LQR Optimal Control for High Precision Rendezvous and Docking

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I. System Description

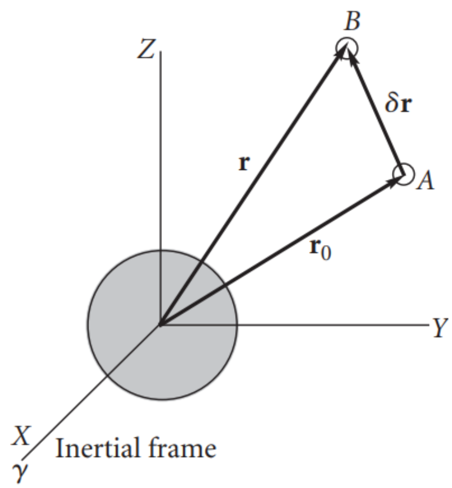

Fig. 1 Orbital framework of Docking: A is target and B is chaser.

A general docking scenario consist of a target vehicle A and a chaser vehicle B. A is in an keplerian circular orbit with radius r_0 while B performs maneuvers at position vector δr from A in an non-keplerian orbit. From the figure,

$$\vec{r} = \vec{r}_0 + \vec{\delta} r$$

with

$$\frac{\delta r}{r_0} << 1$$

We define a moving frame(CWH frame) xyz with origin on A with,

$$\hat{i} = \frac{\vec{r}_0}{r_0}$$

y-axis is the local horizon and $\hat{k} = \hat{i} \times \hat{j}$

Let angular velocity of A(and so of the CWH frame) be Ω

II. Mathematical Model

The equations of motion for the chaser spacecraft are nonlinear and can be expressed in vector form as:

$$\ddot{\vec{r}} = -\mu\frac{\vec{r}}{r^3} + \frac{\vec{F}}{m} \tag{1}$$

$\vec{F}$ is force applied on the spacecraft.

Putting $\vec{r} = \vec{r}_0 + \vec{\delta}r$ in (1),

$$\ddot{\vec{\delta r}} = -\ddot{\vec{r}}_0 - \mu\frac{\vec{r}_0 + \delta\vec{r}}{r^3} + \frac{\vec{F}}{m} \tag{2}$$

Now,

$$r^{-3} = r_0^{-3}\left(1 + \frac{2\vec{r}_0 \cdot \vec{\delta}r}{r_0^2}\right)^{-3/2}$$

by neglecting $\left(\frac{\delta}{r_0}\right)^2$,

Using Binomial theorem this simplifies as:

$$\left(1 + \frac{2\vec{r}_0 \cdot \vec{\delta}r}{r_0^2}\right)^{-3/2} = 1 + \left(\frac{-3}{2}\right)\left(1 + \frac{2\vec{r}_0 \cdot \vec{\delta}r}{r_0^2}\right)$$

$$\therefore\ \frac{1}{r^3} = \frac{1}{r_0^3} - \frac{3}{r_0^5}\vec{r}_0 \cdot \vec{\delta}r$$

Putting this in (2):

$$\ddot{\vec{\delta r}} = \ddot{\vec{r}}_0 - \mu\left(\frac{1}{r_0^3} - \frac{3}{r_0^5}\vec{r}_0 \cdot \vec{\delta}r\right)(\vec{r}_0 + \delta\vec{r}) + \frac{\vec{F}}{m}$$

$$= -\ddot{\vec{r}}_0 - \mu\left[\frac{\vec{r}_0}{r_0^3} + \frac{\vec{\delta}r}{r_0^3} - \frac{3(\vec{r}_0 \cdot \vec{\delta}r)\vec{r}_0}{r_0^5} + \text{Higher order terms in } \delta r \text{ (which we neglect)}\right] + \frac{\vec{F}}{m}$$

$$\therefore\ \ddot{\vec{\delta r}} = -\ddot{\vec{r}}_0 - \mu\left[\frac{\vec{r}_0}{r_0^3} + \frac{\vec{\delta}r}{r_0^3} - \frac{3(\vec{r}_0 \cdot \vec{\delta}r)\vec{r}_0}{r_0^5}\right] + \frac{\vec{F}}{m}$$

Now dynamical equation of the target vehicle is

$$\ddot{\vec{r}}_0 = -\mu\frac{\vec{r}_0}{r_0^3}$$

Substituting above,

$$\ddot{\vec{\delta r}} = -\frac{\mu}{r_0^3}\left[\delta\vec{r} - \frac{3(\vec{r}_0 \cdot \vec{\delta}r)\vec{r}_0)}{r_0^2}\right] + \frac{\vec{F}}{m} \tag{3}$$

Now,

$$\vec{\delta}r = \delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k}$$

Using vector kinematics, we know that,

$$\ddot{\vec{r}} = \ddot{\vec{r}}_0 + \vec{\Omega} \times \delta r + \dot{\vec{\Omega}} \times (\vec{\Omega} \times \vec{\delta r}) + 2\vec{\Omega} \times \delta \vec{v}_{rel} + \vec{\delta} a_{rel}$$

Putting $\vec{r} = \vec{r}_0 + \delta r$ above,

$$\ddot{\vec{\delta r}} = \dot{\vec{\Omega}} \times \vec{\delta r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\delta r}) + 2\vec{\Omega} \times \vec{\delta} v_{rel} + \vec{\delta} a_{rel}$$

Assuming target vehicle to be in **circular orbit**, $\dot{\vec{\Omega}} = 0$,

$$\ddot{\vec{\delta r}} = \vec{\Omega} \times (\vec{\Omega} \times \vec{\delta r}) + 2\vec{\Omega} \times \vec{\delta} v_{rel} + \vec{\delta} a_{rel}$$

Expanding the RHS,

$$\ddot{\vec{\delta r}} = \vec{\Omega}(\vec{\Omega} \cdot \vec{\delta r}) - \Omega^2 \vec{\delta r} + 2\vec{\Omega} \times \vec{\delta} v_{rel} + \vec{\delta} a_{rel}$$

Putting $\vec{\Omega} = n\hat{k}$, $\vec{\delta} v_{rel} = \delta\dot{x}\hat{i} + \delta\dot{y}\hat{j} + \delta\dot{z}\hat{k}$ and $\vec{\delta} a_{rel} = \delta\ddot{x}\hat{i} + \delta\ddot{y}\hat{j} + \delta\ddot{z}\hat{k}$ and recollecting terms this simplifies as,

$$\ddot{\vec{\delta r}} = (-n^2\delta x - 2n\delta\dot{y} + \delta\ddot{x})\hat{i} + (-n^2\delta y - 2n\delta\dot{x} + \delta\ddot{y})\hat{j} + \delta\ddot{z}\hat{k} \tag{4}$$

Now from (3),by substituting values of $\vec{\delta r}$, $\vec{r}_0 \cdot \delta r = r_0 \delta x$ and using $n^2 = \frac{\mu}{r_0^3}$ we get,

$$\ddot{\vec{\delta r}} = -n^2 \left[\delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k} - \frac{3}{r_0^2}(r_0\delta x) r_0 \hat{i} \right] + \frac{F_x}{m}\hat{i} + \frac{F_y}{m}\hat{j} + \frac{F_z}{m}\hat{k} = (2n^2\delta x + \frac{F_x}{m})\hat{i} + (-n^2\delta y + \frac{F_y}{m})\hat{j} + (-n^2\delta z + \frac{F_z}{m})\hat{k} \tag{5}$$

Equating the value of $\ddot{\vec{\delta r}}$ from equation (4) and (5) and collecting terms,

$$(\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y})\hat{i} + (\delta\ddot{y} + 2n\delta\dot{x})\hat{j} + (\delta\ddot{z} + n^2\delta z)\hat{k} = \frac{F_x}{m}\hat{i} + \frac{F_y}{m}\hat{j} + \frac{F_z}{m}\hat{k}$$

This can be simplified to give linear CWH equation:

$$\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y} = \frac{F_x}{m} = u_x \tag{6}$$

$$\delta\ddot{y} + 2n\delta\dot{x} = \frac{F_y}{m} = u_y \tag{7}$$

$$\delta\ddot{z} + n^2\delta z = \frac{F_z}{m} = u_z \tag{8}$$

This model can be formulated as:

$$\dot{X} = AX + BU$$

where X ∈ $\mathcal{R}^6$ is the state vector and U ∈ $\mathcal{R}^3$ is the control vector.

$$X = \begin{pmatrix} \delta x \\ \delta y \\ \delta z \\ \delta \dot{x} \\ \delta \dot{y} \\ \delta \dot{z} \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

A. Controllability:

Now as we have derived our system model, we must check if the system is controllable. This is necessary as system must have ability to move the internal states from any initial state to any final state in a finite time interval using any control input. Control input is subjected to noise from sensor and controllable system guarantees that any final states can be reached. Now,

$$rank\begin{bmatrix} A & A^2B\ldots & A^{(n-1)}B \end{bmatrix} = 6 = rank(A)$$

∴ The derived system satisfies the necessary and sufficient rank criteria for controllability.

III. Optimal Control Design Using Krotov Conditions

LQR control is selected for the controlling the system dynamics. LQR, as an optimal in terms of energy-like regulator, provides robust stability with a minimized energy-like performance index. It is also very computationally efficient as gain matrix can be calculated offline. The cost function used in infinite horizon LQR is,

$$min\ J = \int_0^\infty X^T QX + U^T RU\, dt \tag{9}$$

where Q≥0 and R>0 ∀ t

Although the calculation of variation and HJB equation-based(derived from Dynamic Programming) approaches are widely employed for solving this, there are some assumptions associated with these approaches in their solution procedure. Specifically, the calculus of variation approach uses costates and their relationship with states to compute the optimal control law. Similarly, the HJB equation-based approach requires the existence of the continuously differentiable optimal cost function, and it's gradient with respect to the state is the costate corresponding to the optimal trajectory.So, the information about the optimal cost function must be known a priori.

We now develop optimal control by using Krotov functions which will give sufficient conditions and the control law that will be derived provide global optimum without encountering issues stated above which will be valid for all types of cost functions.

From the equivalency principle, this optimization problem can be restated as:

$$min\ J_{eq} = L(x(0), t)) + \int_0^\infty S(X, U, t))\, dt \tag{10}$$

where L is a Krotov function which when chosen appropriately reduces the non-convex function and

$$s = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial X}[AX + BU] + X^T QX + U^T RU$$

So, the problem reduces to

$$\min_{(X,U)\in\mathcal{R}^6\times\mathcal{R}^3} S(X, U, t), \forall t \in [t, \infty] \tag{11}$$

This gives us a sufficient condition as minimum S will minimise the cost function. As S is non-convex, we set

$L = X^T PX$,

$$\therefore S = X^T \dot{P}X + X^T (P + P^T)[AX + BU] + X^T QX + U^T RU$$

$$= X^T \dot{P}X + X^T PAX + X^T P^T AX + X^T PBX + X^T P^T BX + X^T QX + U^T RU$$

Adding and subtracting $X^T (\frac{1}{2}PBR^{-1}B^T P + \frac{1}{4}PBR^{-1}B^T P^T + \frac{1}{4}P^T BR^{-1}B^T P)X$ and collecting terms:

$$S = X^T (\dot{P} + A^T P + Q - \frac{1}{2}PBR^{-1}B^T P - \frac{1}{4}PBR^{-1}B^T P^T - \frac{1}{4}P^T BR^{-1}B^T P)X + \text{scalar terms}$$

Now S will be convex if,

$$\dot{P} + A^T P + Q - \frac{1}{2}PBR^{-1}B^T P - \frac{1}{4}PBR^{-1}B^T P^T - \frac{1}{4}P^T BR^{-1}B^T P \geq 0$$

We set $\dot{P} = 0$, as a static P is achieved for infinite horizon case and it has been also demonstrated that calculating P iteratively is computationally complex and provides little benfit over static case. For optimality P must satisfy,

$$A^T P + Q - \frac{1}{2}PBR^{-1}B^T P - \frac{1}{4}PBR^{-1}B^T P^T - \frac{1}{4}P^T BR^{-1}B^T P = 0 \tag{12}$$

Therefore The optimal Control solution is:

$$U^* = -\frac{1}{2}R^{-1}B^T (P^T + P)X^* \tag{13}$$

A. Stability Analysis

Docking requires a robust control as parameter error can lead to unwanted deviations from the calculated optimal solution. System is prone to parameter uncertainties and checking stability is necessary for convergence.
For checking stability of the calculated control solution, we choose Lyapunov function $V = X^T (P^T + P)X$, assuming $P + P^T > 0$, then:

$$\dot{V} = X^T (P^T + P)\dot{X} + \dot{X}^T (P^T + P)X$$

Now, using $\dot{X} = AX + BU$ where U is substituted from (13)

$$\dot{V} = 2X^T (-Q - \frac{1}{2}PBR^{-1}B^T P - \frac{1}{4}PBR^{-1}B^T P^T - \frac{1}{4}P^T BR^{-1}B^T P)X \leq 0$$

which is easy to verify since middle terms are negative.
$\therefore$ For, $P + P^T > 0$ the closed loop system is Lyapunov stable for $Q \geq 0$ and asymptotically stable for Q>0

B. Gain Selection

Gain matrix Q and R are selected as diagonal matrix with Q>0 for asymptotic stability.

$$
Q = \begin{pmatrix}
\frac{\alpha_1^2}{(X_1)^2_{max}} & 0 & 0 & 0 & 0 & \\
0 & \frac{\alpha_2^2}{(X_2)^2_{max}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\alpha_3^2}{(X_3)^2_{max}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\alpha_4^2}{(X_4)^2_{max}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\alpha_5^2}{(X_5)^2_{max}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\alpha_2^6}{(X_6)^2_{max}}
\end{pmatrix}
$$

and

$$
R = \begin{pmatrix}
\frac{\beta_1^2}{(U_1)^2_{max}} & 0 & 0 \\
0 & \frac{\beta_2^2}{(U_2)^2_{max}} & 0 \\
0 & 0 & \frac{\beta_3^2}{(U_3)^2_{max}}
\end{pmatrix}
$$

where $\sum_{i=1}^{6} \beta_i^2 = 1$ and $\sum_{i=1}^{6} \alpha_i^2 = 1$ which are used to add an additional relative weighting on the various components of the state/control and $(X_i)_{max}$and $(U_i)_{max}$ represent the largest desired response/control input for that component of the state/actuator signal.

IV. Results