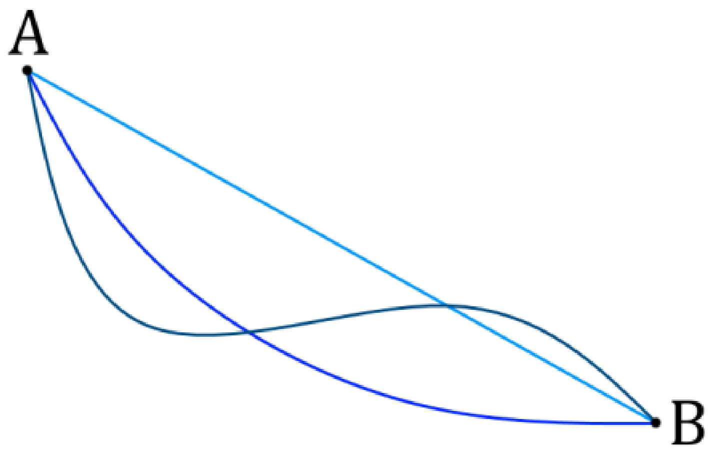
1 Brachistochrone problem

1.1 Problem Statement

Find the minimum time trajectory connecting point A(a,b) and B(c,d) with gravity along negative y-axis.



1.2 Solution

We first formulate the problem as an optimal control problem. Let v(t) be the velocity of the object along the curve y(x). Then,

$$v(t) = \frac{ds}{dt} \tag{1}$$

where ds is the arc length along y(x). Now, for minimising total time T, we choose our cost function as,

$$J = \int_A^B dt = \int_A^B \frac{ds}{v} \tag{2}$$

We can calculate v from conservation of energy.

$$\frac{1}{2}v^2 = gy$$

where g is acceleration due to gravity and y is y-coordinate of object. Hence velocity comes out to be

$$v = \sqrt{2gy} \tag{3}$$

Also ds can be written as,

$$ds = \sqrt{dx^2 + dy^2} \tag{4}$$

Putting in Eq 2, we get our optimal control problem as,

$$J = \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}} = \frac{1}{\sqrt{2g}} \int_a^c \frac{\sqrt{1 + y'^2}}{\sqrt{y}} dx \tag{5}$$

Applying Euler-Lagrange equation on this problem noting that free variable is x and dependent variable is y,

$$\frac{\partial J}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial J}{\partial y'}\right) = 0 \tag{6}$$

Using Eq. 5 for computing each term

$$\frac{\partial J}{\partial y} = -\frac{1}{\sqrt{2g}} \frac{\sqrt{1+(y')^2}}{2y^{3/2}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial J}{\partial y'}\right) = \frac{1}{\sqrt{2g}} \frac{1}{\sqrt{y(1+(y')^2)}} \left(\frac{y''}{1+(y')^2} - \frac{(y')^2}{2y}\right)$$

Putting in Eq. 6 and simplifying, we get the following differential equation,

$$2yy'' + (y')^2 + 1 = 0 \tag{7}$$

Multiplying by y',

$$(y')^3 + 2yy'y'' + y' = 0$$

$$[y + y(y')^2]' = 0$$

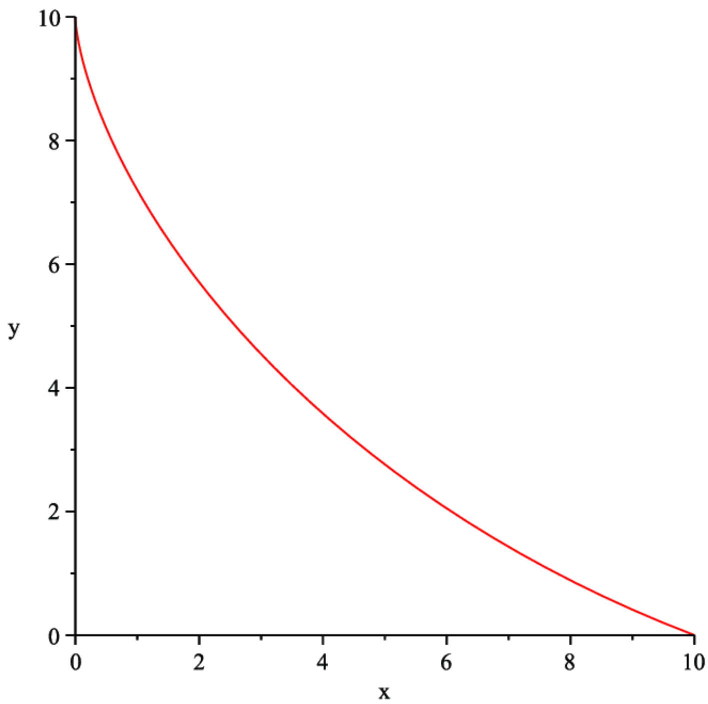
Therefore,

$$y + y(y')^2 = k \quad \text{for some constant k>0.} \tag{8}$$

Eq. 8 can easily be solved using separation of variables to give the parametric curve

$$x = \frac{k}{2}(2\theta - \sin 2\theta) \qquad y = \frac{k}{2}(1 - \cos 2\theta) \tag{9}$$

where θ is the parameter. By using boundary conditions y(a)=b and y(c)=d, we will get the value of k.
This trajectory is of a **cycloid** connecting point A and B.
Solution of Brachistochrone problem with A = (0, 10) and B = (10, 0) is shown below[1]



Hence, the minimum time trajectory is a **cycloid**.

[1] A Discrete Algorithm to the Calculus of Variations - Scientific Figure on ResearchGate. [accessed 15 Aug, 2021]