On J-equation

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Abstract

In this paper, we prove that for any K¨ahler metrics ω0 and χ on M, there exists ω*ϕ* = ω0 + −1∂∂¯ϕ > 0 satisfying the J-equation tr*ωϕ* χ = c if and only if (M, [ω0 ], [χ]) is uniformly J-stable. As a corollary, we can find

√

many constant scalar curvature K¨ahler metrics with c1 < 0. Using the same method, we also prove a similar result for the deformed Hermitian- Yang-Mills equation when the angle is in ( *nπ* − *π* , *nπ* ).

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# Introduction

In this paper, our main goal is to prove the equivalence of the solvability of the J-equation and a notion of stability. Given K¨ahler metrics *ω*0 and *χ* on *M* , the J-equation is defined as

for

tr*ωϕ χ* = *c*

√ ¯

*ωϕ* = *ω*0 + −1*∂∂ϕ >* 0*.*

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In general, the equivalence of the stability and the solvability of an equation is very common in geometry. One of the first results in this direction was the celebrated work of Donaldson-Uhlenbeck-Yau [[20](#_bookmark38), [39](#_bookmark57)] on Hermitian-Yang- Mills connections. Inspired by the study of Hermitian-Yang-Mills connections, Donaldson proposed many questions including the study of J-equation using the moment map interpretation [[21](#_bookmark39)]. It was the first appearance of the J-equation in the literature.

Yau conjectured that the existence of Fano K¨ahler-Einstein metric is also equivalent to some kind of stability [[41](#_bookmark59)]. Tian made it precise in Fano K¨ahler- Einstein case and it was called the K-stability condition [[37](#_bookmark55)]. It was general- ized by Donaldson to the constant scalar curvature K¨ahler (cscK) problem in projective case [[22](#_bookmark40)]. This conjecture has been proved by Chen-Donaldson-Sun [[9](#_bookmark27), [10](#_bookmark28), [11](#_bookmark29)] in Fano K¨ahler-Einstein case. However, there is evidence that this conjecture may be wrong in cscK case [[1](#_bookmark19)]. There is a folklore conjecture that the uniform version of K-stability may be a correct substitution. When restricted to special test configurations called “degeneration to normal cones”, the uniform K-stability is reduced to Ross-Thomas’s uniform slope K-stability [[31](#_bookmark49)]. More

recently, the projective assumption was removed by the work of Dervan-Ross

[[19](#_bookmark37)] and independently by Sjo¨str¨om Dyrefelt [[33](#_bookmark50)].

It is easy to see that cscK metrics are critical points of the K-energy func-

tional [[5](#_bookmark23)]

*K*(*ϕ*) = ∫

log(

*ωn ωn*

*ϕ* ) *ϕ* + J−Ric(*ω* )(*ϕ*)*.*

*M n n*! 0

*ω*

0

The J*χ* functional for any (1,1)-form *χ* is defined by

J*χ*(*ϕ*) =

1 ∫

*n*−1

*ϕ*

Σ

*χ* ∧ *ωk* ∧ *ωn*−1−*k* −

1 ∫

*c*0*ϕ* Σ

*ωk* ∧ *ωn*−*k,*

*n*! *M*

0 *ϕ*

*k*=0

(*n* + 1)! *M*

0 *ϕ*

*k*=0

*n*

where *c*0 is the constant given by

∫ *χ* ∧

*ωn*−1

0

— *c*0

*n*

0 = 0*.*

*ω*

*M* (*n* − 1)! *n*!

When *χ* is a K¨ahler form, it is well known that the critical point of the *χ* func- tional is exactly the solution to the J-equation. It was the second appearance of the J-equation in the literature. Following this formula, using the interpola- tion of the K-energy and the *χ* functional, Chen-Cheng [[6](#_bookmark24), [7](#_bookmark25), [8](#_bookmark26)] proved that the existence of cscK metric is equivalent to the geodesic stability of K-energy. However, the relationship between the existence of cscK metrics and the uniform K-stability is still open.

J

J

When we replace the K-energy by the *χ* functional for a K¨ahler form *χ*, the analogy of the K-stability and the slope stability conditions were proposed by Lejmi and Sz´ekelyhidi [[28](#_bookmark46)]. See also Section 6 of [[19](#_bookmark37)] for the extension to non-projective case. The main theorem of this paper proves the equivalence between the existence of the critical point of J*χ* functional, the solvability of J-equation, the coerciveness of *χ* functional, and the uniform J-stability as well as the uniform slope J-stability.

J

J

Theorem 1.1. *(Main Theorem) Fix a K¨ahler manifold Mn with K¨ahler metrics*

*χ and ω*0*. Let c*0 *>* 0 *be the constant such that*

∫ *χ* ∧

*M*

*ωn*−1

(*n* − 1)!

*ωn*

= *c*0 0 *,*

∫

*M n*!

*then the followings are equivalent:*

0

1. *The*√*re exists a unique smooth function ϕ up to a constant such that*

*ωϕ* = *ω*0 + −1*∂∂*¯*ϕ >* 0 *satisﬁes the J-equation*

tr*ωϕ χ* = *c*0;

1. *The*√*re exists a unique smooth function ϕ up to a constant such that*

*ωϕ* = *ω*0 + −1*∂∂*¯*ϕ >* 0 *satisﬁes the J-equation*

*ωn*−1 *ωn*

*χ* ∧ *ϕ* = *c*0 *ϕ* ;

(*n* − 1)! *n*!

1. *There exists a unique smooth function ϕ up to a constant such that ϕ is the critical point of the* J*χ functional;*
2. *The* J*χ functional is coercive, in other words, there exist ǫ*1*.*1 *>* 0 *and another constant C*1*.*2 *such that* J*χ*(*ϕ*) ≥ *ǫ*1*.*1J*ω*0 (*ϕ*) − *C*1*.*2*;*
3. (*M,* [*ω*0]*,* [*χ*]) *is uniformly J-stable, in other words, there exists ǫ*1*.*1 *>* 0 *such that for all K¨ahler test conﬁgurations* (X *,* Ω) *deﬁned as Deﬁnition 2.10 of [*[*19*](#_bookmark37)*], the numerical invariant J*[*χ*](X *,* Ω) *deﬁned as Deﬁnition 6.3 of [*[*19*](#_bookmark37)*] satisﬁes J*[*χ*](X *,* Ω) ≥ *ǫ*1*.*1*J*[*ω*0](X *,* Ω)*;*
4. (*M,* [*ω*0]*,* [*χ*]) *is uniformly slope J-stable, in other words, there exists*

*ǫ*1*.*1 *>* 0 *such that for any subvariety V of M, the degeneration to normal cone*

(X *,* Ω) *deﬁned as Example 2.11 (ii) of [*[*19*](#_bookmark37)*] satisﬁes J*[*χ*](X *,* Ω) ≥ *ǫ*1*.*1*J*[*ω*0](X *,* Ω)*;*

1. *There exists ǫ*1*.*1 *>* 0 *such that*

∫ (*c*0 − (*n* − *p*)*ǫ*1*.*1)*ωp* − *pχ* ∧ *ωp*−1 ≥ 0

0

0

*V*

*for all p-dimensional subvarieties V with p* = 1*,* 2*, ..., n.*

Remark 1.2. *Lejmi and Sz´ekelyhidi’s original conjecture is that the solvability of*

*is equivalent to*

tr*ωϕ χ* = *c*0

∫ *c*0*ωp* − *pχ* ∧ *ωp*−1 *>* 0

0

0

*V*

*for all p-dimensional subvarieties V with p* = 1*,* 2*, ..., n* 1 *[*[*28*](#_bookmark46)*]. However, it seems that our uniform version is more natural from geometric point of view.*

—

Remark 1.3. *It is well known that there exists a constant C*1*.*3 *depending on*

*n such that the* J*ω*0 *functional*

∫ 1 ∫

*ωn*−1 *ωn*

∫ 1 √ ∫

*tωn*−1

( *ϕ*(*ω*0

∧

*tϕ*

0 *M*

(*n* − 1)! − *n*

*tϕ* ))*dt* = (

*n*! 0

—

1 *∂ϕ* ∧ *∂*¯*ϕ* ∧

*M*

*tϕ*

(*n* − 1)!

)*dt*

*and Aubin’s I-functional*

∫ *ϕ*(*ωn* − *ωn*) =

√ ∫

−1

*∂ϕ* ∧ *∂*¯*ϕ* ∧ Σ

*n*

*ωk* ∧ *ωn*−*k*

0 *ϕ*

*M M*

*satisfy*

0 *ϕ*

*k*=0

*C*−1 ∫

1*.*3

*M*

0

*ϕ*

*ϕ*(*ωn* − *ωn*) ≤ J*ω* (*ϕ*) ≤ *C*1*.*3 ∫

*ϕ*(*ωn* − *ωn*)*.*

*For example, Collins and Sz´ekelyhidi used this fact and their Deﬁnition 20 in*

0

*ϕ*

0

*M*

*[*[*13*](#_bookmark31)*] replaced* J*ω* (*ϕ*) *by* ∫ *ϕ*(*ωn* −*ωn*) *in the deﬁnition of the coerciveness which*

0

*M*

0

*ϕ*

*was called “properness” in [*[*13*](#_bookmark31)*]. By (3) of [*[*2*](#_bookmark20)*], Aubin’s I-functional can also be*

*replaced by Aubin’s J-functional in the deﬁnition of coerciveness. Accordingly, in the deﬁnition of uniform stability, the numerical invariant J*[*ω*0](X *,* Ω) *can be*

*replaced by the minimum norm of* (X *,* Ω) *deﬁned as Deﬁnition 2.18 of [*[*19*](#_bookmark37)*]. By*

*(62) of [*[*16*](#_bookmark34)*], Aubin’s J-functional can be further replaced by the d*1 *distance in the*

*deﬁnition of the coerciveness when ϕ is normalized such that the Aubin-Mabuchi energy of ϕ is 0.*

Remark 1.4. *By Proposition 2 of [*[*5*](#_bookmark23)*], if the solution to the J-equation exists, it is unique up to a constant. It is easy to see that (1) and (2) are equivalent. The equivalence between (2) and (3) follows from the formula*

*d*J*χ dt*

*∂ϕ*

=

∫

*M ∂t*

(*χ* ∧

*ωn*−1

(*n* − 1)!

— *c*0

*ωn*

*ϕ* )*. n*!

*By Proposition 21 and Proposition 22 of [*[*13*](#_bookmark31)*] and Remark* [*1.3*](#_bookmark1)*, (1) and (4)*

*ϕ*

*are equivalent. By Corollary 6.5 of [*[*19*](#_bookmark37)*], (4) implies (5). It is trivial that (5) implies (6). By [*[*28*](#_bookmark46)*], (6) implies (7) in the projective case if ǫ*1*.*1 *is replaced by 0. However, it is easy to see that it is also true in non-projective case and for positive ǫ*1*.*1*. Thus, we only need to prove that (7) implies (1) in Theorem*

* 1. *Remark that there is a simpler proof that (1) implies (7). Let χ* = *δij and*

*ωϕ* = *λiδij, then for any c >* 0*, the condition*

*cωp* − *pχ* ∧ *ωp*−1 ≥ 0

*ϕ ϕ*

*is equivalent to*

*p*

Σ ≤

1 *c*

*j*=1 *λij*

*for all distinct p numbers* {*i*1*, i*2*, ..., ip*} ⊂ {1*,* 2*, ..., n*}*. So* tr*ωϕ χ* = *c*0 *as well as the upper bounds of λi imply that for small enough ǫ*1*.*1 *>* 0*,*

(*c*0 − (*n* − *p*)*ǫ*1*.*1)*ωp* − *pχ* ∧ *ωp*−1 ≥ 0

*ϕ*

*ϕ*

*for all p* = 1*,* 2*, ..., n. (4) follows from the fact that*

∫ (*c*0 − (*n* − *p*)*ǫ*1*.*1)*ωp* − *pχ* ∧ *ωp*−1 = ∫

*V*

0

0

*V*

*ϕ*

*ϕ*

(*c*0 − (*n* − *p*)*ǫ*1*.*1)*ωp* − *pχ* ∧ *ωp*−1*.*

When *c*1(*M* ) *<* 0, we can choose *χ* as a K¨ahler form in −*c*1(*M* ). Since *x* log *x*

is bounded from below for any *x* ∈ R, the entropy ∫*M*

J

*ωn*

log( *ϕ* )

*ωn*

0

*ωn*

*ϕ* is also bounded

*n*!

from below. So the coerciveness of *χ* functional implies the coerciveness of K-

energy. This observation appeared as Remark 2 of [[5](#_bookmark23)]. Using this observation, as a corollary of Theorem 1.3 of [[7](#_bookmark25)] and Theorem [1.1](#_bookmark0), we can find many cscK metrics with *c*1(*M* ) *<* 0.

Corollary 1.5. *If c*1(*M* ) *<* 0*, and ǫ*1*.*1 *>* 0*, then for any K¨ahler class* [*ω*0] *such that*

∫ −*n*[*c*1(*M* )] · [*ω*0]*n*−1

((

[*ω* ]*n* − (*n* − *p*)*ǫ*1*.*1)*ω*0 − *pω*0 ∧ (−*c*1(*M* )) ≥ 0

*V*

0

*p p*−1

*for all p-dimensional subvarieties V with p* = 1*,* 2*, ..., n, there exists a cscK metric in* [*ω*0]*.*

Remark 1.6. *If there exists ωϕ* [*ω*0] *such that* Ric(*ωϕ*) *<* 0 *and ωϕ has constant scalar curvature, then the condition above is also necessary.*

∈

Besides the appearances in the moment map picture and the study of the cscK problem, J-equation also arises from the study of mirror symmetry. In fact, using the following observation of Collins-Jacob-Yau [[12](#_bookmark30)]

*n n*

lim Σ *k*( *π* − arctan(*kλ* )) = Σ 1 *,*

*i*

*k*→∞ *i*=1 2

*i*

*λ*

*i*=1

the J-equation is exactly the limit of the deformed Hermitian-Yang-Mills equa- tion

Σ

*n*

arctan *λi* = constant*,*

*i*=1

where *λi* are the eigenvalues of *ωϕ* with respect to *χ*. It plays an important role in the study of mirror symmetry [[35](#_bookmark53), [29](#_bookmark47)].

Motivated by the J-equation, Collins-Jacob-Yau [[12](#_bookmark30)] conjectured that the solvability of the deformed Hermitian-Yang-Mills equation is also equivalent to a notion of stability. In this paper, we prove the uniform version of their conjecture when the angle is in ( *nπ* − *π , nπ* ):

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Theorem 1.7. *Fix a K¨ahler manifold Mn with K¨ahler metrics χ and ω*0*. Let*

*θ*ˆ ∈ ( *nπ* − *π , nπ* ) *be a constant. Then there exists a unique smooth function ϕ*

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*up to a constant such that*

*n*

arctan *λi* = *θ*ˆ

Σ

*i*=1

*for eigenvalues λi of ωϕ* = *ω*0 + √−1*∂∂*¯*ϕ >* 0 *with respect to χ if and only if*

*there exists a constant ǫ*1*.*1 *>* 0 *and for all p-dimensional subvarieties V with*

*p* = 1*,* 2*, ..., n (V can be chosen as M), there exist smooth functions θV* (*t*) *from*

[1*,* ∞) *to* [*θ*ˆ− (*n*−*p*)*π* + (*n* − *p*)*ǫ*1*.*1*, pπ* ) *such that for all t* ∈ [1*,* ∞)*,*

2 2

∫ (*χ* + √−1*tω* )*p* /= 0*, θ* (*t*) = arg(∫ (*χ* + √−1*tω* )*p*)*,* lim *θ* (*t*) = *pπ .*

0

*V*

0

*t*→∞

*V*

2

*V*

*V*

*Moreover, when V* = *M, it is required that θM* (1) = *θ*ˆ*.*

Remark 1.8. *We only study the case when θ*ˆ ∈ ( *nπ* − *π , nπ* ) *in this paper. So it*

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*is natural to assume that* [*ω*0] *is a K¨ahler class. However, usually we need extra*

*conditions in addition to* [*ω*0] *being K¨ahler to make sure* ∫*V* (*χ* + √−1*tω*0) *is*

*p*

*not 0 so that θV* (*t*) *is well deﬁned. When p* = 1*,* 2*, θV* (*t*) *is always well deﬁned and increasing for t* ∈ (−∞*,* ∞) *without any extra assumption. In addition, θV* (0) = 0*. When p* = 3*,*

∫ (*χ* + √−1*tω*0)3 = ∫

*V*

*V*

*χ* ∧ *ω* + √−1*t*(3 ∫

*V*

0

0

*V*

*χ*3 − 3*t*2 ∫

2 *χ*2 ∧ *ω*0 − *t*2 ∫

*V*

*ω*3)*.*

*So if the inequality*

∫

(

*V*

*χ*3)(

*V*

∫

3 *χ* ∧ *ω*2)(∫

*V*

*V*

*χ*2 ∧ *ω*0)

*in Proposition 3.3 of [*[*14*](#_bookmark32)*] holds, then θV* (*t*) *is well deﬁned for t* ∈ (−∞*,* ∞)*. Moreover, if θV* (1) *> π, then θV* (*t*) *is increasing for t* ∈ [1*,* ∞)*. In addition, θV* (0) = 0*. So the choice of θV* (1) *in this paper is the same as the choice of θV* (1) *in Proposition 8.4 of [*[*15*](#_bookmark33)*]. In higher dimensions, more inequalities are*

*ω* ) *<* 9(∫

0

0

*involved.*

Remark 1.9. *Collins-Jacob-Yau conjectured that θV* (1) *> θ*ˆ − (*n*−*p*)*π for all*

2

*p* = 1*,* 2*, ..., n* 1 *is equivalent to the solvability of the deformed Hermitian-*

—

*Yang-Mills equation [*[*12*](#_bookmark30)*]. However, it seems that our uniform version is more natural because Deﬁnition 8.10 (2) of [*[*15*](#_bookmark33)*] also assumed the uniform positive lower bound.*

Remark 1.10. *By Theorem 1.1 of [*[*26*](#_bookmark44)*], the solution to the deformed Hermitian- Yang-Mills equation is unique up to a constant if it exists. The “only if” part of Theorem* [*1.7*](#_bookmark3) *is a combination of Proposition 3.1 of [*[*12*](#_bookmark30)*] and Remark* [*1.4*](#_bookmark2)*. So we only need to prove the “if” part of Theorem* [*1.7*](#_bookmark3)*.*

Theorem [1.7](#_bookmark3) will be proved in Section 5 using the same method of the proof of Theorem [1.1](#_bookmark0).

Instead of Theorem [1.1](#_bookmark0), we will prove the following stronger statement by induction:

Theorem 1.11. *Fix a K¨ahler manifold Mn with K¨ahler metrics χ and ω*0*. Let*

*c >* 0 *be a constant and f >* − 1 ( 1 )*n*−1 *be a smooth function satisfying*

∫

∫

*χn*

*f*

*M n*!

2*n c*

*n*

*ω*

= *c* 0

*M n*!

— ∫*M*

*ωn*−1

0

(*n* − 1)!

≥ 0*,*

*then there exists ωϕ* = *ω*0 + √−1*∂∂*¯*ϕ >* 0 *satisfying the equation*

*χ* ∧

*χn*

tr*ωϕ χ* + *f n* = *c*

*ω*

*ϕ*

*and the inequality*

*cωn*−1 − (*n* − 1)*χ* ∧ *ωn*−2 *>* 0

*ϕ ϕ*

*if there exists ǫ*1*.*1 *>* 0 *such that*

∫ (*c* − (*n* − *p*)*ǫ*1*.*1)*ωp* − *pωp*−1 ∧ *χ* ≥ 0

0

0

*V*

*for all p-dimensional subvarieties V with p* = 1*,* 2*, ..., n.*

Remark 1.12. *By Remark* [*1.4*](#_bookmark2)*, Theorem* [*1.1*](#_bookmark0) *is a corollary of Theorem* [*1.11*](#_bookmark4) *by choosing f* = 0*.*

Remark 1.13. *When n* = 1*, Theorem* [*1.11*](#_bookmark4) *is trivial. When n* = 2*, Theorem*

[*1.11*](#_bookmark4) *is the statement that the Demailly-Paun’s characterization [*[*18*](#_bookmark36)*] for* [*cω*0 *χ*]

—

*being K¨ahler implies the solvability of the Calabi conjecture*

(*cωϕ* − *χ*)2 = (*cf* + 1)*χ*2

*by Yau [*[*40*](#_bookmark58)*]. In the toric case when f is a non-negative constant, Theorem* [*1.11*](#_bookmark4) *was proved by Collins and Sz´ekelyhidi [*[*13*](#_bookmark31)*].*

There are several steps to prove Theorem [1.11](#_bookmark4). Step 1: Prove the following:

Theorem 1.14. *Fix a K¨ahler manifold Mn with K¨ahler metrics χ and ω*0*. Let*

*c >* 0 *be a constant and f >* − 1 ( 1 )*n*—1 *be a smooth function satisfying*

∫

∫

*χn*

*f*

*M n*!

2*n c*

*n*

*ω*

= *c* 0

*M n*!

— ∫*M*

*ωn*—1

0

(*n* − 1)!

≥ 0*,*

*then there exists ωϕ* = *ω*0 + √−1*∂∂*¯*ϕ satisfying the equation*

*χ* ∧

*χn*

tr*ωϕ χ* + *f n* = *c*

*ω*

*ϕ*

*and the inequality*

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

*ϕ ϕ*

*if*

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0*.*

0

0

We will use the continuity method to prove Theorem [1.14](#_bookmark6). The details will be provided in Section 2.

Remark 1.15. *Let χ* = *δij and ωϕ* = *λiδij, then the equation*

*χn*

tr*ωϕ χ* + *f n* = *c*

*ω*

*ϕ*

*is equivalent to*

*n*

Σ

1

*λ*

*i*

*i*=1

*f*

+ *n*

Q

*i*=1 *λi*

= *c.*

Remark 1.16. *Suppose*  1 *c for all k* = 1*,* 2*, ..., n and*

Σ ≤

*i*/=*k λi*

*n*

Σ

1

*λ*

*i*

*i*=1

*f*

+ *n*

Q

*i*=1 *λi*

= *c,*

*then as long as f >*  1 ( 1 )*n*—1*, it is easy to see that*  1 *< c.*

— Σ

2*n*

*c*

*λi*

*i*/=*k*

Remark 1.17. *When n* = 2*, Theorem* [*1.14*](#_bookmark6) *is the Calabi conjecture solved by Yau [*[*40*](#_bookmark58)*]. When f* = 0*, Theorem* [*1.14*](#_bookmark6) *is a speical case of Song and Weinkove’s*

*ωn*

*result [*[*34*](#_bookmark52)*]. When f is a constant times*  0 *, Theorem* [*1.14*](#_bookmark6) *was proved by Zheng*

*χn*

*[*[*42*](#_bookmark60)*].*

Step 2: Prove the following:

Theorem 1.18. *Fix a K¨ahler manifold Mn with K¨ahler metrics χ and ω*0*. Suppose that for all t >* 0*, there exist a constant ct >* 0 *and a smooth K¨ahler form ωt* ∈ [(1 + *t*)*ω*0] *satisfying*

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

*t t*

*and*

*χn*

tr*ωt χ* + *ct n* = *c.*

*ω*

*t*

*Then there exist a constant ǫ*1*.*4 *>* 0 *and a current ω*1*.*5 ∈ [*ω*0 − *ǫ*1*.*4*χ*] *such that*

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 ≥ 0

1*.*5

*in the sense of Deﬁnition* [*3.3*](#_bookmark11)*.*

1*.*5

Remark 1.19. *In general we can only take the wedge product of ωϕ when ϕ is in C*2*. Bedford-Taylor [*[*3*](#_bookmark21)*] proved that it can also be deﬁned when ϕ is in L*∞*.*

*In our case, ϕ is unbounded, so we have to ﬁgure out the correct deﬁnition of*

*cωp* − *pχ* ∧ *ωp*—1 ≥ 0

*ϕ*

*ϕ*

*for unbounded ϕ and p* = 1*,* 2*, ..., n. This will be done in Deﬁnition* [*3.3*](#_bookmark11)*.*

Remark 1.20. *When n* = 2*, it is same as Theorem 2.12 of [*[*18*](#_bookmark36)*].*

Now let us sketch the proof here. It is analogous to the proof of Theorem

2.12 of Demailly-Paun’s paper [[18](#_bookmark36)]. Consider the diagonal ∆ inside the product manifold *M* × *M* . Cover it by finitely many open coordinate balls *Bj*. Since ∆ is non-singular, we can assume that on *Bj*, *gj,k*, *k* = 1*,* 2*, ...,* 2*n* are coordinates and ∆ = {*gj,k* =Σ0*,* 1 ≤ *k* ≤ *n*}. Assume that *θj* are smooth functions supported

*j*

*n*

in *Bj* such that

*θ*2 = 1 in a neighborhood of ∆. For *ǫ*1*.*6 *>* 0, define

*ψǫ* = log(Σ *θ*2 Σ |*gj,k*|2 + *ǫ*2 )*.*

Define

1*.*6

*j*

*j k*=1

1*.*6

and

*χM*×*M* = *π*1∗*χ* + *π*2∗*χ,*

√ ¯

Let

*χM*×*M,ǫ*1*.*6*,ǫ*1*.*7 = *χM*×*M* + *ǫ*1*.*7

−1*∂∂ψǫ*1*.*6 *.*

*ft,ǫ*

1*.*6

*,ǫ*1*.*7

*χ*2*n ct*

= 1 +

*M*×*M,ǫ*1*.*6*,ǫ*1*.*7 −

*χ*2*n cn*

*M*×*M*

*χ*2*n*

*>* 1*,*

*M*×*M,ǫ*1*.*6 *,ǫ*1*.*7 −

*χ*2*n*

*M*×*M*

then by Lemma 2.1 (ii) of [[18](#_bookmark36)], there exists *ǫ*1*.*7 *>* 0 such that for *ǫ*1*.*6 small enough,

*χ*2*n*

*M*×*M,ǫ*1*.*6 *,ǫ*1*.*7 − −

1 *>*

*χ*2*n*

*M*×*M*

1 1

(

4*n* (*n* + 1)*c*

)2*n*—1*.*

Now we consider *ω*0*,M*×*M,t* = *π*1∗*ωt* + 1 *π*2∗*χ*. By Theorem [1.14](#_bookmark6), there exists

*c*

*ωt,ǫ*1*.*6*,ǫ*1*.*7 ∈ [*ω*0*,M*×*M,t*] such that

*χ*2*n*

tr*ω*

*t,ǫ*1*.*6 *,ǫ*1*.*7

*χM*×*M* + *ft,ǫ*

1*.*6*,ǫ*

*M*×*M* = (*n* + 1)*c.*

*t,ǫ*1*.*6*,ǫ*1*.*7

*ω*

1*.*7 2*n*

Now define *ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7 by

*cn*—1 *n* ∗

∫

*ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7 =

*M*

*nχn* (*π*1)∗(*ωt,ǫ*1*.*6*,ǫ*1*.*7 ∧ *π*2 *χ*)*.*

Fix *ǫ*1*.*7 and let *t* and *ǫ*1*.*6 converge to 0. For small enough *ǫ*1*.*4, let *ω*1*.*5 be the weak limit of *ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7 − *ǫ*1*.*4*χ*. Then we shall expect

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 ≥ 0

1*.*5

1*.*5

in the sense of Definition [3.3](#_bookmark11). The details will be provided in Section 3.

Step 3: Consider the set *I* of *t* ≥ 0 such that there exist a constant *ct* ≥ 0 and a smooth K¨ahler form *ωt* ∈ [(1 + *t*)*ω*0] satisfying

(*cωt* − (*n* − 1)*χ*) ∧ *ωn*—2 *>* 0

*t*

and

*χn*

tr*ωt χ* + *ct n* = *c.*

*ω*

*t*

By Theorem [1.14](#_bookmark6), it suffices to show that 0 *I*. When *t* is large enough, the condition of Theorem [1.14](#_bookmark6) is satisfied. So *t I*. It is easy to see that if *t I*, then for nearby *t*, the condition of Theorem [1.14](#_bookmark6) is also satisfied. So *I* is open.

∈ ∈

∈

Still by Theorem [1.14](#_bookmark6), as long as *t* ∈ *I*, then for all *t*′ ≥ *t*, *t*′ ∈ *I*. Thus, in order to prove the closedness of *I*, it suffices to show that if *t* ∈ *I* for all *t > t*0, then *t*0 ∈ *I*. After replacing (1 + *t*0)*ω*0 by *ω*0, we can without loss of generality assume that *t*0 = 0. In particular we can apply Theorem [1.18](#_bookmark8) to get *ω*1*.*5.

Let *ν*(*x*) be the Lelong number of *ω*1*.*5 at *x*. For *ǫ*1*.*8 *>* 0 to be determined, let *Y* be the set

*Y* = {*x* : *ν*(*x*) ≥ *ǫ*1*.*8}*.*

By the result of Siu [[32](#_bookmark51)], *Y* is a subvariety with dimension *p < n*. Assume that

*Y* is smooth, then by induction hypothesis, we can apply Th√eorem [1.11](#_bookmark4) to *Y* to

obtain a smooth function *ϕ*1*.*9 on *Y* such that *ω*1*.*9 = *ω*0|*Y* + −1*∂∂*¯*ϕ*1*.*9 satisfies

(*c* − (*n* − *p*)*ǫ*1*.*1)*ωp* − *pχ*|*Y* ∧ *ωp*—1 ≥ 0

1*.*9

1*.*9

on *Y* . Then for large enough *C*1*.*10,

*ω*1*.*11 = *ω*0 + √−1*∂∂*¯(Proj∗*Y ϕ*1*.*9 + *C*1*.*10*dχ*(*., Y* )2)

satisfies

(*c* − *n* − *pǫ*

)*ωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

1*.*1

1*.*11

1*.*11

on a tubular neighborhood of *Y* , where Proj*Y* means the projection to *Y* . By a

2

generalization of the result of Bl-ocki and Kol-odziej [[4](#_bookmark22)], w√e can glue the smooth-

ing of *ω*1*.*5 outside *Y* and *ω*1*.*11 near *Y* into *ω*1*.*12 = *ω*0 + −1*∂∂*¯*ϕ*1*.*12 satisfying

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

1*.*12

1*.*12

on *M* . Then we are done by Theorem [1.14](#_bookmark6). In general, *Y* is singular and we need to use Hironaka’s desingularization theorem to resolve it. The details will be provided in Section 4.

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# The analysis part

In this section, we use the continuity method twice to prove Theorem [1.14](#_bookmark6). First of all, for *t* ∈ [0*,* 1], define *χt* by

*c*

*χt* = *tχ* + (1 − *t*) *nω*0

and define *ft* ≥ 0 as the constant such that

∫

∫

∫

∫

∫

*n*

*χ*

*ft t* = *c*

*M n*! *M*

*n*

*n*! −

*ω*

0

*M*

*χt* ∧

*ωn*—1

(*n* − 1)!

= *t*(*c*

*M*

*n*

*n*! − *χ* ∧

*ω*

0

*M*

*ωn*—1

(*n* − 1)!

) ≥ 0*.*

Now we co√nsider the set *I* consisting of all *t* ∈ [0*,* 1] such that there exists

0

0

*ωt* = *ω*0 +

−1*∂∂*¯*ϕt >* 0 for smooth *ϕt* satisfying

*χn*

*t*

tr*ω χt* + *ft* = *c*

*t n*

*ω*

*t*

and

*cωn*—1 − (*n* − 1)*χt* ∧ *ωn*—2 *>* 0*.*

*t*

*t*

Then it is easy to see that 0 ∈ *I*. Remark that the equation is the same as

*ωn ωn*—1 *χn*

*c*  *t* − *χt* ∧ *t* = *ft*  *t .*

*n*! (*n* − 1)! *n*!

The linearization is

1 *n*—1

*n*—2

√ ¯*∂ϕt*

*∂ χn*

*∂χt*

*ωn*—1

(*n* − 1)!

*t*

(*cωt* − (*n* − 1)*χt* ∧ *ωt* ) ∧

−1*∂∂ ∂t*

=  (*ft*  *t* ) +

*∂t n*! *∂t*

∧ (*n* − 1)! *.*

Assume that *t I*, then the left hand side is a second order elliptic equation on

∈

*∂ϕt* . On the other hands, the integrability condition implies that the integral of the right hand side is 0. By standard elliptic theory and the implicit function

*∂t*

theorem, *I* is open when we replace the smoothness assumption of *ϕ* by *C*100*,α*. However, standard elliptic regularity theory implies that any *C*100*,α* solution is automatically smooth. So *I* is in fact open.

Assume that we are able to show the closedness of *I*, then we have proved Theorem [1.14](#_bookmark6) for *f* replaced by *f*1. We can use another continuity path by fixing *χ* and *ω*0 but choosing *f*ˆ*t* = *tf*1 + (1 − *t*)*f* . However, it is the same as before

except that *f*ˆ*t >* − 1 ( 1 )*n*—1 is a function instead of a constant. Thus, we only

2*n*

*c*

need to prove the *a priori* estimate of *ωt* by assuming that *ft >* − 1 ( 1 )*n*—1 is a

2*n*

*c*

function. We start from the following proposition which is analogous to Lemma

3.1 of Song-Weinkove’s paper [[34](#_bookmark52)]:

Proposition 2.1. *Assume that t* ∈ *I and ωt* = *ω*0 + √−1*∂∂*¯*ϕt is the corre- sponding solution, then there exist constants C*2*.*1 *and C*2*.*2 *depending only on c, ω*0*, the C*∞*-norm of χt with respect to ω*0*, the C*2*-norm of* ||*ft*|| *with respect to ω*0 *such that*

tr*χ ωt* ≤ *C*2*.*2*eC*2*.*1(*ϕt*—inf *ϕt*)*.*

*t*

*Proof.* In local coordinates, *χt* = √−1*χ* ¯*dzi* ∧ *dz*¯*j* and *ωt* = √−1*g* ¯*dzi* ∧ *dz*¯*j*.

*ij*

*ij*

Fix any point *x*, choose a *χt*-normal coordinate such that *χi*¯*j* = *δi*¯*j* , *χi*¯*j,k* = *χi*¯*j,k*¯ = 0 and *gi*¯*j* = *λiδi*¯*j* at *x*, where the derivatives are all ordinary derivatives. Then the equation is

Σ *gi*¯*j χ*

+ *f* det *χαβ*¯ = *c.*

Define an operator ∆˜ by

*i,j*

*i*¯*j*

*t* det *g*

*αβ*¯

*i*¯*l*

*g*

*χk*¯*j*

*t*

*,i*¯*l*

∆˜ *u* = Σ(Σ

*g*

*i,l*

*j,k*

*i*¯*j*

*kl*¯

+ *f g* det *χαβ*¯ )*u ,*

*αβ*¯

then it is easy to see that ∆˜ is independent of the choice of local coordinates.

det *g*

At *x*,

∆˜ *u* = Σ( 1 + *f* 1 Q 1 )*u .*

2

*i*

*t λi*

*n*

*α*=1

*,ii*

¯

*λ*

*λα*

*i*

Since 1 *< c* and *ft >* − 1 ( 1 )*n*—1, it is easy to see that

*λα* 2*n c*

1 1 1

*λ*2 + *ft λ* Q*n λ >* 0

*i*

*i*

*α*=1

*α*

for all *i*. So ∆˜ is a second order elliptic operator.

Now we compute ∆˜ (log tr*χt ωt*) = ∆˜ (log(Σ

*ij*

*i,j*

*χi*¯*j g* ¯)). It equals to

Σ Σ (*g* ¯ ¯ + (*χii*¯)

*i ii,kk ,kk*

*i ii,k*

¯*λi*)

| Σ *g* ¯ | 1 1 1

( Σ *λ*

2

— (Σ *λ* )2 )( *λ*2 + *ft λ* Q*n λ* ))

*k i i*

at *x*.

*i i k*

*k α*=1 *α*

Now we differentiate the equation

Σ *gi*¯*j χ*

*i*¯*j*

*i,j*

+ *f* det *χαβ*¯ = *c,*

*αβ*¯

then

*t* det *g*

*t,k*

Σ *gi*¯*j χ*

*i,j*

— Σ *gi*¯*bg*

*i,j,a,b*

*ga*¯*j χ*

+ det *χαβ*¯ (*f*

*i*¯*j*

*αβ*¯

+*ft*

Σ(*χ χ* ¯

*i*¯*j,k*

*a*¯*b,k*

*i*¯*j*

*i*¯*j*

*ij,k*

det *g*

*i,j*

— *g gi*¯*j,k*

)) = 0*.*

So

1 Σ Σ 1

Σ 1

Σ 1 1 2 2

Σ *λ* (

*i*

*i*

*k*

*i*

*i*

*λ χi*¯*i,kk*¯ −

*λ*2 *gi*¯*i,kk*¯ +

*λ*2 *λ*

(|*gi*¯*j,k*|

+ |*gi*¯*j,k*¯| )

+Q 1 (*f* (| Σ( 1 *g* ¯ )|2 + Σ *χ* ¯

*i*

*i*

*λα*

*t*

*λi*

*ii,k*

*ii,kk*

*α*

*i*

*i*

¯ + Σ

*i,j*

1

|*g* ¯

*i,j*

*i*

*j*

*λiλj*

|2 − Σ 1 *g* ¯ ¯)

¯

*i*

+*f* ¯ − *f*

¯ Σ( 1 *g*

) − *f*

Σ( 1 *g* ¯))) = 0

at *x*.

*t,kk*

*t,k*

*i*

*λi ii,k*

*t,k*

*i*

¯

*ij,k*

*λi*

*ii,kk*

*λi ii,k*

By K¨ahler condition, *gi*¯*i,kk*¯ = *gkk*¯*,i*¯*i*, *gi*¯*j,k* = *gk*¯*j,i* and *gi*¯*j,k*¯ = *gik*¯*,*¯*j* . Using the

*i*¯*i*

1

bounds that |*χ* ¯ ¯| + |(*χ* ) ¯| + |*ft,k*| + |*f* ¯| + |*f* ¯| + |*ft*| + *< C*2*.*3 for all

*ii,kk*

*,kk*

*t,k*

*t,kk*

*λi*

*i*, *k*, it is easy to see that by taking the sum of the previous two equations,

˜ Σ | Σ*i gi*¯*i,k*|2 1 1 1

∆(log tr*χωt*) ≥ −*C*2*.*4 − ( (Σ *λ* )2 )( *λ*2 + *ft λ* Q*n λ* )

*k i i k k α*=1 *α*

1 Σ Σ 1 1 2 2 1 Σ 1 2

¯

2

¯

*ij,k*

*t,k*

*ii,k*

+Σ *λ*

*i*

*i*

*k*

*i,j*

*i*

*j*

*α*

*α*

*i*

*i*

*λi*

( *λ*2 *λ*

(|*gi*¯*j,k*|

+ |*gi*¯*j,k*¯| ) + Q

*λ* (*ft*(|

( *λ gi*¯*i,k*)|

+ Σ 1 |*g*

¯

*λiλj*

*t,k*

*λi*

*ii,k*

*i,j*

| ) − *f*

¯ Σ( 1 *g*

*i*

) − *f*

Σ( 1 *g* ¯)))*.*

*i*

Remark that

|Q 1 *f*

*λα*

*α*

¯ Σ( 1 *g* ¯

*i*

1

)| = |Q

*α*

*f* Σ( 1 *g* ¯ ¯)|

*λα*

*t,k*

*λi*

*ii,k*

*i*

= |*f* ¯|| Σ Q 1

*t,k*

*λi*

*ii,k*

*t,k*

*i*

*α*

1 Σ 1

( *g* ¯ )| ≤ *C* | *g* ¯ |

*λ*

*λα*

*λi*

*ii,k*

2*.*5

2

*i*

*ii,k*

*i*

≤ 1 Σ 1 |*g* ¯

2

*λ*

4

| + *C*

Σ 1 ≤ 1 Σ 1 |*g* ¯|2 + *C*

and

3

*i* *i*

Q*ft* Σ 1

*ii,k*

2

2*.*6

*λ* 4

*i* *i*

¯

*λ*

*i*

*nft* Σ 1

3 *ii,k i*

2 1 Σ

2*.*7

1 2

*α λα* |

*i*

—

So

*i*

( *λ gi*¯*i,k*)|

*i*

≤ − Q*α λα*

*λ*2 |*gi*¯*i,k*| ≤ 2

*i* *i*

*i*

*λ*3 |*gi*¯*i,k*¯| *.*

˜ Σ | Σ*i gi*¯*i,k*|2 1 1 1

∆(log tr*χt ωt*) ≥ −*C*2*.*8 − ( (Σ *λ* )2 )( *λ*2 + *ft λ* Q*n λ* )

*k*

*i*

*i*

*k*

*k*

*α*=1

*α*

1 Σ Σ 1 1

Σ

2 *ft*  1 2

¯ ¯

Σ 1 1 |*g* ¯|2 ≥ Σ 1 |*g* ¯|2

+Σ *λ*

( *λ*2 *λ*

|*gij,k*| + Q *λ λ λ* |*gij,k*| )*.*

We have used

¯ ¯

*i i k*

*i,j i j*

*α α i,j i j*

here.

*i*

*i,j*

*λ*2 *λj*

*ij,k*

3 *ii,k*

*i* *i*

*λ*

By Cauchy-Schwarz inequality and the fact that *gi*¯*i,k* = *gk*¯*i,i*,

Σ(| Σ *gi*¯*i,k*

2 1

| )( *λ*2 + *f*

1 1

*t λk* Q*n λα*

)

*k i k*

*α*=1

≤ Σ |*gi*¯*i,k*

||*gj*¯*j,k*

1

|( *λ*2 + *f*

)

1 1

*t λk* Q*n λα*

*i,j,k*

*k α*=1

≤ Σ sΣ |*gi*¯*i,k*

2( 1

*λ*2

|

1 1

+ *f λ* Q*n*

|

*t*

*k*

*α*

*λ* )sΣ |*gj*¯*j,k*

2( 1

*λ*2

1 1

+ *f λ* Q*n λ* )

*t*

*k*

*α*

*i,j k k*

*α*=1 *k*

*k α*=1

= (Σ sΣ |*gi*¯*i,k*

2( 1

*λ*2

|

1 1

+ *f λ* Q*n*

*t*

*k*

))2

*λα*

*i k k*

*α*=1

Σ Σ Σ |*gi*¯*i,k*|2 1 1 1

≤ ( *λi*) *λ* ( *λ*2 + *ft λ* Q*n λ* )

*i*

*i*

*k*

*i*

*k*

*k*

*α*=1

*α*

Σ Σ |*gik*¯*,k*|2 1 1 1

= ( *λi*) *λ* ( *λ*2 + *ft λ* Q*n λ* )

*i*

*i,k*

*k*

*i*

*i*

*α*=1

*α*

Σ Σ |*gi*¯*j,k*|2 1 1 1

≤ ( *λi*) *λ* ( *λ*2 + *ft λ* Q*n λ* )*,*

*i*

*i,j,k*

*j*

*i*

*i*

*α*=1

*α*

so ∆˜ (log tr*χt ωt*) ≥ −*C*2*.*9 at *x*. However, since *x* is arbitrary and ∆˜ is indepen- dent of the local coordinates, we see that ∆˜ (log tr*χt ωt*) *C*2*.*9 on *M* .

≥ −

Choose *ǫ*2*.*10 *<* *c* as a small constant such that

2*n*

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 2*ǫ*2*.*10*ωn*—1*,*

0 0 0

then

*cωn*—1 − (*n* − 1)*χt* ∧ *ωn*—2 *>* 2*ǫ*2*.*10*ωn*—1

0

0

0

by the definition of *χt*. Choose *C*2*.*1 as 2*C*2*.*9 , then at the maximal point of

*ǫ .*

2 10

log tr*χt ωt* − *C*2*.*1*ϕt*,

−∆˜ *ϕ* = − Σ(Σ

*t*

*g*

*g*

*i,l*

*j,k*

*i*¯*j*

*kl*¯

*gi*¯*l* det *χ* ¯

+ *f αβ* )(*g*

*χk*¯*j*

*t*

det *g*

*αβ*¯

0 *ǫ*2*.*10

— *g* ) *< .*

*i*¯*l*

*i*¯*l*

2

*c* − Σ Σ *gi*¯*j gkl*¯*χ* ¯(2*g* ¯ − *g*0 ) *< ǫ ,*

*i,l*

*j,k*

*kj*

*il*

*i*¯*l*

2*.*10

If

by the proof of Lemma 3.1 of [[34](#_bookmark52)], tr*χt ωt* ≤ *C*2*.*11. If

*c* − Σ Σ

0

*g*

*i*¯*l*

2*.*10

*i,l*

*j,k*

*i*¯*j*

*kl*¯

¯

*g*

*χkj* (2*gil*

¯ − *g* ) ≥ *ǫ ,*

then

*i*¯*l*

— Σ *f g* det *χαβ*¯ (*g*

*i*¯*l*

*i*¯*l*

2

*i,j*

*i*¯*j*

2

*t* det *g*

*αβ*¯

*i,l*

— *g*0 ) *<* − *ǫ*2*.*10 + *c* − Σ *gi*¯*j χ*

= − *ǫ*2*.*10 + *f* det *χαβ*¯ *,*

so Q *λ*

*t*

det *g*

*αβ*¯

*i*

*i*

det *χαβ*¯

= det *gαβ*¯ *< C*

. Using the fact that *λ*

*>* 1 , tr

*ω* = Σ *λ* ≤ *C*

is also true. This completes the proof of the proposition.

2*.*12

*i*

*c*

*χt*

*t*

*i*

*i*

2*.*13

By adding a constant if necessary, we can without loss of generality assume that sup*M ϕt* = 0. Then we have the following *C*0 estimate:

Proposition 2.2.

||*ϕt*||*C*0 ≤ *C*2*.*14*.*

*Moreover, C*—1 *χt* ≤ *ωt* ≤ *C*2*.*15*χt.*

2*.*15

*Proof.* Lemma 3.3 and Lemma 3.4 and Proposition 3.5 of [[34](#_bookmark52)] only used the inequality in Proposition [2.1](#_bookmark9). So they are still true in our case.

Proposition 2.3. *I is closed.*

*Proof.* First of all, we want to check the uniform ellipticity and concavity for the Evans-Krylov estimate. The equation is

*i*¯*j*

det *χαβ*¯

−*g χi*¯*j* − *ft* det *g*

*αβ*¯

= −*c.*

View it as a function in terms of *gi*¯*j* , *χi*¯*j* and *ft*, then the partial derivative in the *ga*¯*b* direction is

*i*¯*b*

*a*¯*j*

det *χαβ*¯

*a*¯*b*

At *x*, it equals to

*g g χi*¯*j* + *ft* det *g*

*g .*

*αβ*¯

1 1 *ft*

( *λ*2 + *λ* Q *λ* )*δa*¯*b.*

*a*

*a*

*i*

*i*

It has uniform upper bound and lower bound.

The second order derivative in *ga*¯*b* and *gcd*¯ direction is

*id*¯ *c*¯*b*

*a*¯*j*

*i*¯*b*

*ad*¯ *c*¯*j*

det *χαβ*¯

*a*¯*b*

*cd*¯

det *χαβ*¯

*ad*¯ *c*¯*b*

−*g g g*

*χi*¯*j* − *g g*

*g χi*¯*j* − *ft* det *g*

*g g*

*αβ*¯

— *ft* det *g*

*g g .*

*αβ*¯

At *x*, when taking the product with *wa*¯*bwcd*¯ and summing *a, b, c, d* for any matrix

*wi*¯*j* , we get

Σ 1

2 Σ 1

2 *ft*

Σ *waa*¯ 2

*ft* Σ 1 2

— *λ*2 *λ*

*a,b*

*a*

*b*

|*wa*¯*b*| −

*λ*2*λ*

|*wa*¯*b*|

— Q *λ* (

*λ* ) − Q *λ*

*λ λ* |*wa*¯*b*| *.*

It is easy to see that it is non-positive.

*a,b*

*b*

*a*

*i*

*i*

*a*

*a*

*i*

*i*

*a,b*

*a*

*b*

Thus, if we replace the complex second derivatives by real second derivatives, the uniform ellipticity and concavity for the Evans-Krylov estimate [[23](#_bookmark41), [24](#_bookmark42), [27](#_bookmark45), [38](#_bookmark56)] are satisfied. By checking Evans-Krylov’s estimate carefully, it is easy to see that in our complex case, the estimate

[(*ϕt*)*i*¯*j* ]*Cα* ≤ *C*2*.*16

is still true.

By standard elliptic estimate, ||*ϕt*||*C*101*,α* is bounded. By Arzela-Ascoli the- orem, if *ti t*∞ and *ti I*, then a subsequence of *ϕt* converges to *ϕt*∞ in *C*100*,α*-norm. By Remark [1.16](#_bookmark7),

→ ∈

*cωn*—1 − (*n* − 1)*χt* ∧ *ωn*—2 *>* 0*.*

*t*∞

*t*∞

So by standard elliptic regularity, *ϕt*∞ is smooth. In other words, *t*∞ ∈ *I*. 

# Concentration of mass and its application

In this section, we prove Theorem [1.18](#_bookmark8). However, before that, we need to figure out the correct definition of

when *ω* is only a current.

*cωp* − *pχ* ∧ *ωp*—1 ≥ 0

Recall the following definition of the smoothing:

Definition 3.1. Fix a smooth non-negative function *ρ* supported in [0,1] such that

∫

1

*ρ*(*t*)*t*2*n*—1Vol(*∂B*1(0))*dt* = 1

0

and *ρ* is a positive constant near 0. For any *δ >* 0, the smoothing *ϕδ* is defined by

*ϕ* (*x*) = ∫ *ϕ*(*x* − *y*)*δ*—2*nρ*( *y* )*d*Vol *.*

*δ*

C

*n*

| *δ* |

*y*

We can define the smoothing of a current using simila√r formula. It i√s easy to see

that the smoothing commutes with derivatives. So ( −1*∂∂*¯*ϕ*)*δ* = −1*∂∂*¯(*ϕδ*).

Recall that √ 1*∂∂*¯*ϕ* 0 if and only if √ 1*∂∂*¯*ϕδ* 0 for all *δ >* 0. As an analogy, we can define

— ≥ − ≥

*cωp* − *pχ* ∧ *ωp*—1 ≥ 0

for a closed positive (1,1) current *ω* usi√ng smoothing. Remark that any closed

positive (1,1) current can be written as −1*∂∂*¯ acting on a real function locally.

Definition 3.2. Suppose that *χ* is a K¨ahler form with constant coefficients on an open set *O* ⊂ C*n*. Then we say that

*c*(√−1*∂∂*¯*ϕ*)*p* − *pχ* ∧ (√−1*∂∂*¯*ϕ*)*p*—1 ≥ 0 on *O* if for any *δ >* 0, the smoothing *ϕδ* satisfies

*c*(√−1*∂∂*¯*ϕδ*)*p* − *pχ* ∧ (√−1*∂∂*¯*ϕδ*)*p*—1 ≥ 0

on the set *Oδ* = {*x* : *Bδ*(*x*) ⊂ *O*}.

We can also define it without the constant coefficients assumption.

Definition 3.3. We say that

*cωp* − *pχ* ∧ *ωp*—1 ≥ 0

if on any coordinate chart, for any open subset *O*, as long as *χ* ≥ *χ*0 on *O* for a K¨ahler form *χ*0 with constant coefficients, then

*cωp* − *pχ*0 ∧ *ωp*—1 ≥ 0*.*

Remark 3.4.

√ ¯

√ ¯

*c*( −1*∂∂ϕ*) − *pχ*0 ∧ (

*p*

−1*∂∂ϕ*)*p*—1 ≥ 0

*is a convex property for ϕ. So if ω is smooth, then*

*cωp* − *pχ* ∧ *ωp*—1 ≥ 0

*on O pointwise if and only if it is true on O in the sense of Deﬁnition* [*3.3*](#_bookmark11)*.*

For simplicity, for any positive definite *n* × *n* matrix *A*, we define *PI* (*A*) by

*P* (*A*) = max(Σ 1 ) = max

*I*

*k*

*λj*

*V n*−1⊂C*n*

*V*

*j*/=*k*

(tr(*A*|

)—1)*,*

where *λj* are the eigenvalues of *A*. Then

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 ≥ 0 is equivalent to *Pχ*(*ω*) *c*.

≤

Now we need a lemma:

Lemma 3.5.

*P* (*A* − *CB*—1*C*¯*T* ) + tr(*B*—1) ≤ *P* ( *A C* )

*I*

*I*

*C*¯*T B*

*Proof.* By restricting on the codimension 1 subspaces, it suffices to prove that

*A C* —1

tr((*A* − *CB*—1*C*¯*T* )—1) + tr(*B*—1) ≤ tr(

)*.*

*C*¯*T*

*B*

It is easy to see that

−

*I CB*—1 *A C I O*

*O I C*¯*T B* −*B*—1*C*¯*T I* =

So

*A CB*—1*C*¯*T O*

*O B .*

−

*I O* *A* − *CB*—1*C*¯*T O* —1 *I* −*CB*—1

−*B*—1*C*¯*T I*

=

*O B*

*O* *I*

*A C* —1

*C*¯*T B*

After taking traces, the left hand side equals to

*.*

tr((*A* − *CB*—1*C*¯*T* )—1) + tr(*B*—1) + tr(*B*—1*C*¯*T* (*A* − *CB*—1*C*¯*T* )—1*CB*—1)*.*

Now we start the proof of Theorem [1.18](#_bookmark8). By assumption, for any *t >* 0, there exist *ct >* 0 and *ωt* ∈ [(1 + *t*)*ω*0] satisfying

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

*t t*

and

*χn*

tr*ωt χ* + *ct n* = *c.*

*ω*

*t*

Consider *ω*0*,M*×*M,t* = *π*1∗*ωt* + 1 *π*2∗*χ* and *χM*×*M* = *π*1∗*χ* + *π*2∗*χ*. At each point, diagonalizing them so that *χi*¯*j* = *δi*¯*j* and (*ωt*)*i*¯*j* = *λiδi*¯*j* . Then the eigenvalues on

*c*

the product manifold are *λ*1*, ...λn,* 1 *, ...,* 1 . Their inverses are  1 *, ...,*  1 *, c, ..., c*.

*c c λ*1 *λj*

So the sum of them is at most (*n* + 1)*c* because *ct >* 0. In particular, the sum of (2n-1) distinct elements among them is also at most (*n* + 1)*c*. Define *ft,ǫ*1*.*6*,ǫ*1*.*7 as in Section 1, then there exists *ǫ*1*.*7 *>* 0 such that for *ǫ*1*.*6 small

1 1 2*n*—1

enough, *ft,ǫ ,ǫ >* − ( ) . So we can apply Theorem [1.14](#_bookmark6) to get

1*.*6

1*.*7

4*n*

(*n*+1)*c*

*ωt,ǫ*1*.*6*,ǫ*1*.*7 ∈ [*ω*0*,M*×*M,t*] such that *PχM* ×*M* (*ωt,ǫ*1*.*6*,ǫ*1*.*7 ) *<* (*n* + 1)*c* and

*χ*2*n*

tr*ω*

*t,ǫ*1*.*6 *,ǫ*1*.*7

*χM*×*M* + *ft,ǫ*

1*.*6*,ǫ*

*M*×*M* = (*n* + 1)*c.*

*t,ǫ*1*.*6*,ǫ*1*.*7

*ω*

1*.*7 2*n*

For each point (*x , x* ), we assume that *z*(1)*, ..., z*(1) are the local coordinates

1 2 1 *n*

on *M* × {*x* }, and *z*(2)*, ..., z*(2) are the local coordinates on {*x* } × *M* . Then we

2

1

*n*

1

can express *ωt,ǫ*1*.*6*,ǫ*1*.*7 as

where

*ωt,ǫ*

1*.*6

*,ǫ*1*.*7

(1)

*t,ǫ*1*.*6*,ǫ*1*.*7

= *ω*

*n*

(2)

*t,ǫ*1*.*6*,ǫ*1*.*7

+ *ω*

(1*,*2)

*t,ǫ*1*.*6*,ǫ*1*.*7

+ *ω*

(2*,*1)

*t,ǫ*1*.*6*,ǫ*1*.*7

+ *ω*

*,*

*ω*(1)

= Σ √−1*ω*(1)

*dz*(1) ∧ *dz*¯(1)*,*

*t,ǫ*1*.*6*,ǫ*1*.*7

*i,j*=1 *n*

*t,ǫ*1*.*6*,ǫ*1*.*7*,i*¯*j i j*

*ω*(2)

= Σ √−1*ω*(2)

*dz*(2) ∧ *dz*¯(2)*,*

*t,ǫ*1*.*6*,ǫ*1*.*7

*i,j*=1 *n*

*t,ǫ*1*.*6*,ǫ*1*.*7*,i*¯*j i j*

*ω*(1*,*2)

= Σ √−1*ω*(1*,*2)

*dz*(1) ∧ *dz*¯(2)*,*

*t,ǫ*1*.*6*,ǫ*1*.*7

*i,j*=1

(2*,*1)

*ω*

*t,ǫ*1*.*6*,ǫ*1*.*7

*t,ǫ*1*.*6*,ǫ*1*.*7*,i*¯*j i j*

(1*,*2)

*t,ǫ*1*.*6*,ǫ*1*.*7

= *ω*

*.*

After changing the definition of *z*(2) if necessary, we can assume that

*i*

*n*

and

2 *i* *i*

*i*=1

*π*∗*χ* = √−1 Σ *dz*(2) ∧ *dz*¯(2)

*n*

*ω*(2)

= √−1 Σ *λidz*(2) ∧ *dz*¯(2)

at (*x*1*, x*2).

*t,ǫ*1*.*6*,ǫ*1*.*7

*i* *i*

*i*=1

Now consider *ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7 defined as

*ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7

*cn*—1 *n* ∗

∫

= *nχn* (*π*1)∗(*ωt,ǫ*1*.*6*,ǫ*1*.*7 ∧ *π*2 *χ*)

*M*

*cn*—1

∫

(1)

(2)

*n*—1 ∗

= *nχn* (*π*1)∗(*nωt,ǫ*1*.*6*,ǫ*1*.*7 ∧ (*ωt,ǫ*1*.*6*,ǫ*1*.*7 )

*M*

∧ *π*2 *χ*)

*cn*—1

∫

(1*,*2)

(2*,*1)

(2)

*n*—2 ∗

+ *nχn* (*π*1)∗(*n*(*n* − 1)*ωt,ǫ*1*.*6*,ǫ*1*.*7 ∧ *ωt,ǫ*1*.*6*,ǫ*1*.*7 ∧ (*ωt,ǫ*1*.*6*,ǫ*1*.*7 )

*M*

∧ *π*2 *χ*)*.*

At (*x*1*, x*2),

(1)

*nω*

*t,ǫ*1*.*6*,ǫ*1*.*7

*n*

(2)

*i,j*=1

*t,ǫ*1*.*6*,ǫ*1*.*7*,i*¯*j*

∧ (*ωt,ǫ*1*.*6*,ǫ*1*.*7

)*n*—1 ∧ *π*2∗*χ*

*n*

= Σ √−1*ω*(1)

*i*

*j*

*α*=1

*λα*

*t,ǫ*1*.*6*,ǫ*1*.*7

*dz*(1) ∧ *dz*¯(1) ∧ (Σ 1 )(*ω*(2)

)*n,*

and

(1*,*2)

*n*(*n* − 1)*ω*

*t,ǫ*1*.*6*,ǫ*1*.*7

(2*,*1)

*t,ǫ*1*.*6*,ǫ*1*.*7

∧ *ω*

∧ (*ωt,ǫ*1*.*6 *,ǫ*1*.*7

)*n*—2 ∧ *π*2∗*χ*

*n*

(2)

= − Σ

*t,ǫ*1*.*6*,ǫ*1*.*7*,ik*¯

√−1*ω*(1*,*2)

*ω*(1*,*2)

*dz*(1) ∧ *dz*¯(1) ∧ 1 (Σ 1 )(*ω*(2) )*n*

*n*  *n*

*i,j,k*=1

*t,ǫ*1*.*6*,ǫ*1*.*7*,jk*¯

*i*

*j*

*λk*

*α*/=*k*

*λα*

*t,ǫ*1*.*6*,ǫ*1*.*7

*t,ǫ*1*.*6*,ǫ*1*.*7*,jk*¯

*i*

*j*

*λk*

*α*=1

*λα*

*t,ǫ*1*.*6*,ǫ*1*.*7

≥ − Σ

*i,j,k*=1

*t,ǫ*1*.*6*,ǫ*1*.*7*,ik*¯

√−1*ω*(1*,*2)

*ω*(1*,*2)

*dz*(1) ∧ *dz*¯(1) ∧ 1 (Σ 1 )(*ω*(2)

)*n.*

By Lemma [3.5](#_bookmark12),

*n n*

*λk*

*t,ǫ*1*.*6*,ǫ*1*.*7*,ik*¯

*P* ∗ (√−1 Σ ((*ω*(1)

*t,ǫ*1*.*6*,ǫ*1*.*7*,jk*¯

*i*

*j*

— Σ 1 *ω*(1*,*2)

*ω*(1*,*2)

)*dz*(1) ∧ *dz*¯(1)))

*PχM* ×*M* (*ωt,ǫ*1*.*6*,ǫ*1*.*7 ) tr (2)

≤ − *ω*

*π*1 *χ*

*i,j*=1

*t,ǫ*1*.*6*,ǫ*1*.*7*,i*¯*j*

*k*=1

*t,ǫ*1*.*6 *,ǫ*1*.*7

(*π*2∗*χ*)

(*n* + 1)*c* tr (2)

≤ − *ω*

*t,ǫ*1*.*6 *,ǫ*1*.*7

(*π*2∗*χ*)*.*

Now we view

*cn*—1

∗ (2) *n*

∫*M nχ*

(tr (2)

*t,ǫ*1*.*6 *,ǫ*1*.*7

*n ω*

*π*2 *χ*)(*ωt,ǫ*1*.*6*,ǫ*1*.*7 )

as a measure on {*x*1} × *M* , then it is easy to see that

*cn*—1 ∫

∫*M nχ*

(tr (2)

*t,ǫ*1*.*6 *,ǫ*1*.*7

*π*2 *χ*)(*ωt,ǫ*1*.*6*,ǫ*1*.*7 )

*n*

∗ (2) *n*

*ω*

*cn*—1 ∫

∫

{*x*1}×*M*

=

*M*

*χn*

(*π*2 *χ*) ∧ (*ωt,ǫ*1*.*6*,ǫ*1*.*7 )

∗ (2)

*n*—1

*cn*—1 ∫

∫

{*x*1 }×*M*

*χ n*—1

=

*M χn*

*M*

= 1*.*

*χ* ∧ ( *c* )

By the monotonicity and convexity of *Pχ*,

*π*2 *χ*) (*ωt,ǫ*1*.*6*,ǫ*1*.*7 )

*cn*—1 ∫

*Pχ*(*ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7 ) ≤ (*n* + 1)*c* − ∫ *nχ*

*n*

∗ 2 (2) *n*

*ω*

—1 (∫

*M*

{*x*1}×*M*

(tr (2)

*t,ǫ*1*.*6 *,ǫ*1*.*7

*cn*

{*x*1}×*M*

*ω*(2)

2

*t,ǫ*1*.*6*,ǫ*1*.*7

*t,ǫ*1*.*6 *,ǫ*1*.*7

(tr

*π*∗*χ*)(*ω*(2)

)*n*)2

≤ (*n* + 1)*c* − ∫

*M*

*nχn*

∫{*x*1 }×*M*

(2) *n*

*t,ǫ*1*.*6*,ǫ*1*.*7

(*ω*

)

*ncn*—1 (∫

*M c*

= (*n* + 1)*c* − ∫

∫*M*

*M*

*χn*

( *χ* )*n*—1 ∧ *χ*)2

*c*

( *χ* )*n*

= *c.*

Up to here, we have not used the equation

*χ*2*n*

tr*ω*

*t,ǫ*1*.*6 *,ǫ*1*.*7

*χM*×*M* + *ft,ǫ*

1*.*6*,ǫ*

*M*×*M* = (*n* + 1)*c.*

*t,ǫ*1*.*6*,ǫ*1*.*7

*ω*

1*.*7 2*n*

By the equation, *ω*2*n*

*t,ǫ*1*.*6*,ǫ*1*.*7

*ft,ǫ*1*.*6 *,ǫ*1*.*7 2*n*

. So as in Proposition 2.6 of [[18](#_bookmark36)],

it is easy to see that for any weak limit Θ of *ωn* when *t* and *ǫ*1*.*6 converging

*t,ǫ*1*.*6*,ǫ*1*.*7

(*n*+1)*c*

to 0, 1∆Θ = *ǫ*3*.*1[∆] for a constant *ǫ*3*.*1 *>* 0. Let ∆*ǫ*3*.*2 be the *ǫ*3*.*2 neighborhood of ∆ with respect to *χM*×*M* . Then for any *δ >* 0, for any small enough *ǫ*3*.*2 and *ǫ*3*.*3, the smoothing of

≥ *χ*

*M*×*M*

*cn*—1 ∫

∫*M*

({*x*1}×*M*)∩∆*ǫ*3*.*2

*ωt,ǫ*1*.*6*,ǫ*1*.*7 ∧ *π*2 *χ* − ∫

*nχn ǫ*3*.*1*χ*

*nχn*

*n* ∗ *cn*—1

*M*

is at least *ǫ*3*.*3*χ* for small enough *t* and *ǫ*1*.*6.

Similarly, locally for any *n* + 1 dimensional subvariety *V* containing ∆, for

—

any weak limit Θ′ of *ωn*—1 , 1*V* Θ′ = *ǫ*3*.*4[*V* ] for *ǫ*3*.*4 ≥ 0. Since the dimension

*t,ǫ*1*.*6*,ǫ*1*.*7

of ∆ is strictly smaller then *V* , for any fixed smoothing function, for any *ǫ*3*.*3 *>* 0,

there exists *ǫ*3*.*2 *>* 0 such that the smoothing of

*cn*—1 ∫

∫

({*x*1}×*M*)∩∆*ǫ*3*.*2

*π*1 *χ*((*ωt,ǫ*1*.*6 *,ǫ*1*.*7 )

*M*

*nχn*

∗ (2)

*n*—1 ∗

∧ *π*2 *χ*)

is at most *ǫ*3*.*3*χ* for small enough *t* and *ǫ*1*.*6.

Now let *ǫ*3*.*5 be an arbitrary small positive number. Then we can choose *ǫ*3*.*3

such that *ǫ*3*.*3 + *ǫ*

*<* 1 , *cn*−1 *ǫ*

. Then we choose the number *ǫ*

depending

*ǫ*3*.*5

3*.*3

2 *M nχn*

3*.*1

3*.*2

on *ǫ*3*.*3. For any K¨ahler form *ω* restricted to the first *n* coordinates of *M* × *M* ,

after choosing a good coordinate, assume that *π*1∗*χ* = *δi*¯*j* and *ωi*¯*j* = *λiδi*¯*j* . We

define the truncation *Tπ*∗*χ* (*ω*) by (*Tπ*∗*χ* (*ω*))*i*¯*j* = min{*λi,*  1 }*δi*¯*j* . Now consider

∗

( *π*1 *χ* )

the truncation *ω ǫ*3*.*5

1

*ǫ*3*.*5

1

*ǫ*3*.*5

defined as

*ǫ*3*.*5

1*,t,ǫ*1*.*6*,ǫ*1*.*7

*cn*—1

(*π*1)∗(*T* (*ω*˜*t,ǫ*

(1)

(2)

*n*—1 ∗

∫*M nχn*

*π*∗*χ*

1

*ǫ*3*.*5

1*.*6

*,ǫ*1*.*7

) ∧ (*ωt,ǫ*

1*.*6

*,ǫ*1*.*7 )

∧ *π*2 *χ*)*,*

where the (1,1)-form *ω*˜(1)

*t,ǫ*1*.*6*,ǫ*1*.*7

is defined by

(1)

*nω*

*t,ǫ*1*.*6*,ǫ*1*.*7

∧ (*ωt,ǫ*1*.*6*,ǫ*1*.*7

)*n*—1 ∧ *π*2∗*χ*

(2)

(1*,*2)

(2)

+ *n*(*n* − 1)*ω*

*t,ǫ*1*.*6*,ǫ*1*.*7

(2)

(2*,*1)

*t,ǫ*1*.*6*,ǫ*1*.*7

∧ *ω*

∧ (*ωt,ǫ*1*.*6*,ǫ*1*.*7

)*n*—2 ∧ *π*2∗*χ*

(1)

= *ω*˜

*t,ǫ*1*.*6*,ǫ*1*.*7

The smoothing of

∧ (*ωt,ǫ*1*.*6*,ǫ*1*.*7

)*n*—1 ∧ *π*2∗*χ.*

∗

( *π*1 *χ* )

*cn*—1

*ǫ*3*.*5

*ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7 − *ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7 − ∫ *nχn ǫ*3*.*1*χ*

is at least − *ǫ*3*.*3 *χ* − *ǫ*3*.*3*χ* for small enough *t* and *ǫ*1*.*6. In fact, the sum of the first two terms is non-negative outside ∆3*.*2, the sum of the first and third term inside ∆3*.*2 is at least −*ǫ*3*.*3*χ* and the second term inside ∆3*.*2 is at least − *ǫ*3*.*3 *χ*.

*ǫ*3*.*5

*ǫ*3*.*5

*M*

By the choice of *ǫ*3*.*3, the smoothing of

∗

( *π*1 *χ* )

1 *cn*—1

*ǫ*3*.*5

*ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7 − *ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7 − 2 ∫ *nχn ǫ*3*.*1*χ*

is nonnegative for small enough *t* and *ǫ*1*.*6. On the other hands,

*M*

*Pπ*∗*χ*(*Tπ*∗ *χ* (*ω*)) − *Pπ*∗*χ*(*ω*) ≤ (*n* − 1)*ǫ*3*.*5

1 1 1

*ǫ*3*.*5

for any (1,1)-form *ω* on the first *n* coordinates of *M* × *M* . So using the estimate of *Pχ*(*ω*1*,t,ǫ*1*.*6*,ǫ*1*.*7 ), it is easy to see that

∗

( *π*1 *χ* )

*ǫ*3*.*5

*Pχ*(*ω*

1*,t,ǫ*1*.*6*,ǫ*1*.*7

) ≤ *c* + (*n* − 1)*ǫ*3*.*5*.*

So if *χ χ*0 on the support of the smoothing function for a K¨ahler form *χ*0 with

≥

∗

( *π*1 *χ* )

constant coefficients, then *Pχ*0

1*,t,ǫ*1*.*6*,ǫ*1*.*7

3*.*5

*χ*0

1*,t,ǫ*1*.*6*,ǫ*1*.*7

3*.*1

acting on the smoothing of *ω ǫ*3*.*5

is at most

*c* + (*n* − 1)*ǫ*

. So *P*

acting on the smoothing of *ω*

— 1 , *cn*−1 *ǫ*

1*,t,ǫ*1*.*6*,ǫ*1*.*7

2

*nχn*

*χ* is

also at most *c*+(*n*−1)*ǫ*

3*.*5

1*.*5

2

*nχn*

3*.*1

. Let *ω*

be a weak limit of *ω*

— 1 , *cn*−1 *ǫ χ*,

then *ω*

*M*

1*.*4

∈ [*ω* − *ǫ*

*χ*], where *ǫ*

= 1 , *cn*−1 *ǫ*

. Moreover, *P*

acting on the

smoothing of *ω*1*.*5 is at most *c* + (*n* 1)*ǫ*3*.*5. Since *ǫ*3*.*5 is arbitrary, it is at most

—

*M*

1*.*5

0

1*.*4

2

*M*

*nχn*

3*.*1

*χ*0

*c*. This completes the proof of Theorem [1.18](#_bookmark8).

# Regularization

In this section, we prove Theorem [1.11](#_bookmark4). By Remark [1.13](#_bookmark5), the *n* = 1 and *n* = 2 cases have been proved. By induction, we can assume that Theorem [1.11](#_bookmark4) has been proved in dimension 1, 2, ..., *n* 1. By Section 1, we can in addition assume that the condition of Theorem [1.18](#_bookmark8) are satisfied. So by Theorem [1.18](#_bookmark8), there exist a constant *ǫ*1*.*4 *>* 0 and a current *ω*1*.*5 ∈ [*ω*0 − *ǫ*1*.*4*χ*] such that

—

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 ≥ 0

1*.*5

1*.*5

in the sense of Definition [3.3](#_bookmark11).

Pick small enough *ǫ*4*.*1 *<*  1 such that

10000

*ω*0 − 100*ǫ*4*.*1*ω*0 ≥ (1 + *ǫ*4*.*1)2(*ω*0 − *ǫ*1*.*4*χ*)*.*

Then there exists a current *ω*4*.*2 = *ω*0 − 100*ǫ*4*.*1*ω*0 + √−1*∂∂*¯*ϕ*4*.*2 such that

*c ωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 ≥ 0

(1 + *ǫ*4*.*1)2

in the sense of Definition [3.3](#_bookmark11).

4*.*2

4*.*2

Now we pick a finite number of coordinate balls *B*2*r*(*xi*) such that *Br*(*xi*) is a cover of *M* . Moreover, we require that

*χi < χ <* (1 + *ǫ*4*.*1)*χi*

0 0

on *B*2*r*(*xi*) for K¨ahler forms *χi* with constant coefficients. We also assume that (1 − *ǫ*4*.*1)√−1*∂∂*¯|*z*|2 ≤ *ω*0 ≤ (1 + *ǫ*4*.*1)√−1*∂∂*¯|*z*|2

0

on *B*2*r*(*xi*). Let *ϕi*

*ω*0

we also assume that

*ω*0

be potential such that √−1*∂∂*¯*ϕi*

= *ω*0 on *B*2*r*(*xi*). Then

*i* 2 2

|*ϕω*0 − |*z*| | ≤ *ǫ*4*.*1*r .*

Let *ϕi* be the smoothing of *ϕ*4*.*2 + (1 − 100*ǫ*4*.*1)*ϕi* . When *δ < r* , it is well

*δ*

*ω*0

5

defined on *B* 9 (*xi*). By assumption, it is easy to see that

*r*

5

*c* (√−1*∂∂*¯*ϕi* )*n*—1 − (*n* − 1)*χi* ∧ (√−1*∂∂*¯*ϕi* )*n*—2 ≥ 0*.*

(1 + *ǫ*4*.*1)2

So *c*

*δ*

√ ¯

0 *δ*

√ ¯

*δ*

(

*δ*

1 + *ǫ*4*.*1

−1*∂∂ϕi* )*n*—1 − (*n* − 1)*χ* ∧ (

−1*∂∂ϕi* )*n*—2 *>* 0*.*

*i i* *i*

Now define the function *ϕ*4*.*3 from *B* 9 *r* (*xi*) to R as *ϕδ* − *ϕω*0 , our goal is to show

that for any *x* ∈ *M* ,

5

*ǫ*4*.*1*r*2 + max

{*i*:*x*∈*B* 9 *r* (*xi*)\*B* 8 *r* (*xi*)}

*i*

4*.*3

*ϕ*

(*x*) *<* max

{*i*:*x*∈*Br* (*xi*)}

*i*

4*.*3

*ϕ*

(*x*)*.*

5 5

If this is true, then for the smooth function *ϕ*4*.*4 = m˜ax*ϕi*

4*.*3

on *M* , where m˜ax

means the regularized maximum by choosing the parameters “*ηi*” in Lemma

I.5.18 of [[17](#_bookmark35)] to be smaller than *ǫ*4*.*1*r*2 , the K¨ahler form *ω*

3

4*.*4

= *ω*0

+ √−1*∂∂*¯*ϕ*4*.*4

will satisfy *cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0. In general, this is not true. However,

4*.*4

4*.*4

using the proof of the results of Bl-ocki and Kol-odziej [[4](#_bookmark22)], it is in fact true if the

Lelong number is small enough. The details consist the rest of this section.

*r i j*

It is easy to see that if *x* ∈ *B* 9 *r* (*xi*) ∩ *B* 9 *r*(*xj* ) and *δ <* 10 , *Bδ* (*x*) ⊂ *Bδ* (*x*).

5 5 2

*r*  *i*

For any *δ <* 20 and *x* ∈ *B* 9 *r*(*xi*), we define *ϕ*ˆ*δ* by

5

*ϕ*ˆ*i* (*x*) = sup (*ϕ*4*.*2 + (1 − 100*ǫ*4*.*1)*ϕi* )

*δ ω*0

*Bi* (*x*)

*δ*

and define *νi*(*x, δ*) by

*i*

*i* (*x*) − *ϕ*ˆ*i* (*x*)

*ν* (*x, δ*) = 16 *.*

*ϕ*ˆ

*δ*

log( *r* ) − log *δ*

*r*

16

Then *νi*(*x, δ*) is monotonically non-decreasing in *δ*. Recall that the Lelong number is defined by

*νi*(*x*) = lim *νi*(*x, δ*)*.*

*δ*→0

It is independent of *i* and can be denoted as *ν*(*x*) instead. Recall the definition of *ρ* in Definition [3.1](#_bookmark10). Let

*ǫ*4*.*5 =

*ǫ*4*.*1*r*2

5(∫ 1 log( 1 )Vol(*∂B* (0))*t*2*n*—1*ρ*(*t*)*dt* + log 2 + 32*n*−1 log 2)

then by the result of Siu [[32](#_bookmark51)], the set *Y* = *x* : *ν*(*x*) *ǫ*4*.*5 is a subvariety.

{ ≥ }

0

*t*

1

22*n*−3

*,*

For simplicity, we assume that *Y* is smooth. The singular case will be done at the end of this section.

Since *Y* is smooth by our assumption, as in Section 1, there exists a smooth function *ϕ*1*.*11 in a neighborhood *O* of *Y* such that

(*c* − *n* − *pǫ*

2

1*.*1

)*ωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

1*.*11

1*.*11

on *O*. Now we pick smaller neighborhoods *O*′ and *O*′′ such that *O*′ ⊂ *O* and

*O*′′ ⊂ *O*′. We need to prove the following proposition:

Proposition 4.1. *(1) For small enough δ <* *r , if*

20

max

{*i*:*x*∈*B* 9 *r* (*xi*)}

5

*νi*(*x, δ*) ≤ 2*ǫ*4*.*5*,*

*then*

sup *ϕ*1*.*11 + 3*ǫ*4*.*5 log *δ* + *ǫ*4*.*1*r*2 *<* max (*ϕi* (*x*) − *ϕi*

1. )*.*

*O*′

* 1. *For small enough δ <* *r , if*

20

*δ ω*0

{*i*:*x*∈*B* 9 *r* (*xi*)}

5

inf *ϕ*1*.*11 + 3*ǫ*4*.*5 log *δ* − *ǫ*4*.*1*r*2 ≤ max (*ϕi* (*x*) − *ϕi*

(*x*))*,*

*O* {*i*:*x*∈*B*

′

}

*δ ω*0

9 *r* (*xi*)

5

*then*

max

{*i*:*x*∈*B* 9 *r* (*xi*)}

5

*νi*(*x, δ*) *<* 4*ǫ*4*.*5*.*

* 1. *For small enough δ <* *r , if*

20

max

{*i*:*x*∈*B* 9 *r* (*xi*)}

5

*νi*(*x, δ*) ≤ 4*ǫ*4*.*5*.*

*then*

max

(*ϕi* (*x*) − *ϕi*

(*x*)) + *ǫ*4*.*1*r*2 *<* max

(*ϕi* (*x*) − *ϕi*

1. )*.*

*δ ω*0

{*i*:*x*∈*B* 9 *r* (*xi*)\*B* 8 *r* (*xi*)}

*δ ω*0

{*i*:*x*∈*Br* (*xi*)}

5 5

If Proposition [4.1](#_bookmark13) is true, for small enough *δ*, we can define *ϕ*1*.*12 as the

5 *r*

regularized maximum of *ϕ*1*.*11(*x*) + 3*ǫ*4*.*5 log *δ* on *O*′ and *ϕi* − *ϕi*

*δ*

*ω*0

on *B* 9

(*xi*).

Since *ν*(*x*) *< ǫ*4*.*5 for *x* /∈ *Y* , for small enough *δ*, max

{*i*:*x*∈*B* 9 *r* (*xi*)}

*νi*(*x, δ*) ≤ 2*ǫ*4*.*5

for all *x* /∈ *O*′′. So by Proposition [4.1](#_bookmark13) (1), we do not need to worry about the

5

discontiuty near the boundary of *O*′. By Proposition [4.1](#_bookmark13) (2) and (3), there is

also no need to worry about the discontinuity near the boundary of *B* 9 (*xi*). In

*r*

5

conclusion, *ϕ*1*.*12 will be smooth and satisfy

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

1*.*12

1*.*12

on *M* as long as *Y* is smooth and Proposition [4.1](#_bookmark13) is true.

In order to prove Proposition [4.1](#_bookmark13), we need the following lemma of Bl-ocki and Kol-odziej [[4](#_bookmark22)].

Lemma 4.2. *For any δ <* 20 *and x* ∈ *B* 9 *r* (*xi*)*, the following estimates hold:*

5

*r*

*i*

* 1. 0 ≤ *ϕ*ˆ*i* − *ϕ*ˆ*i* ≤ *ν* (*x, δ*) log *a for all a* ≥ 1*,*

*δ δ*

*a* ∫ 1

*(2)* 0 ≤ *ϕ*ˆ*i* − *ϕi* ≤ *νi*(*x, δ*)(

log( 1 )Vol(*∂B*1(0))*t*2*n*—1*ρ*(*t*)*dt* + 3 log 2)*.*

*δ*

*δ*

0

2*n*−1

*t*

22*n*−3

*Proof.* For readers’ convenience, we almost line by line copy the paper [[4](#_bookmark22)] here:

1. It follows from the logarithmical convexity of *ϕ*ˆ*i*

*δ*

and the definition of

*νi*(*x, δ*).

1. Define another regularization *ϕ*˜*i* by

*δ*

*ϕ*˜*i* (*x*) = 1 ∫

*δ*

Vol(*∂Bδ*(*x*))

(*ϕ* + (1 − 100*ǫ*

)*ϕi*

4*.*1

*ω*0

)*d*Vol*.*

*∂Bδ*(*x*)

Then by the Poisson kernel for subharmonic functions [[4](#_bookmark22)] and the estimate in (1),

4*.*2

*i i* 32*n*—1 *i i* 32*n*—1 *i*

*ϕ*ˆ*tδ* (*x*) − *ϕ*˜*tδ* (*x*) ≤ 22*n*—2 (*ϕ*ˆ*tδ* − *ϕ*ˆ*tδ/*2) ≤ ( 22*n*—2 log 2)*ν* (*x, tδ*)

for all *t* ∈ (0*,* 1]. By monotonicity,

*i i* 32*n*—1 *i* 32*n*—1 *i*

Define

*ϕ*ˆ*tδ* (*x*) − *ϕ*˜*tδ* (*x*) ≤ ( 22*n*—2 log 2)*ν* (*x, tδ*) ≤ ( 22*n*—2 log 2)*ν* (*x, δ*)*.*

*ρ*˜(*t*) = Vol(*∂B*1(0))*t*2*n*—1*ρ*(*t*)*,*

then ∫ 1 *ρ*˜(*t*) = 1. So

0

*i i* ∫ 1 *i* *i*

*ϕ*˜*δ* − *ϕδ* =

(*ϕ*˜*δ* − *ϕ*˜*tδ* )*ρ*˜(*t*)*dt* ≤

0

∫ 1 *i* *i*

0

32*n*—1 *i*

(*ϕ*ˆ*δ* − *ϕ*ˆ*tδ* )*ρ*˜(*t*)*dt* + ( 22*n*—2 log 2)*ν* (*x, δ*)*.*

By the estimate in (1) again,

*ϕ*ˆ*i* − *ϕ*ˆ*i* ≤ *i*

*δ*

*tδ*

*ν* (*x, δ*) log( )*.*

*t*

1

The other side of inequality 0 ≤ *ϕ*ˆ*i* − *ϕi* is trivial.

*δ*

*δ*

It is easy to see that there exists a constant *C*4*.*6 such that for any *δ <* *r*

20

and *x B* 9 (*xi*), *ν* (*x, δ*) *< C*4*.*6. Now we are ready to prove Proposition [4.1](#_bookmark13).

5

*r*

*i*

∈ *r*

(1) Suppose *δ <* 20 , *x* ∈ *B* 9 *r* (*xi*) and

5

*i* (*x*) *ϕ*ˆ*i* (*x*)

*ϕ*ˆ

*δ*

*r*

—

*i*

log( *r* ) − log *δ*

then

*ν* (*x, δ*) = 16 ≤ 2*ǫ*4*.*5*,*

16

*i i r*

*ϕ*ˆ*δ* (*x*) ≥ *ϕ*ˆ *r* (*x*) + 2*ǫ*4*.*5(log *δ* − log( 16 )) ≥ −*C*4*.*7 + 2*ǫ*4*.*5 log *δ.*

16

By Lemma [4.2](#_bookmark14) (2),

*ϕi* (*x*) ≥ −*C*4*.*8 + 2*ǫ*4*.*5 log *δ.*

*δ*

It is easy to see that for *δ* small enough,

sup *ϕ*1*.*11 + 3*ǫ*4*.*5 log *δ* + *ǫ*4*.*1*r*2 *< ϕi* (*x*) − *ϕi*

(*x*)

*δ ω*0

*O*′

5

because *ϕω*0 is uniformly bounded on *B* 9 *r*(*xi*).

*r*

*i*

(2) Suppose *δ <* 20 , *x* ∈ *B r* (*x* ) and

9 *i*

5

inf *ϕ*1*.*11 + 3*ǫ*4*.*5 log *δ* − *ǫ*4*.*1*r*2 ≤ *ϕi* (*x*) − *ϕi*

(*x*)*,*

*O*′

then as before

*δ ω*0

*ϕ*ˆ*i* (*x*) ≥ *ϕi* (*x*) ≥ −*C*4*.*9 + 3*ǫ*4*.*5 log *δ.*

*δ*

*δ*

By Lemma [4.2](#_bookmark14) (1) and the definition of *ϕ*ˆ*i* (*x*),

*δ*

2

sup

*Bi* (*x*)

*δ*

2

*ϕ*4*.*2 ≥ −*C*4*.*10 + 3*ǫ*4*.*5 log *δ.*

*i j*

If *x* ∈ *B* 9 *r* (*xj*), then *Bδ* (*x*) ⊂ *Bδ* (*x*) and therefore

5 2

sup *ϕ*4*.*2 ≥ sup *ϕ*4*.*2 ≥ −*C*4*.*10 + 3*ǫ*4*.*5 log *δ.*

*Bj* (*x*)

*δ*

*δ*

*Bi* (*x*)

2

By the definition of *ϕ*ˆ*j* (*x*) and *νj*(*x, δ*), it is easy to see that *νj*(*x, δ*) *<* 4*ǫ*4*.*5 if

*δ*

*δ* is small enough.

1. Suppose *δ <* 20 , *x* ∈ (*B* 9 *r*(*xi*) \ *B* 8 *r*(*xi*)) ∩ *Br*(*xj* ) and

*r*

5

max

{*i*:*x*∈*B* 9 *r* (*xi*)}

5

5

*νi*(*x, δ*) ≤ 4*ǫ*4*.*5*,*

then

*i* *i*

*ϕ*ˆ (*x*) − *ϕ*

*δ ω*0

2

(*x*) sup *ϕ*4*.*2

*Bi* (*x*)

*δ*

≤

2

+ 2*ǫ*

4*.*1

7

*r*2 + (2*r* + *δ*)2 − (2*r*)2 − 100*ǫ*4*.*1( *r*)2*,*

6

and

5

sup *ϕ*

≤ *ϕ*ˆ (*x*) − *ϕ*

(*x*) + 2*ǫ*

*r*2 + (2*r* + *δ*)2 − (2*r*)2 + 100*ǫ* 2

*Bj* (*x*)

*j j*

*δ*

4*.*2 *δ ω*0

4*.*1

4*.*1( 5 *r*) *.*

By Lemma [4.2](#_bookmark14) (1),

*i*

*ϕ*ˆ*i* − *ϕ*ˆ*i*

≤ *ν* (*x, δ*) log 2 ≤ 4*ǫ*

log 2*.*

By Lemma [4.2](#_bookmark14) (2), *ϕi* ≤ *ϕ*ˆ*i* and

*δ*

*δ*

2

4*.*5

*δ δ*

*j j* ∫ 1 1

*ϕ*ˆ*δ* − *ϕδ* ≤ 4*ǫ*4*.*5(

log( *t* )Vol(*∂B*1(0))*t*

*ρ*(*t*)*dt* + 22*n*—3 log 2)*.*

0

2*n*—1

32*n*—1

Since sup*Bi* (*x*) *ϕ*4*.*2 ≤ sup*Bj* (*x*) *ϕ*4*.*2, by summing everything together, for *δ* small

*δ*

2

enough, *ϕi* (*x*) − *ϕi*

*δ*

*ω*0

*δ*

(*x*) + *ǫ*4*.*1*r*2 *< ϕj* (*x*) − *ϕj*

*δ*

*ω*0

(*x*). We are done if *Y* is smooth.

In general *Y* is singular. By Hironaka’s desingularization theorem, there exists a blow-up *M*˜ of *M* obtained by a sequence of blow-ups with smooth centers

such that the proper transform *Y*˜

of *Y* is smooth. Without loss of generality,

assume that we only need to blow up once. Let *π* be the projection of *M*˜ to

*M* . Let *E* be the exceptional divisor. Let *s* be the defining section of *E*. Let *h*

be any smooth metric on the line bundle [*E*], then √—1 *∂∂*¯ log |*s*|2 = [*E*] + *ω*4*.*11

2*π*

*h*

by the Poincar´e-Lelong equation. Then it is well known that the smooth (1,1)-

form *ω*4*.*11 ∈ −[*E*] on *M*˜ and *ω*4*.*11 *>* −*C*4*.*12*π*∗*ω*0. For example, see Lemma 3.5

of [[18](#_bookmark36)] for the explanation. Define *ω*4*.*13 = *C*4*.*12*π*∗*ω*0 + *ω*4*.*11. Then *ω*4*.*13 is a K¨ahler form on *M*˜ .

Lemma 4.3. *Let C*4*.*14 =  6*n . Then for all small enough t and q-dimensional*

*ǫ .*

1 1

*subvarieties V of M*˜ *, as long as q < n,*

(*c n* − *q ǫ*

—

∫

*V* 3*n*

1*.*1

)((1 + *C*4*.*14

*t*)*π*∗*ω*0

+ *C*4*.*14

*t*2*ω*

4*.*13)*q*

≥ ∫*V*

*q*((1 + *C*4*.*14*t*)*π*∗*ω*0 + *C*4*.*14*t*2*ω*4*.*13)*q*—1 ∧ (*π*∗*χ* + *t*2*ω*4*.*13)*.*

*Proof.* By assumption,

3

0

0

∫ (*c* − *ǫ*1*.*1 )*π*∗*ωq* − *qπ*∗*ωq*—1 ∧ *π*∗*χ* = ∫

3

0

0

*π*(*V* )

*V*

(*c* − *ǫ*1*.*1 )*ωq* − *qωq*—1 ∧ *χ* ≥ 0*.*

So

∫ (*c* − *ǫ*1*.*1 )((1 + *C*

3

4*.*14

*V*

*t*)*π*∗*ω* )*q* − *q*((1 + *C*

*t*)*π*∗*ω* )*q*—1 ∧((1 + *C*

0

4*.*14

*t*)*π*∗*χ*) ≥ 0*.*

It suffices to show that

0

4*.*14

∫ (*c* − *ǫ*1*.*1 )((1 + *C t*)*π*∗*ω* + *C t*2*ω* )*q*

*V*

3

4*.*14

0

4*.*14

4*.*13

— *q*((1 + *C*4*.*14*t*)*π*∗*ω*0 + *C*4*.*14*t*2*ω*4*.*13)*q*—1 ∧ (*π*∗*χ* + *t*2*ω*4*.*13)

≥ ∫ (*c* − *ǫ*1*.*1 )((1 + *C t*)*π*∗*ω* )*q*

*V*

3

4*.*14

0

— *q*((1 + *C*4*.*14*t*)*π*∗*ω*0)*q*—1 ∧ ((1 + *C*4*.*14*t*)*π*∗*χ*)*.*

Since it only depends on the cohomology classes, we want to replace *ω*0 by a better representative in its cohomology class. Remark that *π*(*E*) is smooth by assumption. So we can apply Theorem [1.11](#_bookmark4) to *π*(*E*). As in Section 1, there

exists a smoo√th function *ϕ*4*.*15 on a neighborhood *O*4*.*16 of *π*(*E*) in *M* such that

*ω*4*.*15 = *ω*0 +

−1*∂∂*¯*ϕ*4*.*15 satisfies

(*c* − *ǫ*1*.*1 )*ωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

on *O*

. Define *ϕ*

2 4*.*15

= *π* log |*s*|2

*h*

4*.*15

on *M* \ *π*(*E*). Recall the definition of the

4*.*16

4*.*17

∗ 4*πC*4*.*12

regularized maximum in Lemma I.5.18 of [[17](#_bookmark35)]. For large enough *C*4*.*18, let *ϕ*4*.*19

be the regularized max√imum of *ϕ*4*.*17 + *C*4*.*18 and *ϕ*4*.*15. Then *ϕ*4*.*19 is smooth

on *M* and *ω*4*.*19 = *ω*0 + −1*∂∂*¯*ϕ*4*.*19 *>* 0 on *M* . Moreover, there exists a smaller

neighborhood *O*4*.*20 of *π*(*E*) such that *ϕ*4*.*19 = *ϕ*4*.*15 on *O*4*.*20 ⊂ *O*4*.*16.

After replacing *ω*0 by *ω*4*.*19, it suffices to show that

(*c* − *ǫ*1*.*1 ) Σ *q*! ((1 + *C*

*q*

*t*)*π*∗*ω*

)*q*—*i* ∧ (*C*

*t*2*ω* )*i*

3

*i*=1

*q*

*i*!(*q* − *i*)!

4*.*14

4*.*19

4*.*14

4*.*13

−*q* Σ (*q* − 1)! ((1 + *C*

*t*)*π*∗*ω*

)*q*—*i* ∧ *Ci*—1 (*t*2*ω* )*i*

*i*=1 *q*—1

(*q* − *i*)!(*i* − 1)!

4*.*14

4*.*19

4*.*14

4*.*13

−*q* Σ (*q* − 1)! ((1 + *C*

*i*!(*q* − 1 − *i*)!

4*.*14

4*.*19

*i*=1

*t*)*π*∗*ω*

)*q*—1—*i*

∧ (*C*4*.*14*t*2*ω*4*.*13)*i* ∧ (1 + *C*4*.*14*t*)*π*∗*χ*

+*q*((1 + *C*4*.*14*t*)*π*∗*ω*4*.*19 + *C*4*.*14*t*2*ω*4*.*13)*q*—1 ∧ *C*4*.*14*tπ*∗*χ*

≥0*.*

By definition of *C*4*.*14,

*q*  (*q* − 1)! *< ǫ*1*.*1 *q*! *C*

(*q* − *i*)!(*i* − 1)!

6 *i*!(*q* − *i*)!

4*.*14

for all *i* = 1*,* 2*, ..., q*. So we can combine the first term and the second term. If the point is inside *π*—1(*O*4*.*20), then for all *i* = 1*,* 2*, ..., q* − 1,

(*c ǫ*1*.*1 )(*π*∗*ω* 2

—

4*.*19

)*q*—*i* ≥ (*q* − *i*)(*π*∗*ω*

4*.*19

)*q*—1—*i* ∧ *π*∗*χ*

because

*ǫ*1*.*1

*n*—1

*n*—2

(*c* − 2 )*ω*4*.*19 ≥ (*n* − 1)*ω*4*.*19 ∧ *χ*

on *O*4*.*20. So the sum of the first three terms is non-negative if *i* = 1*,* 2*, ..., q* 1. So we are done because the *i* = *q* term and the fourth term are non-negative. If the point is outside *π*—1(*O*4*.*20), then there exists *C*4*.*21 such that

—

*C*4*.*21*π*∗*χ > ω*4*.*13 *> C*—1 *π*∗*χ*

4*.*21

and

*C*4*.*21*π*∗*χ > π*∗*ω*4*.*19 *> C*—1 *π*∗*χ*

4*.*21

on *M*˜ \ *π*—1(*O*4*.*20). The only first order term in *t* is

*qπ*∗*ωq*—1 ∧ *C*4*.*14*tπ*∗*χ.*

4*.*19

Since it is positive, for small enough *t*, we also get the required inequality.

Now we pick *t >* 0 such that *t* satisfies Lemma [4.3](#_bookmark15) and

*c >* max{*c* − *ǫ*1*.*1 *,*  *c* }*.*

1 + *C*4*.*14*t* + *C*4*.*12*C*4*.*14*t*2

4*n* 1 + *ǫ*4*.*1

We apply Theorem [1.11](#_bookmark4) to the lower dimensional smooth manifold *Y*˜ with the K¨ahler forms (1 + *C*4*.*14*t*)*π*∗*ω*0 + *C*4*.*14*t*2*ω*4*.*13 and *π*∗*χ* + *t*2*ω*4*.*13. As in Section

1, there exists a smooth function *ϕ*4*.*22 on a neighborhood of *Y*˜ in *M*˜ such that

*ω*4*.*22 = (1 + *C*4*.*14*t*)*π*∗*ω*0 + *C*4*.*14*t*2*ω*4*.*13 + √−1*∂∂*¯*ϕ*4*.*22

satisfies

(*c* −

*ǫ*1*.*1

4*n*

)*ωn*—1 − (*n* − 1)(*π*∗*χ* + *t*2*ω*4*.*13) ∧ *ωn*—2 *>* 0

near *Y*˜ . Similarly, let *ϕ*4*.*23 be the potential near *E*. For large enough constant

4*.*22

4*.*22

*C*4*.*24, define

and

*ϕ*4*.*25 = m˜ax{*ϕ*4*.*23*, ϕ*4*.*22 + *C*—1 *π*∗*ϕ*4*.*17 + *C*4*.*24}

√ ¯

4*.*24

*ω*4*.*25 = (1 + *C*4*.*14*t*)*π*∗*ω*0 + *C*4*.*14*t*2*ω*4*.*13 +

−1*∂∂ϕ*4*.*25*.*

Then

*ǫ*1*.*1

*n*—1

4*n*

*n*—2

(*c* −

)*ω*4*.*25 − (*n* − 1)(*π*∗*χ* + *t*2*ω*4*.*13) ∧ *ω*4*.*25 *>* 0

on a neighborhood *O* of *Y*˜ ∪ *E* in *M*˜ . Since *t*2*ω*4*.*13 *>* 0, it is easy to see that

(*c* − *ǫ*1*.*1 )(*π ω*

4*n*

∗

4*.*25

∗

)*n*—1 − (*n* − 1)*χ* ∧ (*π ω*

)*n*—2 *>* 0

4*.*25

on *π*(*O* \ *E*). Now we choose neighborhoods *O*′ and *O*′′ of *Y* ∪ *π*(*E*) in *M* such that *O*′ ⊂ *π*(*O*) and *O*′′ ⊂ *O*′. Then as before, for small enough *δ*, we can define *ϕ*4*.*26 as the regularized maximum of *π*∗*ϕ*4*.*25 + 3*ǫ*4*.*5 log *δ* on *O*′ \ *π*(*E*)

*i* *i*

and *ϕδ* − *ϕω*0 on *B* 9 *r* (*xi*). Then *ϕ*4*.*26 is smooth and bounded on *M* \ *π*(*E*).

5

Moreover, for

*ω*4*.*26 = (1 + *C*4*.*14*t*)*ω*0 + *C*4*.*14*t*2*π*∗*ω*4*.*13 + √−1*∂∂*¯*ϕ*4*.*26

= (1 + *C*4*.*14*t* + *C*4*.*12*C*4*.*14*t*2)*ω*0 + *C*4*.*14*t*2*π*∗*ω*4*.*11 + √−1*∂∂*¯*ϕ*4*.*26*,*

it is easy to see that

(max{*c* − *ǫ*1*.*1 *,*  *c* })*ωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

4*n* 1 + *ǫ*4*.*1

4*.*26

4*.*26

on *M* \ *π*(*E*) because *C*4*.*14*tω*0 + *C*4*.*14*t*2*π*∗*ω*4*.*13 *>* 0. Now we define

*ω*4*.*26

*C*4*.*14*t*2*π*∗*ω*4*.*11 + √−1*∂∂*¯*ϕ*4*.*26

*ω*4*.*27 =

1 + *C*

4*.*14

*t* + *C*4*.*12

*C*4*.*14

*t*2 = *ω*0 +

1 + *C*4*.*14

*t* + *C*4*.*12

*C*4*.*14

*t*2 *,*

then by the choice of *t*,

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

4*.*27

4*.*27

on *M* \ *π*(*E*). For large enough constant *C*4*.*28, define

*C*4*.*14 *t*2 *π*

log |*s*|2 + *ϕ*

*ϕ* = m˜ax{ ∗

2*π*

*h* 4*.*26 + *C*

*, ϕ* }*,*

4*.*29

1 + *C*4*.*14*t* + *C*4*.*12*C*4*.*14*t*2

4*.*28

4*.*15

then *ϕ*4*.*29 is smooth on *M* and *ω*4*.*29 = *ω*0 + √−1*∂∂*¯*ϕ*4*.*29 satisfies

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 *>* 0

on *M* . We are done.

4*.*29

4*.*29

# Deformed Hermitian-Yang-Mills Equation

In this section, we prove Theorem [1.7](#_bookmark3). The equation

*n*

Σ

arctan *λi* = *θ*ˆ

*i*=1

for eigenvalues *λi* of *ωϕ* = *ω*0 + √ 1*∂∂*¯*ϕ >* 0 with respect to *χ* is the same as the equation

—

*n*

Σ arctan( 1 ) = *nπ* − *θ*ˆ*.*

*i*=1

*λi*

2

To simplify the notations, define *θ*0 = *nπ* − *θ*ˆ. Then the equation is equivalent

2

to the inequality

arctan( 1 ) *< θ*

*λi* 0

Σ

*i*/=*j*

for *j* = 1*,* 2*, ..., n* and the equation

Im(*ωϕ* + √−1*χ*)*n* = tan(*θ*0)Re(*ωϕ* + √−1*χ*)*n.*

Inspired by the work of Pingali in the toric case [[30](#_bookmark48)], the analogy of Theorem

[1.11](#_bookmark4) is the following:

Theorem 5.1. *Fix a K¨ahler manifold Mn with K¨ahler metrics χ and ω*0*. Let*

*θ*0 ∈ (0*, π* ) *be a constant and f >* − 1 *be a smooth function satisfying*

4

*χn*

*f* = (

*n*!

100*n*

tan(*θ*0)Re(*ωϕ* + √−1*χ*)*n*

∫

∫

*n*!

Im(*ω* + √ 1*χ*)*n*

*n*! ) ≥ 0*,*

—

*ϕ* −

*M M*

*then there exists a smooth function ϕ satisfying the equation*

Im(*ωϕ* + √−1*χ*)*n* + *fχn* = tan(*θ*0)Re(*ωϕ* + √−1*χ*)*n*

*and the inequality*

Σ

arctan( 1 ) *< θ*

*λi* 0

*i*/=*j*

*for j* = 1*,* 2*, ..., n and eigenvalues λi of ωϕ* = *ω*0 + √−1*∂∂*¯*ϕ >* 0 *with respect to χ if there exists a constant ǫ*1*.*1 *>* 0 *and for all p-dimensional subvarieties V*

*with p* = 1*,* 2*, ..., n (V can be chosen as M), there exist smooth functions θV* (*t*)

*from* [1*,* ∞) *to* [ *pπ* − *θ*0 + (*n* − *p*)*ǫ*1*.*1*, pπ* ) *such that for all t* ∈ [1*,* ∞)*,*

2 2

∫ (*χ* + √−1*tω* )*p* /= 0*, θ* (*t*) = arg(∫ (*χ* + √−1*tω* )*p*)*,* lim *θ* (*t*) = *pπ .*

0

*V*

0

*t*→∞

*V*

2

*V*

*V*

When *n* = 1, it is trivial. In higher dimensions, we need to prove it by induction.

Inspired by the work of Collins-Jacob-Yau [[12](#_bookmark30)], the analogy of Theorem [1.14](#_bookmark6) is the following:

Theorem 5.2. *Fix a K¨ahler manifold Mn with K¨ahler metrics χ and ω*0*. Let*

*θ*0 ∈ (0*, π* ) *be a constant and f >* − 1 *be a smooth function satisfying*

4

*χn*

*f* = (

*n*!

100*n*

tan(*θ*0)Re(*ωϕ* + √−1*χ*)*n*

∫

∫

*n*!

Im(*ω* + √ 1*χ*)*n*

*n*! ) ≥ 0*,*

—

*ϕ* −

*M M*

*then there exists a smooth function ϕ satisfying the equation*

Im(*ωϕ* + √−1*χ*)*n* + *fχn* = tan(*θ*0)Re(*ωϕ* + √−1*χ*)*n*

*and the inequality*

Σ

arctan( 1 ) *< θ*

*λi* 0

*i*/=*j*

*for j* = 1*,* 2*, ..., n and eigenvalues λi of ωϕ* = *ω*0 + √ 1*∂∂*¯*ϕ >* 0 *with respect to*

—

*χ if*

Σ

arctan( 1 ) *< θ*

*λi,*0 0

*i*/=*j*

*for j* = 1*,* 2*, ..., n and eigenvalues λi,*0 *of ω*0 *>* 0 *with respect to χ.*

We will use the continuity method three times to prove Theorem [5.2](#_bookmark16). Let *ω*˜0 be the form *ω*0 and *f*˜ be the function *f* in Theorem [5.2](#_bookmark16). There exists a constant *C*5*.*1 such that *ω*˜0 ≥ *C*5*.*1*χ*. We start from *f* = 0 and *ω*0 = cot( *θ*0 )*χ*. In this case, *ϕ* = 0 is the solution. Then we let *ω*0 = *t* cot( *θ*0 )*χ* + (1 − *t*)(*C*—1 cot( *θ*0 ) + 1)*ω*˜0

*n*

*n*

5*.*1

*n*

and *f* be the non-negative constant satisfying the integrability condition as the

first path. It will imply the result for *ω*0 = (*C*—1 cot( *θ*0 ) + 1)*ω*˜0. Then we let

5*.*1 *n*

*ω*0 = *tω*˜0 and *f* be the constant satisfying the integrability condition as the

second path. *f* must be non-negative because

*d*

∫

(

*dt M*

∫

=

(

*M*

tan(*θ*0)Re(*tω*0 + √−1*χ*)*n*

*n*!

—

tan(*θ*0)Re(*tω*0 + √−1*χ*)*n*—1

(*n* − 1)! −

Im(*tω*0 + √−1*χ*)*n*

*n*!

)

Im(*tω*0 + √−1*χ*)*n*—1

(*n* − 1)! ) ∧ *ω*0 *>* 0

by the assumption on *ω*0 and Lemma 8.2 of [[12](#_bookmark30)]. The continuity method will imply the result for *ω*˜0 when *f* is the non-negative constant *f*0 satisfying the integrability condition. Finally, we let *ω*0 = *ω*˜0 and *f* = *tf*˜+ (1 *t*)*f*0 be the

—

third continuity path. It will imply Theorem [5.2](#_bookmark16).

It is easy to see the openness along the paths. Thus, we only need to prove the *a priori* estimate along the paths. It will be achieved by Sz´ekelyhidi’s estimates in [[36](#_bookmark54)]. First of all, we need to rewrite the equation.

The equation

Im(*ωϕ* + √−1*χ*)*n* + *fχn* = tan(*θ*0)Re(*ωϕ* + √−1*χ*)*n*

can be written as

*n n*

Im Y(*λi* + √−1) + *f* = tan(*θ*0)Re Y(*λi* + √−1)*.*

*i*=1

It is equivalent to

*i*=1

Σ 1 *f* Σ 1

*n*

*n*

sin(

Q*n*

*λi* + 1

arctan( )) +

*λ*

*i*=1

*i*

*i*=1

*i*

√ 2 = tan(*θ*0) cos(

*i*=1

arctan( ))*, λ*

which is the same as

Q*n*

Σ 1

*n*

*f* cos(*θ*0)

sin(*θ*0 −

*i*

arctan( )) =

*λ*

*i*=1

√ 2 *.*

*i*=1

Let Γ be the region consisting of (*λ*1*, ..., λn*) ∈ R*n* such that *λi >* 0 and

*λi* + 1

Σ arctan( 1 ) *< θ*

*λi*

0

*i*/=*j*

for all *j* = 1*,* 2*, ..., n*, then we want to study the function

Σ 1 *f* cos(*θ*0)

*n*

*F* (*f, λ*1*, ..., λn*) = sin(*θ*0 − arctan( *λ* )) − Q*n* √ 2

*i*=1

*i*

*i*=1

*λi* + 1

on (− 1 *,* ∞) × Γ¯, where Γ¯ is the closure of Γ. For any K¨ahler form *ω*, we say *ω* ∈ Γ*χ* if the eigenvalues of *ω* with respect to *χ* is in Γ. Similarly, we define *Fχ*(*f, ω*) as *F* (*f, λi*) for eigenvalues *λi* of *ω* with respect to *χ*.

100*n*

In order to apply Sz´ekelyhidi’s estimates in [[36](#_bookmark54)], we claim the following:

Proposition 5.3. *Assume that n* ≥ 2*. If f >* − 1 *, then*

100*n*

1. *∂ F* (*f, λ*) *>* 0 *if λ* ∈ Γ*;*

*∂λi*

1. *∂*2 *F* (*f, λ*) ≤ − cos(*θ* ) *λiδij if λ* ∈ Γ *and F* (*f, λ*) = 0*;*

*i*

*∂λi∂λj*

0

2(*λ*2+1)2

1. *∂ F* (*f, λ*) ≤ *∂ F* (*f, λ*) *if λ* ∈ Γ *and λi* ≥ *λj;*

*∂λi*

*∂λj*

1. *If λ* ∈ *∂*Γ*, then F* (*f, λ*) *<* 0*;*
2. *For any λ* ∈ Γ*, the set*

{*λ*′ ∈ Γ : *F* (*f, λ*′) = 0*, λ*′*i* ≥ *λi*}

*is bounded. Proof.* (1)

*∂*

cos(*θ*0 − Σ*n*

arctan(  1 ))

*f* cos(*θ* ) *λ*

*F* (*f, λ*) =

*∂λ*

*i*

*k*=1

*λ*2 + 1 Q*n* 2

*λk* +

*i*

*k*=1

*λk* + 1

0

√

*i .*

*λ*2 + 1

*i*

Using the definition of Γ, it is easy to see that

*n*

0 *<* Σ arctan( 1 ) *<*  *nθ*0

on Γ. So

*k*=1

*n*

*λk n* − 1

cos(*θ*0 −

*k*Σ=1

1

arctan(

*λk*

1

)) *>* cos(*θ*0) *>* √2 *.*

It is easy to see that *∂ F* (*f, λ*) *>* 0 if *λ* ∈ Γ.

*∂λi*

(2)

*∂*2

*F* (*f, λ*) = − cos(*θ*

— Σ arctan( 1 )) 2*λiδij*

*∂λi∂λj*

*n*

0

*k*=1

*n*

*i*

*λk* (*λ*2 + 1)2

— sin(*θ* − Σ arctan( 1 )) 1

0

*λk*

(*λ*2 + 1)(*λ*2 + 1)

*k*=1

*i*

*j*

*f* cos(*θ*0) *λi λj*

*f* cos(*θ*0)

1 − *λ*2

— Q*n* √*λ*2 + 1 *λ*2 + 1 *λ*2 + 1 + Q*n* √*λ*2 + 1 (*λ*2 + 1)2 *δij.*

*i*

*k*=1

*k*

*i*

*j*

*k*=1

*k*

*i*

The first term is at most − cos(*θ* ) 2*λiδij* . When *F* (*f, λ*) = 0, the second term

equals to −Q *f* co√s(*θ*0)

*i*

*n*

1

0 (*λ*2+1)2

. When *f* ≥ 0, it is a non-negative definite

*λ*2 +1 (*λ*2+1)(*λ*2+1)

*i*

*j*

*k*=1

*k*

matrix. When 0 *> f >* − 1 , for any *ξ*1*, ...ξn* ∈ R,

100*n*

Σ *f* cos(*θ*0) *ξiξj*

*n*

— Q*n* √*λ*2 + 1 (*λ*2 + 1)(*λ*2 + 1)

*i,j*=1

*n*

*k*=1

*k*

*i*

*j*

≤ Σ 1 cos(*θ* ) |*ξiξj*|

100*n*

0

(*λ*2 + 1) 5 (*λ*2 + 1) 5

*i,j*=1

*i* 4 *j* 4

*n*

= 1 cos(*θ* )(Σ |*ξi*| )2

100*n*

0

(*λ*2 + 1) 5

*i*=1 *i* 4

*n* 2

≤ 1 cos(*θ* ) Σ *ξi*

100

0

(*λ*2 + 1) 5

*i*=1 *i* 2

*n n*

100

0

*λi*(*λ*2 + 1)2

≤ 1 cos(*θ* ) Σ Σ *ξiξjδij .*

Since *λi >* cot(*θ*0) *>* 1 on Γ, it is easy to see that the second term is at most

*i*

*i*=1 *j*=1

4 times the first term. Similarly the third term and the fourth term are also

1

—

1

at most − times the first term.

4

1. It suffices to show that

*d* 1 *f d x*

−2*x*

*f* 1 − *x*2

*dx x*2 + 1 Q*n* √*λ*2 + 1 *dx x*2 + 1 = (*x*2 + 1)2 + Q*n* √*λ*2 + 1 (*x*2 + 1)2

+

*k*=1

*k*

*k*=1

*k*

is non-positive for all *x* ∈ [*λj, λi*]. When *f* ≥ 0, it is trivial. When 0 *> f >*

1

— 100*n* , it is at most

2 2

−2*x* + 1 *x* − 1 ≤ −2*x* + 1 *x* − 1 *.*

(*x*2 + 1)2 100*nλi* (*x*2 + 1)2 (*x*2 + 1)2 *x* (*x*2 + 1)2

This is indeed non-positive because *x* ≥ *λj >* 1.

1. The point *λi* = cot( *θ*0 ) belongs to *∂*Γ and *F* at this point is negative

because

1

*n*—1

(cot2( *θ*0 ) + 1) *n θ* 1

*<*

100*n*

*n*—1

cos(*θ*0)

2

sin(

*n*

0 ) =

*n* − 1

cos(*θ*0) sin 2 —1

( *θ*0 ) *.*

*n*—1

because

100*n*

1

If 0 *> f >* − 1 , then 0 ≥ *θ*0 −

Thus, it suffices to prove that if *F* (Σ*f, λ*) = 0, then *λ* /∈ *∂*Γ. If *f* ≥ 0, it is obvious.

*<* ( inf

(√*x*2

*n*

*i*=1

arctan(  1 ) ≥ − *θ*0 ≥ − *π* . So *λi* /∈ *∂*Γ

+ 1 arctan(

4

*λi*

1

)))(

*n*—1

inf

4

sin *x*

)*.*

100*n*

1. is obvious.

*x*∈(1*,*∞)

*x x*∈[— *π ,*0] *x*

Compared to Sz´ekelyhidi’s conditions in [[36](#_bookmark54)], there are three major differ- ences. First of all, *F* also depends on *f* . Second of all, Γ does not contain the positive orthant. Finally, even if we fix the *f* variable, *F* is only concave when *F* = 0. However, we will show that his works still survive without much changes.

Proposition 5 of [[36](#_bookmark54)] only requires the concavity of *F* when *F* = 0. So it still holds. Sz´ekelyhidi’s *C*0 estimate relies on the variant of Alexandroff-Bakelman- Pucci maximum principle similar to Lemma 9.2 of [[25](#_bookmark43)]. Clearly it does not take derivatives of *f* . So Sz´ekelyhidi’s *C*0 estimate is still true.

The next step is to prove that

|√−1*∂∂*¯*ϕ*|*χ* ≤ *C*5*.*2(1 + sup |∇*ϕ*|2 )*.*

*χ*

*M*

We will use the same notations as in [[36](#_bookmark54)] except that letter *f* in [[36](#_bookmark54)] is replaced by *F* , the letter *F* is replaced by *Fχ* and the letter *u* is replaced by *ϕ*. It is easy to see that (78) of [[36](#_bookmark54)] still holds. Now we differentiate the equation *Fχ*(*f, ωϕ*) = 0. We see that

*Fijgi*¯*j*1 + *Ff f*1 = 0*,*

*χ χ*

and

*Fpq,rsgpq*¯1*grs*¯1¯ + *Fkkg* ¯ ¯ + *Fkk,f g* ¯ *f*1¯ + *Fkk,f g* ¯¯*f*1 + *Ff f*1¯1 = 0

*χ χ kk*11

*χ kk*1

*χ kk*1 *χ*

because *Fff* = 0. Since |*Ff* | = |− Q cos√(*θ*0)

*i*

*χ*

*χ*

*n i*=1

*λ*2+1

| ≤ 1, the term *F f*1¯1 is bounded.

So the only additional term in (85) of [[36](#_bookmark54)] is −*C*0*λ*1—1|*F kk,f gkk*¯1| on the right

*f*

*χ*

*χ*

hand side. Instead of (94) of [[36](#_bookmark54)], we get

*Fkkg* ¯ + *Ff fp* = 0*.*

*χ kkp χ*

Since |*Ff fp*| is bounded, the estimate in (95) still holds. So the only additional term in (99) and (104) of [[36](#_bookmark54)] is −*C*0*λ*1—1|*F kk,f gkk*¯1| on the right hand side.

*χ*

*χ*

The case 1 in [[36](#_bookmark54)] will not happen. The additional term in (120) of [[36](#_bookmark54)] is also

−*C*0*λ*—1|*Fkk,f g* ¯ |. However, recall that (67) of [[36](#_bookmark54)] is that

1 *χ kk*1

*ij,rs*

−*F g g*

¯

*χ*

*ij*1

*rs*¯1

¯ ≥ −*F*

*g* ¯ *g* ¯¯ − Σ *F*1 − *Fi* |*g* ¯ |*.*

*i>*1

(Remark that the letter *f* in [[36](#_bookmark54)] is replaced by *F* and the letter *F* is replaced by *Fχ*.) The term −*Fijgi*¯*i*1*gj*¯*j*¯1 was thrown away. However, this term is at least

*ij*

*ii*1

*jj*1

*λ*1 − *λi*

*i*11

*λi*  2

2(*λ*2 +1)2 |*gi*¯*i*1| . The term

*i*

*kk,f*

*f* cos(*θ*0) *λi λi*

−*C*0|*Fχ gkk*¯1| = −*C*0|Q*n* √ 2 *λ*2 + 1 *gii*¯1| ≥ −*C*0|*f* | 2

*k*=1

*λk* + 1

*i*

(*λi* + 1) 2

3 |*gi*¯*i*1|

is at least − *λi* |*g* ¯ |2 − *C* . So Sz´ekelyhidi’s estimate

*i*

2(*λ*2+1)2

*ii*1

5*.*3

|√−1*∂∂*¯*ϕ*|*χ* ≤ *C*5*.*2(1 + sup |∇*ϕ*|2 )

*χ*

*M*

still holds.

Sz´ekelyhidi used the property that Γ contains the positive orthant to prove the *C*2 estimate [[36](#_bookmark54)]. We do not have this property. However, we can use Proposition 5.1 of [[12](#_bookmark30)] to achieve this.

The Evans-Krylov estimate requires the uniform ellipticity and concavity of *Fχ*(*f, .*). Its relationship with the function *F* was cited as (63) and (64) of [[36](#_bookmark54)]. By Proposition [5.3](#_bookmark17), the conditions for the Evans-Krylov estimate are indeed true. The higher order estimate follows from standard elliptic theories. Finally, *ωϕ* will stay in the region Γ*χ* along the continuity paths by Proposition [5.3](#_bookmark17) (4).

The analogy of Theorem [1.18](#_bookmark8) is the following:

Theorem 5.4. *Fix a K¨ahler manifold Mn with K¨ahler metrics χ and ω*0*. Sup- pose that for all t >* 0*, there exist a constant ct >* 0 *and a smooth K¨ahler form ωt* ∈ [(1 + *t*)*ω*0] *satisfying ωt* ∈ Γ*χ, and*

Im(*ωϕ* + √−1*χ*)*n* + *ctχn* = tan(*θ*0)Re(*ωϕ* + √−1*χ*)*n.*

*Then there exist a constant ǫ*5*.*4 *>* 0 *and a current ω*5*.*5 ∈ [*ω*0 − *ǫ*5*.*4*χ*] *such that*

*ω*5*.*5 ∈ Γ¯*χ in the sense of current.*

The definition of a current being in Γ¯*χ* is similar to Definition [3.3](#_bookmark11) except that we replace the condition

*cωn*—1 − (*n* − 1)*χ* ∧ *ωn*—2 ≥ 0

by *ω* ∈ Γ¯*χ* for K¨ahler form *ω*. To simplify notations, for a K¨ahler form *ω*, we define *Pχ*(*ω*) and *Qχ*(*ω*) by

1. (*ω*) = max(Σ arctan( 1 ))*,*

*χ*

*j λi*

*i*/=*j*

and

1. (*ω*) = arctan( 1 )*,*

*χ λ*

Σ

*i*

*i*

where *λi* are the eigenvalues of *ω* with respect to *χ*. Then *ω* ∈ Γ¯*χ* is equivalent to *Pχ*(*ω*) *θ*0.

≤

The analogy of Lemma [3.5](#_bookmark12) is the following:

Lemma 5.5. *Suppose that*

*A C* *>* *I O* *.*

*Then*

*C*¯*T B O I*

*P* (*A* − *CB*—1*C*¯*T* ) + *Q* (*B*) ≤ *P* ( *A C* )*.*

*I*

*I*

*I*

*C*¯*T B*

*Proof.* It is easy to see that *B > I*. Moreover, for any *ξ* /= 0 ∈ C*n*,

*ξ*¯*T* (*A* − *CB*—1*C*¯*T* )*ξ* = *ξ*¯*T* −*ξ*¯*T CB*—1 *A C* *ξ* ¯

*>* |*ξ*|2 + |*B*—1*C*¯*T ξ*|2*,*

*C*¯*T B*

−*B*—1*CT ξ*

so *A CB*—1*C*¯*T > I*. Therefore, both hand sides of the inequality are well defined.

—

By restricting on the codimension 1 subspaces, it suffices to prove that

*I* *I*

*Q* (*A* − *CB*—1*C*¯*T* ) + *Q* (*B*) ≤ *Q* ( *A C* )*.*

For any *s* ∈ [0*,* 1], it is easy to see that

√

*I C*¯*T B*

det *A* + −1*I s*√*C*

*s*¯*CT B* + −1*I*

*A* − *s*2*C*(*B* + √−1*I*)—1*C*¯*T* + √−1*I O*√

= det

*O B* +

*.*

−1*I*

So we need to compute det(*A s*2*C*(*B* + √ 1*I*)—1*C*¯*T* + √ 1*I*).

− − −

We already know that *B > I*, so

(*B* + √−1*I*)—1 = *B*—1(*I* + √−1*B*—1)—1

∞ ∞

= Σ(−1)*kB*—2*k*—1 − √−1 Σ(−1)*kB*—2*k*—2*.*

Therefore

*k*=0

*k*=0

*A* − *s*2*C*(*B* + √−1*I*)—1*C*¯*T* + √−1*I*

∞ ∞

*k*=0

*k*=0

= *A* − *s*2(Σ(−1)*kCB*—2*k*—1*C*¯*T* ) + √−1(*I* + *s*2 Σ(−1)*kCB*—2*k*—2*C*¯*T* )*.*

The real part is at least *A s*2*CB*—1*C*¯*T > I* and the imaginary part is also at least *I*. So

—

√

arg det *A* + −1*I s*√*C*

*s*¯*CT B* + −1*I*

= arg det(*A* − *s*2*C*(*B* + √−1*I*)—1*C*¯*T* + √−1*I*) + arg det(*B* + √−1*I*)*.*

if we define arg det(*X* + √ 1*Y* ) as *QY* (*X*) for *X, Y >* 0. In fact, this is true up to an integer times 2*π*. However, both hand sides are continuous with respect to *s* and this equation holds for *s* = 0. So it holds for all *s* [0*,* 1].

—

∈

Now it suffices to show that

arg det(*A* − *C*(*B* + √−1*I*)—1*C*¯*T* + √−1*I*) ≥ arg det(*A* − *CB*—1*C*¯*T* + √−1*I*)*.* It follows from the facts that

∞ ∞

Σ(−1)*kCB*—2*k*—2*C*¯*T* ≥ Σ(−1)*k*+1*CB*—2*k*—1*C*¯*T* ≥ 0

and

*k*=0

*k*=1

*A* − *CB*—1*C*¯*T > I.*

Choose *C*5*.*6 large enough such that *θ*0 + *n* arctan( 1 ) *< π* . The definitions

*C*5*.*6 4

of *χM*×*M* , *χM*×*M,ǫ*5*.*7 *,ǫ*5*.*8 and *ft,ǫ*5*.*7*,ǫ*5*.*8 are still the same as in Section 1. As

before, there exists *ǫ*5*.*8 *>* 0 such that for *ǫ*5*.*7 small enough, *ft,ǫ*

*>* − .

5*.*7

*,ǫ*5*.*8

1 200*n*

Now we consider *ω*0*,M*×*M,t* = *π*1∗*ωt* + *C*5*.*6*π*2∗*χ*. By Theorem [5.2](#_bookmark16), there exists

*ωt,ǫ*

5*.*7

*,ǫ*5*.*8

∈ [*ω*0*,M*×*M,t*] such that *PχM* ×*M*

(*ωt,ǫ*

5*.*7

*,ǫ*5*.*8

) *< θ*0 + *n* arctan( 1 ) and

5 6

*C .*

Im(*ωt,ǫ*

5*.*7

*,ǫ*5*.*8

+ √−1*χM*×*M* )2*n* + *ft,ǫ*

5*.*7

*,ǫ*5*.*8

*χ*2*n*

*M*×*M*

= tan(*θ*0

1

+ *n* arctan( ))Re(*ω C*5*.*6

*t,ǫ*5*.*7*,ǫ*5*.*8

+ √−1*χ*

*M*×*M*

)2*n.*

Define *ω*1*,t,ǫ*5*.*7*,ǫ*5*.*8 by

Σ⌊ *n*−1 ⌋

2

*k*=0

*ω*1*,t,ǫ*5*.*7*,ǫ*5*.*8 =

*k*  *n*!

(−1)

*χ* + √−1*χ*)*n .*

(*n*—2*k*)!(2*k*+1)!

*n*—2*k*

Im(*C*5*.*6

5*.*7

∗ 2*k*+1

5*.*8

Fix *ǫ*5*.*8 and let *t* and *ǫ*5*.*7 converge to 0. For small enough *ǫ*5*.*4, we shall expect

(*π*1)∗(*ωt,ǫ*

*,ǫ*

∧ *π*2 *χ*

)

∫*M*

*ω*5*.*5 to be the weak limit of *ω*1*,t,ǫ*5*.*7*,ǫ*5*.*8 − *ǫ*5*.*4*χ*.

As before, we write *ωt,ǫ*5*.*7*,ǫ*5*.*8 as

*ωt,ǫ*

5*.*7

*,ǫ*5*.*8

(1)

*t,ǫ*5*.*7*,ǫ*5*.*8

= *ω*

(2)

*t,ǫ*5*.*7*,ǫ*5*.*8

+ *ω*

(1*,*2)

*t,ǫ*5*.*7*,ǫ*5*.*8

+ *ω*

(2*,*1)

*t,ǫ*5*.*7*,ǫ*5*.*8

+ *ω*

*,*

and assume that

*π*∗*χ* = √−1 Σ *dz*(2) ∧ *dz*¯(2)

*n*

2

*i*

*i*

*i*=1

and

*ω*(2)

= √−1 Σ *λidz*(2) ∧ *dz*¯(2)

at (*x*1*, x*2).

*t,ǫ*5*.*7*,ǫ*5*.*8

*i* *i*

*i*=1

*n*

To simplify notations, we omit *t, ǫ*5*.*7*, ǫ*5*.*8, then

Σ⌊ *n*−1 ⌋

2

*k*=0

∫

(−1)*k*(*π*1)∗*ω*ˆ*k*

where *ω*ˆ*k* equals to

*ω*1 =

*M*

Im(*C*5*.*6

*χ* + √−1*χ*)*n ,*

*n*! *ω*(1) ∧ (*ω*(2))*n*—2*k*—1 ∧ *π*∗*χ*2*k*+1

2

(*n* − 2*k* − 1)!(2*k* + 1)!

+ *n*! *ω*(1*,*2) ∧ *ω*(2*,*1) ∧ (*ω*(2))*n*—2*k*—2 ∧ *π*∗*χ*2*k*+1

2

(*n* − 2*k* − 2)!(2*k* + 1)!

*n* √ (1)

(1)

(1)

= Σ −1*ωi*¯*j dzi* ∧ *dz*¯*j*

(2*k* + 1)!

∧ ( Σ

1 )(*ω*(2))*n*

*n* √ (1*,*2) (1*,*2) (1)

*i,j*=1

*α ,...α*

distinct *λα*1 *...λα*2*k*+1

1 2*k*+1

(1)

— Σ −1*ωi*¯*l ωjl*¯

(2*k* + 1)!

*,l*distinct *λlλα*1 *...λα*2*k*+1

*dzi* ∧ *dz*¯*j* ∧ Σ

1 (*ω*(2))*n.*

*i,j,l*=1

Remark that

*α ,...α*

*n*−1

1 2*k*+1

⌊ 2 ⌋

Σ

*k*=0

(−1)*k* Σ

(2*k* + 1)!

*α ,...α*

1 − Σ 1 )

*α ,...α*

= − 1 Re(1 + √−1 1 )*...*(1 + √−1 1 )(1 + √−1 1 )*...*(1 + √−1 1 )

(

*,l*distinct *λα*1 *...λα*2*k*+1

distinct *λα*1 *...λα*2*k*+1

1

2*k*+1

1

2*k*+1

*λl*

≤ 0*,* and

*λ*1

*n*−1

⌋

⌊

2 *k*

*λl*—1

*λl*+1 *λn*

Σ (−1)

(2*k* + 1)!

*α ,...α*

*k*=0

Σ 1 (*ω*(2))*n*

1

2*k*+1

= Im(1 + √−1 1 )*...*(1 + √−1 1 )(*ω*(2))*n*

distinct *λα*1 *...λα*2*k*+1

*λ*1 *λn*

= Im(*ω*(2) + √−1*π*2∗*χ*)*n.*

By Lemma [5.5](#_bookmark18),

*n n*

*π*1 *χ*

*i*¯*j*

*λk*

*ik*¯

*jk*¯

*i*

*j*

*P* ∗ (√−1 Σ ((*ω*(1) − Σ 1 *ω*(1*,*2)*ω*(1*,*2))*dz*(1) ∧ *dz*¯(1)))

≤ *PχM* ×*M*

*i,j*=1

*k*=1

(*ω*) − *Qπ*∗ *χ*(*ω*(2)) 1

(2)

2

*< θ*0 + *n* arctan(

2

*C*

5*.*6

) − *Qπ*∗*χ*(*ω* )*.*

So by the monotonicity and convexity of *Pχ*,

*Pχ*(*ω*1)

∫

{*x*1}×*M C*5*.*6 2

(*θ*0 + *n* arctan( 1 ) − *Qπ*∗*χ*(*ω*(2)))Im(*ω*(2) + √−1*π*2∗*χ*)*n*

*<* √

∫

*M* Im(*C*5*.*6*χ* + −1*χ*)*n*

1 ∫ (*Qπ*∗ *χ*(*ω*(2)))Im(*ω*(2) + √−1*π*2∗*χ*)*n*

{*x*1}×*M* 2

= *θ*0 + *n* arctan( )

*C*

—

5*.*6

∫*M* Im(*C*5*.*6

*χ* + √−1*χ*)*n .*

Using the convexity of *x* arctan *x* and the fact that

*Qπ*∗*χ*(*ω*

(2)

) = arctan(

Im(*ω*(2) + √ 1*π*∗*χ*)*n*

)*,*

√

— 2

2

it is easy to see that the minimum of

Re(*ω*(2) +

−1*π*2∗*χ*)*n*

∫{*x*1}×*M*

(*Qπ*∗*χ*(*ω*(2)))Im(*ω*(2) + √−1*π*2∗*χ*)*n*

is achieved if *Qπ*∗*χ*(*ω*(2)) is a constant, which must be *n* arctan( 1 ). Thus

2

2 *C*5*.*6

*Pχ*(*ω*1) *< θ*0. Back to our original notations, it means that *Pχ*(*ω*1*,t,ǫ*5*.*7*,ǫ*5*.*8 ) *< θ*0.

The rest part of Section 3 still holds because *ωn*—*k* has no concentration

*t,ǫ*5*.*7*,ǫ*5*.*8

of mass on the diagonal if *k* 1. Most part of Section 4 also holds because Γ¯*χ*

≥

is convex. We only need to prove the following analogy of Lemma [4.3](#_bookmark15):

Lemma 5.6. *Let C*5*.*9 = cot( *ǫ*1*.*1 )*. Then for all small enough t and all q- dimensional subvarieties V of M*˜ *, as long as q < n,*

6*n*

3

5*.*9

4*.*13

Im ∫

*e*√—1( *ǫ*1*.*1 —*θ*0)((1 + *C*

*t*)*π*∗*ω* + *C*

0

5*.*9

*t*2*ω*

+ √−1(*π*∗*χ* + *t*2*ω*

))*q* ≤ 0*.*

4*.*13

*V*

*Proof.* We already know that

Im ∫

*e*√—1( *ǫ*1*.*1 —*θ*0)((1 + *C*

*t*)*π*∗*ω*

+ √−1(1 + *C*

*t*)*π*∗*χ*)*q* ≤ 0*.*

*V*

So it suffices to show that

3

5*.*9

0

5*.*9

Im ∫

*e*√—1( *ǫ*1*.*1 —*θ*0)((1 + *C*

*t*)*π*∗*ω* + *C*

*t*2*ω*

+ √−1(*π*∗*χ* + *t*2*ω*

))*q*

Im ∫

*V*

3

5*.*9

≤

3

*e*√—1( *ǫ*1*.*1 —*θ*0)((1 + *C*

*t*)*π*∗*ω*

0

+ √−1(1 + *C*

*t*)*π*∗*χ*)*q.*

5*.*9

4*.*13

4*.*13

*V*

As before, there exists a smoo√th function *ϕ*5*.*10 on a neighborhood *O*5*.*11 of

5*.*9

0

5*.*9

*π*(*E*) in *M* such that *ω*5*.*10 = *ω*0 + −1*∂∂*¯*ϕ*5*.*10 *>* 0 satisfies *Pχ*(*ω*5*.*10) *< θ*0 − *ǫ*1*.*1

2

on *O*5*.*11. We define *ϕ*4*.*17 as before on *M* \ *π*(*E*). For any√*s >* 0, by our

assumption, there exists *ϕ*5*.*12 such that *ω*5*.*12 = (1 + *s*)*ω*0 + −1*∂∂*¯*ϕ*5*.*12 *>* 0

satisfies *Pχ*(*ω*5*.*12) *< θ*0 *< π* . Choose *s* small enough such that *Pχ*( *ω*5*.*12 ) *< π* .

4 1+*s* 4

Choose a large enough constant *C*5*.*13. Let *ϕ*5*.*14 be the regularized maximum

of *ϕ*5*.*10 and *ω*5*.*12 + *C*—1 *ϕ*4*.*17 + *C*5*.*13. Then it is easy to see that *ϕ*5*.*14 is

1+*s*

5*.*13

√ ¯

smooth and *Pχ*(*ω*5*.*14) *< π* on *M* for *ω*5*.*14 = *ω*0 + −1*∂∂ϕ*5*.*14 *>* 0. Moreover,

4

there exists a smaller neighborhood *O*5*.*15 of *π*(*E*) such that *ϕ*5*.*14 = *ϕ*5*.*10 on

*O*5*.*15 ⊂ *O*5*.*11.

After replacing *ω*0 by *ω*5*.*14, it suffices to show that

5*.*9

4*.*13

4*.*13

Im(*e*√—1( *ǫ*1*.*1 —*θ*0)((1 + *C t*)*π*∗*ω*

3

5*.*9

+ *C t*2*ω* + √−1(*π*∗*χ* + *t*2*ω*

))*q*)

≤ Im(*e*√—1( *ǫ*1*.*1 —*θ*0)((1 + *C*

5*.*14

*t*)*π*∗*ω*

+ √−1(1 + *C*

*t*)*π*∗*χ*)*q*)*.*

3

First of all,

Im(*e*√—1( *ǫ*1*.*1 —*θ*0)((1 + *C*

3

5*.*9

5*.*9

*t*)*π*∗*ω*

5*.*14

+ √−1(1 + *C*

5*.*14

5*.*9

5*.*9

*t*)*π*∗*χ*)*q*)

— Im(*e*√—1( *ǫ*1*.*1 —*θ*0)((1 + *C*

*t*)*π*∗*ω*

+ √−1*π*∗*χ*)*q*)

3

∫ 1+*C*5*.*9 *t*

√ *ǫ*1*.*1

5*.*9

5*.*14

√

= Re(*qe*

1

≥ 0

—1(

3 —*θ*0)((1 + *C*5*.*9*t*)*π*∗*ω*5*.*14 +

−1*τπ*∗*χ*)*q*—1 ∧ *π*∗*χ*)*dτ*

using a calculation similar to Lemma 8.2 of [[12](#_bookmark30)] because *Pχ*(*ω*5*.*14) *< π* on *M* . If the point is outside *π*—1(*O*5*.*15), then as before, we get the required inequality if *t* is small enough. If the point is inside *π*—1(*O*5*.*15), then

4

Im(*e*√—1( *ǫ*1*.*1 —*θ*0)((1 + *C*

3

5*.*9

*t*)*π*∗*ω*

+ *C t*2*ω* + √−1(*π*∗*χ* + *t*2*ω*

))*q*)

−Im(*e*√—1( *ǫ*1*.*1 —*θ*0)((1 + *C t*)*π*∗*ω*

5*.*14

5*.*9

4*.*13

4*.*13

+ √−1*π*∗*χ*)*q*)

3 5*.*9 5*.*14

*q*—1

√ *ǫ* Σ *q*! √

3

=Im(*e*

—1( 1*.*1 —*θ*0)

*k*!(*q* − *k*)!

((1 + *C*5*.*9

*t*)*π*∗*ω*

5*.*14

+

−1*π*∗*χ*)*k*

∧ (*C*5*.*9*t*2*ω*4*.*13 + √−1*t*2*ω*4*.*13)*q*—*k*)

*k*=0

≤0

using a calculation similar to Lemma 8.2 of [[12](#_bookmark30)] because *Pχ*(*ω*5*.*14) *< θ*0 − *ǫ*1*.*1

2

on *O*5*.*15. We are done.

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