume that f is algebraic over K. Then also $f - a_0$ would be algebraic. But that cannot be the case since $\omega(f - a_0) > 0$ and so f is not algebraic. □

Suppose that K is a field of characteristic 0 and that it is complete with respect to a valuation, v. Say that we have a power series $f \in K[[X]]$,

$$f(X) = \sum_{n=0} a_n X^n.$$

The set C containing all points $x \in K$ so that $f(x)$ converges is called the region of convergence for f. We get an induced function $f : C \to K$ given by $x \mapsto f(x)$ and this function is continuous. Differentiating the power series, f, induces another function, which also has C as its region of convergence (REFER TO SOMETHING HERE). (REMARK THAT WE CANNOT EXPECT IT TO BE THE CASE THAT h(X) = f(g(X)) WHEN REGARDING THEM AS FUNCTIONS). Example where this goes wrong in $\mathbb{Q}_2$: $f(x) = \exp(x)$ and $g(x) = 2x^2 - 2x$ and $h(x) = f(g(x))$

4 Forms and Modules

A form $F(x_1, ..., x_m)$ in m variables of degree k is a homogenous polynomial in the variables $x_1, ..., x_m$ of degree k with coefficients in $\mathbb{Q}$. If it splits into linear factors in some extension, K, of $\mathbb{Q}$, then it is called decomposable and in this case K can in fact be chosen to be a number field [See Thm 1 p 79 Borevich]. The following proposition provides some examples of forms that are decomposable.

(HAVE A LOOK AT THIS AGAIN)

Proposition 4.1. *Let $F(x, y) = \sum_{i=0}^{m} a_i x^{m-i} y^i$ be a form of degree m where $a_0 \neq 0$. Then over the splitting field for $F(x, 1)$ we have the factorization*

$$F(x, y) = a_0(x - \alpha_1)...(x - \alpha_m)$$

Proof. Since $a_0 \neq 0$ we must have $y \nmid F(x, y)$. Let $G(t) = F(t, 1)$. This is a polynomial in $\mathbb{Q}[t]$ with degree m. We can now write $F(x, y) = y^m F(\frac{x}{y}, 1) = y^m G(\frac{x}{y})$. As G is just a polynomial we can factor it into linear terms over its splitting field, K

$$G(t) = a_0(t - \alpha_i)...(t - \alpha_m)$$

This means that

$$F(x,y) = y^m G(\frac{x}{y}) = a_0(x+\alpha_i y)...(x-\alpha_m y)$$

Thus, $F(x,y)$ is decomposable as it splits into linear factors over K. □

Remark 4.1. *Consider the special case where $F(x,y)$ is also irreducible and $a_0 = 1$. It then follows that $F(x,1)$ is also irreducible of degree m so adjoining any of its roots α to $\mathbb{Q}$ gives an extension, K, of degree m. The α_i in the factorization of G above are then the conjugates of α so setting $\theta = -\alpha$ we obtain*

$$F(x,y) = N_{K/\mathbb{Q}}(x+y\theta)$$

We shall use this fact once we start looking at Thue's theorem.

A form F is called reducible if it can be written as a product of two forms of degree strictly larger than 0. Otherwise it is called irreducible. Two forms of the same degree are called integrally equivalent if one can be obtained from the other and vice versa by a linear change of variables with coefficients in $\mathbb{Z}$. Two forms being integrally equivalent clearly defines an equivalence relation on the set of forms having some fixed degree. If $F(x_1,...,x_m)$ and $G(y_1,...,y_l)$ are integrally equivalent, then knowing all the solutions to $F(x_1,...,x_m) = a$ is exactly the same as knowing all the solutions to $G(y_1,...,y_l) = a$.

Let us assume that K is a number field of degree n over $\mathbb{Q}$ and let $\mu_1,...,\mu_k$ be elements in K. The set, M, consisting of all $\mathbb{Z}$-linear combinations of these is called a module in K and we will write $M = \{\mu_1,...,\mu_k\}$ for this set. The μ_i's are called generators for the module and we can use them to construct a form; As K is in particular a finite separable extension of $\mathbb{Q}$, the primitive element theorem states that there is $\theta \in K$ so that $K = \mathbb{Q}(\theta)$. Let f be the minimal polynomial for θ over $\mathbb{Q}$. Adjoining all the roots of f to $\mathbb{Q}$, we get a field, N, which has finite degree, n, over $\mathbb{Q}$ and which is splitting field for the irreducible polynomial f. Hence N is a number field and is Galois over $\mathbb{Q}$. Let $G = \mathrm{Gal}(N/\mathbb{Q})$ and $H = \mathrm{Gal}(N/K)$. Now $n = [K:\mathbb{Q}] = [G:H]$ so we can find $\sigma_1,...,\sigma_n$ so that $G = \cup_{i=1}^n \sigma_i H$ is a disjoint union. For $x_1,...,x_k \in \mathbb{Z}$ we now set $F(x_1,...,x_k) = N_{K/\mathbb{Q}}(x_1\mu_1 + ... + x_k\mu_k)$ and we observe that

$$\begin{aligned} F(x_1,...,x_k) &= \prod_{i=1}^n \sigma_i(x_1\mu_1 + ... + x_k\mu_k) \\ &= \prod_{i=1}^n (x_1\sigma_i(\mu_1) + ... + x_k\sigma_i(\mu_k)) \end{aligned}$$

When expanding out the parentheses we see that F is a homogenous polynomial of degree n in the variables $x_1,...,x_k$. Since the norm, $N_{K/\mathbb{Q}}$, sends elements in K to $\mathbb{Q}$ we know that $F(x_1,...,x_n)$ is fixed by every single $\sigma \in G$ for all $x_1,...,x_n \in \mathbb{Z}$. This implies that all the coefficients of F are fixed by every $\sigma \in G$, so they all belong to $\mathbb{Q}$. Therefore F is a form of degree n.

Definition 4.1. *Suppose that $M = \{\mu_1, ..., \mu_m\}$ and let $F(x_1, ..., x_m)$ be a form. If $F(x_1, ..., x_m) = N_{K/\mathbb{Q}}(x_1\mu_1 + ... + x_m\mu_m)$ then F is called a norm form and F is said to be connected to the module M.*

Norm forms are decomposable as they clearly factor into linear factors over N.

Fact 4.1. *All forms connected to a module M are integrally equivalent.*

Proof. Suppose that M is generated by $\mu_1, ..., \mu_m$ and also by $\rho_1, ..., \rho_l$. We get the following form

$$G(y_1, ..., y_s) = N_{K/\mathbb{Q}}(y_1\rho_1 + ... + y_s\rho_l).$$

For every $j = 1, ..., l$ we can find integers $c_{j1}, ..., c_{jk}$ so that $\rho_j = \sum_{k=1}^{m} c_{jk}\mu_k$ since $\rho_j \in M$ and the μ_i generate M. We then have

$$\sum_{j=1}^{l} y_j\rho_j = \sum_{j=1}^{l} y_j \sum_{k=1}^{m} c_{jk}\mu_k = \sum_{k=1}^{m}(\sum_{j=1}^{l} c_{jk}y_j)\mu_k = \sum_{k=1}^{m} x_k\mu_k,$$

where $x_k = \sum_{j=1}^{l} c_{jk}y_j$ for $k = 1, ..., m$. Taking norms on both sides of the above equation, we see that the form F can be obtained from G by a linear change of variables with integer coefficients. □

If M is a module and it contains a basis for the vector space $K/\mathbb{Q}$, then it is called a **full module**. Otherwise it is called a **nonfull module**. A form which is connected to a module is correspondingly called full or nonfull. By definition M is a finitely generated abelian group and by the structure theorem, M determines uniquely integers $r, s \geq 0$ and $d_1 \mid ... \mid d_s$, $d_i \geq 2$ such that

$$M \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/d_s\mathbb{Z}$$

But M lives inside a field, which has no zero divisors, so M must be a torsion-free and so $s = 0$. Thus, $M \cong \mathbb{Z}^r$, which means that M is a free abelian group of rank r. The modules in K can therefore be thought of as finitely generated free abelian groups, which means that concepts such as rank and basis now make sense for modules. The full modules inside K are exactly the modules having rank n and the nonfull ones have rank strictly smaller than that.

Fact 4.2. *If $\mu_1, ..., \mu_m$ is a basis for a module M then these elements are linearly independent over $\mathbb{Q}$.*

Proof. Suppose that we have a $\mathbb{Q}$-linear combination

$$a_1\mu_1 + ... + a_m\mu_m = 0$$

Set c to be the product of all the denominators of the a_i. Then $c \neq 0$ and $ca_i \in \mathbb{Z}$ so we must have $a_i = 0$ as the μ_i are linearly independent over $\mathbb{Z}$. Thus the a_i are all 0. □

5 Coefficient rings and orders

In the following we assume that K is an algebraic number field of degree n over $\mathbb{Q}$. An **order** in K is a full module in K which is also a ring with unity. We will now give a way of constructing such a ring. Given a full module M in K, we can consider an element α in K so that $\alpha M \subseteq M$. Such an element is called a **coefficient** of M, and the set of all of these is called the **coefficient ring** of M, which we will denote by $\mathfrak{D}_M$, or simply $\mathfrak{D}$, when it is clear from the context what is meant. It would be strange to call this object a ring, if it wasn't a ring, so let us check that it is. It is clear that $1 \in \mathfrak{D}$ since $1M \subseteq M$ and so $\mathfrak{D}$ is non-empty. Let now α, β in $\mathfrak{D}$ and take any element in x in M. We have $\alpha x, \beta x \in M$ so

$$(\alpha - \beta)x = \alpha x - \beta x \in M$$

Thus, $\alpha - \beta \in \mathfrak{D}$. Checking that we have closure under multiplication is similar and so by the subring criterion $\mathfrak{D}$ is a subring of K. In fact, $\mathfrak{D}$ is also a full module. To justify this, we must show that $\mathfrak{D}$ is a module and that it contains a basis for K as a vector space over $\mathbb{Q}$. If γ is any non-zero element of M, then $\gamma\mathfrak{D}$ is a group under addition and we have that $\gamma\mathfrak{D} \subseteq M$. Thus, $\gamma\mathfrak{D}$ is a free abelian group, since it is contained in the free abelian group M which means that it is a module in K. But then also $\mathfrak{D} = \gamma^{-1}\gamma\mathfrak{D}$ is a module. Let us show that $\mathfrak{D}$ is also full. Since M is full we can find inside it a basis, $\mu_1, ..., \mu_n \in M$, for K as a vector space over $\mathbb{Q}$ which is also a basis for M. (MAYBE JUSTIFY THIS) We need the following lemma

Lemma 5.1. *For any $\alpha \in K$ we have that $\alpha\mu_i$ is in M for all the μ_i if and only if α is in $\mathfrak{D}$.*

Proof. Take any $x \in M$ and write $x = \sum_{i=1}^{n} a_i\mu_i$ where the a_i are integers. Multiplying by α we get

$$\alpha x = \sum_{i=1}^{n} a_i(\alpha\mu_i)$$

So if the $\alpha\mu_i$ are all in M, this is just a finite sum of elements in M, meaning that the entire sum is in M. Hence, $\alpha M \subseteq M$. The other direction is clear. □

From this lemma it follows that if $\alpha \in K$ there exists an integer $c \neq 0$ so that $c\alpha \in \mathfrak{D}$. This is because we each μ_i can find a $\mathbb{Q}$-linear combination

$$\alpha\mu_i = \sum_{j=1}^{n} a_{ij}\mu_j,$$

since the μ_i constitute a basis for K over $\mathbb{Q}$. Choose now an integer, $c \neq 0$ so that ca_{ij} is an integer for all i, j. This implies that $c\alpha\mu_i$ is in M for all the μ_i which by lemma 5.1 means that $c\alpha$ is in $\mathfrak{D}$. From this it is clear that we can pick an integer $b \neq 0$ so that we for all μ_i have that $c\mu_i$ is in $\mathfrak{D}$. The elements $c\mu_1, ..., c\mu_n$ is of course still is a basis for K over $\mathbb{Q}$ so $\mathfrak{D}$ is full. Hence $\mathfrak{D}$ is an order in K. In fact $\mathfrak{D}$ is contained in O_K, the ring of integers of K. Because

if $\alpha \in \mathfrak{D}$ then we have $\mathbb{Z}[\alpha] \subseteq \mathfrak{D}$ so $\mathbb{Z}[\alpha]$ has rank smaller than or equal to n. Hence $\mathbb{Z}[\alpha]$ is in particular finitely generated and that is the same as saying that $\alpha \in O_K$. Let us summarize what we have found in the following proposition

Proposition 5.1. *The coefficient ring $\mathfrak{D}$ is an order of K and is a subring of the ring of integers of K.*

6 Only finitely many solutions up to associates

Theorem 6.1 (Dirichlet's unit theorem)**.** *Let $\mathfrak{D}$ be an order in some number field K of degree n and let r and $2s$ be the number of real and complex embeddings into $\mathbb{C}$, respectively. Then*

$$\mathfrak{D}^* = W \oplus V$$

where W is a finite cyclic group consisting of all roots of unity of $\mathfrak{D}$ and V is a free abelian group of rank $t = r + s - 1$.

Proof. Let O be the ring of algebraic integers in K. For the order O we know that the above theorem holds, so we get

$$O^* = W \oplus V$$

with W and V as above. Since $\mathfrak{D}$ is a subring of O we also have $\mathfrak{D}^* \subseteq O^*$. Hence,

$$\mathfrak{D}^* = W' \oplus V'$$

where $W' \trianglelefteq W$ is finite cyclic and $V' \trianglelefteq V$ is free abelian of rank $t' \leq t$. We wish to show two things; That W' does indeed consist of all roots of unity of $\mathfrak{D}$ and that $t' = t$. For the first claim, if we have any root of unity $\xi \in \mathfrak{D}$, then ξ has finite order so it cannot possibly belong to V'. Thus the only possibility is that ξ is in W'. For the second claim, consider the quotient of groups $O/\mathfrak{D}$. Both of these have rank n, so this quotient is finite, and so we know that $f := [O : \mathfrak{D}] < \infty$. Thus, if $x \in O$ then $\overline{fx} = 0$ in $O/\mathfrak{D}$ so $fx \in \mathfrak{D}$, so $fO \subseteq \mathfrak{D}$. Of course fO is also a free abelian group of rank n, so again $R = O/fO$ is finite. But fO is also an ideal of the ring O, so in fact R is a finite ring. Consider now any unit $\epsilon \in V$. Then ϵ is in O^*, so $\overline{\epsilon} \in R$ is also a unit, since ring maps preserve units. Set now $k = \#R^*$. Then $\overline{\epsilon^k} = \overline{1}$ and $\overline{\epsilon^{-k}} = \overline{(\epsilon^{-1})^k} = \overline{1}$. Together, these equalities give us

$$\epsilon^k = 1 + f\alpha$$
$$\epsilon^{-k} = 1 + f\beta$$

where $\alpha, \beta \in O$. But as we argued above, $f\alpha$ and $f\beta$ both belong to $\mathfrak{D}$ and so both ϵ^k and ϵ^{-k} belong to $\mathfrak{D}$ which means that $\epsilon^k \in \mathfrak{D}^*$. Thus, ϵ^k is either in W' or V' and the first option is impossible as that would imply that ϵ^k would be in W. Thus, ϵ^k is in V' so V/V' is finite meaning that $t' = t$. □

In particular 8.6 says that there are units $\epsilon_1, ..., \epsilon_t$ such that every element in V has a unique representation of the form $\epsilon_1^{u_1}...\epsilon_t^{u_t}$, where all the u_i belong to $\mathbb{Z}$. These units are called a fundamental system for $\mathfrak{D}$.

We say that two elements, α, β in a a full module M are **associated** if there is a unit $\epsilon \in \mathfrak{D}$ so that $\alpha = \epsilon\beta$. Note that when M is equal to its own coefficient ring, this concept is exactly the same as that of being associated in rings. Being associated elements in M defines an equivalence relation on M, and from now on we will denote this relation as $\sim$. Define now for $c \in \mathbb{N}$ the subsets

$$M_c = \{\alpha \in M \mid N(\alpha) = c\}$$
$$\overline{M}_c = \{\alpha \in M \mid |N(\alpha)| = c\}$$

We are now ready to formulate the following theorem.

Theorem 6.2. *Let M be a full module of K. Then the quotient set $\overline{M}_c/\sim$ is finite for any $c \in \mathbb{N}$. In particular $M_c/\sim$ is finite.*

Proof. We first consider the special case where $M = \mathfrak{D}$. The ring $\mathfrak{D}$ is a full module so it is a free abelian group of rank n, hence isomorphic to $\mathbb{Z}^n$. Considering $\mathfrak{D}$ as an abelian group with respect to addition we get the isomorphisms

$$\mathfrak{D}/c\mathfrak{D} \cong \mathbb{Z}^n/c\mathbb{Z}^n \cong (\mathbb{Z}/c\mathbb{Z})^n$$

Now, $\mathbb{Z}/c\mathbb{Z}$ contains c elements, which means that

$$c^n = \#(\mathbb{Z}/c\mathbb{Z})^n = \#\mathfrak{D}/c\mathfrak{D}$$

Denote by $\bar{\alpha}$ as the image of the canonical projection of α in $\mathfrak{D}/c\mathfrak{D}$ and denote by $[\alpha]$ an equivalence class in $\overline{M}_c/\sim$, represented by $\alpha \in \overline{M}_c$. We show that there is a well-defined surjective function of sets

$$\phi : \overline{M}_c/c\mathfrak{D} \twoheadrightarrow \overline{M}_c/\sim,$$

given by $\phi(\bar{\alpha}) = [\alpha]$. Suppose $\bar{\alpha}, \bar{\beta}$ are in $\overline{M}_c/c\mathfrak{D}$ so that $\bar{\alpha} = \bar{\beta}$, where α, β are in $\overline{M}_c$, so $|N_{K/\mathbb{Q}}(\alpha)| = |N_{K/\mathbb{Q}}(\beta)| = c$. Let us show that $[\alpha] = [\beta]$ - in other words we show that α and β are associates. Since $M = \mathfrak{D}$ by assumption, we know that $\alpha, \beta \in \mathfrak{D}$ so $\overline{\alpha} = \overline{\beta}$ implies that

$$\alpha = \beta + c\gamma = \beta + |N_{K/\mathbb{Q}}(\beta)|\gamma,$$

for some γ in $\mathfrak{D}$. We claim that β divides $N_{K/\mathbb{Q}}(\beta)$ in $\mathfrak{D}$ - the reason for this is as follows. We know that β is an algebraic integer since it belongs to $\mathfrak{D}$, so its minimal polynomial $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_0$ has integer coefficients. Now $a_0 = -(\beta^m + a_{m-1}\beta^{m-1} + ... + a_1\beta)$ so $\beta \mid a_0$ in $\mathfrak{D}$ and therefore β also divides a_0^m, where $m = [K : \mathbb{Q}(\alpha)]$. By lemma 8.2 we have $N_{K/\mathbb{Q}}(\beta) = \pm a_0^m$ and the claim follows. We therefore also know that β divides $|N_{K/\mathbb{Q}}(\beta)|$ in $\mathfrak{D}$. Hence, β divides α in $\mathfrak{D}$ and similarly α divides β in $\mathfrak{D}$. Thus, α and β are associates, showing that ϕ is well-defined. It is surjective simply because if $[\alpha] \in \overline{M}_c/\sim$,

then α is in $\overline{M}_c$ so $\phi(\bar{\alpha}) = [\alpha]$. That ϕ is a surjection implies that $\#(\overline{M}_c/\sim) \leq \#\overline{M}_c/c\mathfrak{D}$, since each element in $\overline{M}_c/\sim$ has at least one preimage. Now the inclusion $\overline{M}_c/c\mathfrak{D} \subseteq \mathfrak{D}/c\mathfrak{D}$ implies that $\#(\overline{M}_c/\sim) \leq \#\overline{M}_c/c\mathfrak{D} \leq \#\mathfrak{D}/c\mathfrak{D} = c^n$. We will now prove the general statement. Suppose that M is a full module and that $\mathfrak{D}$ is the coefficient ring of M. Then $\overline{\mathfrak{D}}_c/\sim$ has finitely many elements. Take now a non-zero integer b so that $bM \subseteq \mathfrak{D}$. We then obtain injective maps

$$M \hookrightarrow bM \hookrightarrow \mathfrak{D},$$

where the first map is multiplication by b and the second one is inclusion. It is clear that if α and β are associated then also $b\alpha$ and $b\beta$ are associated. Hence we get the induced injective maps

$$(\overline{M}_c/\sim) \hookrightarrow (b\overline{M}_c/\sim) \hookrightarrow (\overline{\mathfrak{D}_c}/\sim)$$

Which means that

$$\#(\overline{M}_c/\sim) \leq \#(b\overline{M}_c/\sim) \leq \#(\overline{\mathfrak{D}_c}/\sim) \leq c^n$$

The last claim now follows since $M_c \subseteq \overline{M}_c$.

□

We now present a result that allows to find all the elements of M_c if we know the elements of $M_c/\sim$ and all the units with norm 1 in $\mathfrak{D}$.

Lemma 6.1. *Assume that the elements of $M_c/\sim$ are $[\gamma_1], ..., [\gamma_k]$ and that $\alpha \in M$. We then have that $\alpha \in M_c$ if and only if there is a uniquely determined i such that $\alpha = \epsilon\gamma_i$ where ϵ is a unit in $\mathfrak{D}$ with norm 1.*

Proof. If $\alpha \in M_c$ then, there is a unique γ_i such that $\alpha \in [\gamma_i]$. This means that $\alpha = \epsilon\gamma_i$ for some unit ϵ in $\mathfrak{D}$. But then

$$c = N(\alpha) = N(\epsilon\gamma_i) = N(\epsilon)N(\gamma_i) = N(\epsilon)c$$

So we must have that $N(\epsilon) = 1$. □

Note that an element $\epsilon \in \mathfrak{D}$ is a unit if and only if $N_{K/\mathbb{Q}}(\epsilon) = \pm 1$ simply because ϵ is an algebraic integer. In light of the above proposition we are now interested in figuring out when ϵ has norm 1. This is what the next theorem is about.

Lemma 6.2. *Let K be a number field of degree n over $\mathbb{Q}$. Suppose n is odd. Then the only roots of unity in O_K are ± 1 and we have $N(1) = 1$ and $N(-1) = -1$. On the other hand, if n is even, then all the roots of unity in O_K have norm 1.*

Proof. Let ζ be a primitive kth root of unity in O_K. To start with, let us suppose that n is odd. Then

$$\mathbb{Q} \subseteq \mathbb{Q}(\zeta) \subseteq K$$

Denote by φ Euler's totient function. We have $\varphi(k) = [\mathbb{Q}(\zeta) : \mathbb{Q}]$, we have $\varphi(k) \mid n$. Thus, $\varphi(k)$ has to be odd. But this happens only when k is 1 or 2 so $\zeta = \pm 1$. We see that $N(-1) = (-1)^n = -1$ and the fact that $N(1) = 1$ is always true. Next, assume that n is even. We then clearly have $1 = N(1) = N(-1)$. Then any embedding $\sigma : K \hookrightarrow \mathbb{C}$ must send ζ to a primitive kth root of unity in $\mathbb{C}$. Suppose that $k \geq 3$. In this case $\sigma(\zeta)$ is an imaginary number and therefore σ must be a complex embedding. Since σ is arbitrary we have $n = 2s$, where s is the number of pairs of complex conjugate embeddings. Thus,

$$N(\zeta) = \prod_{i=1}^{s} \sigma_i(\zeta)\overline{\sigma_i}(\zeta) = \prod_{i=1}^{s} |\sigma_i(\zeta)|^2 = 1$$

□

Theorem 6.3. *Let K be a number field of degree $n = 2s + r$ over the rationals and let $c \in \mathbb{Z}$. Assume further that M is a full module with ring of coefficients $\mathfrak{D}$ and set $t = r+s-1$. Then there exists a fundamental system of units $\epsilon_1, ..., \epsilon_t$ for $\mathfrak{D}$, and a finite number of elements $\gamma_1, ..., \gamma_k$ in M_c such that every $\alpha \in M_c$ can be written uniquely as*

$$\alpha = \gamma_i \epsilon_1^{u_1} ... \epsilon_t^{u_t}$$

for some γ_i and where $u_1, ..., u_t \in \mathbb{Z}$.

Proof. By Dirichlet's unit theorem, we know that we can find a fundamental system of units, $\epsilon_1, ..., \epsilon_t$, where $t = r + s - 1$ such that every element $\epsilon \in \mathfrak{D}^*$ can be written uniquely as

$$\epsilon = \zeta \epsilon_1^{u_1} ... \epsilon_t^{u_t}$$

where the u_i are integers and $\zeta \in \mathfrak{D}$ is a root of unity. Let us find the subgroup of $\mathfrak{D}^*$ containing only elements of norm 1. Suppose first that n is odd. Then we know from lemma 6.2 that there are only two roots of unity, namely 1 and -1 and that $N(-1) = -1$. If need be, let us swap out all ϵ_i that have norm -1 with $-\epsilon_i$. This leaves us with a new system of fundamental units and they generate the subgroup of $\mathfrak{D}$ having norm 1. Suppose now n is even. Then by lemma 6.2 all the roots of unity have norm 1, so if it happens that all the ϵ_i have norm 1, then all units of $\mathfrak{D}$ will have norm 1. So suppose that not all the ϵ_i have norm 1. We can without loss of generality assume that $1 = N(\epsilon_1) = ... = N(\epsilon_q)$ and $-1 = N(\epsilon_{q+1}) = ... = N(\epsilon_t)$, where $q < t$. Define $\mu_i = \epsilon_i$ for $i \in \{1, ..., q\}$ and $\mu_i = \epsilon_i \epsilon_t$ for $i \in \{q + 1, ..., t - 1\}$. We now have a new fundamental system of units, namely $\mu_1, ..., \mu_{t-1}, \epsilon_t$ and only the last unit, ϵ_t, has norm -1. By setting $\mu_t = \epsilon_t^2$, all units of norm 1 in $\mathfrak{D}$ can now be written as $\zeta \mu_1^{u_1} ... \mu_t^{u_t}$, where ζ is a root of unity in $\mathfrak{D}$.

Let now $[\gamma_1'], ..., [\gamma_l']$ be all the elements of $M_c/\sim$ and let $\alpha \in M_c$. By lemma 6.1, there is a unique γ_i' and a unit $\epsilon \in \mathfrak{D}$ of norm 1 so that $\alpha = \gamma_i' \epsilon$. For odd n the statement in the proposition now follows once we set $\gamma_i = \gamma_i'$ and $k = l$. When n is even, we can consider all products $\zeta \gamma_i'$, where $\zeta \in \mathfrak{D}$ is a root of unity. There are only finitely many of these products, and we can denote them by $\gamma_1, ..., \gamma_k$. All of these belong to M_c since the roots of unity have norm 1 and the statement now follows for when n is even. □

Remark 6.1. *Observe that if $t = 0$ then the above theorem says that there can only be finitely many elements in M_c. Since $t = r + s - 1$, this happens if and only if K is an imaginary quadratic field or $\mathbb{Q}$.*

7 Fields with an absolute value

Definition 7.1 (Absolute value)**.** *Let K be a field. A function $|\cdot| : K \to \mathbb{R}$, is called an absolute value if it happens to satisfy the properties*

- *$|x| \geq 0$ for every $x \in K$. (Non-negativity)*
- *$|x| = 0$ if and only if $x = 0$. (Positive definiteness)*
- *$|xy| = |x||y|$ for every $x, y \in K$. (Multiplicativity)*
- *$|x + y| \leq |x| + |y|$ for every $x, y \in K$. (Triangle inequality)*

When the triangle inequality can be upgraded to the stronger condition

$$|x + y| \leq \max\{|x|, |y|\} \quad \text{for every } x, y \in K,$$

*the absolute value is said to be **non-archimedian**. Otherwise it is called **archimedian**. If $|\cdot|$ is indeed an absolute value then the field K is called a valued field. A field with a non-archimedian absolute value is called a non-archimedian valued field and similarly a field with an archimedian absolute value is called an archimedian valued field.*

We could easily have made this definition more general by allowing K to be any ring. In that case, absolute values would only exists on domains. For if we had x, y both non-zero so that $xy = 0$ then $|x|, |y|$ would also be non-zero but $0 = |0| = |xy|$. But $|x||y| \neq 0$, so $|\cdot|$ cannot be an absolute value. The usual absolute value on $\mathbb{R}$, which we will denote by $|\cdot|_\infty$, is an example of an archimedian absolute value, so $(\mathbb{R}, |\cdot|)$ is an example of a valued field. The function sending everything to 1 except for 0 which is sent to 0 is also an example of an absolute value, and it is called the trivial absolute value. Note that for an arbitrary absolute value we have for all $x \in K$ that

- If $x^n = 1$ then $|x| = 1$.
- $|x| = |-x|$

In particular, $1 = |1| = |-1|$ and if K is a finite field then $|\cdot|$ is the trivial absolute value, since every non-zero element in such a field has finite order.

Once we have an absolute value on field K, we can use it to define a metric, $d : K^2 \to \mathbb{R}$, by setting $d(x, y) = |x - y|$. As such, K can now be thought of as a metric space, and we therefore now have a topology on K, generated by d. We will also refer to this topology as the topology induced by $|\cdot|$. It might very well happen that two absolute values, $|\cdot|_1$ and $|\cdot|_2$, induce the same topology on K. In fact, all of these are equivalent

- $|\cdot|_1$ and $|\cdot|_2$ are equivalent
- $|x|_1 = |x|_2^s$ for all $x \in K$ for some $s > 0$
- $|x|_1 < 1 \iff |x|_2 < 1$
- If (x_n) is a sequence in K then $x_n \to a$ with respect to $|\cdot|_1$ if and only if $x_n \to a$ with respect to $|\cdot|_2$.

, which is again the same as saying that

$$|x|_1 < 1 \iff |x|_2 < 1$$

If one of these equivalent conditions are satisfied we say that the absolute values are equivalent. This defines an equivalence relation on the set of absolute values on K, and the equivalence classes are called places of K. The non-archimedian absolute values induce a topology that is quite different compared to what one perhaps might expect. For example, triangles are isosceles - that is to say, any triangle will always have two sides of the same length. (SAY THAT WE ARE WORKING WITH NON ARCHIMEDIAN FIELDS)

Proposition 7.1. *Suppose $x, y \in K$ and that $|x| \neq |y|$. Then $|x + y| = \max\{|x|, |y|\}$. In particular, all triangles are isosceles.*

Proof. Assume without loss of generality that $|x| > |y|$. Then we have

$$|x + y| \leq |x| = \max\{|x|, |y|\}$$

Also, $|x| = |(x + y) - y| \leq \max\{|x + y|, |y|\} = |x + y|$, since $|x| > |y|$. Thus, $|x| = |x+y|$. For the last claim suppose that $z \in K$ and consider the three sides $A = x - y$, $B = y - z$ and $C = x - z$ and suppose that two of these sides, say A and B have different length, that is, $|A| \neq |B|$. We clearly have that $A+B = C$ and so by what we have just shown we have that

$$|C| = |A + B| = \max\{|A|, |B|\}$$

But that means that C has the same length as A or B and we conclude that two sides in the triangle must have the same length. □

We also have the following lemma which does not hold for archimedian absolute values

Lemma 7.1. *Let $|\cdot|$ be a non-achimedian absolute value on a field K. Suppose (x_n) is a sequence in K. Define the sequence $y_n = x_{n+1} - x_n$. The following are equivalent*

1. *x_n is Cauchy.*
2. *$y_n \to 0$ for $n \to \infty$.*

Proof. A sequence being Cauchy clearly implies that y_n converges to 0. On the other hand let $N \in \mathbb{N}$ be so large that $|y_n| < \epsilon$ for all $n \geq N$. Suppose now $n > m > N$. We obtain

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - ... + x_{m+1} - x_m| \\ &= |y_n + y_{n-1} + ... + y_m| \leq \max\{|y_n|, ..., |y_m|\} < \epsilon \end{aligned}$$

□

This lemma can be really helpful in situations where the sequence (x_n) has the form $x_n = \sum_{i=1}^{n} z_n$, since in this case we can determine if (x_n) is cauchy simply by checking if z_n converges to 0. This is certainly not something we can do in the archimedian setting though. For example if $z_n = \frac{1}{n}$ then x_n is the harmonic series and we know that it diverges with respect to $|\cdot|_\infty$. However, z_n certainly converges to 0.

Proposition 7.2. *An absolute value $|\cdot|$ on a field K is uniformly continuous.*

Proof. Let $\epsilon > 0$ and pick $\delta = \epsilon$. For any $x, y \in K$ so that $|x - y| < \delta$ we have

$$||x| - |y||_\infty \leq |x - y| < \delta = \epsilon$$

where we have used the reverse triangle inequality. □

Uniformly continuous functions have the nice property that they preserve Cauchy sequences. This implies that if we have some Cauchy sequence, (x_n) in K, then $(|x_n|)$ is a Cauchy sequence in $\mathbb{R}$. We will use this fact later on.

Definition 7.2. *Let K, L be valued fields and suppose that $\phi : K \to L$ is a function. We say that ϕ preserves the absolute value if $|\phi(x)|_L = |x|_K$.*

It is worth pointing out that if ϕ is as above and is also a homomorphism, then ϕ is actually uniformly continuous. This is simply because we have for all $x, y \in K$ that

$$|\phi(x) - \phi(y)|_F = |\phi(x - y)|_F = |x - y|_K$$

We now turn to a notion that is closely related to non-archimedian absolute values - namely valuations.

Definition 7.3 (Valuation)**.** *A valuation on a field K is a function $v : K \to \mathbb{R} \cup \{\infty\}$ with the following properties*

1. *$v(x) = \infty$ if and only if $x = 0$.*
2. *$v(xy) = v(x) + v(y)$ for every $x, y \in K$.*
3. *$v(x + y) \geq \min\{v(x), v(y)\}$ for every $x, y \in K$.*

The symbol ∞ is just an element so that for all $x \in \mathbb{R}$ we have $\infty > x$, $\infty + x = x + \infty = \infty$ and $\infty + \infty = \infty$.

We stipulated above that valuations and absolute values have something to do with each other. Let us see why. Suppose we have access to a valuation, v on a field K. Then for any $q > 1$ we get what we will call a corresponding absolute value on K by setting $|x| = q^{-v(x)}$. This absolute value is in fact non-archimedian because of the third property in the definition above and all absolute values corresponding to v are equivalent. In other words, they all correspond to the same place. Because if we take $p, q > 1$ and suppose that $p > q$ then we get two absolute values $|x|_1 = p^{-v(x)}$ and $|x|_2 = q^{-v(x)}$. Setting $s = \log \frac{p}{q}$ we have $s > 0$ and we obtain $p = q^s$ and so

$$|x|_1 = |x|_2^s$$

We will say that two valuations v_1 and v_2 are equivalent if $v_1 = sv_2$ for some $s > 0$. In this case any absolute value corresponding to v_1 will be equivalent to any absolute value corresponding to v_2. Because if we have an arbitrary absolute value corresponding to v_1, that is to say, if we have $|x| = q^{-v_1(x)}$ for some $q > 1$ then we can write

$$|x| = q^{-v_1(x)} = q^{-sv_2(x)} = (q^s)^{-v_2(x)}$$

and as $q^s > 1$ it follows from the argument above that any absolute value corresponding to v_2 has to be equivalent to $|x|$. On the other hand, if we start with a non-archimedian absolute value, $|\cdot|$, on a field K then we can obtain a valuation v by fixing some $q > 1$ and setting $v(x) = -\frac{\log |x|}{\log q}$ for $x \in K^*$ and $v(0) = \infty$. We will call v a valuation corresponding to $|\cdot|$ and we see that there is a bijective correspondence between non-archimedian absolute values and valuations. As such, fields with non-archimedian absolute value and fields with a valuation are really the same objects and we can therefore freely choose to work with valuations or absolute values as we see fit. When working with valuations it is useful to have the following in mind. Assume that K is a valued field with non-archimedian absolute value $|\cdot|$ and v is a corresponding valuation. If $a \in K$ and (x_n) is a sequence in K, then x_n converges to a if and only if $v(x_n - a) \to \infty$. Let us now present different objects attached to a valuation. Some of these don't change when the valuation is swapped out with another equivalent one. The object we define now, the so-called value group, does not have this property though.

Definition 7.4. *Let v be a valuation on a field K. The group $\Gamma_v = v(K^*)$ is called the value group of v and if there is $s > 0$ so that $\Gamma_v = s\mathbb{Z}$ we say that v is discrete. If $s = 1$ then v is called normalized and in this case an element $\pi \in K$ so that $v(\pi) = 1$ is called a prime element.*

Definition 7.5 (Valuation ring)**.** *Let O be an integral domain with field of fractions K. If for all $x \in K$ we have that $x \in O$ or $x^{-1} \in O$ then O is called a valuation ring.*

A valuation ring O is a local ring and it has maximal ideal $\mathfrak{p} = \{x \in K \mid x^{-1} \notin O\}$ since the set $O \setminus \mathfrak{p}$ consists of all units of O. Hence $O/\mathfrak{p}$ is a field

and it is called the residue field of O. Also, O is integrally closed; Suppose that $x \in K$ is integral over O. That is we have $x^n + a_{n-1}x^{n-1} + ... + a_0 = 0$ where the a_i all belong to O. Assume now for contradiction that $x \notin O$. That means that $x^{-1} \in O$ and so we can solve for x in the equation above, meaning that $x \in O$, which is a contradiction. A valuation on a field can be used to construct a valuation ring.

Proposition 7.3. *Let K be a non-archimedian valued field with valuation v and let $|\cdot|$ be a corresponding absolute value. Then*

$$O := \{x \in K \mid v(x) \geq 0\} = \{x \in K \mid |x| \leq 1\}$$

is a valuation ring and is called the valuation ring of K with respect to v (or equivalently, with respect to $|\cdot|$). The ring O has field of fractions K and it has unique maximal ideal

$$\mathfrak{p} = \{x \in K \mid v(x) > 0\} = \{x \in K \mid |x| < 1\}$$

Its units are

$$O^* = \{x \in K \mid v(x) = 0\} = \{x \in K \mid |x| = 1\}$$

If v is discrete and normalized and $\pi \in O$ is a prime element then we have

- $K^* = \{\pi^k \mid k \in \mathbb{Z}\} \times O^*$
- *The non-zero ideals of O are given by $\mathfrak{p}^n = \{x \in K : v(x) \geq n\}$ for $n \geq 0$, and $\mathfrak{p} = \pi O$. So O is a PID.*
- $O/\mathfrak{p} \cong \mathfrak{p}^n/\mathfrak{p}^{n+1}$ *for* $n \geq 0$

Proof. Much of this is clear, so we will only deal with the part where v is assumed to be discrete and normalized. Assume $x \in K^*$ and let $m = v(x)$. We have $m \in \mathbb{Z}$, since $v(K^*) = \mathbb{Z}$ by assumption, and so $v(x\pi^{-m}) = v(x) - mv(\pi) = 0$ meaning that $x\pi^{-1} \in O^*$. Thus $x\pi^{-1} = u$ for some $u \in O^*$ which means that $K^* = (\pi) \times O^*$. Take now an ideal $I \neq 0$ of O. Since v is discrete we can choose $x \in I$, where $x \neq 0$ so that $v(x) = \min_{z \in I} v(z)$. Let $n := v(x)$. By what we have already shown there is $u \in O^*$ so that $x = u\pi^n$. If now $z \in \pi^n O$ we have that $z = xy$ for some $y \in O$ and since $x \in I$ and I is an ideal also $z \in I$. Hence $\pi^n O \subseteq I$. For the other inclusion, take $z \in I$ and write $z = u\pi^m$ for some $u \in O^*$. We have $v(z) = m \geq n$ so we can write $z = (u\pi^{m-n})\pi^n$ which is an element of $\pi^n O$. Thus, $I = \pi^n O$. Thus all the non zero ideals of O are $\pi O, \pi^2 O, ...$, so O is a PID. Among these ideals, πO is the largest with respect to inclusion, so must be the maximal ideal, $\mathfrak{p}$. Consider the homomorphism

$$\begin{aligned} \phi : \mathfrak{p}^n &\to O/\mathfrak{p} \\ a\pi^n &\mapsto \overline{a}, \text{ for all } a \in O \end{aligned}$$

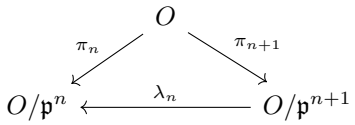
This map is well-defined since $a\pi^n = a'\pi^n$ implies $a = a'$ meaning that $\overline{a} = \overline{a}'$. Now $\phi(a\pi^n) = 0$ if and only if $a \in \mathfrak{p}$ if and only if $a\pi^n \in \mathfrak{p}^{n+1}$ so $\ker \phi = \mathfrak{p}^{n+1}$. This map is also clearly surjective and so the first isomorphism theorem gives the desired result.

□

Suppose K is complete with respect to a discrete valuation. For each n we have the natural projection

$$O \xrightarrow{\pi_n} O/\mathfrak{p}^n.$$

We have the inclusions $\mathfrak{p} \supset \mathfrak{p}^2 \supset ...$, which implies that every element in $\mathfrak{p}^{n+1}$ is sent to zero under this map, giving us a commutative diagram



Thus, we get a sequence of maps

$$O/\mathfrak{p} \xleftarrow{\lambda_1} O/\mathfrak{p}^2 \xleftarrow{\lambda_2} O/\mathfrak{p}^3 \xleftarrow{\lambda_3} ...$$

This leads us the following definition.

Definition 7.6. *The set*

$$\lim_{\leftarrow} O/\mathfrak{p}^n = \{(x_n) \in \prod_{n=1}^{\infty} O/\mathfrak{p}^n : \lambda_n(x_{n+1}) = x_n\}$$

is a subring of $\prod_{n=1}^{\infty} O/\mathfrak{p}^n$ *and is called the projective limit of the system of rings* $O/\mathfrak{p}^n$.

Because of the commutative diagram above we obtain a ring map

$$O \to \lim_{\leftarrow} O/\mathfrak{p}^n$$

which just sends an element in O to its respective equivalence class in the projective limit. Equipping each of the $O/\mathfrak{p}^n$ with the discrete topology, the projective limit becomes a closed subset of the product topology on $\prod_{n=1}^{\infty} O/\mathfrak{p}^n$. We have the following theorem

Theorem 7.1. *The map* $O \to \lim_{\leftarrow} O/\mathfrak{p}^n$ *is an isomorphism of rings and a homeomorphism.*

8 Completions of fields

Let $(K, |\cdot|)$ be a valued field. A Cauchy sequence in K is a sequence (x_n) in K so that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ so that if $m, n \geq N$ then

$$|x_n - x_m| \leq \epsilon$$

A null sequence is a sequence in K converging to 0.

Definition 8.1. *We say that a metric space X with metric d is complete if every Cauchy sequence in X converges to an element in X with respect to d. A valued field $(K, |\cdot|)$ is called a complete field if it is complete as metric space with respect to $d(x, y) = |x - y|$.*

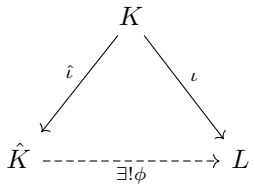
(Maybe we should say here that $\mathbb{Q}$ is dense in $\mathbb{R}$).

Recall that the closed sets in a metric space are characterized as the sets containing all their limit points. In a complete metric space we have the following characterization of closed sets.

Proposition 8.1. *Suppose X is a complete metric space. Then a subset $A \subseteq X$ is closed if and only if A is complete.*

Proof. Suppose first that A is closed and let (x_n) be any Cauchy sequence in A. Then this sequence converges to some point $a \in X$ since X is complete. But as A is closed in X that means by definition that $a \in A$. On the other hand, assume that A is complete and let (x_n) be a sequence in A converging to some element $a \in X$. But then (x_n) is in fact a Cauchy sequence, hence $a \in A$. □

Definition 8.2 (Completion)**.** *Suppose $(K, |\cdot|)$ is a valued field. A completion of K is a complete valued field $\hat{K}$ together with a homomorphism $\hat{\iota} : K \to \hat{K}$ preserving the absolute value, satisfying the following universal property; If L is any other complete valued field and $\iota : K \to L$ is a homomorphism preserving the absolute value, then there is a unique homomorphism $\phi : \hat{K} \to L$, which preserves the absolute value, making the following diagram commute*

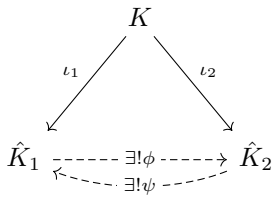


The map ϕ in the diagram above is actually uniformly continuous since it preserves absolute values and is a homomorphism. Note also that we talk about 'a' completion, insinuating that there are multiple different ones. This is true, but it turns out that they are all unique up to unique isomorphism. This is a quite strong statement - much stronger than just being unique up to isomorphism.

Proposition 8.2. *Suppose K is a valued field and that $(\hat{K}_1, \iota_1)$ and $(\hat{K}_2, \iota_2)$ are completions of K. Then there is a unique absolute value preserving isomorphism, $\phi : \hat{K}_1 \to \hat{K}_2$ so that $\phi \circ \iota_1 = \iota_2$. In other words, the completion of K is unique up to unique isomorphism.*

Proof. Let us invoke the universal property on both $\hat{K}_1$ and $\hat{K}_2$ to obtain the

diagram



By definition both ϕ and ψ preserve the absolute value. Writing out the two commutativity conditions we get

$$\iota_1 = \psi \circ \iota_2$$
$$\iota_2 = \phi \circ \iota_1$$

Combining these we obtain

$$\iota_1 = \psi \circ (\phi \circ \iota_1) = (\psi \circ \phi) \circ \iota_1$$

So $\psi \circ \phi$ has to be the identity on $\hat{K}_1$ and similarly $\phi \circ \psi$ is the identity on $\hat{K}_2$, hence ϕ is an isomorphism and it is the only one satisfying the equation $\iota_2 = \phi \circ \iota_1$. □

Because of this proposition we will more often than not say "the completion" rather than "a completion". So far we haven't demonstrated that the completion actually exists. This will be our next goal. The process of constructing such an object is very similar to the idea of the concrete construction of, for example, the tensor product - construct a huge object that sort of does what you want, then sculpt it into the desired object by modding out by a suitable relation.

Theorem 8.1. *Let K be a valued field and R be the set of all Cauchy sequences of K. Then R is a ring and the set $\mathfrak{m}$ of all null sequences of R is a maximal ideal.*

Proof. The addition and multiplication operations on R are defined element wise and it is easily seen that the Cauchy property is preserved under these operations. Rules such as the distributive law are inherited directly from the ring structure on K, and so R is a ring. The set $\mathfrak{m}$ is non-empty, as it most certainly contains the constant sequence $(0, 0, 0, ...)$. It is also contained in R since every convergent sequence is a Cauchy sequence so in particular the null sequences are Cauchy. Furthermore the difference of two null sequences is again a null sequence and the product of any sequence by a null sequence is also a null sequence. Thus, $\mathfrak{m}$ is an ideal of R. Let us now show that it is a maximal ideal. Take a sequence $(x_n) \in R$ which is not also in $\mathfrak{m}$. Such an element exists since R contains, for example, the constant sequence $1 = (1, 1, 1, ...)$ which is clearly not a null sequence. We wish to show that the ideal $J = (x_n) + \mathfrak{m}$ contains 1, i.e, that this ideal is in fact the entire ring R. The sequence $|x_n|$ is a Cauchy

sequence in $\mathbb{R}$ with respect to the usual absolute value. Hence there is an N so that if $n \geq N$ then $|x_n| \geq c > 0$. Thus we can define

$$y_n = \begin{cases} \frac{1}{x_n} & \text{if } n \leq N \\ 0 & \text{if } n > N \end{cases}$$

Let now $\epsilon > 0$ be given and choose M so that $|x_n - x_m| < c^2\epsilon$ whenever $n, m \geq M$. For $n, m \geq \max\{N, M\}$ we now have

$$|y_n - y_m| = |\frac{1}{x_n} - \frac{1}{x_m}| = |\frac{x_m - x_n}{x_n x_m}| \leq \frac{1}{c^2}|x_m - x_n| \leq \frac{c^2\epsilon}{c^2} = \epsilon$$

But that means that y_n belongs to R and so $x_n y_n$ is in J. Denote by 1 the identity in R, i.e, the constant sequence $(1, 1, 1, ...)$. We now have $1 - x_n y_n \in J$ and so 1 is in J. □

From this it follows that $\hat{K} = R/\mathfrak{m}$ is a field. Define now

$$\hat{\iota}(x) = (x, x, x, ...) + \mathfrak{m} \in \hat{K}$$

for $x \in K$. This map is a homomorphism of fields and it is certainly not the zero map, which implies that it is injective. We can therefore think of K as a subfield of $\hat{K}$.

let us first recall the following proposition

Proposition 8.3. *Suppose X and Y are topological spaces and that $f : A \to Y$ is map defined on a dense subset A of X. Then f extends uniquely to a continuous map $\tilde{f} : X \to Y$.*

Theorem 8.2 (Existence of completion)**.** *Suppose $(K, |\cdot|)$ is a valued field and let R the set of all Cauchy sequences of K with respect to $|\cdot|$ and $\mathfrak{m} \subseteq R$ the maximal ideal consitsting of all null sequences. The function $f : R \to \mathbb{R}$, given by*

$$(x_n) \mapsto \lim_{n\to\infty} |x_n|$$

descends to the function, $|\cdot|' : \hat{K} \to \mathbb{R}$ given by

$$|\overline{(x_n)}|' = f((x_n))$$

We have the following properties

- *$(\hat{K}, |\cdot|')$ is a valued field*
- *The homomorphism $\hat{\iota}$ preserves the absolute value.*
- *K is dense in $\hat{K}$*
- *$\hat{K}$ is complete with respect to $|\cdot|'$*

Furthermore, the pair $(\hat{K}, \hat{\iota})$ is the completion of K.

Proof. Let us first show that f descends to a function on the quotient. First of all, if $(x_n) \in R$, then (x_n) is a Cauchy sequence in K. Hence $(|x_n|)$ is a Cauchy sequence in $\mathbb{R}$, hence it converges to an element in $\mathbb{R}$, since $\mathbb{R}$ is complete with respect to $|\cdot|_\infty$, which is shown in .

Hence the map f is well defined. Take now two elements x and y in $\hat{K}$ with $x = y$ and suppose they are represented by the Cauchy sequences (x_n) and (y_n) in K, respectively. By the reverse triangle inequality we have

$$||x_n| - |y_n||_\infty \leq |x_n - y_n|$$

But $x_n - y_n$ is a null sequence and hence the left hand side converges to 0. This means that $\lim_{n\to\infty}(|x_n| - |y_n|) = 0$ and therefore

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |y_n|$$

This shows that the map f descends to the quotient as desired. Let us move on to justifying that $|\cdot|'$ is actually an absolute value on $\hat{K}$. This is essentially inherited from the absolute value on K. So let us just show that $|\cdot|'$ possesses the zero-detection property. Suppose $x = \overline{(x_n)} \in \hat{K}$ so that $|x|' = 0$. We have

$$0 = |x|' = \lim_{n\to\infty} |x_n|$$

But this just says that (x_n) is a null sequence so $x = 0$. On the other hand, if we already know that $x = 0$ then it is clear that x_n is a null sequence and therefore $|x|' = 0$. It is also easily seen that $|\cdot|'$ extends $|\cdot|$. For if $x \in K$ then

$$|x| = \lim_{n\to\infty} |x| = f(\hat{\iota}(x)) = |\overline{\hat{\iota}(x)}|'$$

Let us now show that K is dense in $\hat{K}$. What we mean by this is that the image of K under the $\hat{\iota}$ is dense in $\hat{K}$. Let $\epsilon > 0$, take $\lambda \in \hat{K}$ and consider the open ball $B(\lambda, \epsilon)$. Let us show that we can find a constant sequence inside of this ball. Let (x_n) be a Cauchy sequence representing λ and find $N \in \mathbb{N}$ so that $|x_n - x_m| < \epsilon/2$ for all $m, n \geq N$. Define now the constant sequence $y = \hat{\iota}(x_N)$. Using the definition of the absolute value on $\hat{K}$ we get

$$|\lambda - y|' = \lim_{n\to\infty} |x_n - x_N| \leq \epsilon/2 < \epsilon$$

This shows that $y \in B(\lambda, \epsilon)$ and we conclude that K is dense in $\hat{K}$. Now we are ready to show that $\hat{K}$ is complete with respect to $|\cdot|'$. To do this, suppose that we have a Cauchy sequence (λ_n) in $\hat{K}$. As we have just seen, K is dense in $\hat{K}$. So for every λ_i, we can find an element $y_i \in K$ so that $|\lambda_i - \hat{\iota}(y_i)|' < \frac{1}{i}$. This way we get a sequence $(\hat{\iota}(y_n))$ in $\hat{K}$ and we observe that

$$\lim_{n\to\infty} |\lambda_n - \hat{\iota}(y_n)|' = 0$$

So by (**??**), $(\hat{\iota}(y_n))$ is also Cauchy. But then also the sequence (y_n) in K is Cauchy since $|\cdot|'$ extends the absolute value on K. But that means that

(y_n) represents an element in $\hat{K}$. Call this element λ and let us show that $\lim_{n\to\infty} \lambda_n = \lambda$. To do this, it suffices to show that $\lim_{n\to\infty} \tilde{y}_n = \lambda$. As (y_n) is Cauchy, we can find N so that if $n, m \geq N$ then $|y_n - y_m| < \epsilon/2$. For any $n \geq N$ we now have

$$|\tilde{y}_n - \lambda|' = \lim_{m\to\infty} |y_n - y_m| \leq \epsilon/2 < \epsilon$$

This shows that

$$\lim_{n\to\infty} |\tilde{y}_n - \lambda|' = 0$$

which shows that $\hat{K}$ is complete with respect to $|\cdot|'$. Moving on to the last statement, let us see why $(\hat{K}, \hat{\iota})$ is the completion of K. So let L be a complete valued field and $\iota : K \to L$ a homomorphism preserving the absolute value. Let us define a function $\phi : \hat{K} \to L$. Pick an element $\lambda \in \hat{K}$. Because $\hat{\iota}(K)$ is dense in $\hat{K}$, there is Cauchy sequence, (y_n), in $\hat{\iota}(K)$ that converges to λ. Hence there is a sequence (x_n) in K so that $(\hat{\iota}(x_n)) = (y_n)$ and as $\hat{\iota}$ preserves the absolute value the sequence (x_n) is also Cauchy. As ι preserves the absolute value, we now have a Cauchy sequence $(\iota(x_n))$ in L and the fact that L is complete ensures that it converges to an element, $\lambda' := \lim_{n\to\infty} \iota(x_n)$. By uniqueness of limits we therefore obtain a well defined map $\phi : \hat{K} \to L$ mapping the element λ to λ'. It preserves absolute values since we have for $x \in \hat{K}$ that

$$\begin{aligned}
|\phi(x)|_L &= |\lim_{n\to\infty} \iota(x_n)|_L \\
&= \lim_{n\to\infty} |\iota(x_n)|_L \\
&= \lim_{n\to\infty} |x_n|_K \\
&= \lim_{n\to\infty} |\hat{\iota}(x_n)|_{\hat{K}} \\
&= |\lim_{n\to\infty} \hat{\iota}(x_n)|_{\hat{K}} = |x|_{\hat{K}}
\end{aligned}$$

Therefore f is continuous and it satisfies $\phi \circ \hat{\iota} = \iota$. This means that it is in particular defined on the dense subset $\hat{\iota}(K)$ which makes it uniquely determined by proposition 8.3. The map ϕ is also a ring map because the limit of the sum of two convergent sequences is the sum of the limits of the sequences and similarly for the product. Thus the pair $(\hat{K}, \hat{\iota})$ is the completion of K. □

Remark 8.1. *Because $|\cdot|'$ extends $|\cdot|$, we will, for ease of notation, simply denote $|\cdot|'$ by $|\cdot|$. Also, if we knew that $|\cdot|$ was non-archimedian then, as proposition 8.4 spells out, the sequence $(|x_n|)$ would eventually become constant. Thus we would not have needed to invoke the fact that $\mathbb{R}$ is complete to conclude that $(|x_n|)$ converges.*

Proposition 8.4. *Let $(K, |\cdot|)$ be a non-archimedian valued field. If (x_n) is a Cauchy sequence which is not a null sequence, then there exists an N so that $|x_n| = |x_m|$ for all $m, n \geq N$. In other words, the sequence must eventually become constant.*

Proof. Since (x_n) does not converge to 0, there exists an $N_1 \in \mathbb{N}$ and $c > 0$ so that $|x_n| \geq c > 0$ for all $n \geq N_1$. As the sequence is Cauchy, we can find $N_2 \in \mathbb{N}$ so that $|x_n - x_m| < c$ for all $n, m \geq N_2$. Set now $N := \max\{N_1, N_2\}$ and suppose $n, m \geq N$. Then

$$|x_n - x_m| < c \leq \min\{|x_n|, |x_m|\}$$

This means that both $|x_n|$ and $|x_m|$ are different from $|x_n - x_m|$ and hence $|x_n| = |x_m|$ since triangles are isosceles by proposition 7.1. □

Let $(K, |\cdot|)$ be a nonarchimedian valued field with $\hat{K}$ its completion and let v be a valuation on K corresponding to $|\cdot|$. We can extend the valuation on K to a valuation on $\hat{K}$; Let $x \in \hat{K}$ be represented by the Cauchy sequence (x_n) in K and suppose that $x = 0$. Then (x_n) converges to 0 and that means that $v(x_n) \to \infty$. On the other hand, if (x_n) does not converge to 0, then by what we showed above, the sequence $(|x_n|)$ must eventually become a non-zero constant and hence the sequence $(v(x_n))$ becomes constant at some point as well and therefore it converges. Hence we obtain a valuation, $\hat{v}$, on $\hat{K}$ by defining $\hat{v}(x_n) = \lim_{x\to\infty} v(x_n)$. Note that this shows that $\hat{v}$ and v have the same value group and in particular, if v is discrete then so is $\hat{v}$. We will sometimes speak about the completion with respect to a valuation v. By this we mean the completion of the field with respect to an absolute value corresponding to v.

Lemma 8.1 (Hensel's Lemma)**.** *Let K be a field with valuation and let O be its corresponding valuation ring and $\mathfrak{p}$ the unique maximal ideal of O. Denote by $\bar{\cdot}$ the canonical map $O[x] \twoheadrightarrow \kappa[x]$, where $\kappa = O/\mathfrak{p}$. Suppose that $f \in O[x]$ such that $\overline{f} \neq 0$ and that there are relatively prime polynomials $g_0, h_0 \in \kappa[x]$ so that $\overline{f} = g_0 h_0$. Then there are polynomials $g, h \in O[x]$ so that $f = gh$ with $\deg(g) = \deg(g_0)$ and $\overline{g} = g_0$ and $\overline{h} = h_0$*

Proof. MISSING □

Lemma 8.2. *(DOES THIS ONLY APPLY FOR CHARACTERIC 0?) Consider the field extension K/F and suppose that $\alpha \in K$ is algebraic over F. Write*

$$f(x) = x^n + a_{n-1}x^{n-1} + ... + a + a_0$$

for the minimal polynomial of α over F. Then $N_{F(\alpha)/F}(\alpha) = \pm a_0$. If K/F is finite then $N_{K/F}(\alpha) = \pm a_0^m$ where $m = [K : F(\alpha)]$

Proof. Denote by $\alpha_1, ..., \alpha_n$ the n conjugates of α. Then we can write

$$f(x) = (x - \alpha_1)...(x - \alpha_n),$$

and we see that

$$a_0 = \pm\alpha_1...\alpha_n = N_{F(\alpha)/F}(\alpha)$$

Now if K/F is finite and $m = [K : F(\alpha)]$ then

$$N_{K/F}(\alpha) = N_{F(\alpha)/F}(N_{K/F(\alpha)}(\alpha)) = N_{F/K}(\alpha^m) = N_{F/K}(\alpha)^m = \pm a_0^m$$

□

Theorem 8.3. *Suppose $(K, |\cdot|)$ is a complete valued nonarchimedian field and that L/K is a finite field extension of degree n. Then $|\cdot|$ can be extended in a unique way to a nonarchimedian absolute value on L given explicitly as*

$$|\alpha| = \sqrt[n]{|N_{L/K}(\alpha)|}, \text{ for all } \alpha \in L \qquad (\star)$$

Furthermore $(L, |\cdot|)$ is complete.

Proof. Let us first make sure that the proclaimed function defined in eq. $(\star)$ is actually well defined. For all $\alpha \in L$ we know that $N_{L/K}(\alpha) \in K$. Hence it makes sense to use the absolute value on K on it. After doing so we get a real number greater than or equal to 0, and so it makes sense to take the nth root of this number. Let us start with the existence part. Let o be the valuation ring of K and O the integral closure of o in L. To help us establish that eq. $(\star)$ is an absolute value on L we will first show that

$$O = \{\alpha \in L \mid N_{L/K}(\alpha) \in o\}. \qquad (\triangle)$$

First, suppose that $\alpha \in O$. Then α is integral over o and hence so are all the conjugates of α. But then the product of all of the conjugates of α is also integral over o and this product is equal to $N_{L/K}(\alpha)$, which we know is an element of K. Since o is integrally closed in K it now follows that $N_{L/K}(\alpha)$ is in o. This shows one of the inclusions. Let us now show the other. So assume that $\alpha \in L^*$ and that $N_{L/K}(\alpha) \in o$. Since L/K is a finite extension, L is algebraic over K and therefore α is in particular algebraic over K. Hence we can consider its minimal polynomial, f, over K. Then by lemma 8.2 we have that

$$N_{L/K}(\alpha) = \pm a_0^m$$

where $m = [L : F]$ and a_0 is the constant term of f. But that means that $a_0^m \in o$ so $|a_0|^m = |a_0^m| \leq 1$, so $|a_0| \leq 1$ so $a_0 \in o$. If we can show that $f \in o[x]$ we are done, as that would establish that α is integral over o meaning that $\alpha \in O$. For contradiction, let us assume that $f \notin o[x]$ and let $\mathfrak{p}$ be the maximal ideal of o. We have $a := \max\{|a_d|, |a_{d-1}|..., |a_0|\} > 1$, where $a_d = 1$, since at least one of the coefficients of f does not belong to o. We can now find the smallest i so that $|a_i| = a$. This implies that $a \neq 0$ and so we can now define $g(x) = \frac{1}{a}f = \frac{1}{a}x^d + ... + \frac{a_0}{a}$ which is clearly in $o[x]$ and $\overline{g}$ is non-zero in $o/\mathfrak{p}$, since the ith coefficient of g is $1 \notin \mathfrak{p}$. Now by the choice of i we now have that $|\frac{a_0}{a}| < 1$, ..., $|\frac{a_{i-1}}{a}| < 1$. Thus, $\frac{a_d}{a}, ..., \frac{a_{i+1}}{a}$ all belong to $\mathfrak{p}$ and therefore they are all zero in the quotient $o/\mathfrak{p}$. Hence the terms in g having degree $0, ..., i-1$ all vanish in $o/\mathfrak{p}[x]$ and so we get a factorization $\overline{g(x)} = x^i h(x)$ where $h \in o/\mathfrak{p}[x]$ and $x \nmid h(x)$. Hence x^i and h are coprime and so by Hensel's lemma we now get a factorization of g, so g is not irreducible. But this cannot be the case since f is irreducible. Hence we get a contradiction, and we conclude that we must have $f \in o[x]$. We have now established eq. $(\triangle)$.

Let us now use this result to show that

$$O = \{\alpha \in L \mid |\alpha| \leq 1\}$$

So suppose $\alpha \in L$. We have that $|\alpha| \leq 1$ is the same as saying that $|N_{L/K}(\alpha)| \leq 1$ which is the same as saying that $N_{L/K}(\alpha)$ belongs to the valuation ring o which, by what we have just shown is the same as $\alpha \in O$. Note that this means that once we have shown that eq. $(\star)$ is a nonarchimedian absolute value on L, we can conclude that its valuation ring is O. So let us show this. Let us first show that eq. $(\star)$ satisfies the strong triangle inequality. Since O is a ring we know that if $\alpha \in O$ then also $\alpha + 1 \in O$. From what we just showed this immediately implies that if $|\alpha| \leq 1$ then $|\alpha + 1| \leq 1$. From this we can now deduce the strong triangle inequality. Assume that $\alpha, \beta \in L$ and that $|\alpha| \leq |\beta|$. Note that if $\beta = 0$ then the strong triangle inequality is already satisfied, so suppose that $\beta \neq 0$. Then $|\beta| \neq 0$ and we have $\frac{\alpha}{\beta} \in L$ and $|\frac{\alpha}{\beta}| \leq 1$. Thus

$$|\frac{\alpha}{\beta} + 1| \leq 1$$

and after multiplication by $|\beta|$ we are left with the inequality

$$|\alpha + \beta| \leq |\beta|$$

Because we assumed that $|\alpha| \leq |\beta|$ we have $|\beta| = \max\{|\alpha|, |\beta|\}$. We conclude that the strong triangle inequality is satisfied, namely $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$ for all $\alpha, \beta \in L$. We still have some things to check before we can conclude that $|\cdot|$ is an absolute value on L, but these are easier to do. Firstly if $\alpha \in L$ then we have $|N_{L/K}(\alpha)| = 0$ if and only if $N_{L/K}(\alpha) = 0$ if and only if $\alpha = 0$. The fact that $|\alpha\beta| = |\alpha||\beta|$ for all $\alpha, \beta \in L$ follows from the fact that all the functions involved in the right hand side of eq. $(\star)$ are multiplicative. We conclude that eq. $(\star)$ is an absolute value on L. It also extends the absolute value on K since if $\alpha \in K$ then $N_{L/K}(\alpha) = \alpha^n$.

Let us now move on to proving the uniqueness of eq. $(\star)$. So let $|\cdot|'$ be another nonarchimedian absolute value on L which extends the absolute value on K. Denote by O' the valuation ring with respect to $|\cdot|'$ and let $\mathfrak{p}$ and $\mathfrak{p}'$ be the maximal ideal of O and O' respectively. Let us show that $O \subseteq O'$. Assume for contradiction that $\alpha \in O$ but $\alpha \notin O'$. Then $\alpha \neq 0$ and we have that $|\alpha|' > 1$ and thus $\alpha^{-1} \in O'$. Since $\alpha \notin O'$ then α^{-1} cannot be a unit of O' and hence α^{-1} belongs to $\mathfrak{p}'$. As $\alpha \in O$ and O is the integral closure of o in L, we can find an equation

$$\alpha^d + a_{d-1}\alpha^{d-1} + ... + a_0 = 0$$

where all the a_i belong to o. But o is contained in O' since if $x \in o$ then $x \in K$ so $|x|' = |x| \leq 1$. Hence all the a_i belong to O'. Multiplying the above equation by α^{-1} and moving things around we get

$$1 = -(a_{d-1}\alpha^{-1} + ... + a_0\alpha^{-d})$$

But each term $a_{d-i}\alpha^{-i}$ is in $\mathfrak{p}'$ since $a_{d-i} \in O'$ and $\alpha^{-i} \in \mathfrak{p}'$ combined with the fact that $\mathfrak{p}'$ is an ideal. But that means that $1 \in \mathfrak{p}'$ which is impossible. Hence $O \subseteq O'$. What this means is that if $|\alpha| \leq 1$ then $|\alpha|' \leq 1$ and this implies that $|\cdot|$ and $|\cdot|'$ are equivalent (TODO). But then there exists some fixed $s \geq 1$ so

that $|x|' = |x|^s$ for all $x \in L$ and as $|\cdot|$ and $|\cdot|'$ are equal on K we must have $s = 1$. Hence $|x|' = |x|$, proving uniqueness.

We now only need to show that L is complete with respect to $|\cdot|$. Let $1, \beta_2, ..., \beta_n \in L$ be a basis for L over K. Then for any $z \in L$ there exists unique $x_1, ..., x_n \in K$ so that $z = x_1 + x_2\beta_2 + ... + x_n\beta_n$, and so we get a map, $\phi : L \to \mathbb{R}$ by sending z to $\max\{|x_1|, ..., |x_n|\}$. This defines a non-archimedian absolute value on L and L is complete with respect to it.

Observe that if $z \in K$ then we have $0 = x_2 = ... = x_n$, showing that ϕ extends the absolute value on K. Hence by the uniqueness property we have just shown we now have that $\phi = |\cdot|$ and therefore $(L, |\cdot|)$ is complete. □

Proposition 8.5. *Let (K, v) be a valued field and K_v the completion of K with respect to v. We have*

$$O_{\hat{v}}/\mathfrak{p}_{\hat{v}} \cong O_v/\mathfrak{p}_v$$

If v is discrete then for all $n \geq 1$ we have

$$O_{\hat{v}}/\mathfrak{p}_{\hat{v}}^n \cong O_v/\mathfrak{p}_v^n$$

Proof. □

Lemma 8.3. *Let K be a number field. The field of fractions of the ring of integers of K, O_K, is equal to K. In other words, every element of K can be written as $\frac{\alpha}{\beta}$ for some $\alpha, \beta \in O$, where $\beta \neq 0$.*

Proof. Let F be the field of fractions of O_K. Then $F \subseteq K$ as F is the smallest field containing O_K. For the other inclusion, suppose for contradiction that $[K : F] > 1$. Take then an element, x, which is in K but not in F. Next, find an integer d so that $d\alpha$ is an algebraic integer. What this means is that $d\alpha \in O_K$, and therefore $d\alpha \in F$. But then $\alpha \in F$, contrary to our initial assumption. Hence $[K : F] = 1$ and so $F = K$. □

Recall that for a commutative ring R with unity we can localize at any given prime ideal $\mathfrak{p} \subseteq R$ and obtain the local ring $R_\mathfrak{p} = \{\frac{\alpha}{\beta} \mid \alpha \in R, \beta \in R \setminus \mathfrak{p}\}$ with two elements $\frac{\alpha}{\beta}$ and $\frac{\alpha'}{\beta'}$ in $R_\mathfrak{p}$ being equal if an only if there exists an element $s \in R \setminus \mathfrak{p}$ so that $s\alpha\beta' = s\alpha'\beta$. The ring $R_\mathfrak{p}$ has unique maximal ideal $\mathfrak{M} = \{\frac{\alpha}{\beta} \mid \alpha \in \mathfrak{p}, \beta \in R \setminus \mathfrak{p}\}$ and the localization map is the homomorphism sending an element $r \in R$ to $\frac{r}{1} \in R_\mathfrak{p}$.

Lemma 8.4. $\mathrm{Frac}(R/\mathfrak{p})$ *is canonically isomorphic to $R_\mathfrak{p}/\mathfrak{M}$. In particular, if $\mathfrak{p}$ is a maximal ideal then $R/\mathfrak{p} \cong R_\mathfrak{p}/\mathfrak{M}$.*

Proof. Firstly, $R/\mathfrak{p}$ is an integral domain as $\mathfrak{p}$ is a prime, so it makes sense to take the field of fractions of this ring. Consider the composition of maps $R \hookrightarrow R_\mathfrak{p} \twoheadrightarrow R_\mathfrak{p}/\mathfrak{M}$, with the first one being the localization map and the second one the natural projection. Then $r \in R$ is sent to zero if and only if $\frac{r}{1} \in \mathfrak{M}$ which is equivalent to $r \in \mathfrak{p}$. Hence we get an induced injective homomorphism, ϕ

$$R/\mathfrak{p} \hookrightarrow R_\mathfrak{p}/\mathfrak{M}$$

which is given by $\phi(r+\mathfrak{p}) = \frac{r}{1} + \mathfrak{M}$. We know that $R_\mathfrak{p}/\mathfrak{M}$ is a field since $\mathfrak{M}$ is a maximal ideal, so it follows from the universal property for the field of fractions that there is a unique injective homomorphism $\sigma : \mathrm{Frac}(R/\mathfrak{p}) \to R_\mathfrak{p}/\mathfrak{M}$, extending ϕ, which is given by $\sigma(\frac{\alpha}{\beta}) = \phi(\alpha)\phi(\beta)^{-1}$. Let us show that this map is surjective. Suppose that $\frac{a}{b} + \mathfrak{M} \in R_\mathfrak{p}/\mathfrak{M}$ with $a \in R$ and $b \notin \mathfrak{p}$. Thus, $b+\mathfrak{p}$ is non-zero in $R/\mathfrak{p}$. We can now evaluate

$$\sigma(\frac{a+\mathfrak{p}}{b+\mathfrak{p}}) = \phi(a+\mathfrak{p})\phi(b+\mathfrak{p})^{-1} = (\frac{a}{1}+\mathfrak{M})(\frac{b}{1}+\mathfrak{M})^{-1} = \frac{a}{b}+\mathfrak{M}$$

For the last statement, suppose that $\mathfrak{p}$ is maximal. Then $R/\mathfrak{p}$ is a field, so it is isomorphic to its field of fractions. □

Proposition 8.6. *Let O be a Dedekind domain and K its field of fractions. For any prime $\mathfrak{p}$ of O there is a function, $v_\mathfrak{p} : O \to \mathbb{R} \cup \{\infty\}$ on O so that $v_\mathfrak{p}(0) = \infty$ and that*

$$\forall x \in O \setminus \{0\} : v_\mathfrak{p}(x) = m \iff (x) = \mathfrak{p}^m A,\ A \text{ an ideal of } O \text{ and } \mathfrak{p} \nmid A. \quad (8.1)$$

The function v extends to a discrete valuation on K, which we will also call $v_\mathfrak{p}$, which for $\frac{\alpha}{\beta} \in K$, $\alpha, \beta \in O$ and $\beta \neq 0$ is given by

$$v_\mathfrak{p}(\frac{\alpha}{\beta}) = v_\mathfrak{p}(\alpha) - v_\mathfrak{p}(\beta)$$

The valuation $v_\mathfrak{p}$ extends to the completion $K_\mathfrak{p}$ of K with respect to $v_\mathfrak{p}$ and is also denoted by $v_\mathfrak{p}$. Let O_{v_p} be the valuation ring of K and let $\mathfrak{P}_{v_\mathfrak{p}}$ be its unique maximal ideal. Similarly, let $O_\mathfrak{p}$ be the valuation ring of $K_\mathfrak{p}$ and $\hat{\mathfrak{p}}$ its unique maximal ideal. We have the following;

1. *The localization of O at $\mathfrak{p}$, that is, the ring*

$$O_{(\mathfrak{p})} = \{\frac{\alpha}{\beta} \in K \mid \alpha \in O, \beta \notin \mathfrak{p}\} = \{\frac{\alpha}{\beta} \in K \mid \alpha \in O, v_\mathfrak{p}(\beta) = 0\}$$

   is the same ring as $O_{v_\mathfrak{p}}$.

2. *The natural inclusions*

$$O \hookrightarrow O_{(\mathfrak{p})} \hookrightarrow O_\mathfrak{p},$$

   give rise to the equalities $\mathfrak{P}_{v_\mathfrak{p}} = \mathfrak{p}O_{(\mathfrak{p})}$ and $\hat{\mathfrak{p}} = \mathfrak{p}O_\mathfrak{p}$, as well as the isomorphisms

$$O/\mathfrak{p} \cong O_{(\mathfrak{p})}/\mathfrak{P}_{v_\mathfrak{p}} \cong O_\mathfrak{p}/\hat{\mathfrak{p}}$$

In particular, $K_\mathfrak{p}$ is a local field if $O/\mathfrak{p}$ is finite.

Proof. The function $v_\mathfrak{p}$ on O with the mentioned property is well defined since we have unique factorization of prime ideals in Dedekind domains. Let us now see that the function $v_\mathfrak{p}$ we have defined on K is actually well defined. If

$\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'}$ then $\alpha\beta' = \alpha'\beta \in O$ and applying $v_\mathfrak{p}$ on both sides and using that it takes multiplication to addition we obtain

$$v_\mathfrak{p}(\alpha) - v_\mathfrak{p}(\beta) = v_\mathfrak{p}(\alpha') - v_\mathfrak{p}(\beta')$$

which shows that it is well defined. If we can show that all the properties in definition 7.3 are satisfied for the restriction of $v_\mathfrak{p}$ to O, we are done. It is immediate that $v_\mathfrak{p}(x) = \infty$ if and only if $x = 0$. Let now $x, y \in O$ and suppose both x and y are non-zero. We then have $v_\mathfrak{p}(x) = m$ and $v_\mathfrak{p}(y) = n$ and we can write

$$(x) = \mathfrak{p}^m A, (y) = \mathfrak{p}^n B$$

Thus, $(xy) = (x)(y) = \mathfrak{p}^{m+n}AB$ and $\mathfrak{p} \nmid AB$, so $v_\mathfrak{p}(xy) = m + n$. For the last property, let $d = \min\{m, n\}$. Then

$$(x + y) \subseteq (x) + (y) = \mathfrak{p}^d(\mathfrak{p}^{m-d}A + \mathfrak{p}^{n-d}B)$$

From this we see that $\mathfrak{p}^d \mid (x+y)$ and so $d \leq v_\mathfrak{p}(x+y)$. Note that if either x or y are 0, then these two properties follow immediately. We conclude that $v_\mathfrak{p}$ is a valuation on K. It is discrete since it maps O into $\mathbb{N}_0 \cup \{\infty\}$ so it maps K into $\mathbb{Z} \cup \{\infty\}$. Because of **??** the valuation on $K_\mathfrak{p}$ is discrete as well. For (2.), it is clear that we have the inclusion, $O_{(\mathfrak{p})} \subseteq O_{v_\mathfrak{p}}$. So let $\frac{x}{y} \in O_{v_\mathfrak{p}}$ with $x, y \in O$ and both x and y non-zero and let $u = v_\mathfrak{p}(x)$, $v = v_\mathfrak{p}(y)$. Since $v_\mathfrak{p}(\frac{x}{y}) \geq 0$ we have $u \geq v$. We can now write

$$xO = \mathfrak{p}^u A, \text{ and } yO = \mathfrak{p}^v B$$

where A, B are two ideals of O so that $\mathfrak{p} \nmid A$ and $\mathfrak{p} \nmid B$. This means that $\mathfrak{p} \not\supseteq B$, which implies that we can find $c \in B$ such that $c \notin \mathfrak{p}$. But then $v_\mathfrak{p}(c) = 0$ and hence we are done if we can show that there is $b \in O$ so that $xc = yb$ as that would imply that $\frac{x}{y} = \frac{b}{c}$. Now, $xc \in xO$ and also $xc \in B$, meaning that $xc \in xO \cdot B$. But,

$$xO \cdot B = \mathfrak{p}^u AB = \mathfrak{p}^{u-v} A\mathfrak{p}^v B = \mathfrak{p}^{u-v} A \cdot yO$$

So $xc \in \mathfrak{p}^{u-v}A \cdot yO$ which means that we can find $b \in \mathfrak{p}^{u-v}A$ such that $xc = by$. We conclude that $O_{v_\mathfrak{p}} = O_{(\mathfrak{p})}$. As for (3.), consider the natural inclusions

$$O \hookrightarrow O_{(\mathfrak{p})} \hookrightarrow O_\mathfrak{p},$$

Since O is a Dedekind domain, $\mathfrak{p}$ is in fact a maximal ideal so by lemma 8.4 it follows that $O/\mathfrak{p} \cong O_{(\mathfrak{p})}/\mathfrak{P}_{v_\mathfrak{p}}$. The other isomorphism is just proposition 8.5.

(((Apparently this valuation is always discrete? :OOOO)))

For the last claim, if K is discrete then by (1) in the proposition we know that the valuation on $K_\mathfrak{p}$ is also discrete. If furthermore $O/\mathfrak{p}$ is finite then so is the residue field $K_\mathfrak{p}/O_\mathfrak{p}$ because of the isomorphisms from (3). Hence $K_\mathfrak{p}$ is in this case a local field. □

Remark 8.2. *Note that we have shown that the localization of Dedekind domain at a prime is a local Dedekind domain. It can be shown that local Dedekind domains are PIDs which implies that $O_{(\mathfrak{p})}$ is in fact a discrete valuation ring.*

Let us now see some examples using this construction.

Example 8.1. *Let K be a number field and pick a prime $\mathfrak{p}$ of O_K. From lemma 8.3 we know that $K = \mathrm{Frac}(O_K)$. Hence from we obtain the so-called $\mathfrak{p}$-adic valuation $v_\mathfrak{p} : K \to \mathbb{Z} \cup \{\infty\}$ which extends to a discrete valuation on the completion, $K_\mathfrak{p}$. The elements of $K_\mathfrak{p}$ and $O_\mathfrak{p}$ are called $\mathfrak{p}$-adic numbers and $\mathfrak{p}$-adic integers, respectively. We know that $O_K/\mathfrak{p}$ is finite so $K_\mathfrak{p}$ is a local field. In the special case where $K = \mathbb{Q}$ we have $\mathfrak{p} = (p)$ for some prime $p \in \mathbb{Z}$. We denote $K_\mathfrak{p}$ as $\mathbb{Q}_p$ and we call this ring the p-adic numbers. The ring O_p is denoted by $\mathbb{Z}_p$ and called the p-adic integers.*

Lemma 8.5. *A PID is a Dedekind domain.*

Proof. Assume R is a PID. Then any ideal I of R is generated by a single element, so in particular it is finitely generated. Also, we know that if I is prime then it is in fact maximal. R is in particular a UFD and we claim that these are integrally closed. Let K be the field of fractions of R and suppose $\frac{a}{b} \in K$ with $a, b \in R$, $b \neq 0$ so that $\gcd(a, b) = 1$. Suppose that we have

$$(\frac{a}{b})^n + c_{n-1}(\frac{a}{b})^{n-1} + ... + c_1(\frac{a}{b}) + c_0 = 0$$

where the c_i are in R. Multiplying by b^n we get

$$a^n + c_{n-1}a^{n-1}b + ... + c_1ab^{n-1} + c_0b^n = 0$$

which shows that $b \mid a^n$ in R, meaning that $b \mid a$ in R. But since $\gcd(a, b) = 1$, b has to be a unit and therefore $\frac{a}{b} \in R$. We conclude that R is a Dedekind domain. □

Example 8.2. *Let F be a field. Then $O := F[x]$ has field of fractions $K := F(x)$. Since F is a field O is a PID so by lemma 8.5 O is a Dedekind domain. The ideal $\mathfrak{p} = (x)$ is a maximal ideal since $F[x]/(x) \cong F$ so it is in particular a prime ideal so by proposition 8.6 we get a discrete valuation $v_\mathfrak{p}$ on $K := F(x)$ that extends to a discrete valuation on $K_\mathfrak{p}$. The field $K(x)$ is dense in $K((x))$, $K((x))$ is complete and the inclusion $K(x) \hookrightarrow K((x))$ preserves absolute values. Thus $K((x)) \cong K_\mathfrak{p}$. Also, $O_\mathfrak{p} = K[[x]]$ and $\hat{\mathfrak{p}} = (x)O_\mathfrak{p}$ so $K((x))$ has residue field $O_\mathfrak{p}/\hat{\mathfrak{p}} \cong K$. Hence $K((x))$ is a local field if and only K is a finite field.*

9 Local manifold

(PROVE THIS OR PROVIDE A REFERENCE)

Suppose U is an open neighborhood around a point $x_0 \in K^n$ and that we have a function $f : U \to K$ and a formal power series $g \in K[[X_1, ..., X_n]]$ that

converges to $f(x)$ for all $x \in U$. Then the function f is called an analytic function and is said to be represented by the formal power series g. If $f' : U' \to K$ is another analytic function, with U' another open neighborhood around x_0, and also represented by g, we will consider f and f' to be equal if $U \subseteq U'$ or $U' \subseteq U$.

Next, let us define what is called a local manifold. These are closely related to varieties known from algebraic geometry. Let $|\cdot|$ be an absolute value corresponding to v. Denote by $\tilde{K}^n$ the set of elements $(\alpha_1, ..., \alpha_n)$ so that each α_1 belongs to K or a finite field extension of K. We will refer to the elements as points. For $\epsilon > 0$ we call the set of points where $|\alpha_i| < \epsilon$ for all $i = 1, ..., n$, an ϵ-neighborhood of the origin and we will write U_ϵ for this set. Let D be the set containing all $f \in K[[X_1, ..., X_n]]$ that converge in some ϵ-neighborhood of the origin. This set is in fact a subring of $K[[X_1, ..., X_n]]$.

Definition 9.1 (Local manifold)**.** *Assume $f_1, ..., f_m \in D$ all of which have zero constant term. The set V of points $(\alpha_1, ..., \alpha_n) \in \tilde{K}^n$ belonging to some ϵ-neighborhood of the origin so that*

$$f_1(\alpha_1, ...\alpha_n) = ... = f_m(\alpha_1, ..., \alpha_n) = 0$$

is called a local manifold. We say that the power series $f_1, ..., f_m$ determine V.

If V and W are two local manifolds we will by $V \subseteq W$ mean that there is an ϵ-neighborhood around the origin, U, so that $U \cap V \subseteq U \cap W$. We write $V = W$ if and only if both $V \subseteq W$ and $W \subseteq V$. Consider the set, I_V, of elements $f \in D$ that vanish on V in some ϵ-neighborhood of the origin. This set is an ideal of D.

Definition 9.2 (Curve)**.** *A curve in $\tilde{K}^n$ is a collection of n integral (QUESTION: WHAT IS MEANT BY THIS) formal power series, $\omega_1(X), ..., \omega_n(X)$ having coefficients in K or a finite extension of K, not all identically zero, and with constant term zero. We say the curve lies on a manifold V, if for every power series $f \in I_V$ we have*

$$f(\omega_1(X), ..., \omega_n(X)) = 0$$

Note that a curve in the sense of this definition above is not a set of points. It is merely a collection of formal power series. There is an easier way to test whether a curve is on a local manifold

Lemma 9.1. *Suppose that V is a local manifold and $\omega_1(X), ..., \omega_n(X)$ is a curve in $\tilde{K}^n$ and suppose V is determined by the power series $f_1, ..., f_m$. If $f_i(\omega_1(X), ..., \omega_n(X)) = 0$ for every $i \in \{1, ..., m\}$ then the curve $\omega_1(X), ..., \omega_n(X)$ is on V.*

Proof. Lemma 3, p. 303 Borevich states that the ideal I_V is finitely generated and we can therefore assume that $f_1, ..., f_m$ generate I_V (QUESTION: Why can we do this?). Take any $f \in I_V$. Then we can write $f = h_1 f_1 + ... + h_m f_m$ where the h_i are in D. But then $f(\omega_1(X), ..., \omega_m(X)) = 0$ so the curve is on V. $\square$

We need the following theorems. Note that if $V = V(f_1, ..., f_n)$ is a local manifold then certainly, $f_1, ..., f_n \in I_V$. Thus if $\omega_1(X), ..., \omega_n(X)$ is a curve on V then we have in particular that $f_i(\omega_1(X), ..., \omega_n(X)) = 0$ for all $i \in \{1, ..., n\}$.

Theorem 9.1. *A local manifold is either equal to the set containing just the origin, or it contains a curve.*

Theorem 9.2. *Suppose V and V' are local manifolds in $\overline{K}^n$ such that $V \nsubseteq V'$. Then there is a curve on V that does not lie on V'.*

Remark 9.1. *Note that by contraposition, we have the following: If for all curves, C, we have that C is not on V or C is on V', then $V \subseteq V'$. This is the same as saying: If every curve on V is also on V' then $V \subseteq V'$.*

10 Local fields

In algebraic number theory we study finite extensions of $\mathbb{Q}$, known as number fields. Similarly, it turns out to be fruitful to consider the finite extensions of $\mathbb{Q}_p$, the field of p-adic numbers. These fields are called local fields and we will take a look at them now. We start out by giving another definition of these fields.

Definition 10.1. *A field K is called a **local field** if it is complete with regards to a discrete valuation and has finite residue field.*

Proposition 10.1. *A local field K of characteristic 0 is the same thing as a finite extension of $\mathbb{Q}_p$. Such a field is called a $\mathfrak{p}$-adic number field.*

Proof. Suppose first that $K/\mathbb{Q}_p$ is a finite extension of degree n. Then by theorem 8.3 the absolute value on $\mathbb{Q}_p$ extends uniquely to the nonarchimedian absolute value $|\alpha| = \sqrt[n]{|N_{K/\mathbb{Q}_p}|}$ on K and K is complete with respect to it. The absolute value on $\mathbb{Q}_p$ is discrete, so this must also be the case for the absolute value on K. Let κ be the residue field of K and set $m = [\kappa : \mathbb{F}_p]$. To show that κ has finitely many elements it suffices to show that $m \leq n$ as that implies that $\#\kappa \leq p^n$. Suppose that $x_1, ..., x_n \in K$ are linearly dependent over $\mathbb{Q}_p$. Then we can find $\lambda_1, ..., \lambda_n \in \mathbb{Q}_p$ with at least one $\lambda_i \neq 0$ so that

$$\lambda_1 x_1 + ... + \lambda_n x_n = 0.$$

The coefficient, λ_i, having the largest absolute value has to have absolute value strictly larger than 0, as otherwise $\lambda_i = 0$ which would imply that all coefficients are 0. Hence we can divide the above linear combination by λ_i to achieve a linear combination where all coefficients have absolute value smaller than or equal to 1. In other words, a linear combination with coefficients in $\mathbb{Z}_p$, the valuation ring of $\mathbb{Q}_p$. But then the ith coefficient is 1, so this linear combination is still non-trivial when mapped to κ. (NOT COMPLETELY DONE WITH THIS ONE. ASK ABOUT IT)

(NOT VERY SURE ABOUT THIS PART EITHER)

On the other hand, suppose that K is a local field with characteristic 0 and discrete valuation v. Let O be the valuation ring of K and $\mathfrak{p}$ be its unique maximal ideal. The residue field of K is then $\kappa = O/\mathfrak{p}$ and as K is a local field, we know that κ is a finite field so its characteristic is a prime number, say p. But that means that p is zero in κ, which is to say that $p \in \mathfrak{p}$, meaning that $v(p) > 0$. This implies (TODO: SHOW WHY) that v and v_p are equivalent valuations when restricted to $\mathbb{Q}$. The closure of $\mathbb{Q}$ in K is the smallest closed containing $\mathbb{Q}$ and as K is complete, it is therefore the smallest complete set containing $\mathbb{Q}$

therefore $\mathbb{Q}_p$ and is contained in K. As K is a metric space it is Hausdorff and by **??** K is locally compact. It therefore follows by a theorem which can be found HERE that the $\mathbb{Q}$-vector space K has finite dimension. □

Lemma 10.1. *Let G be a group of order n. Then $x^n = 1$ for all $x \in G$.*

Proof. Let $x \in G$ and consider the cyclic subgroup H generated by x. As G is finite so is H so by Lagrange's theorem, $|H|$ divides n meaning that $n = |H|k$ for some $k \in \mathbb{Z}$. As H is cyclic we have $x^{|H|} = 1$ and so $x^n = x^{|H|k} = 1^k = 1$. □

Proposition 10.2. *Let K be a local field, O its valuation ring, $\mathfrak{p}$ the maximal ideal of O, κ its residue field and $q = \#\kappa$. Then K^* contains μ_{q-1}, the group of $(q-1)$st roots of unity and we have*

$$K^* = \{\pi^k \mid k \in \mathbb{Z}\} \times \mu_{q-1} \times U^{(1)}$$

Proof. Because of proposition 7.3 we really only need to show that $O^* = \mu_{q-1} \times U^{(1)}$. From the natural projection $O \twoheadrightarrow \kappa$ we obtain a group homomorphism $\phi : O^* \to \kappa^*$. Let us first show that O^* contains μ_{q-1}. First, κ^* is a group of order $q-1$ so if z is any element in κ^* then by lemma 10.1 we have $z^{q-1} = 1$ and so $f(x) = x^{q-1} - 1 \in O[x]$ splits into distinct linear factors over κ. Thus we have the factorization $f(x) \equiv (x-z)g(x) \pmod{\mathfrak{p}}$ with $x - z$ and g being coprime and both belonging to $\kappa[x]$. By Hensel's Lemma we get a factorization $f(x) = \alpha(x)\beta(x)$ where both α and β are in $O[x]$ and where $\deg \alpha = \deg(x-z) = 1$. Hence $\alpha(x) = x - \zeta$ for some $\zeta \in O$. Hensel's Lemma also tells us that $x - \zeta \equiv x - z \pmod{\mathfrak{p}}$, so $z \equiv \zeta \pmod{\mathfrak{p}}$. In other words, $\phi(\zeta) = z$. Now since $\alpha(\zeta) = 0$ we have $f(\zeta) = 0$, that is, ζ is a $q-1$st root of unity and therefore in particular we have $\zeta \in O^*$. As z was an arbitrary element of κ^*, we conclude that f has all of its $q-1$ roots in O^*, all of which are distinct so O^* contains μ_{q-1} and these are mapped bijectively onto κ^* since $q - 1 = \#\kappa = \#\mu_{q-1}$. To show that $O^* = U^{(1)} \times \mu_{q-1}$ it suffices to show that $U^{(1)} \cap \mu_{q-1} = \{1\}$ and $O^* = U^{(1)}\mu_{q-1}$. To do this, let us first show that $\ker \phi = U^{(1)}$. Saying that $\phi(x) = 1$ for some $x \in O^*$ is the same as saying that $1 \equiv x \pmod{\mathfrak{p}}$ which is equivalent to $x - 1 \in \mathfrak{p}$ which is also equivalent to $x \in 1 + \mathfrak{p} = U^{(1)}$, so $\ker \phi = U^{(1)}$. Now if $x \in U^{(1)} \cap \mu_{q-1}$ then $\phi(x) = 1$ as $x \in U^{(1)}$ and as $x \in \mu_{q-1}$ we must have $x = 1$ since ϕ maps μ_{q-1} bijectively onto κ^* and $\phi(1) = 1$ simply because ϕ is a homomorphism so $U(1) \cap \mu_{q-1} = \{1\}$. Suppose that $u \in O^*$.

Then $\phi(u) \in \kappa^*$ and we can find $\zeta^{-1} \in \mu_{q-1}$ so that $1 = \phi(u)\phi(\zeta^1)$. But then $u\zeta^{-1} \in \ker \phi = U^{(1)}$, so we can write $u\zeta^{-1} = v$ for some $v \in U^{(1)}$. Hence $u = v\zeta$ showing that $O^* = U^{(1)}\mu_{q-1}$. We conclude that $O^* = U^{(1)} \times \mu_{q-1}$. □

Lemma 10.2. *Let K be a local field with residue field $\kappa = O/\mathfrak{p}$, and let $q = \#\kappa$. For any $n \in \mathbb{N}$ we have $\#(O/\mathfrak{p}^n) = q^n$.*

Proof. Since the valuation on K is discrete, we know that for any $k \in \mathbb{N}$ we have

$$\mathfrak{p}^k/\mathfrak{p}^{k+1} \cong \kappa,$$

as groups under addition. We prove the statement using induction on n. The base case $n = 1$ is clear. So suppose that $\#(O/\mathfrak{p}^n) = q^n$. We have the isomorphism

$$(O/\mathfrak{p}^{n+1})/(\mathfrak{p}^n/\mathfrak{p}^{n+1}) \cong O/\mathfrak{p}^n$$

But since $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ and $O/\mathfrak{p}^n$ have finite order, also $O/\mathfrak{p}^{n+1}$ must have finite order. By Lagrange's theorem, it now follows that

$$\#(O/\mathfrak{p}^{n+1}) = \#(O/\mathfrak{p}^n) \cdot \#(\mathfrak{p}^n/\mathfrak{p}^{n+1}) = q^{n+1}$$

□

Proposition 10.3. *Let K be a valued field and (x_n) and (y_n) sequences of K. If (x_n) is Cauchy and $\lim_{n\to\infty} |x_n - y_n| = 0$ then (y_n) is also Cauchy. If it is also known that $\lim_{n\to\infty} x_n = a$ then $\lim_{n\to\infty} y_n = a$.*

Proof. Observe that

$$|y_n - y_m| = |x_n - y_n + x_m - y_m + x_n - x_m| \tag{10.1}$$

A three fold application of the triangle inequality on the right hand side basically tells us that (y_n) is Cauchy. For the last part, use the triangle inequality on the right hand side of

$$|y_n - a| = |x_n - y_n + x_n - a|$$

and take the limit. □

Theorem 10.1. *The valuation ring of a local field K is compact.*

Proof. The valuation ring can be thought of as the closed unit ball around 0 with respect to the absolute value on K. Hence it is closed and is homeomorphic to $\varprojlim O/\mathfrak{p}^n$ which is then of course also closed. This inverse limit is contained in $\prod_{n=1}^{\infty} O/\mathfrak{p}^n$, which is compact by Tychonoff's theorem since all the $O/\mathfrak{p}^n$ are finite by lemma 10.2, and hence they are compact. It follows that O is compact. □

Remark 10.1. *Since the absolute value K induces a metric on K, it means that compactness is equivalent to sequential compactness. Thus every sequence in O has a convergent subsequence.*

11 Logarithms and Exponentials

In this section, K is a $\mathfrak{p}$-adic field. As such it is by proposition 10.1 a finite extension of $\mathbb{Q}_p$ and by theorem 8.3 the valuation v_p on $\mathbb{Q}_p$ therefore extends uniquely to K. This valuation is discrete so there is $\pi \in K$ so that $v_p(\pi) = \frac{1}{e} > 0$ is the smallest attainable strictly positive value for this function. Hence defining $v_\mathfrak{p} = ev_p$ we get a normalized valuation on K. Let us also denote by O the valuation ring with respect to $v_\mathfrak{p}$ and $\mathfrak{p}$ its unique maximal ideal. In this section we will see how one can define logarithm- and exponential functions on K. We start by looking at the formal power series

$$\log(1+X) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{X^k}{k} \tag{11.1}$$

$$\exp(X) = \sum_{k=1}^{\infty} \frac{X^k}{k!} \tag{11.2}$$

To emphasize that these are really just formal power series, we have written the variable in uppercase. As the coefficients of both belong to $\mathbb{Q}$ these are certainly elements of $K[[X]]$.

(ADD SOME TEXT HERE EXPLAINING THAT WE WANT TO USE THESE AS FUNCTIONS ON K)

We have the following equalities in formal power series

$$\exp(\log(1+X)) = 1 + X \tag{11.3}$$

$$\log(\exp(X)) = X \tag{11.4}$$

Proofs for these can be found here: [Integrand, 2016, Earnest, 2022].

We define $U^{(k)} := 1 + \mathfrak{p}^k$ for $k \in \mathbb{N}_0$. Each of these are multiplicative groups and we have $U^{(0)} = O^*$. The group $U^{(1)}$ is called the principal units. Let us first show that log converges on $U^{(1)}$.

Proposition 11.1. *(BE MORE PRECISE HERE) The power series,* $\log(1+z)$*, defined in eq.* (11.1) *converges for all* $1 + z \in U^{(1)}$*.*

Proof. Suppose $(1+z) \in U^{(1)}$. Then $z \in \mathfrak{p}$ and so $v_p(z) > 0$, which means that $c = p^{v_p(z)} > 0$. Thus we can apply the usual logarithm and get $v_p(z) = \frac{\ln c}{\ln p}$. If k is any natural number, then we always have

$$p^{v_p(k)} \leq k,$$

since $p^{v_p(k)}$ divides k. Applying ln to both sides of this inequality is valid, as both sides are strictly positive and from doing so we get

$$v_p(k) \ln p \leq \ln k$$

and so,

$$v_p(k) \leq \frac{\ln k}{\ln p}$$

Now for any $k \in \mathbb{N}$ we get

$$\begin{aligned} v_p(\frac{z^k}{k}) &= v_p(z^k) - v_p(k) \\ &= kv_p(z) - v_p(k) \\ &\geq k\frac{\ln c}{\ln p} - \frac{\ln k}{\ln p} \\ &= \frac{\ln c^k/k}{\ln p} \end{aligned}$$

Clearly, $\ln c^k/k \to \infty$ as $k \to \infty$. Hence, $v_p(\frac{z^k}{k}) \to \infty$ as $k \to \infty$ and by lemma 7.1 we are now done. □

(STATE HOW IT IS TEMPTING TO JUST INVOKE THE EQUALITIES WE ALREADY HAVE FOR POWER SERIES)

(REFER TO RESULTS USED IN THIS PROOF)

Proposition 11.2. *We have* $\log((1+x)(1+y)) = \log(1+x) + \log(1+y)$ *for all* $x, y \in \mathfrak{p}$.

Proof. First let us define

$$f(x) = \log(1+x)$$

We want to show that $f(y + (1+y)x) = f(x) + f(y)$. First we see that

$$f'(x) = \sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}$$

This sum converges on $\mathfrak{p}$ because f converges on this set. Fix some $y \in \mathfrak{p}$. We set

$$g(x) = f(y + (1+y)x)$$

This also converges for all $x \in \mathfrak{p}$. Using the chain rule we get

$$g'(x) = f'(y + (1+y)x)(1+y) = \frac{1+y}{1+y+(1+y)x} = \frac{1}{1+x}$$

Thus $g(x) = f(x) + c$ for some $c \in K$. As $f(0) = 0$ we now have $f(y) = g(0) = f(0) + c = c$ meaning that $g(x) = f(x) + f(y)$, which is what we wanted to show. □

Proposition 11.3. *(COULD REFORMULATE A BIT) There is a uniquely determined group homomorphism taking multiplication to addition,*

$$\log : K^* \to K$$

so that $\log p = 0$ *and for* $(1+z) \in U^{(1)}$ *this function is given by*

$$\log(1+z) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{k}$$

Proof. We only need to show existence and uniqueness. Suppose that $\alpha \in K^*$ and that $\pi \in K$ is an element so that $v_{\mathfrak{p}}(\pi) = 1$. By proposition 10.2 there exists a unique elements $k_\alpha \in \mathbb{Z}$, $\zeta_\alpha \in \mu_{q-1}$ and $u_\alpha \in U^{(1)}$ (WHAT IS q??) so that $\alpha = \pi^{k_\alpha}\zeta_\alpha u_\alpha$, and we have $v_{\mathfrak{p}}(\alpha) = k_\alpha$. $v_{\mathfrak{p}} = ev_p$. Note that $v_{\mathfrak{p}}(p) = e$. So as $p \in K^*$ we can write $p = \pi^e \zeta_p u_p$. Let us now define $\log \pi = -\frac{1}{e}\log u_p$. Since log is 0 on all of $U^{(1)}$ we now get an extension of log to all of K^* by setting

$$\log \alpha = v_{\mathfrak{p}}(\alpha)\log(\pi) + \log(u_\alpha)$$

This function is continuous because addition of continuous functions are continuous and it is also a homomorphism by inheritance. Furthermore we have that

$$\log p = e\log(\pi) + \log(u_p) = \frac{-e\log u_p}{e} + \log(u_p) = 0$$

Suppose that we have another extension, $\lambda : K^* \to K$, of log so that $\lambda(p) = 0$ and which is also a homomorphism. Note first that for any $\alpha \in K^*$ we have

$$\lambda(\alpha) = \lambda(\pi^{v_{\mathfrak{p}}(\alpha)}\zeta_\alpha u_\alpha) = v_{\mathfrak{p}}(\alpha)\lambda(\pi) + \lambda(\zeta_\alpha) + \lambda(u_\alpha)$$

By assumption we know that $\log(u_\alpha) = \lambda(u_\alpha)$ since $u_\alpha \in U^{(1)}$. Hence

$$\lambda(\alpha) - \log(\alpha) = v_{\mathfrak{p}}(\alpha)(\lambda(\pi) - \log(\pi)) + \lambda(\zeta_\alpha)$$

So to show that $\lambda = \log$ it suffices to show that λ vanishes on μ_{q-1} and that $\lambda(\pi) = \log(\pi)$. Take any $\zeta \in \mu_{q-1}$. Since $\zeta^{q-1} = 1$ and $\lambda(1) = 0$ we see that

$$\lambda(\zeta) = \frac{1}{q-1}\lambda(\zeta^{q-1}) = 0$$

By assumption $0 = \lambda(p) - \log(p)$. Since $\lambda(\zeta_p) = 0$ and $v_{\mathfrak{p}}(p) = e \neq 0$ we must have $\lambda(\pi) - \log(\pi) = 0$. □

Lemma 11.1. *(Legendre's formula) Suppose we have $k \in \mathbb{N}$. Then*

$$v_p(k!) = \sum_{i=1}^{\infty} \left\lfloor \frac{k}{p^i} \right\rfloor$$

Proof. First of all, there are only finitely many terms in the sum since $\left\lfloor \frac{k}{p^i} \right\rfloor$ is eventually zero when i is large enough so it converges. For natural numbers q and n we define the function

$$f_q(n) = \begin{cases} 1 & \text{if } q \mid n \\ 0 & \text{otherwise} \end{cases}$$

We then have for any $m \in \mathbb{N}$ that

$$v_p(m) = \sum_{i=1}^{\infty} f_{p^i}(m)$$

Thus,

$$\begin{aligned} v_p(k!) &= \sum_{j=1}^{k} v_p(j) \\ &= \sum_{j=1}^{k} \sum_{i=1}^{\infty} f_{p^i}(j) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{k} f_{p^i}(j) \end{aligned}$$

The swaps in summation are allowed here since all sums involved are finite. But clearly, $\sum_{j=1}^{k} f_{p^i}(j) = \left\lfloor \frac{k}{p^i} \right\rfloor$, so we get the result. □

Proposition 11.4. *Let $x \in K$ and suppose $f(X)$ and $g(X)$ are formal power series. If both $f(x)$ and $g(x)$ converge then the formal series $(f+g)(x)$ and $(fg)(x)$ converge to $f(x)+g(x)$ and $f(x)g(x)$ respectively.*

Proof. Suppose f and g have coefficients a_n and b_n respectively. We will only prove that $(fg)(x)$ converges to $f(x)g(x)$. RPOBLEM 150 □

(GIVE AN EXAMPLE WHERE THE FIRST 2 CONDITIONS ARE SATISFIED BUT THE LAST ONE ISNT AND WHERE THE CONCLUSION GOES WRONG) (UNIFORM CONVERGENCE OF SEQUENCE) We will say that a sequence $b_{ij} \in K$ converges to 0 uniformly in j if for every j and $\epsilon > 0$ there exists an $N \in \mathbb{N}$, which does not depend on j, so that if $i \geq N$ then $|b_{ij}| < \epsilon$. In notation, we will write $\lim_{j\to\infty} b_{ij} = 0$.

Proposition 11.5. *Assume that $b_{ij} \in K$ and assume that*

1. $\lim_{j\to\infty} b_{ij} = 0$
2. $\lim_{i\to\infty} b_{ij} = 0$ *uniformly in j*

Then the following two sums converge and are equal

$$\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} b_{ij}), \quad \sum_{j=0}^{\infty}(\sum_{i=0}^{\infty} b_{ij}) \tag{11.5}$$

For a proof, see [Marcus, 2018][Proposition 5.1.4]. The idea is to first show that the inner sums in eq. (11.5) converge and then use that to show that the double sums converge. To show that they converge to the same thing, look at the absolute value of their difference.

Theorem 11.1. *Suppose we have the power series*

$$f(X) = \sum_{n=0}^{\infty} a_n X^n, \quad g(X) = \sum_{n=1}^{\infty} b_n X^n$$

Let $h(X) = f(g(X))$ and let $x \in K$. Assume

1. *$g(x)$ converges*
2. *$f(g(x))$ converges, that is, $g(x)$ is in the region of convergence of $f(X)$*
3. *$|b_n x^n| \leq |g(x)|$ for all $n \in \mathbb{N}$*

Then $h(x)$ converges and we have $h(x) = f(g(x))$

Proof. We can write $g(X)^m = \sum_{n=m} d_{m,n} X^n$ with

$$d_{m,n} = \begin{cases} \sum_{i_1+\ldots+i_m=n} b_{i_1} \ldots b_{i_m} \text{ if } n \geq m \\ 0 \text{ otherwise} \end{cases}$$

Thus we get

$$h(X) = a_0 + \sum_{n=1}^{\infty} \big(\sum_{m=1}^{n} a_m d_{m,n} \big) X^n$$

By assumption we know that $g(x)$ converges and so by an inductive argument and using proposition 11.4 we have that

$$g(x)^m = \sum_{n=m}^{\infty} d_{m,n} x^n$$

Suppose that $n \geq m$. Using the ultrametric property on $|\cdot|$ and the assumption that $|b_n x^n| \leq |g(x)|$ for all $n \in \mathbb{N}$ we obtain

$$\begin{aligned} |d_{m,n} x^n| &= |\sum_{i_1+\ldots+i_m=n} b_{i_1} x^{i_1} \ldots b_{i_m} x^{i_m}| \\ &\leq \max_{i_1+\ldots+i_m=n} \{|b_{i_1} x^{i_1}| \ldots |b_{i_m} x^{i_m}|\} \\ &\leq |g(x)|^m \end{aligned} \tag{11.6}$$

If instead $n < m$ we have $d_{m,n} = 0$ so we conclude that $|d_{m,n} x^n| \leq |g(x)|^m$ for all $m, n \in \mathbb{N}$.

This means that we now have

$$\begin{aligned} f(g(x)) &= a_0 + \sum_{m=1}^{\infty} a_m g(x)^m \\ &= a_0 + \sum_{m=1}^{\infty} a_m \big(\sum_{n=m}^{\infty} d_{m,n} x^n \big) \\ &= a_0 + \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} a_m d_{m,n} x^n \end{aligned}$$

Let n and $\epsilon > 0$ be given. Since $f(g(x))$ converges by assumption we know that $a_m g(x)^m$ converges to 0. Hence we can pick N so that if $m \geq N$ then $|a_m g(x)^m| < \epsilon$. Using eq. (11.6) we now have $|a_m d_{m,n} x^n| \leq |a_m g(x)^m| < \epsilon$.

This shows that $\lim_{m\to\infty} a_m d_{m,n} x^n = 0$ uniformly in n. Let now m be given. Since $g(x)^m$ converges we know that $d_{m,n}x^n$ converges to 0 as $n \to \infty$. Thus $a_m d_{m,n} x^n$ converges to 0 as $n \to \infty$. It now follows from proposition 11.5 that the sum

$$h(x) = a_0 + \sum_{n=1}^{\infty} \big(\sum_{m=1}^{n} a_m d_{m,n} \big) x^n$$

converges and that it equals $f(g(x))$. □

Proposition 11.6. *The power series defined in eqs.* (11.1) *and* (11.2) *induce, for* $m > \frac{e}{p-1}$*, isomorphisms*

$$U^{(m)} \underset{\exp}{\overset{\log}{\rightleftarrows}} \mathfrak{p}^m$$

and these are inverses of one another.

Proof. We have already established that the power series for the logarithm converges on $U^{(1)}$, hence it also converges on $U^{(m)}$ for all $m \in \mathbb{N}$. Our first goal is to show that log maps $U^{(m)}$ into $\mathfrak{p}^m$ for all $m > \frac{e}{p-1}$. To do this we need a little help from the following inequality. We have for $k \in \mathbb{N}$ that

$$\frac{v_p(k)}{k-1} \leq \frac{1}{p-1},$$

Let us show that this actually holds. Write $k = p^a k_0$, where $p \nmid k_0$. Then $v_p(k) = a$ and so the inequality is satisfied when $a = 0$. So suppose $a > 0$. Then we can write

$$p^a - 1 = (p-1)(p^{a-1} + ... + p + 1).$$

We have $a \leq p^{a-1} + ... + p + 1$, since the right side of the inequality consists of a total of a terms all of which are larger than 1. Now

$$\frac{v_p(k)}{k-1} = \frac{a}{p^a k_0} \leq \frac{a}{p^a - 1} = \frac{1}{p-1} \cdot \frac{a}{p^{a-1} + ... + p + 1} \leq \frac{1}{p-1}$$

Suppose now that $1 + z \in U^{(m)}$ with $m > \frac{1}{p-1}$. Then $z \in \mathfrak{p}^m$ which is the same as saying that $v_{\mathfrak{p}}(z) \geq m$, since $v_{\mathfrak{p}}$ is normalized and so $v_{\mathfrak{p}}(z) > \frac{e}{p-1}$, which means that $v_p(z) > \frac{1}{p}$. Let us show that $\log(1+z) \in \mathfrak{p}^m$ by showing that $v_{\mathfrak{p}}(z) = v_{\mathfrak{p}}(\log(1+z))$. By what we showed above we have

$$v_p(\frac{z^k}{k}) - v_p(z) = (k-1)v_p(z) - v_p(k) > (k-1)(\frac{1}{p-1} - \frac{v_p(k)}{k-1}) \geq 0$$

Hence $v_{\mathfrak{p}}(\frac{z^k}{k}) > v_{\mathfrak{p}}(z)$ for all $k \in \mathbb{N}$. Thus, for all $s \in \mathbb{N}$ we have

$$v_{\mathfrak{p}}(A_s) \geq \min\{v_{\mathfrak{p}}(\frac{z^2}{2}), ..., v_{\mathfrak{p}}(\frac{z^s}{s})\}$$

with $A_s = \sum_{i=2}^{s} \frac{z^i}{i}(-1)^{(i-1)}$ meaning that $v_\mathfrak{p}(A_s) > v_\mathfrak{p}(z)$. It now follows from proposition 7.1 that

$$v_\mathfrak{p}(z + A_s) = \min\{v_\mathfrak{p}(z), v_\mathfrak{p}(A_s)\} = v_\mathfrak{p}(z)$$

From this it follows that $v_\mathfrak{p}(z + A_s) \to v_\mathfrak{p}(z)$. We also know that $z + A_s \to \log(1 + z)$ as $s \to \infty$ and $v_\mathfrak{p}$ is continuous so $v_\mathfrak{p}(z + A_s) \to v_\mathfrak{p}(\log(1 + z))$. By uniqueness of limits we now have $v_\mathfrak{p}(z) = v_\mathfrak{p}(\log(1 + z))$.

Let us now show that the power series for the exponential function converges on $\mathfrak{p}^m$ for $m > \frac{e}{p-1}$. Using Legendre's formula we get the following inequality

$$v_\mathfrak{p}(k!) = e\sum_{i=1}^{\infty} \lfloor \frac{k}{p^i} \rfloor < ek \sum_{i=1}^{\infty} \frac{1}{p^i} = \frac{ek}{p-1}$$

Let us again assume that $z \in \mathfrak{p}^m$. Then

$$v_\mathfrak{p}(\frac{z^k}{k!}) = kv_\mathfrak{p}(z) - v_\mathfrak{p}(k!) > k(v_\mathfrak{p}(z) - \frac{e}{p-1})$$

But $v_\mathfrak{p}(z) - \frac{e}{p-1} \geq 0$, so $v_\mathfrak{p}(\frac{z^k}{k!}) \to \infty$ for $k \to \infty$. Let us now move on to showing that exp maps $\mathfrak{p}^m$ to $U^{(m)}$. Maintaining the assumption that $z \in \mathfrak{p}^m$ we will do this by showing that $v_\mathfrak{p}(\exp(z) - 1) = v_\mathfrak{p}(z)$ since this implies that $\exp(z) \in U^{(m)}$. Let us show that $v_\mathfrak{p}(\frac{z^k}{k!}) > v_\mathfrak{p}(z)$ for all $k > 1$. We have

$$v_\mathfrak{p}(\frac{z^k}{k!}) - v_\mathfrak{p}(z) = (k-1)v_\mathfrak{p}(z) - v_\mathfrak{p}(k!) > (k-1)v_\mathfrak{p}(z) - \frac{ek}{p-1} \geq 0$$

Hence $v_\mathfrak{p}(\frac{z^k}{k!}) > v_\mathfrak{p}(z)$ for all $k > 1$ and it now follows by an argument similar to one we gave a moment ago that $v_\mathfrak{p}(\exp(z) - 1) = v_\mathfrak{p}(z)$, which is what we wanted to show. To show that exp and log are inverses of one another observe that we now have the following

1. $\exp(z)$ converges,
2. $\log(\exp(z))$ converges since $\exp(z) \in \mathfrak{p}^m$,
3. $v_\mathfrak{p}(\frac{z^k}{k!}) \geq v_\mathfrak{p}(z) = v_\mathfrak{p}(\exp(z))$ for all $k \geq 1$.

Combining this with eq. (11.4) and theorem 11.1 we see that we indeed have $\log \exp z = z$. A similar argument shows that we also have $\exp \log(1 + z) = 1 + z$. The last thing we need to do is showing that exp is a homomorphism. If $x, y \in \mathfrak{p}^m$ then

$$\log(\exp(x + y)) = x + y = \log(\exp(x)) + \log(\exp(y)) = \log(\exp(x)\exp(y))$$

Since log is injective on $U^{(m)}$ it now follows that exp is a homomorphism. $\square$

Example 11.1. *Let $K = \mathbb{Q}_2$ and let $g(X) = 2X^2 - 2X$ and $f(X) = \exp(X)$. Conditions 1 and 2 in theorem 11.1 are clearly satisfied. We claim that $h(1) \neq f(g(1))$. Computing the first few terms of $h(X) := \sum_{n=0} a_n X^n$ using the following python code*

```
import sympy as sp

x = sp.Symbol('x')
series = sp.series(sp.exp(x), x, 0, 10)
g = 2*x**2 - 2*x
composition_series = series.subs(x, g)
expanded_terms = composition_series.expand()
expanded_terms
```

we obtain

$$h(X) = 1 - 2x + 4x^2 - \frac{16}{3}x^3 + \frac{20}{3}x^4 - \frac{104}{15}x^5 + \frac{304}{45}x^6 - \frac{1856}{315}x^7 + \frac{1528}{315}x^8 - \frac{2096}{567}x^9 + \dots$$

All of these terms, except for the first 2, have numerators which are divisible by 4 and this will in fact remain true for all subsequent terms. To prove this, one can show that we have. Get back to this...

12 Skolem's Method

Lemma 12.1. *Let K be a field of characteristic 0 and let $n, n_1, n_2 \in \mathbb{N}$ so that $n_1 = n-2$ and $n_2 \geq 2$ and suppose we have formal power series, $P_1(t), ..., P_n(t)$, with coefficients in K, and with zero constant term so that*

$$\sum_{j=1}^{n} a_{ij} \exp P_j(t) = 0, \text{ for all } i \in \{1, ..., n_1\}$$

$$\sum_{j=1}^{n} b_{ij} P_j(t) = 0, \text{ for all } i \in \{1, ..., n_2\},$$

with the a_{ij} and b_{ij} in K and where both matrices (a_{ij}) and (b_{ij}) have K-linearly independent rows. Then there are two indices $k \neq l$ so that $P_k(t) = P_l(t)$.

Proof. Let $V = \text{span}\{\exp P_1, ..., \exp P_n\}$ and consider the linear map

$$T : K^n \to V$$

given by

$$T(a_1, ..., a_n) = \sum_{i=1}^{n} a_i \exp P_i$$

By assumption we have $n_1 = n - 2$ linearly independent elements in $\ker T$. Hence nullity$(T) \geq n-2$ so by rank-nullity we get rank$(T) \leq 2$. Thus, it has to

be the case that two elements among $\{\exp P_1, ..., \exp P_n\}$ span all of V. After potentially renumbering these elements we can safely say that these elements are $\exp P_{n-1}$ and $\exp P_n$. Thus, we can, in particular, for any $i = 1, ..., n$ find elements $a_i, b_i \in K$ so that

$$\exp P_i = a_i \exp P_{n-1} + b_i \exp P_n \tag{12.1}$$

It is easy to see that a_i and b_i should be when $i \in \{n-1, n\}$. So let us restrict ourselves to only look at $i \in \{1, ..., n-2\}$. In this case, let us suppose that $a_i = 0$. Then $\exp P_i = b_i \exp P_n$. But the formal power series, $\exp t$, has constant term 1, so for any formal power series, F, without constant term, also $\exp F$ has constant term 1. Thus $\exp P_i$ and $\exp P_n$ also have constant term 1. Therefore, $b_i = 1$ and so $P_i = P_n$, in which case we are done. We can therefore from now on assume that $a_i \neq 0$ for all $i = 1, ..., n_1$. Let us now define, for $i = 1, ...n-1$,

$$Q_i = P_i - P_n, \tag{12.2}$$

If we had $Q_i = 0$ we would again be done, so let us assume this never happens. Combining eqs. (12.1) and (12.2) we have for $i = 1, ..., n-1$ that

$$\begin{aligned}
\exp Q_i &= \exp P_i \exp(-P_n) \\
&= (a_i \exp P_{n-1} + b_i \exp P_n) \exp(-P_n) \\
&= a_i \exp(P_{n-1} - P_n) + b_i \\
&= a_i \exp Q_{n-1} + b_i
\end{aligned}$$

Differentiating both sides of this equation and using the chain rule for formal power series we obtain

$$Q_i' \exp Q_i = a_i Q_{n-1}' \exp Q_{n-1}$$

As both $\exp Q_i$ has non-zero constant term it is a unit in $K[[t]]$, meaning that we can divide by it. Solving for Q_i' we obtain

$$\begin{aligned}
Q_i' &= \frac{(a_i Q_{n-1}' \exp Q_{n-1})}{\exp Q_i} \\
&= \frac{a_i Q_{n-1}' \exp Q_{n-1}}{a_i \exp Q_{n-1} + b_i} \\
&= \frac{Q_{n-1}' \exp Q_{n-1}}{\exp Q_{n-1} + c_i}
\end{aligned} \tag{12.3}$$

where $c_i = a_i^{-1} b_i$. In case where $i = n-1$ we choose $a_i = 1$ so that $b_i = 0$. That way we have $c_i = 0$. Let us now deal with the next group of equations. Let $U = \text{span}\{P_1, ..., P_n\}$ and $U' = \text{span}\{P_1 - P_n, ..., P_{n-1} - P_n\}$ and define the linear maps $T : K^n \to U$ and $T' : K^{n-1} \to U'$ as

$$\begin{aligned}
T(a_1, ..., a_n) &= \sum_{i=1}^{n} a_i P_i \\
T'(a_1, ..., a_{n-1}) &= \sum_{i=1}^{n-1} a_i Q_i
\end{aligned}$$

By assumption, we know that nullity$(T) \geq 2$. Using rank-nullity on T we have

$$n = \text{rank}(T) + \text{nullity}(T) \geq \text{rank}(T) + 2.$$

So rank$(T) \leq n - 2$. Since $U' \subseteq U$ and both T and T' are surjective, we must have rank$(T') \leq$ rank$(T) \leq n - 2$ and by rank-nullity used now on T' we get

$$n - 1 = \text{rank}(T') + \text{nullity}(T') \leq n - 2 + \text{nullity}(T')$$

Hence nullity$(T') \geq 1$, meaning that we can find a non-trivial linear combination,

$$\sum_{i=1}^{n-1} d_i Q_i = 0$$

where the d_i are in K. Differentiating this sum and using eq. (12.3) we obtain

$$\sum_{i=1}^{n-1} d_i Q_i' = Q_{n-1}' \exp Q_{n-1} \sum_{i=1}^{n-1} \frac{d_i}{\exp Q_{n-1} + c_i} = 0$$

It must be the case that $Q_{n-1}' \neq 0$. Otherwise, we would have $Q_{n-1} \in K$, and since the P_i have zero constant term, we would then have $Q_{n-1} = 0$, which is a contradiction. Also, $\exp Q_{n-1} \neq 0$ so we must have

$$\sum_{i=1}^{n-1} \frac{d_i}{c_i \exp Q_{n-1}} = 0 \tag{12.4}$$

Next let us show that this implies that the rational function

$$f(z) = \sum_{i=1}^{n-1} \frac{d_i}{c_i + z}$$

is the zero function on K. As this is a rational function we can find polynomials $\phi, \psi \in K[z]$ so that $f(z) = \frac{\phi(z)}{\psi(z)}$, where $\psi(z)$ is not the zero polynomial. Let us for contradiction suppose that f is not identically zero. Then ϕ cannot be identically 0. But then because of eq. (12.4) we now have

$$0 = f(\exp Q_{n-1}) = \frac{\phi(\exp Q_{n-1})}{\psi(\exp Q_{n-1})},$$

meaning that $\phi(\exp Q_{n-1}) = 0$. But because of lemma 3.1 this is only possible if ϕ is the zero polynomial, which is a contradiction. Hence f is identically 0. Now

$$g(z) := f(z) \prod_{j=1}^{n-1} (c_j + z) = \sum_{i=1}^{n-1} \prod_{j \neq i} (c_j + z)$$

The polynomial g is also identically 0 so for any c_k we get

$$0 = g(-c_k) = \prod_{j \neq k}^{n-1} (c_j - c_k)$$

Hence we must have $c_j = c_k$ for some $j \neq k$. This means that $b_k = \frac{b_j a_k}{a_j}$ and from eq. (12.1) we get

$$\begin{aligned}\frac{a_k}{a_j}\exp P_j &= \frac{a_k}{a_j}(a_j \exp P_{n-1} + b_j \exp P_n) \\ &= a_k \exp P_{n-1} + \frac{b_j a_k}{a_j}\exp P_n \\ &= \exp P_k\end{aligned}$$

But as we previously mentioned, both $\exp P_k$ and $\exp P_j$ have constant term 1 so we must have $\frac{a_k}{a_j} = 1$ and so $\exp P_j = \exp P_k$. Now

$$P_j = \log(1 + (\exp P_j - 1)) = \log(1 + (\exp P_k - 1)) = P_k$$

□

Suppose that F is an irreducible, decomposable, nonfull form of degree m. This means that there is a number field, K, of degree $n > m$ and a module M of rank m inside K so that F is connected to M. Let $\mu_1, ..., \mu_m \in K$ be a basis for M. In the following it suffices to look at F up to integral equivalence so because of fact 4.1 we can assume without loss of generality that

$$F(x_1, ..., x_m) = N_{K/\mathbb{Q}}(x_1\mu_1 + ... + x_m\mu_m)$$

Fix now some $c \in \mathbb{Q}$. We see that finding the solutions in integers to the equation

$$F(x_1, ..., x_m) = c \tag{12.5}$$

is exactly the same as finding the elements $\alpha \in M$ so that

$$N_{K/\mathbb{Q}}(\alpha) = c \tag{12.6}$$

Note that if $c = 0$ then $\alpha = 0$, so the above equation is not really interesting in this case. Thus we will assume that $c \neq 0$.

By fact 4.2, the elements $\mu_1, ..., \mu_m$ are linearly independent over $\mathbb{Q}$, so we can extend them to a basis $\mu_1, ..., \mu_m, \mu_{m+1}, ...\mu_n$ for K over $\mathbb{Q}$. We then get a module $M' = \{\mu_1, ..., \mu_n\}$ and we clearly have $M \subseteq M'$. Similar to what we have done previously, we define

$$\begin{aligned}M'_c &= \{\alpha \in M' \mid N_{K/\mathbb{Q}}(\alpha) = c\} \\ M_c &= \{\alpha \in M \mid N_{K/\mathbb{Q}}(\alpha) = c\}\end{aligned}$$

Let now $\sigma_1, ..., \sigma_n$ be the n embeddings of K into $\mathbb{C}$. Let us write $n = 2s + r$ where s is the number of pairs of complex conjugate embeddings and r is the number of real embeddings. By the primitive element theorem we know that $K = \mathbb{Q}(\beta)$ for some $\beta \in K$. Letting $N = \mathbb{Q}(\sigma_1(\beta), ..., \sigma_n(\beta))$ we get a subfield of $\mathbb{C}$ which is splitting field for the minimal polynomial for β over $\mathbb{Q}$ meaning that N is Galois over $\mathbb{Q}$, and it also has finite degree over $\mathbb{Q}$, so N is a number field. Let $\mathfrak{p}$ be a prime ideal of N. By example 8.1 we get the local field $N_\mathfrak{p}$ with valuation ring $O_\mathfrak{p}$, which has unique maximal ideal $\hat{\mathfrak{p}}$. The k'th higher unit group, $U^{(k)}$, is then given by $U^{(k)} = 1 + \hat{\mathfrak{p}}^k$.

Lemma 12.2. *Let $\alpha \in N$ and assume $\hat{\mathfrak{p}} \nmid \alpha$. For any $k \in \mathbb{N}$ we can find $q \in \mathbb{N}$ so that $\alpha^q \in U^{(k)}$.*

Proof. Let $q = \#(O_{\mathfrak{p}}/\hat{\mathfrak{p}}^k)^*$. This is a natural number by lemma 10.2. The condition $\hat{\mathfrak{p}} \nmid \alpha$ means that $\hat{\mathfrak{p}}$ means that $\gcd(\alpha O_N, \hat{\mathfrak{p}}) = O_N$ so also $\gcd(\alpha O_N, \hat{\mathfrak{p}}^k) = O_N$. But that means that $\alpha\beta + l = 1$ for some $l \in \hat{\mathfrak{p}}^k$ with $\beta \in O_N$, and so α is a unit in $O_{\mathfrak{p}}/\hat{\mathfrak{p}}^k$. Thus $\alpha^q \equiv 1 \pmod{\hat{\mathfrak{p}}^k}$ and so $\alpha^q \in U^{(k)}$. □

Let now k be a natural number so that the pair of maps $\log : U^{(k)} \to \hat{\mathfrak{p}}^k$ and $\exp : \hat{\mathfrak{p}}^k :\to U^{(k)}$ are mutual inverses (this is the number m in proposition 11.6). Let $\mathfrak{D}$ be the coefficient ring of M'. Using theorem 6.3 we can pick $\phi_1, ..., \phi_t \in \mathfrak{D}$, $t = r + s - 1$, to be a fundamental system of units for $\mathfrak{D}$, and a finite subset $\Gamma' \subseteq M'_c$ so that any $\alpha \in M_c$ can be written as

$$\alpha = \gamma' \phi_1^{v_1} ... \phi_t^{v_t},$$

for some $\gamma' \in \Gamma'$ and $v_i \in \mathbb{Z}$. Since $\mathfrak{D} \subseteq M'$ by proposition 5.1 we have that ϕ_i is also a unit in O_K. Thus $\sigma_j(\phi_i)$ is a unit in O_N so $\sigma_j(\phi_i)O_N = O_N$ meaning that $\hat{\mathfrak{p}} \nmid \sigma_j(\phi_i)$. Using lemma 12.2 there is a number q so that $\sigma_j(\phi_i)^q \in U^{(k)}$, so setting $\epsilon_i = \phi_i^q$ we now have $\sigma_j(\epsilon_i) \in U^{(k)}$. Let now $\Delta = \{\prod_{i=1}^t \phi_i^{\rho_i} \mid \rho_i \in \{0, ..., q-1\}\}$ and define $\Gamma = \{\delta\gamma \mid (\delta, \gamma) \in \Delta \times \Gamma'\}$. We see that we can now write the element α from above as

$$\alpha = \gamma \epsilon_1^{u_1} ... \epsilon_t^{u_t} \tag{12.7}$$

For an appropriate choice of u_i and $\gamma \in \Gamma$. Consider now the special case where $\alpha \in M_c \subseteq M'_c$ and suppose that it is written as in eq. (12.7). Since $\alpha \in M'$ we can also write $\alpha = x_1\mu_1 + ... + x_n\mu_n$ and since $\alpha \in M$, we must have

$$x_{m+1} = ... = x_n = 0 \tag{12.8}$$

This restriction can be reformulated to something that is more convenient to work with. Let $\mu_1^*, ..., \mu_n^*$ be the dual basis of $\mu_1, ..., \mu_n$. The following computation shows that we can recover the x_i's.

$$\mathrm{Tr}_{K/\mathbb{Q}}(\mu_i^* \alpha) = \mathrm{Tr}_{K/\mathbb{Q}}(\sum_{j=1}^n \mu_i^* x_j \mu_j) = \sum_{j=1}^n x_j \,\mathrm{Tr}_{K/\mathbb{Q}}(\mu_i^* \mu_j) = x_i$$

Therefore the above restriction is the same as the requirement that

$$\mathrm{Tr}_{K/\mathbb{Q}}(\mu_{m+1}^* \alpha) = ... = \mathrm{Tr}_{K/\mathbb{Q}}(\mu_n^* \alpha) = 0$$

Which, by use of eq. (12.7) and the definition of the trace, amounts to the equations

$$\sum_{i=1}^n \sigma_j(\gamma \mu_i^*) \sigma_j(\epsilon_1)^{u_1} ... \sigma_j(\epsilon_t)^{u_t} = 0 \text{ for all } i \in \{1, ..., n\} \tag{12.9}$$

Define now the function

$$\begin{aligned}\varphi : U^{(k)} \times O_{\mathfrak{p}} &\to U^{(k)} \\ (\alpha, u) &\mapsto \exp(u \log \alpha)\end{aligned}$$

This function is well defined since if $\alpha \in U^{(k)}$ and $u \in O_{\mathfrak{p}}$ then $u \log \alpha \in \hat{\mathfrak{p}}^k$ as $\hat{\mathfrak{p}}^k$ is an ideal. Note that if $u \in \mathbb{Z} \subseteq O_{\mathfrak{p}}$ we get $\varphi(\alpha, u) = \exp(\log \alpha^u) = \alpha^u$. For this reason we will simply write $\alpha^u := \varphi(\alpha, u)$. Assume now that all the u_i are in $O_{\mathfrak{p}}$. Then eq. (12.9) still makes sense since we have made sure that $\sigma_j(\epsilon_i) \in U^{(k)}$. We can now define

$$\begin{aligned}L_j(u_1, ..., u_t) &= \sum_{i=1}^{t} u_i \log \sigma_j(\epsilon_i), \text{ for } j \in \{1, ..., n\} \\ A_{ij} &= \sigma_j(\gamma \mu_i^*), \text{ for } (i, j) \in \{1, ..., n\}^2\end{aligned}$$

Note that the L_j can be thought of as power series in $P := N_{\mathfrak{p}}[[u_1, ..., u_t]]$. These have constant term 0 so it makes sense to consider the formal power series $\exp L_j(u_1, ..., u_t) \in P$. This means that we can define

$$G_i(u_1, ..., u_t) = \sum_{j=1}^{n} A_{ij} \exp L_j(u_1, ..., u_t) \text{ for } i \in \{m+1, ..., n\}$$

Note that the G_i all belong to P and also have constant term 0. The G_i also converge for all $(u_1, ..., u_t) \in O_{\mathfrak{p}}^t$, so these are analytic functions. Observe that we can now rewrite eq. (12.9) to

$$G_i(u_1, ..., u_t) = 0, \text{ for all } i \in \{m+1, ..., n\} \tag{12.10}$$

Assume that there are infinitely many solutions in integers to eq. (12.5). Thus we have infinitely many solutions of the shape eq. (12.7). But as Γ has only finitely many elements, there has to exists some $\gamma \in \Gamma$ giving rise to an infinite subset of solutions,

$$S_\gamma := \{\gamma \epsilon_1^{u_1} ... \epsilon_t^{u_t} \mid (u_1, ..., u_t) \in \mathbb{Z}^t\} \subseteq M_c$$

As the ϵ_i are independent units, we get an injective homomorphism $\iota : S_\gamma \to O_{\mathfrak{p}}^t$, given by $\gamma \epsilon_1^{u_1} ... \epsilon_t^{u_t} \mapsto (u_1, ..., u_t)$. Now take a sequence of distinct elements, (α_s), in S_γ. We get a sequence $U_s = \iota(\alpha_s) \in O_{\mathfrak{p}}^t$ and as ι is injective, the elements of this sequence are also distinct. As the elements α_s all belong to M_c the tuples U_s satisfy eq. (12.10). Therefore we have $G_i(U_s) = 0$ for all the G_i and all $s \in \mathbb{N}$. Since $O_{\mathfrak{p}}$ is compact by theorem 10.1, it follows from Tychonoff's theorem that $O_{\mathfrak{p}}^t$, equipped with the product topology, is also compact. But $O_{\mathfrak{p}}$ is a metric space so $O_{\mathfrak{p}}^t$ is metrizable, (STATE THIS IN A THEOREM SOMEWHERE), and therefore it is sequentially compact. This means that the sequence U_s has a convergent subsequence, U_s^*, converging to a point, call it $u^* = (u_1^*, ..., u_t^*) \in O_{\mathfrak{p}}^t$. As the G_i are analytic they are continuous (NEED A PROOF FOR THIS), we

know that $G_i(U_s^*)$ converges to $G_i(u^*)$ for $s \to \infty$. But also $G_i(U_s^*)$ converges to 0 since $G_i(U_s^*) = 0$ for all $s \in \mathbb{N}$. By uniqueness of limits in metric spaces we therefore have $G_i(u^*) = 0$. Also any neighborhood of u^* contains infinitely many solutions to eq. (12.10); Take any open ϵ-ball, B, with center u^*. Because U_s^* converges to u^*, we know that there is an $N \in \mathbb{N}$ so that $U_s^* \in B$ for all $s \geq N$, all of which are distinct and are solutions to eq. (12.10). Let us now write $u_i = u_i^* + v_i$, where $v_i \in O_{\mathfrak{p}}$. We get

$$\begin{aligned} G_i(u_1, ..., u_r) &= \sum_{j=1}^{n} A_{ij} \exp L_j(u_1^* + v_1, ..., u_r^* + v_r) \\ &= \sum_{j=1}^{n} A_{ij} \exp L_j(u_1^*, ..., u_r^*) \exp L_j(v_1, ..., v_r) \end{aligned}$$

Setting $A_{ij}^* = A_{ij} \exp L_j(u_1^*, ..., u_r^*)$ and $H_i(v_1, ..., v_r) = \sum_{j=1}^{n} A_{ij}^* L_j(v_1, ..., v_r)$, we get $G_i(u_1, ..., u_r) = H_i(v_1, ..., v_r)$. We get yet another reformulation of eq. (12.8)

$$H_i(v_1, ..., v_t) = 0 \text{ for all } i \in \{m+1, ..., n\} \tag{12.11}$$

As the G_i are formal power series with constant term 0, the H_i are also formal power series with constant term 0. These converge on $O_{\mathfrak{p}}^t$ hence also on $\hat{\mathfrak{p}}^t$, which is an ϵ-neighborhood of the origin (with $\epsilon = 1$). Hence we get a local manifold, V, determined by eq. (12.11). From what we showed above there are infinitely many solutions in any neighborhood of the origin. Therefore, by theorem 9.1, V contains an analytic curve. We summarize what we have found in the following theorem. (QUESTION: DO WE EVEN NEED TO SAY THAT IT CONTAINS AN ANALYTIC CURVE)

Theorem 12.1. *If eq.* (12.5) *has infinitely many solutions in integers, then there exists $\gamma \in \Gamma$ which determines a local manifold, V, given by eq.* (12.11) *and it contains an analytic curve.*

Corollary 12.1. *Let $\omega_1(X), ..., \omega_t(X)$ be any curve on the manifold V determined by eq.* (12.11) *and define $P_j = L_j(\omega_1(X), ..., \omega_t(X))$ for $j \in \{1, ..., n\}$. These are power series with coefficients in a finite extension, $N_\bullet$ of $N_{\mathfrak{p}}$. There exists a matrix B_{ij} with coefficients in $N_\bullet$ so that the P_j satisfy the system of equations*

$$\begin{aligned} \sum_{j=1}^{n} A_{ij}^* \exp P_j(t) = 0, \text{ for all } i \in \{m+1, ..., n\} \\ \sum_{j=1}^{n} B_{ij} P_j(t) = 0, \text{ for all } i \in \{1, ..., n-t\}, \end{aligned} \tag{12.12}$$

The matrix (A_{ij}^*) has $N_\bullet$-linearly independent rows and (B_{ij}) can be chosen so that it also has $N_\bullet$-linearly independent rows.

Proof. Since $\omega_1(X), ..., \omega_t(X)$ is a curve on V each $\omega_i(X)$ has coefficients in a finite extension $N_\bullet$ of $N_{\mathfrak{p}}$ and therefore the P_j have coefficients in $N_\bullet$. Because

$\omega_1(X), ..., \omega_t(X)$ is a curve on V we know in particular that

$$H_i(\omega_1(X), ..., \omega_t(X)) = 0 \text{ for all } i \in \{m+1, ..., n\} \tag{12.13}$$

Considering how the H_i are defined we see that the above is just a different way of spelling out the first equation in eq. (12.12). The matrix (A_{ij}) is invertible because the $\gamma\mu_i^*$ constitute a basis for K over $\mathbb{Q}$ so the discriminant of this basis is non-zero and it equals $|\sigma_j(\gamma\mu_i^*)|^2 = |A_{ij}|^2$ so $|A_{ij}| \neq 0$. Since exp sends elements to $U^{(k)}$ and $0 \notin U^{(k)}$ we have $\exp L_j(v_1, ..., v_t) \neq 0$ for all $(v_1, ..., v_t) \in O_{\mathfrak{p}}^t$. Since $A_{ij}^* = A_{ij} \exp L_j(u_1^*, ..., u_t^*) \in N_{\mathfrak{p}} \subseteq N_\bullet$.

$$|A_{ij}^*| = |A_{ij}| \prod_{j=1}^{n} L_j(u_1^*, ..., u_t^*) \neq 0$$

Thus the rows of (A_{ij}^*) are linearly independent over $N_\bullet$.

Let $U = \text{span}_{N_\bullet}\{\omega_1(X), ..., \omega_t(X)\}$ and notice that each P_j is in U. This means we can define the following linear map

$$T : N_\bullet^n \to U$$
$$T(a_1, ..., a_n) = \sum_{i=1}^{n} a_i P_i$$

The dimension of U is at most t, so the rank of T is also at most t. By rank nullity, the nullity of T is therefore larger than or equal to $n-t$. In other words, there are at least $n-t$ linearly independent elements in the kernel of T and we can therefore construct an $n-t \times n$ matrix, (B_{ij}) with $N_\bullet$-linearly independent rows. □

We have yet another corollary

Corollary 12.2. *Keep the setup from above theorem 12.1 and consider the special case where $m = 2$, $n \geq 3$ and where at least one of the σ_i is a complex embedding. Then for any curve $\omega_1(X), ..., \omega_t(X)$ on V there exists indices k and l with $k \neq l$ so that $L_k(\omega_1(X), ..., \omega_t(X)) = L_l(\omega_1(X), ..., \omega_t(X))$.*

Proof. The existence of a complex embedding is the same as saying that $2s \geq 1$ which is the same as saying that $n - t \geq 2$. Setting $n_1 = n - m$ and $n_2 = n - t$ the result follows from combining lemma 12.1 and corollary 12.1. □

Theorem 12.2 (Thue)**.** *Suppose $f(x, y)$ is an irreducible form with integer coefficients, of degree $n \geq 3$ and that $f(x, 1)$ has at least one complex root. Then there are only finitely many integer solutions to the equation $f(x, y) = c$, for some fixed $c \in \mathbb{Z}$.*

Proof. To prove this, we want to use the results obtained in this section and to do that it suffices to show that f is decomposable and nonfull. Since f is a form of degree n we know that we can write

$$f(x, y) = \sum_{i=0}^{n} a_i x^{n-i} y^i$$

where the a_i are in $\mathbb{Z}$. We must have that $a_0 \neq 0$, since otherwise y would divide $f(x,y)$ which is impossible since f is irreducible. Let $h(x) = f(x,1)$. We have that h is a polynomial of degree n with leading coefficient a_0. The leading term of $a_0^{n-1}h(x)$ is $a_0^n x^n = (a_0 x)^n$, so h can be regarded as a monic polynomial in the variable t using the substitution $a_0 x \mapsto t$. So assume without loss of generality that h is monic, i.e. that $a_0 = 1$. It now follows from remark 4.1 that

$$f(x,y) = N_{K/\mathbb{Q}}(x + \theta y) \tag{12.14}$$

where $K = \mathbb{Q}(\theta)$ is a number field of degree n with embeddings $\sigma_1, ..., \sigma_n$ from K into $\mathbb{C}$. Consider the module M generated by $\mu_1 = 1$ and $\mu_2 = \theta$. These are linearly independent over $\mathbb{Q}$ so the rank, m, of this module is 2 and since $n \geq 3$ we have $m < n$. This shows that f is a nonfull form. Assume now that $f(x,y) = c$ has infinitely many solutions in integers. Then we cannot have $c = 0$ as the only solution in this case is $(x,y) = (0,0)$ so assume $c \neq 0$. We have just verified that f is a decomposable and nonfull form and it is irreducible by assumption. Using the notation from the proof of theorem 12.1 we now obtain $\gamma \in \Gamma$ and independent units $\epsilon_1, ..., \epsilon_t \in \mathfrak{D}$ giving rise to a local manifold V determined by eq. (12.13). We get an injective map $\iota : S_\gamma \to O_\mathfrak{p}^t$ and a sequence (α_s) of distinct elements in S_γ so that $U_s^* = \iota(\alpha_s)$ and $U_s^* \to (u_1^*, ..., u_t^*) \in O_\mathfrak{p}^t$. Write $U_s^* = (u_{1s}, ..., u_{ts})$ and $u_{is} = u_i^* + v_{is}$ and define the sequence $V_s = (v_{1s}, ..., v_{ts})$. We have $V_s \in V$ and it converges to the origin. Let W be the local manifold given by

$$\prod_{i \leq k < l \leq n} (L_k(v_1, ...v_t) - L_l(v_1, ..., v_t)) = 0$$

If we have any curve, $\tau_1(X), ..., \tau_t(X)$ on V then we can invoke corollary 12.2 to conclude that there exists distinct k and l so that $L_k(\tau_1(X), ..., \tau_t(X)) = L_l(\tau_1(X), ..., \tau_t(X))$. This implies that the curve $\tau_1(X), ..., \tau_t(X)$ is also on W. (QUESTION: Why? Do we use that I_A is generated by the equations that define it? (A is some manifold)). Using remark 9.1 we get that $V \subseteq W$, that is there is an ϵ-neighborhood around the origin, U, so that $U \cap V \subseteq U \cap W$. Since (V_s) converges to the origin there is an $N \in \mathbb{N}$ so that for all $s \geq N$ we have $V_s \in W$. Thus we can find distinct k and l so that $L_k(V_s) = L_l(V_s)$ for infinitely many $s \geq N$. Therefore there are infinitely many $s_1, s_2 \in \mathbb{N}$, $s_1 \neq s_2$ so that $L_k(V_{s_1}) = L_l(V_{s_2})$. Choose any such s_1 and s_2 and set $\alpha = \alpha_{s_1}$ and $\alpha' = \alpha_{s_2}$. Let us write $\alpha = x + y\theta$ and $\alpha' = x' + y'\theta$. Furthermore, since $\alpha \in S_\gamma$, we can write $\alpha = \gamma \epsilon_1^{u_1} ... \epsilon_t^{u_t}$. Set now

$$\begin{aligned} u_i &= u_i^* + v_i \\ c_j &= \sigma_j(\gamma)\sigma_j(\epsilon_1)^{u_1^*} ... \sigma_j(\epsilon_t)^{u_t^*} \end{aligned}$$

We obtain

$$\sigma_j(\alpha) = c_j \exp L_j(v_1, ..., v_t)$$

And in a similar fashion we get

$$\sigma_k(\alpha) = c_k \exp L_k(v_1, ..., v_t)$$

Since $L_j(v_1, ..., v_t) = L_k(v_1, ..., v_t)$ we have $\exp L_j(v_1, ..., v_t) = \exp L_k(v_1, ..., v_t)$. Also both c_j and c_k are different from 0 since otherwise we would have $\gamma = 0$ which would imply $c = 0$. Thus we have

$$\frac{\sigma_j(\alpha)}{c_j} = \frac{\sigma_k(\alpha)}{c_k}$$

Similarly we obtain

$$\frac{\sigma_j(\alpha')}{c_j} = \frac{\sigma_k(\alpha')}{c_k}$$

Since α and α' have norm $c \neq 0$ they are both non-zero. Hence we now have

$$\frac{x + y\sigma_j(\theta)}{x' + y'\sigma_j(\theta)} = \frac{x + y\sigma_k(\theta)}{x' + y'\sigma_k(\theta)}$$

So $(xy' - x'y)(\sigma_k(\theta) - \sigma_j(\theta)) = 0$. But σ_k and σ_j are distinct embeddings since $k \neq j$ and they are determined by their action on θ. Hence $\sigma_k(\theta) - \sigma_j(\theta) \neq 0$ so we must have

$$xy' - x'y = 0$$

Thus there is $d \in \mathbb{Q}$ so that $\alpha = d\alpha'$, so

$$N_{K/\mathbb{Q}}(\alpha) = N_{K/\mathbb{Q}}(d)N_{K/\mathbb{Q}}(\alpha')$$

But $N_{K/\mathbb{Q}}(\alpha) = N_{K/\mathbb{Q}}(\alpha')$ so $1 = N_{K/\mathbb{Q}}(d) = d^n$. Hence $d = \pm 1$ so $\alpha = \pm\alpha'$. But then it is impossible for all the elements of (α_s) to be distinct contrary to our assumption so S_γ cannot be infinite, meaning that there has to be only finitely many solutions to $f(x, y) = c$. □

Discussion of Thue's theorem

A natural question to ask when looking at any theorem is to ask which assumptions can be dropped or altered without changing the conclusion of the theorem too much. This is what we will explore in the case of theorem 12.2.

The number of variables

The following example shows that the theorem no longer holds if we swap out the form $f(x, y)$ with a form with more variables.

Example 12.1. *Consider the field $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. This field has degree 4 over $\mathbb{Q}$ and*

$$1, \sqrt{2}, \sqrt{3}, \sqrt{6}$$

is a basis for it by [Dummit and Foote, 2004, Example (2), p. 526]. Note that K is splitting field for the separable polynomial $(x^2 - 2)(x^2 - 3)$ so K is Galois over $\mathbb{Q}$. The automorphisms on K are determined by their action on $\sqrt{2}$ and $\sqrt{3}$ and these must be sent to $\pm\sqrt{2}$ and $\pm\sqrt{3}$ respectively. There are four ways of

doing this, and as K is Galois all of these possibilities yield an automorphism. Using this information we find that $F(x,y,z) := N_{K/\mathbb{Q}}(x+y\sqrt{2}+z\sqrt{3})$ is equal to

$$(x+y\sqrt{2}+z\sqrt{3})(x-y\sqrt{2}+z\sqrt{3})(x+y\sqrt{2}-z\sqrt{3})(x-y\sqrt{2}-z\sqrt{3}) \quad (12.15)$$

which after a (cumbersome) computation turns to be equal to

$$x^4+4y^4+9z^4-4x^2y^2-6x^2z^2-12y^2z^2$$

The module $\{1,\sqrt{2},\sqrt{3}\}$ has rank 3 as the generators are linearly independent over $\mathbb{Q}$, so this module is nonfull and hence F is a nonfull form. It is also clearly decomposable as F splits into linear factors over K. Furthermore F is irreducible; Clearly, none of the linear factors in eq. (12.15) *are forms. Hence if F were reducible, then there has to be a pair of these linear factors that multiply to a form. Consider for example*

$$(x+y\sqrt{2}+z\sqrt{3})(x-y\sqrt{2}+z\sqrt{3}) = x^2-2y^2+2\sqrt{3}xz+3z^2$$

This is certainly not a form as the coefficients are not in $\mathbb{Q}$. As the isomorphisms in $\mathrm{Gal}(K/\mathbb{Q})$ permute the linear factors in eq. (12.15) *this shows that no pair of linear factors multiply to a form and therefore F must be irreducible. Observe that*

$$\begin{aligned} F(x,y,0) &= x^4+4y^4-4x^2y^2 = (x^2-2y^2)^2 \\ &= N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(x+y\sqrt{2})^2 \end{aligned} \quad (12.16)$$

Consider the full module $\{1,\sqrt{2}\}$ inside $\mathbb{Q}(\sqrt{2})$. Let's look at the equation

$$N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(x+y\sqrt{2}) = 1$$

Applying lemma 6.1 we have $t = r+s-1 = 2+0-1 = 1$ so there has to be infinitely many solutions in integers to this equation and therefore, because of eq. (12.16) *there are infinitely many solutions to the equation $F(x,y,z)=1$.*

In the above example we got an infinite number of solutions because the module $\{1,\sqrt{2},\sqrt{3}\}$ inside $\mathbb{Q}(\sqrt{2},\sqrt{3})$ contains the submodule $\{1,\sqrt{2}\}$, which is a full module when regarded as a module inside $\mathbb{Q}(\sqrt{2})$. The fact that we get infinitely many solutions can, more abstractly, be attributed to the fact that $\{1,\sqrt{2},\sqrt{3}\}$ is what is called a degenerate module. Let's define what we mean by this

Definition 12.1. *Let K be a number field and M a module with generators $\mu_1,...,\mu_m$ and consider the vector space $L = \mathrm{span}_{\mathbb{Q}}\{\mu_1,...,\mu_m\}$. If L contains a subspace L' so that $\gamma K' = L'$ for some subfield K' of K and $\gamma \in K$ and K' is neither $\mathbb{Q}$ or a quadratic imaginary field then we say that M is degenerate. Otherwise the module is called non degenerate.*

We see that the module $\{1, \sqrt{2}, \sqrt{3}\}$ inside $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is certainly degenerate since for example $\mathbb{Q}(\sqrt{2}) = \text{span}_{\mathbb{Q}}\{1, \sqrt{2}\}$. Also all full modules in a field different from $\mathbb{Q}$ or an imaginary quadratic field are degenerate - just consider the span over $\mathbb{Q}$ of any basis of such a module. For an example of a non-degenerate module look at the module $M = \{1, \beta\}$ from the proof of theorem 12.2. One can of course argue for this using the definition directly. But it also follows from the following fact combined with the statement of theorem 12.2.

Fact 12.1. *Assume K is a number field and M is a degenerate module. There exists $c \in \mathbb{Z}$ so that there are infinitely many solutions, $\alpha \in M$, to the equation*

$$N_{K/\mathbb{Q}}(\alpha) = c$$

Proof. By assumption we have that $K' = \gamma L'$, where K', L' and $\gamma \in K$ are as in definition 12.1. Set $M' = L' \cap M$. Then $\gamma^{-1}M'$ is a full module inside K' and since K' is neither $\mathbb{Q}$ or a quadratic imaginary field, we know from theorem 6.3 and remark 6.1 that there exists $c' \in \mathbb{Z}$ so that

$$N_{K'/\mathbb{Q}}(\xi) = c'$$

has infinitely many solutions $\xi \in \gamma^{-1}M'$ (take for example $c' = 1$). Take any such solution, ξ, and also set $c_\bullet = N_{K/\mathbb{Q}}(\gamma)c_\bullet^m$, where $m = [K : K']$. We get

$$N_{K/\mathbb{Q}}(\xi\gamma) = N_{K/\mathbb{Q}}(\xi)N_{K/\mathbb{Q}}(\gamma) = (N_{K'/\mathbb{Q}}(\xi))^m N_{K/\mathbb{Q}}(\gamma) = c$$

Since $\xi\gamma \in M' \subseteq M$ the result now follows. □

This raises the question: If M is a non degenerate module, is it then always the case that there are only finitely many solutions to $N_{K/\mathbb{Q}}(\alpha) = c$ for $\alpha \in M$? If this turned out to be true we would obtain a generalization of theorem 12.2.

Improvements made by Alan Baker

Alan Baker improved theorem 12.2 significantly. Quoting [Baker, 1975, Theorem 4.1] he showed the following

Theorem 12.3. *Let K be a number field of degree d, let $\alpha_1, ..., \alpha_n$ be $n \geq 3$ distinct algebraic integers in K and let μ be a non-zero algebraic integer. The equation*

$$f(x, y) := (x - \alpha_1 y)...(x - \alpha_n y) = \mu$$

has only finitely many solutions in algebraic integers x and y in K and these can be effectively determined.

As we can see we obtain a much stronger result that what we arrived at in theorem 12.2 and under much milder conditions too. Of course f will still be a homogenous polynomial of degree n, but the coefficients need not be in $\mathbb{Q}$ so f need not be a form. In the proof of theorem 12.2 we made heavy use of the fact that $f(x, y)$ was a norm form. We have no guarantee that f in the theorem above is such a form. As such it might not come as a surprise that the proof of the above theorem relies on very different tools than those covered in this text.

Baker

If one could prove that there are only a finite number of solutions to , where α is in some non degenerate

is a number field of degree $n \geq 3$ and module $\{\mu_1, \mu_2\}$ of rank 2 inside a number field that has degree $n \geq 3$.

Talk about similar modules here

This is something that never happens for the modules of rank 2. a theta1 + b theta2 = a + theta1

ca theta1 + cb theta2 = c

x t1 + y t2 = 1/c

It might be the case that forms connected to modules which don't have a property like this always have a finite number of solutions.

A natural question to ask is which assumptions in theorem 12.2 can be dropped to achieve the same conclusion. As lemma 6.1 illustrates there can only be finitely many solutions in a full module if $t = 0$. This happens exactly when $r = 0$ and $s = 1$ or $r = 1$ and $s = 0$.

It is clear that forms of degree 1 can have infinitely many solutions. What about the forms of degree 2? Consider the form $F(x, y) = x^2 - 2y^2$ and the equation

$$F(x, y) = 7$$

Suppose we already have a solution, (x, y). Then also $(3x + 4y, 2x + 3y)$ is a solution. Thus a single solution can inductively produce infinitely many distinct. But $(3, 1)$ is indeed a solution, and therefore $F(x, y) = 7$ has infinitely many solutions. This highlights the importance of the assumption that the degree of F must be greater than or equal to 3.

One might ask if the assumption that F has a complex root is really necessary and perhaps more interestingly, if all the solutions can be found. A big limiting factor about the statement we have proved is that we never know when we have found all solutions. This is of course expected, since this is just the nature of contradiction proofs - there are not constructive. But both of these questions can actually be answered if we look to Baker [1975], **Theorem 4.1** on page 38. Here we see that we can simply drop the assumption that there has to be a complex root and it is even possible to find all solutions.

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