6. Skolem's Method

In this section we study the application of Skolem's method to equations of the form

$$F(x_1, \ldots, x_m) = c, \tag{1}$$

where F is an irreducible, decomposable, nonfull form (Section 1.3 of Chapter 2), and c is a rational number. This method is based on some simple properties of local analytic manifolds over p-adic fields, which will be proved in the next section. An example which illustrates the idea of Skolem's method was given at the beginning of this chapter.

6.1. Representation of Numbers by Nonfull Decomposable Forms

In Section 1.3 of Chapter 2 we saw that (6.1) can be written in the form

$$N(x_1\mu_1 + \cdots + x_m\mu_m) = a \tag{2}$$

or

$$N(\alpha) = a, \quad (\alpha \in M), \tag{3}$$

where $\mu_1, \ldots, \mu_m$ are numbers of some algebraic number field k, and $M = \{\mu_1, \ldots, \mu_m\}$ is the module generated by these numbers (a is a rational number). Replacing, if necessary, the form F by a form integrally equivalent to it, we may assume that the numbers $\mu_1, \ldots, \mu_m$ of the module M are linearly independent over the field R of rational numbers. Since M is nonfull, $m < n = (k : R)$.

In Chapter 2 we saw how to find all solutions to (6.3) when M is a full module of k. It is thus natural to embed M in a full module M, and to use the methods of Chapter 2 to find all solutions of the equation $N(\alpha) = a, \alpha \in M$, and then to pick out those solutions which lie in M.

It is clear that any module of k can be embedded in a full module. To do this it suffices to extend the linearly independent set $\mu_1, \ldots, \mu_m$ to a basis $\mu_1, \ldots, \mu_n$ of the field k and to set $M = \{\mu_1, \ldots, \mu_n\}$.

If all $\alpha \in M$ for which $N(\alpha) = a$ have already been found, then we shall obtain all solutions of (6.3) if we can isolate those solutions for which in the representation

$$\alpha = x_1\mu_1 + \cdots + x_n\mu_n \tag{4}$$

the coefficients $x_{m+1}, \ldots, x_n$ are equal to zero. To express the conditions $x_{m+1} = \cdots = x_n = 0$ directly in terms of α, it is convenient to use the dual basis $\mu_1^*, \ldots, \mu_n^*$ (see Section 2.3 of the Supplement). Since the trace Sp $\mu_i^*\mu_j^* \neq 0$ for $i \neq j$ and 1 for $i = j$, then $x_i = \text{Sp } \alpha\mu_i^*(1 \leq i \leq n)$. It follows that the