AE 691 Assignment-2

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(1)

Finite time Linear quadratic regulator :-

System : $\dot{x}(t) = A(t)x(t) + B(t)u$

Cost function:

$$J = \frac{1}{2} x_f^T S_f x_f + \frac{1}{2} \int_{t_0}^{t_f} [x^T Q x + u^T R u] dt$$

initial condition, $x(t_0) = x_0$, t_f is specified, x_f is free.

→ *We omit explicit time dependence representation.

We defined Hamiltonian as,

$$H(x, u, \lambda) = \frac{1}{2}\{x^T Q x + u^T R u\} + \lambda^T [\dot{x}]$$

$$= \frac{1}{2}\{x^T Q x + u^T R u\} + \lambda^T \{Ax + Bu\}$$

→ Optimal conditions:

i) $\left.\frac{\partial H}{\partial u}\right|_* = 0 \Rightarrow Ru^* + B^T \lambda^* = 0$

$$\therefore u^*(t) = -R^{-1}(t) B^T(t) \lambda(t) \quad \text{—(1)}$$

ii) $\dot{x}^*(t) = \left.\frac{\partial H}{\partial \lambda}\right|_* \leftrightarrow \dot{x}^*(t) = A(t)x(t) + B(t)u(t)$ —(11)

which gives dynamics eqn.

(iii) $\dot{\lambda}^*(t) = -\left(\frac{\partial H}{\partial x}\right)_*$

$$\dot{\lambda}^*(t) = -Q(t)x^*(t) - A^T(t)\lambda^*(t) \quad \text{— (iii)}$$

From general boundary condition,

$$\left(H^* + \frac{\partial S}{\partial t}\right)_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial x^*}\right) - \lambda^*(t)\right]_{t_f} \delta x_f = 0$$

as t_f is specified, $\delta t_f = 0$.

$$\therefore \lambda^*(t) = \left.\frac{\partial S}{\partial x^*}\right|_{t_f} = \frac{\partial\left[\frac{1}{2}x_f^T F(t_f)\,x(t_f)\right]}{\partial x_f}$$

$$= F(t_f)\;x^*(t_f)$$

This gives a TPBVP.

For solving, we assume

$$\lambda(t) = P(t)\,x(t) \quad \text{— (iv)}$$

with $\quad P(t_f) = F(t_f)$

~~Putting (iv) in~~

Now,

$$\dot{\lambda}(t) = \dot{P}(t)\,x(t) + P(t)\,\dot{x}(t)$$

$$= \dot{P}(t)\cdot x(t) + P(t)\,(A(t)x(t) + B(t)u(t))$$

putting in (iii)

$$\dot{P}x + P(Ax+Bu) = -Qx - A^T P x$$

puttin $u = -R^{-1}B^T\lambda$
$= -R^{-1}B^T P x$ above,

$$\dot{P}x + P(Ax - BR^{-1}B^T P x) = -Qx - A^T P x.$$

Rearranging terms,

$$(\dot{P} + PA + A^T P + Q - PBR^{-1}B^T P)x = 0.$$

Hence, this gives us DRE or differential Riccati eqn.

$$\boxed{\dot{P}(t) + P(t)A(t) + A^T(t)P(t) + Q(t) - P(t)B(t)R^{-1}B^T(t)P(t) = 0}$$

with B.C $\Rightarrow$ $P(t_f) = F(t_f)$

Solving this eqn gives $P(t)$, which can be substituted in (1) to give feedback control $u^*(t)$ as

$$\boxed{u^*(t) = -R^{-1}B^T P x^*}$$

Infinite -time LQR

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

$$J = \frac{1}{2}\int_{t_0}^{\infty} [x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt$$

Here, terminal constraint is not present as $\lim_{t\to\infty} x_f = 0$

Hence proceeding as before,

$$u^*(t) = -R^{-1}(t)B^T(t)P(t)x(t)$$

where, $\lim_{t\to\infty} P(t) = 0$.

Now, if the system is controllable and $P(t \to t_f) = 0$, then

P approaches a fixed ~~fixed~~ constant value $\bar{P}$ as $t \to \infty$.

Using this in DRE,

$$\dot{\bar{P}} = 0 = -\bar{P}A - A^T\bar{P} + \bar{P}BR^{-1}B^T\bar{P} - Q$$

$$\therefore \boxed{\bar{P}A + A^T\bar{P} + Q - \bar{P}BR^{-1}B^T\bar{P} = 0}$$

This gives ARE (Algebric Ricatti Equation)

and, $\boxed{u^* = -R^{-1}B^T\bar{P}x}$

(1) Given system is

$$\begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

let

$$M = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

we choose state X as $X = \begin{bmatrix} \phi \\ \theta \\ \psi \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$

then,

$$\dot{X} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

This can be written as

$$\dot{X} = AX + BU, \quad \text{where } U = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

Now, <u>finite time LQR</u>

$$J = S(X(t_f)) + \int_{t_0}^{t_f} X^T Q X + U^T R U \, dt$$

s.t. $\dot{X} = AX + BU$

$X(t_0) = X_0$

$t_f \to$ fixed

$X_f \to$ free

$Q \in R^{6\times6}, \; R \in R^{3\times3}$

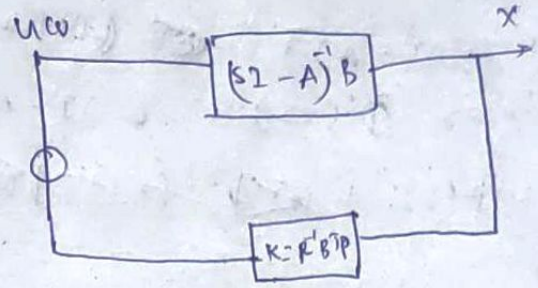
Infinite time LQR

$$J = \int_{t_0}^{\infty} (x^T Q x + u^T R u) \, dt$$

→ Terminal Cost will not be there in the cost function as infinite time is there.

$$\text{s.t. } \dot{x} = Ax + Bu.$$

(3) Tranfer function



we know,

$$u^* = -Kx = -R^{-1}B^TPx.$$

From the block diagrame

Close loop tranfer function $= \dfrac{Kx\,(sI-A)^{-1}B}{1+K(sI-A)^{-1}B}$

Stability analysis

From the block diagram, loop tranfer function

$$G_{EQ} = K\,(sI-A)^{-1}B.$$

and open loop tranfer function is

$$G_{OL} = C(sI-A)^{-1}B$$

we state the Kalman equality theorem.

Theorem:- $[I + G_{LQ}(-s)]^T R\,[I + G_{LQ}(s)]$

$$= R + G_{OL}^T(-s)\,G_{OL}(s)$$

Proof:- From ARE,

$$-[PA + A^TP + Q - PBR^{-1}B^TP] = 0.$$

Add subtract sP,

$$sP - PA - sP - A^TP - Q + PBR^{-1}B^TP = 0.$$

Thus

$$P(sI - A) + [-sI - A^T]P + K^T R K = Q.$$

Premultiply by $B^T \Phi'(-s)$ and post multiply by $\Phi(s) B$, ($\Phi(s) = (sI - A)^{-1}$, $\Phi'(-s) = (-sI - A^T)^{-1}$)

we get

$$B^T \Phi^T(-s)\, Q\, \Phi'(s) B + R$$
$$= [I + K \Phi(-s) B]^T R [I + K \Phi(s) B]$$

~~puttis~~ s.

This gives us the required proof.

Putting $s = j\omega$ in kalman equality and rearranging, we get,

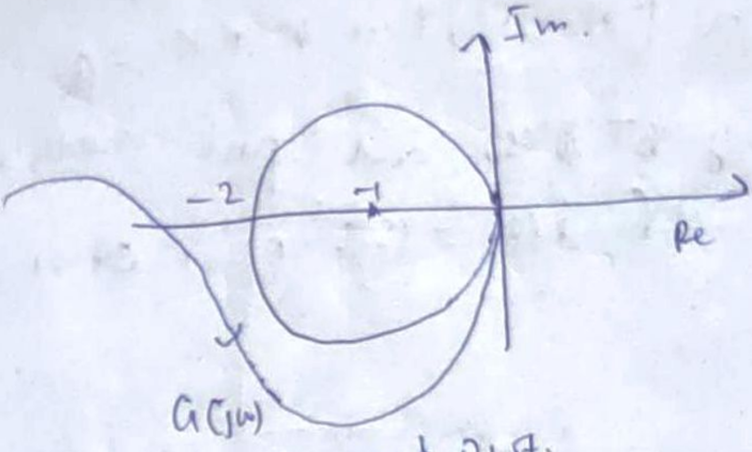
$$\| I + K^T [j\omega I - A]^{-1} B\|^2 = I + \| C^T [j\omega I - A]^{-1} B\|^2.$$

Hence,

$$(1 + \bar{k} [j\omega I - A]^{-1}) \geq 1.$$

So the gain margin is ∞. as the distance b/twn critical point $-1 + j0$ is always at least 1 in Nyquist plot.

- Phase margin is atleast $60°$ as for $\theta \in [-60, 60]$ rotation, the number of encirclements dont change.



Nyquist plot.

(4)

Given:

$$J = \frac{1}{2}\int_0^{T_f} u^2 \, dt.$$

$$\dot{\theta} = q$$

$$I\dot{\omega} + \omega \times I\omega = M. \qquad M = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$$T = \nu k \, l \, (C_{T1} + C_{T2} + C_{T3} + C_{T4})$$

$$l = \nu k l \, (C_{T1} - C_{T2} - C_{T3} + C_{T4})$$

$$m = \nu k l \, (C_{T1} + C_{T2} - C_{T3} - C_{T4})$$

$$n = \frac{\nu k l}{\sqrt{2}} \left(C_{T1}^{3/2} - C_{2}^{3/2} + C_{T3}^{3/2} - C_{T4}^{3/2} \right)$$

We want to have an flip maneuver.

then

$$\theta(0) = 0 \qquad \theta(t_f) = -\pi$$

Other angles ϕ and ψ will be maintained at zero. $(\dot{\phi} = \dot{\psi} = 0)$

Now,

let $y = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$

then, $\dot{y} = \Phi \omega$ where ω is angular velocity vector. and Φ is matrix depends on (ϕ, θ, ψ).

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

~~Also,~~

~~Differentiate,~~

$$\cancel{\ddot{y} = \dot{\Phi}\omega + \Phi\dot{\omega}}$$

$$\cancel{\therefore \dot{\omega} = \Phi^{-1}(\ddot{y} - \dot{\Phi}\omega)}$$

Given $\dot{\theta} = q$, $\dot{\phi} = 0$, $\dot{\psi} = 0$, $\phi = 0$, $\psi = 0$.

putting in $\dot{y} = \Phi \omega$, we get.

$$\omega_2 = \dot{\theta} = q$$

$$\omega_1 + \tan\theta\, \omega_3 = 0.$$

$$\frac{\omega_3}{\cos\theta} = 0.$$

$$\therefore \left.\begin{matrix} \omega_2 = q. \\ \omega_1 = 0 \\ \omega_3 = 0 \end{matrix}\right]$$

~~Now from Euler equation,~~

Now from Euler I_{yy} Equation,

Here, $\dot{\omega}_2 = \dot{q}_2 = \dfrac{m}{I_{y_3}}$

From given equation,

$$I_{yy}\dot{q}_2 = m = \nu k l \,(C_{T_1} + C_{T_2} - C_{T_3} - C_{T_4}).$$

$$\therefore \boxed{C_{T_1} + C_{T_2} - C_{T_3} - C_{T_4} = \frac{I_{y_3}\dot{q}_2}{\nu k l.}}$$

Now we have to minimi

$$J = \frac{1}{2}\int^{T_f} u^2\, dt$$

$$J = \frac{1}{2} \int_0^{T_f} \langle C, C \rangle \, dt$$

where $C = \begin{bmatrix} C_{T1} \\ C_{T2} \\ C_{T3} \\ C_{T4} \end{bmatrix}$.

$$\therefore J = \frac{1}{2} \int_0^{T_f} \left(C_{T1}^2 + C_{T2}^2 + C_{T3}^2 + C_{T4}^2 \right) dt.$$

→ T_f can be calculated as follows,

$\theta(0) = 0 \qquad \theta(T_f) = -\pi.$

$\therefore$

$$\boxed{-\pi = q T_f + \frac{1}{2} \dot{q} T_f^2}$$

Now for minimizing J, we have to minimize $\left(C_{T1}^2 + C_{T2}^2 + C_{T3}^2 + C_{T4}^2 \right)$.

with constraint.

$$C_{T1} + C_{T2} - C_{T3} - C_{T4} = \frac{I_{yy} \dot{q}_i}{v k l}.$$

For maximum,

$$\boxed{\begin{aligned} C_{T1} &= C_{T2} = \frac{I_{yy} \dot{q}_i}{v k l} \\ C_{T3} &= C_{T4} = \frac{1}{2} \frac{I_{yy} \dot{q}_i}{v k l} \end{aligned}}$$

Ans