Practice Problems

Note: Some problems are computationally longer than what you will see on the exam, but they are still good practice!

1. Suppose we are given an ordering of the numbers 1 through 6 uniformly at random. An inversion is an ordered pair of numbers (i, j) such that $i < j$ but j comes before i in the random order.

   For example, if we are given $3, 5, 1, 4, 2, 6$ then the inversions are:

   $$(1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (4, 5)$$

   (a) How many different orderings are there of 1 through 6?
   (b) Fix two numbers a and b. What is the probability that a comes before b in a random ordering?
   (c) What is the expected number of inversions in a random ordering?

2. Suppose Alice and Bob each choose a point uniformly and independently from the interval $[0, 1]$.

   (a) What is the joint distribution of the two points chosen?

   (b) If A is the point Alice chooses and B is the point Bob chooses, what is the cumulative distribution function for the sum $C = A + B$ of the two points chosen? (Be careful of your bounds.)

   (c) What is the probability density function for the sum?

3. Suppose random variables X and Y have the following joint density:

   $$f(x, y) = \begin{cases} 6(1 - y) & 0 \leq x \leq y \leq 1, \\ 0 \text{ otherwise} \end{cases}$$

   (a) Find the Covariance Cov(X,Y).
   (b) Are X and Y independent?

4. (a) The amount of time it takes Prof. Klivans to walk to the train station is random with a standard deviation of 6 minutes. Suppose she walked to the train station (an impressive) 81 times. The average amount of time it takes for her to get to the train station is 18 minutes.

Give a 95% confidence interval for the mean walking time.

(b) Assuming the average remained 18 minutes and the standard deviation 6, how many times would Prof. Klivans need to have walked to the train station so that the total length of the 95% confidence interval was less than 2 minutes?

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Geometric distribution. Find the maximum likelihood estimator for p.

6. The number of breakdowns per week for a type of minicomputer is a random variable Y with a Poisson distribution and mean λ. A random sample $Y_1, Y_2, \ldots, Y_n$ of observations on the weekly number of breakdowns is available.

   (a) Suggest an unbiased estimator for λ.
   (b) The weekly cost of repairing these breakdowns is $C = 3Y + Y^2$. Show that $E(C) = 4\lambda + \lambda^2$.
   (c) Find a function of $Y_1, \ldots, Y_n$ that is an unbiased estimator of $E(C)$. (Hint: consider $\bar{Y}$ and $(\bar{Y})^2$.

7. Would you rather take a multiple-choice test or a full recall test? If you have no knowledge of the test material, you will score zero on a full-recall test. However, if you are given 5 choices for each multiple-choice question, you have at least one chance in five of guessing each correct answer! Suppose that a multiple-choice exam contains 100 questions, each with 5 possible answers, and guess the answer to each of the questions.

   (a) What is the expected value of the number Y of questions that will be answered correctly?
   (b) Find the standard deviation of Y.
   (c) Calculate the intervals $\mu \pm 2\sigma$ and $\mu \pm 3\sigma$.
   (d) If the results of the exam are curved so that 50 correct answers is a passing score, are you likely to receive a passing score? (What does Chebyshev's inequality give you?)

8. Suppose we assume the ages of people in a region are Normally distributed and the variance of the ages is known to be 25. We sample

100 people and find an average age of 29.

(a) At the .05 level, is there evidence to suggest the average is < 30?
(Use a one-sided statistical test.)
(b) What is the p-value of this test?
(c) If our alternative hypothesis is that the average is 29, what is β?