Proposition 0.1. *(Universal property for quotients) Let R, S be a rings and $I \subseteq R$ an ideal. Suppose we have a function $f : R \to S$, which vanishes on I and is an additive group homomorphism when restricted to I. Then there exists a uniquely determined function*

$$\bar{f} : R/I \to S$$

such that the following diagram commutes

$$\begin{tikzcd} & R \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ R/I \arrow[rr, dashed, "\exists! \bar{f}"'] & & S \end{tikzcd}$$

In this case we say that f descends to the quotient, R/I. If f is a ring map then so is $\bar{f}$.

Proof. If there is a map $\bar{f}$ so that the diagram commutes then what that means is that we have, for all $r \in R$

$$\bar{f}(\pi(r)) = f(r)$$

But π is surjective so this condition forces how $\bar{f}$ is defined, and hence $\bar{f}$ is unique if it exists. Suppose now that $x, y \in R$ so that $\bar{x} = \bar{y}$. Then $x - y \in I$ and so $f(x-y) = 0$, so $f(x) = f(y)$ as f is an additive homomorphism when restricted to I. Hence $\bar{f}$ is well defined. If f is a ring map, then $\bar{f}$ also be a ring map because the diagram commutes. □

Suppose that v is a valuation as above. Let us briefly go through some important properties. Note first that property 2. above makes v into a homomorphism $v : K^* \to \mathbb{R}$. Thus, if $x \in K^*$ has finite order, then also $v(x)$ has finite order. But then $v(x) = 0$ as 0 is the only element in $\mathbb{R}$ that has finite order with respect to addition. In particular, $v(-1) = 0$ so $v(-x) = v(-1) + v(x) = v(x)$ for all $x \in K$. It follows that $v(x+y) = v(y)$ if $v(x) > v(y)$, since

$$v(y) = v(x + y - x) \geq \min\{v(x+y), v(x)\} \geq \min\{v(x), v(y)\} = v(y)$$

There is another, perhaps more down-to-earth way of characterizing the completion of a field. But

Now for the alternative characterization.

Proposition 0.2. *Let K be a valued field, $\hat{K}$ a complete valued field and $\hat{\imath} : K \to \hat{K}$ a homomorphism preserving the absolute value. Then $(\hat{K}, \hat{\imath})$ is a completion of K if and only if K is dense in $\hat{K}$.*

Proof. Assume first that the pair $(\hat{K}, \hat{\imath})$ is in fact the completion of K and let us show that K is dense in $\hat{K}$, by which we of course mean that the image $\hat{\imath}(K)$

is dense in $\hat{K}$. Now, as $\hat{K}$ is complete also $\overline{K}(=\overline{\hat{\imath}(K)})$ is complete since it is closed and contained in $\hat{K}$. Also, as K is dense in $\overline{K}$ the inclusion is a subfield of $\hat{K}$ (SHOW THIS). Thus, we have the inclusion map $\psi : \overline{K} \to \hat{K}$. This shows that $(\overline{K}, \hat{\imath})$ satisfies the same universal property as $(\hat{K}, \hat{\imath})$ and hence $(\overline{K}, \hat{\imath})$ is the completion of K. (THIS PART IS UNFINISHED)

Let us now prove the converse. So suppose that $\hat{\imath}(K)$ is dense in $\hat{K}$ and that (L, ι) is a pair as in (**??**). $\square$

Let us now look at some examples. We have already mentioned that $\mathbb{R}$ is the completion of $\mathbb{Q}$. We have the inclusion $\mathbb{Q} \to \mathbb{R}$ which preserves absolute values, so this statement follows if we are willing to accept that $\mathbb{R}$ is complete and that $\mathbb{Q}$ is dense in $\mathbb{R}$. Here is another example. Suppose that K is a field and consider the formal power series $K[[x]]$. As we have mentioned already this is a local ring with maximal ideal $\mathfrak{p} = (x)$. Consider the valuation $v_m frakp$ on $K[[x]]$ defined by

Theorem 0.1 (Ostrowski)**.** *Suppose that K is field which is complete with respect to an archimedean valuation. Then there is an isomorphism σ from K into $\mathbb{R}$ or $\mathbb{C}$ and a constant $s \in (0, 1]$ so that*

$$|x| = |\sigma(x)|^s$$

for all $x \in K$.

A nonarchimedean absolute value $|\cdot|$ on a field K extends to a nonarchimedian absolute value on $K(t)$ by setting $|f| = \max |a_0|, ..., |a_n|$ where $f \in K[x]$ and $f(x) = a_n x^n + ... + a_0$. For an arbitrary element $\frac{g}{h} \in K(x)$ where $h \neq 0$ we then define $|\frac{g}{h}| = |g| - |h|$.