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Homework Assignment 1
AE691A-OCRL

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1 Definitions with Examples

1.1 Infimum

Let $S \subset \mathbb{R}$ be a set bounded from below. Then their exist a unique lower bound α_0 for S such that if, α is a lower bound then $\alpha \leq \alpha_0$. Thus α_0 is greatest lower bound and is called infimum of Sdenoted by inf S.
Example:

- For the set (-2,$\inf$) and [-2,$\inf$], -2 is infimum as for any lower bound α, -2 $\geq \alpha$.
- For the set $\{$-1,0,1,...$\}$, -1 is infimum as -1 is greatest lower bound of the set.

1.2 Minimum

If α_0=inf S∈ S, then α_0 is called minimum of set S denoted by min S.
Example:

- For the set (-2,$\inf$) and [-2,$\inf$], -2 is minimum for second set as -2 $\in$ [-2, $\inf$] but it is not minimum for the first set.
- For the set $\{$-1,0,1,...$\}$, -1 is minimum as -1 is greatest lower bound of the set and -1 belongs to the set.

1.3 Supremum

Let $S \subset \mathbb{R}$ be a set bounded from above. Then their exist a unique upper bound β_0 for S such that if, β is a upper bound then $\beta \geq \beta_0$. Thus _0 is lowest upper bound and is called supremum of S denoted by sup S.
Example:

- For the set (-$\inf$,2) and [-$\inf$,2], 2 is supremum as for any upper bound β, 2 $\leq \beta$.
- For the set $\{$-1,-2,-3,...$\}$, -1 is supremum as -1 is lowest upper bound of the set.

1.4 Maximum

If β_0=sup S∈ S, then β_0 is called maximum of set S denoted by max S.
Example:

- For the set (-$\inf$,2) and [-$\inf$,2], 2 is maximum for second set as 2 $\in$ [-$\inf$,2] but it is not maximum for the first set.
- For the set $\{$-1,-2,-3,...$\}$, -1 is maximum as -1 is lowest upper bound of the set and -1 belongs to the set.

1.5 Compact Set

A set S of real numbers is called compact if every sequence in S has a sub sequence that converges to an element again contained in S. Examples:

- Set [0,1] is compact since every sequence contained in it is bounded, so we can extract a convergent sub sequence by the Bolzano-Weierstrass theorem. Using the theorem on intermediate and boundary points and noting that the set [0, 1] is closed, the limit of this sub sequence must be contained in [0, 1]. Hence, the set is compact by definition.
- The Set [0, 1) is not compact. Consider the sequence { 1 - 1/n }. Then that sequence is contained in [0, 1), and converges to 1. Therefore, every sub sequence of it must also converge to 1, which is not part of the original set. Therefore, the set can not be compact.
- set of Natural number $\mathbb{N}$ is not compact since the sequence $\{$n$\}$ of natural numbers converges to infinity, and so does every sub sequence. But infinity is not part of the natural numbers.

Proposition: A set S of real numbers is compact if and only if it is closed and bounded.

1.6 Convex set

A set C is convex if the line segment between any two points in C lies in C, *i.e.* $\forall\, x_1, x_2 \in C$, $\forall\, \theta \in [0,1]$:

$$\theta x_1 + (1-\theta)x_2 \in C$$

Examples:

- The empty set $\varnothing$, the singleton set $\{x_0\}$, and the complete space $\mathbb{R}^n$.
- Line $a^T x = b$
- $f : \mathbb{R}^2 \mapsto \mathbb{R}$ with f(x,y)= $x^2 + y^2$

2 Weierstrass's theorem

Theorem: Let $D \subset \mathbb{R}^m$ be compact and $f : D \mapsto \mathbb{R}$ be continuous, then f has a global minimizer. In addition, if D is convex set and f is strictly convex function, then f has a strict global minimizer.

This results do not hold in general if the assumption about D and f is weakened.
For example, consider $D = \mathbb{R}$ and $f(x) = x + e^x - 1$, where x $\in$ D. Here note that f is strictly convex but D is not compact since $\mathbb{R}$ is closed but not bounded. So $f(x)$ does not have a global minimizer in D.

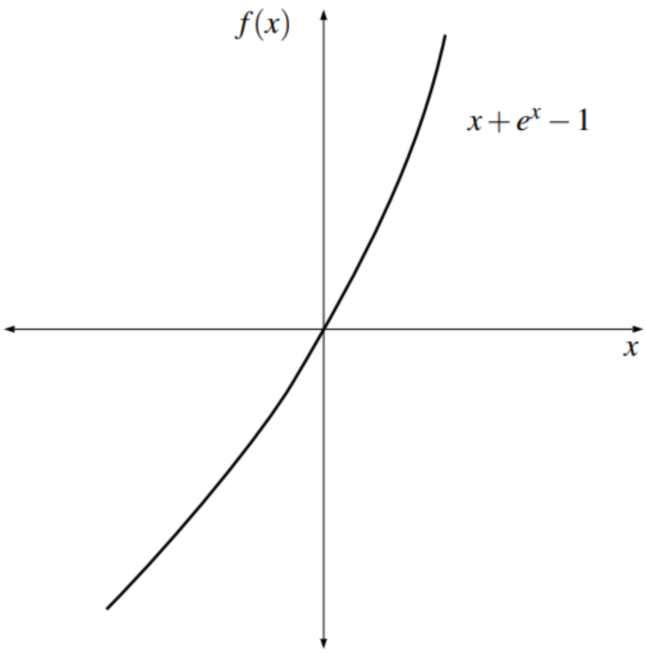

Abbildung 1: $f(x) = x + e^x - 1$, strictly convex but no global minimizer in $\mathbb{R}$