A Greedy Approximation for Minimum Connected Dominating Sets

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Abstract

Given a graph, a connected dominating set is a subset of vertices such that every vertex is either in the subset or adjacent to a vertex in the subset and the subgraph induced by the subset is connected. A minimum connected dominating set is such a vertex subset with minimum cardinality. In this paper, we present a new one-step greedy approximation with performance ratio $\ln \delta + 2$ where δ is the maximum degree in the input graph. The interesting aspect is that the greedy potential function of this algorithm is not supmodular while all previously-known one-step greedy algorithms with similar performace have supmodular potential functions.

Key Words: Connected dominating set, greedy approximation.

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1 Introduction

A *dominating set* of a graph is a subset of vertices such that every vertex is either in the subset or adjacent to (a vertex in) the subset and a *connected* dominating set has an additional condition that the subgraph induced by the dominating set is connected. Given a graph, we are interested in finding a connected dominating set with minimum cardinality. The optimal solution is called a *minimum* connected dominating set. Recently, the minimum connected dominating set problem received a lot of attention in study of wireless networks [1, 3, 7, 8, 2].

The minimum connected dominating set problem is NP-complete [4]. Moreover, Guha and Khuller [5] showed that there does not exist a polynomial-time approximation with performance ratio $\rho H(\delta)$ for $0 < \rho < 1$ unless $NP \subseteq TIME(n^{o(\log \log n)})$, where δ is the maximum degree in the input graph. Guha and Khuller [5] also gave a two-stage greedy algorithm with performance ratio $3 + \ln \delta$. In this paper, we present a new greedy approximation, which is one-stage, with performance ratio $2 + \ln \delta$. The greedy potential function of this algorithm is not supmodular. Therefore, the performance analysis of this algorithm is quite interesting.

2 Preliminary

Consider a graph G and a subset C of vertices in G. All vertices in G can be divided into three classes with respect to C: Vertices belong to C, which for convinience are called *black* vertices. Vertices are not in C but adjacent to C, which are called *grey* vertices. Vertices are not in C and not adjacent to C neither, which are called *white* vertices.

Clearly, C is a connected dominating set if and only if there is no white vertex and the subgraph induced by black vertices is connected. Namely, the number of white vertices plus the number of connected components of the subgraph induced by black vertices, called *black components*, equals one. This suggests a greedy algorithm with the potential function equal to the number of white vertices plus the number of black components as follows.

Greedy Algorithm. For a given connected graph G, do the following:

Set $w := 1$;

while $w = 1$ **do**

if there exists a white or grey vertex such that coloring it in black and its adjacent white vertices in grey would reduce the value of potential function

then choose such a vertex to make the value of potential function reduced in a maximum amount

else set $w := 0$;

Clearly, when the while-loop ends, no white vertex would exist, that is, all black vertices form a dominating set; however, the subgraph induced by black vertices may not be connected. An example is shown in Fig. 1. In fact, what appeared in this example is a

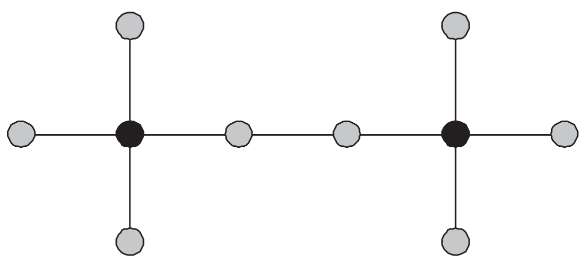

Figure 1: Black components.

typic case. Namely, if the number of subgraph induced by black vertices is not connected, then those black components are connected together by chains of two grey vertices. Based on this observation, Guha and Khuller [5] at end of the above algorithm color, at each time, two grey vertices in black to reduce the number of black components and finally obtain a connected dominating set. This results in a greedy approximation with performance ratio $3 + \ln \delta$.

3 Main Results

We are going to modify the potential function. For each vertex subset C, let $p(C)$ denote the number of connected components of the subgraph induced by black vertices. Let $D(C)$ be the set of all edges incident to vertices in C. Denote by $q(C)$ the number of connected components of the subgraph with vertex set V and edge set $D(C)$, denoted by $(V, D(C))$. Define $f(C) = p(C) + q(C)$.

Lemma 3.1 *Suppose G is a connected graph with at least three vertices. Then, C is a connected dominating set if and only if $f(C \cup \{x\}) = f(C)$ for every $x \in V$.*

Proof. If C is a connected dominating set, then $f(C) = 2$, which reaches the minimum value. Therefore, $f(C \cup \{x\}) = f(C)$ for every $x \in V$.

Coversely, suppose $f(C \cup \{x\}) = f(C)$ for every $x \in V$. First, C cannot be the empty set. In fact, for contradiction, suppose $C = \emptyset$. Since G is a connected graph with at least three vertices, there must exist a vertex x with degree at least two and for such a vertex x, $f(C \cup \{x\}) < f(C)$, a contradiction. Now, we may assume $C \neq \emptyset$. Consider a connected component of the subgraph induced by C. Let B denote its vertex set which is a subset of C. For every gray vertex y adjacent to B, if y is adjacent to a white vertex or a grey vertex not adjacent to B, then we must have $p(C \cup \{y\}) < p(C)$ and $q(C \cup \{y\}) \leq q(C)$; if y is adjacent to a black vertex not in B, then we must have $p(C \cup \{y\}) \leq p(C)$ and $q(C \cup \{y\}) < q(C)$; hence in all cases, $f(C \cup \{y\}) < f(C)$, a contradiction. Therefore, every grey vertex adjacent to B cannot be adjacent to any vertex neither in B nor adjacent to B. Since G is connected, it follows that every vertex of G must belong to B or adjacent to B. That is, $B = C$ is a connected dominating set. $\square$

This lemma means that when the Greedy Algorithm with potential function f ends, all black vertices form a connected dominating set C_G. To establish an upper bound for $|C_G|$, we first study a property of function q. For any $A \subseteq V$ and $y \in V$, denote

$$\Delta_y q(A) = q(A) - q(A \cup \{y\}).$$

Then, we have

Lemma 3.2 *If $A \subset B$, then $\Delta_y q(A) \geq \Delta_y q(B)$.*

Proof. Note that each connected component of graph $(V, D(B))$ is constituted by one or more connected components of graph $(V, D(A))$ since $A \subset B$. Thus, the number of connected components of $(V, D(B))$ dominated by y is no more than the number of connected components of $(V, D(A))$ dominated by y. Therefore, the lemma holds. □

Let C^* be a minimum connected dominating set for G. Let a_i denote the value of potential function f when i vertices have been colored in black in the Greedy Algorithm. Initially, $a_0 = n$ where n is the number of vertices in G.

Lemma 3.3 *For $i = 1, 2, \ldots, |C|$,*

$$a_i \leq a_{i-1} - \frac{a_{i-1} - 2}{|C^*|} + 1.$$

Proof. First, consider $i \geq 2$. Let $x_1, x_2, \ldots, x_{|C|}$ be elements of C_G in the ordering of their appearance in the Greedy Algorithm. Denote $C_i = \{x_1, x_2, \ldots, x_i\}$. Then

$$a_i = f(C_i) = a_{i-1} - \Delta_{x_i} f(C_{i-1})$$

where

$$\Delta_{x_i} f(C_{i-1}) = \max_y \Delta_y f(C_{i-1}).$$

Since C^* is a connected dominating set, we can always arrange elements of C^* in an ordering $y_1, y_2, \ldots, y_{|C^*|}$ such that y_1 is adjacent to a vertex in C_{i-1} and for $j \geq 2$, y_j is adjacent to a vertex in $\{y_1, \ldots, y_{j-1}\}$. Denote $C_j^* = \{y_1, y_2, \ldots, y_j\}$. Denote

$$\Delta_{C^*} f(C_{i-1}) = \sum_{j=1}^{|C^*|} \Delta_{y_j} f(C_{i-1} \cup C_{j-1}^*).$$

Note that

$$\Delta_{y_j} p(C_{i-1} \cup C_{j-1}^*) \leq \Delta_{y_j} p(C_{i-1}) + 1.$$

In fact, y_j can dominate at most one additional connected component in the subgraph induced by $C_{i-1} \cup C^*_{j-1}$ than in the subgraph induced by C_{i-1}, since all $y_1, \ldots, y_{j-1}$ are connected. Moreover, by Lemma 3.2,

$$\Delta_{y_j} q(C_{i-1} \cup C^*_{j-1}) \leq \Delta_{y_j} q(C_{i-1}).$$

Therefore,

$$\Delta_{y_j} f(C_{i-1} \cup C^*_{j-1}) \leq \Delta_{y_j} f(C_{i-1}) + 1.$$

It follows that

$$\begin{aligned} a_{i-1} - 2 &= \Delta_{C^*} f(C_{i-1}) \\ &\leq \sum_{j=1}^{|C^*|} (\Delta_{y_j} f(C_{i-1}) + 1). \end{aligned}$$

There exists $y_j \in C^*$ such that

$$\Delta_{y_j} f(C_{i-1}) + 1 \geq \frac{a_{i-1} - 2}{|C^*|}.$$

Hence,

$$\Delta_{x_i} f(C_{i-1}) \geq \frac{a_{i-1} - 2}{|C^*|} - 1.$$

It implies that

$$a_i \leq a_{i-1} - \frac{a_{i-1} - 2}{|C^*|} + 1.$$

For $i = 1$, the proof is similar, we only need to note a difference that y_1 can be chosen arbitrarily. $\square$

Theorem 3.4 *The Greedy algorith with potential function f produces an approximation solution for minimum connected dominating set with performance ratio $2 + \ln \delta$ where δ is the maximum vertex-degree in input graph.*

Proof. By Lemma 3.3,

$$a_i - 2 \leq (a_{i-1} - 2)(1 - \frac{1}{|C^*|} + 1$$

$$
\begin{aligned}
&\leq (a_0 - 2)(1 - \frac{1}{|C^*|})^i + \sum_{k=0}^{i-1}(1 - \frac{1}{|C^*|})^k \\
&= (a_0 - 2)(1 - \frac{1}{|C^*|})^i + |C^*|(1 - (1 - \frac{1}{|C^*|})^i) \\
&= (a_0 - 2 - |C^*|)(1 - \frac{1}{|C^*|})^i + |C^*|.
\end{aligned}
$$

Since $a_i \leq a_{i-1} - 1$ and $a_{|C_G|} = 2$, we have $a_{|C_G| - 2|C^*|} \geq 2|C^*| + 2$. Set $i = |C_G| - 2|C^*|$. Then

$$2|C^*| \leq (n - 2 - |C^*|)(1 - \frac{1}{|C^*|})^i + |C^*|.$$

Since $(1 - 1/|C^*|)^i \leq e^{-i/|C^*|}$, we obtain

$$i \leq |C^*| \ln \frac{n - 2 - |C^*|}{|C^*|}.$$

Note that each vertex can dominate at most $\delta + 1$ vertices. Hence, $n/|C^*| \leq \delta + 1$. Therefore, $|C_G| = i + 2|C^*| \leq |C^*|(2 + \ln \delta)$. □

4 Discussion

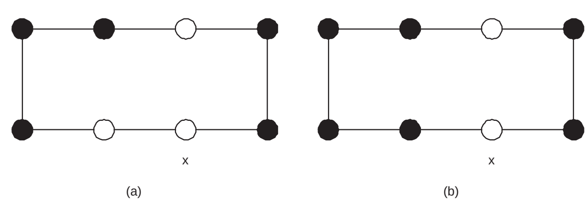

Figure 2: f is not supmodular.

If $f(C)$ is supmodular, then Theorem 3.4 can be derived from a general result [6]. However, it is a very interesting aspect that the potential function $f(C)$ is not supmodular. To see this, let us consider the graph in Fig. 2. The black vertices in Fig. 2(a) form set A and the black vertices in Fig. 2(b) form set B. Clearly, $A \subset B$ and $\Delta_x(A) = 0 > \Delta_x(B) = -1$.

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