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The Stability of Linear Time-dependent Control Systems

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ABSTRACT

It is shown that the trivial solution of the equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, where the dash denotes the time derivative, is asymptotically stable if the rate of change of the elements $a_{ij}(t)$ of $\mathbf{A}(t)$ is sufficiently slow. An explicit bound for the $a_{ij}'(t)$ is obtained when the matrix $\mathbf{A}$ has a special form.

§ 1. INTRODUCTION

IT commonly occurs that the equations of a controlled system can be linearized in a given operating condition, and can be written in the canonical form:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad (1)$$

where $\mathbf{x}$ is an n-vector and the dash denotes the derivative with respect to time. The criterion for asymptotic stability is then simply that every eigenvalue λ of $\mathbf{A}$ should have a negative real part.

If the operating conditions of the system vary, the matrix $\mathbf{A}$ may be time-dependent:

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}. \qquad (2)$$

It is well known that stability is not then ensured by having

$$\operatorname{Re}\lambda(\mathbf{A}) \leqslant -\epsilon < 0$$

for all t: the following counter-example is quoted by Zubov (1962) from Vinogradov:

$$\left.\begin{aligned} x_1' &= (-1 - 9\cos^2 6t + 12\sin 6t\cos 6t)x_1 \\ &\quad + (12\cos^2 6t + 9\sin 6t\cos 6t)x_2, \\ x_2' &= (-12\sin^2 6t + 9\sin 6t\cos 6t)x_1 \\ &\quad - (1 + 9\sin^2 6t + 12\sin 6t\cos 6t)x_2. \end{aligned}\right\} \qquad (3)$$

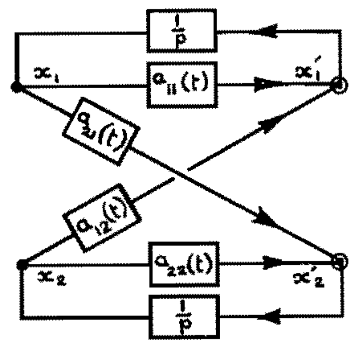
Here the eigenvalues are -1 and -10 for all t, yet the solution‡ is:

$$\left.\begin{aligned} x_1 &= a_1\exp(2t)(\cos 6t + 2\sin 6t) + a_2\exp(-13t)(\sin 6t - 2\cos 6t), \\ x_2 &= a_1\exp(2t)(2\cos 6t - \sin 6t) + a_2\exp(-13t)(2\sin 6t + \cos 6t), \end{aligned}\right\} \qquad (4)$$

† Communicated by the Author.
‡ An error in the second equation as quoted by Zubov (1962) has been corrected.

and the system is unstable. Equations (3) correspond to the system shown in fig. 1.

Fig. 1



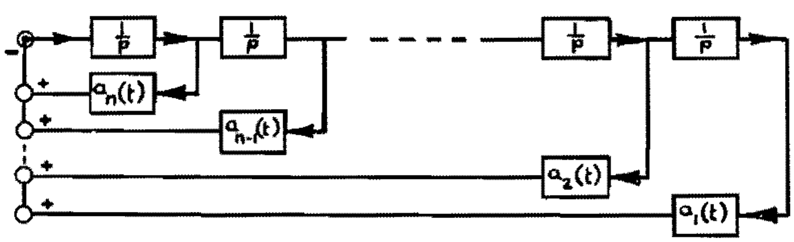
Intuition suggests that though the condition $\mathrm{Re}\,\lambda(\mathbf{A}) \leqslant -\epsilon < 0$ is not in itself sufficient to ensure stability when $\mathbf{A}$ is time-dependent, yet it will do so if the variation of $\mathbf{A}$ is sufficiently slow. This can be proved, as below, but the difficulty remains of setting bounds to the permitted rate of variation of $\mathbf{A}$. Such bounds will be developed here for the particular form of $\mathbf{A}$ given by:

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_1(t) & -a_2(t) & -a_3(t) & \dots & -a_n(t) \end{bmatrix} \quad . \quad . \quad (5)$$

and corresponding to the equation

$$x^{(n)} + a_n(t)x^{(n-1)} + \dots + a_2(t)x' + a_1(t)x = 0. \quad . \quad . \quad . \quad . \quad (6)$$

Fig. 2



Equation (5) corresponds to the system shown in fig. 2 and is sufficiently general to include some problems of practical interest. With this special form for $\mathbf{A}$, the eigenvalues are uniquely related to the elements

$$a_1(t),\ a_2(t),\ \dots,\ a_n(t).$$

The bounds on the rate of change of $\mathbf{A}$ may therefore be replaced by bounds on the rates of change of the eigenvalues.

§ 2. Analysis

It is assumed that for all $t \geqslant t_0$, the matrix $\mathbf{A}(t)$ in eqn. (2) has every $\operatorname{Re}\lambda(\mathbf{A}) \leqslant -\epsilon < 0$. Consider the quadratic form:

$$V = \mathbf{x}^{\mathrm{T}}\mathbf{P}(t)\mathbf{x}, \qquad (7)$$

where the symmetric matrix $\mathbf{P}(t)$ will be defined later. Then if $\mathbf{R}(t) = \mathbf{P}(t) - \epsilon_1\mathbf{I}$:

$$V' = \mathbf{x}'[\mathbf{A}^{\mathrm{T}}(t)\mathbf{R}(t) + \mathbf{R}(t)\mathbf{A}(t)]\mathbf{x} + \epsilon_1\mathbf{x}^{\mathrm{T}}[\mathbf{A}^{\mathrm{T}} + \mathbf{A}]\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{P}'\mathbf{x}. \qquad (8)$$

For any given t, and for a given symmetric matrix $\mathbf{C}(t)$, the equation

$$\mathbf{A}^{\mathrm{T}}(t)\mathbf{R}(t) + \mathbf{R}(t)\mathbf{A}(t) = -\mathbf{C}(t) \qquad (9)$$

can be solved for $\mathbf{R}$ according to a theorem of Lyapunov. The matrix $\mathbf{R}$ so defined will be positive definite for every fixed $t \geqslant t_0$ if $\mathbf{C}(t)$ is positive definite for each fixed $t \geqslant t_0$. With $\epsilon_1 > 0$ the form $\mathbf{x}^{\mathrm{T}}[\mathbf{R}(t) + \epsilon_1\mathbf{I}]\mathbf{x} = V(\mathbf{x}, t)$ will be positive definite as a function of $t(\geqslant t_0)$ and $\mathbf{x}$. An explicit expression for $\mathbf{R}$ (cf. La Salle and Lefschetz 1961) is:

$$\mathbf{R} = \int_0^\infty \exp\{\mathbf{A}^{\mathrm{T}}(t)u\}\mathbf{C}(t)\exp\{\mathbf{A}(t)u\}\,du. \qquad (10)$$

Suppose now that $\mathbf{C}$ is a constant matrix. Then

$$\mathbf{A}^{\mathrm{T}}\mathbf{R}' + \mathbf{R}'\mathbf{A} = -(\mathbf{A}'^{\mathrm{T}}\mathbf{R} + \mathbf{R}\mathbf{A}'), \qquad (11)$$

and this equation also can be solved for $\mathbf{R}'$, giving:

$$\mathbf{P}' = \mathbf{R}' = \int_0^\infty \exp(\mathbf{A}^{\mathrm{T}}u)(\mathbf{A}'^{\mathrm{T}}\mathbf{R} + \mathbf{R}\mathbf{A}')\exp(\mathbf{A}u)\,du. \qquad (12)$$

Equation (12) shows that each element p_{ij}' of $\mathbf{P}'$ is a continuous function of the a_{ij}' and r_{ij}. Equation (10), with constant $\mathbf{C}$, shows that each element r_{ij} of $\mathbf{R}$ is a continuous function of the a_{ij}.

Now let $|a_{ij}| \leqslant a$, $|a_{ij}'| \leqslant \epsilon_2$ for all $t \geqslant t_0$ and all i, j. The continuous function $|p_{ij}'(\mathbf{A}, \mathbf{A}')|$, which is zero when $\mathbf{A}' = 0$, has a maximum value η_{ij} in the closed region so defined: let $\eta(\epsilon_2)$ be the greatest of the η_{ij}. Then η is a continuous function of ϵ_2 and $\eta(0) = 0$. It follows that ϵ_2 can be chosen small enough to ensure that $-\mathbf{I} + \mathbf{P}'$ is negative definite. Also, subject to $|a_{ij}| \leqslant a$, all i, j, it is possible to choose ϵ_1 so that $-\mathbf{I} + \epsilon_1(\mathbf{A}^{\mathrm{T}} + \mathbf{A})$ is negative definite.

Now let $\mathbf{C} = 3\mathbf{I}$, and choose ϵ_1, ϵ_2 as described. Then

$$\begin{aligned} V'(\mathbf{x}, t) &= \mathbf{x}^{\mathrm{T}}[-3\mathbf{I} + \epsilon_1(\mathbf{A}^{\mathrm{T}} + \mathbf{A}) + \mathbf{P}']\mathbf{x} \\ &\leqslant -\mathbf{x}^{\mathrm{T}}\mathbf{x}. \end{aligned} \qquad (13)$$

Hence we have $V(\mathbf{x}, t)$ positive definite and $V'(\mathbf{x}, t)$ negative definite.

Since the r_{ij} are continuous functions of the a_{ij}, there is a maximum value ρ_{ij} of $|r_{ij}|$ in the closed region defined by $|a_{ij}| \leqslant a$. If ρ is the greatest of the ρ_{ij}, it follows that $V(\mathbf{x}, t) \leqslant (\epsilon_1 + n^2\rho)\mathbf{x}^T\mathbf{x}$. Thus all the conditions are fulfilled which make V a Lyapunov function, and the trivial solution $\mathbf{x} = 0$ is asymptotically stable. Since eqn. (2) is linear, asymptotic stability holds in the large.

This result may be restated:

Theorem 1

Let $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, where for all $t \geqslant t_0$ every element $a_{ij}(t)$ of $\mathbf{A}(t)$ is differentiable and satisfies $|a_{ij}| \leqslant a$ and every eigenvalue λ of $\mathbf{A}$ satisfies

$$\operatorname{Re}\lambda(\mathbf{A}) \leqslant -\epsilon < 0.$$

Then there is some $\delta > 0$ (independent of t) such that if every $|a_{ij}'| \leqslant \delta$ the point $\mathbf{x} = 0$ is asymptotically stable.

Attention will now be restricted to the special form of $\mathbf{A}$ given in eqn. (5). It will also be assumed that the eigenvalues $\lambda(\mathbf{A})$ are all distinct. The eigenvectors of $\mathbf{A}$ are then:

$$\mathbf{u}^{(i)} = (1, \lambda_i, \lambda_i^2, \ldots, \lambda_i^{n-1}), \quad i = 1, 2, \ldots, n. \qquad (14)$$

Let $\mathbf{H}$ be the matrix of eigenvectors:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{bmatrix}. \qquad (15)$$

It follows that

$$\mathbf{H}^{-1}\mathbf{A}\mathbf{H} = \mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \qquad (16)$$

and from eqn. (10):

$$\mathbf{H}^*\mathbf{R}\mathbf{H} = \int_0^\infty \mathbf{H}^* \exp(\mathbf{A}^T u)(\mathbf{H}^*)^{-1}\mathbf{H}^*\mathbf{C}\mathbf{H}\mathbf{H}^{-1}\exp(\mathbf{A}u)\mathbf{H}\, du, \qquad (17)$$

where $\mathbf{H}^*$ is the transposed complex conjugate of $\mathbf{H}$. Hence

$$\mathbf{H}^*\mathbf{R}\mathbf{H} = \int_0^\infty \exp(\mathbf{D}^*u)\mathbf{H}^*\mathbf{C}\mathbf{H}\exp(\mathbf{D}u)\, du. \qquad (18)$$

Now make the particular choice of $\mathbf{C}(t)$ defined by:

$$\mathbf{H}^*\mathbf{C}\mathbf{H} = -(\mathbf{D}^* + \mathbf{D}) \qquad (19)$$

$$= -2\operatorname{diag}(\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2, \ldots, \operatorname{Re}\lambda_n). \qquad (20)$$

This makes $\mathbf{C}$ positive definite, and it will be shown below that $\mathbf{C}$ is real. Then:

$$\mathbf{H}^*\mathbf{R}\mathbf{H} = -\int_0^\infty \exp(\mathbf{D}^*u)(\mathbf{D}^* + \mathbf{D})\exp(\mathbf{D}u)\, du \qquad (21)$$

$$= -\int_0^\infty (\mathbf{D}^* + \mathbf{D})\exp(\mathbf{D}^* + \mathbf{D})u\, du \qquad (22)$$

$$= \mathbf{I}, \qquad (23)$$

since the diagonal matrices $\mathbf{D}$, $\mathbf{D}^*$ and $\exp(\mathbf{D}^*u)$ commute.

Hence

$$\mathbf{R} = (\mathbf{H}^*)^{-1}\mathbf{H}^{-1} = (\mathbf{H}\mathbf{H}^*)^{-1}, \qquad (24)$$

and from eqns. (8), (9) and (19):

$$V' = \mathbf{x}^{\mathrm{T}}\left[(\mathbf{H}^*)^{-1}(\mathbf{D}^* + \mathbf{D})\mathbf{H}^{-1} + \frac{d}{dt}(\mathbf{H}\mathbf{H}^*)^{-1} + \epsilon_1(\mathbf{A}^{\mathrm{T}} + \mathbf{A})\right]\mathbf{x}. \qquad (25)$$

Now putting $\mathbf{H}\mathbf{H}^* = \mathbf{S}$,

$$\frac{d}{dt}\mathbf{S}\mathbf{S}^{-1} = \left(\frac{d\mathbf{S}}{dt}\right)\mathbf{S}^{-1} + \mathbf{S}\left(\frac{d\mathbf{S}^{-1}}{dt}\right) = 0, \qquad (26)$$

$$\frac{d\mathbf{S}^{-1}}{dt} = -\mathbf{S}^{-1}\mathbf{S}'\mathbf{S}^{-1}, \qquad (27)$$

so that

$$V' = \mathbf{x}^{\mathrm{T}}[(\mathbf{H}^*)^{-1}(\mathbf{D}^* + \mathbf{D})\mathbf{H}^{-1} - \mathbf{S}^{-1}\mathbf{S}'\mathbf{S}^{-1} + \epsilon_1(\mathbf{A}^{\mathrm{T}} + \mathbf{A})]\mathbf{x} \qquad (28)$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{S}^{-1}[\mathbf{H}(\mathbf{D}^* + \mathbf{D})\mathbf{H}^* - \mathbf{S}' + \epsilon_1\mathbf{S}(\mathbf{A}^{\mathrm{T}} + \mathbf{A})\mathbf{S}]\mathbf{S}^{-1}\mathbf{x} \qquad (29)$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{S}^{-1}[\mathbf{H}\mathbf{H}^*(\mathbf{H}^*)^{-1}\mathbf{D}\mathbf{H}^* + \mathbf{H}\mathbf{D}\mathbf{H}^{-1}\mathbf{H}\mathbf{H}^* - \mathbf{S}' + \epsilon_1\mathbf{S}(\mathbf{A}^{\mathrm{T}} + \mathbf{A})\mathbf{S}]\mathbf{S}^{-1}\mathbf{x} \qquad (30)$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{S}^{-1}[\mathbf{S}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathbf{S} - \mathbf{S}' + \epsilon_1\mathbf{S}(\mathbf{A}^{\mathrm{T}} + \mathbf{A})\mathbf{S}]\mathbf{S}^{-1}\mathbf{x}, \qquad (31)$$

which will be negative for all $\mathbf{x} \neq 0$ and some $\epsilon_1 > 0$ if every $|a_i| \leqslant a$ and if

$$\mathbf{L} = \mathbf{S}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathbf{S} - \mathbf{S}' + \eta\mathbf{I} \qquad (32)$$

is negative definite for every fixed $t \geqslant t_0$ and some $\eta > 0$. For

$$V' = \mathbf{x}^{\mathrm{T}}\mathbf{S}^{-1}[\mathbf{L} + \epsilon_1\mathbf{S}(\mathbf{A}^{\mathrm{T}} + \mathbf{A})\mathbf{S} - \eta\mathbf{I}]\mathbf{S}^{-1}\mathbf{x} \qquad (33)$$

and if every $|a_i| \leqslant a$, ϵ_1 can be chosen so that $\epsilon_1\mathbf{S}(\mathbf{A}^{\mathrm{T}} + \mathbf{A})\mathbf{S} - \frac{1}{2}\eta\mathbf{I}$ is negative definite, and $V'(\mathbf{x}, t)$ is then negative definite.

By eqn. (24), $V = \mathbf{x}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{x} + \epsilon_1\mathbf{x}^{\mathrm{T}}\mathbf{x}$ and if every element σ_{ij} of $\mathbf{S}^{-1}$ satisfies $|\sigma_{ij}| \leqslant \sigma$ for all $t \geqslant t_0$, $V(\mathbf{x}, t) \leqslant (\epsilon_1 + n^2\sigma)\mathbf{x}^{\mathrm{T}}\mathbf{x}$. The conditions for V to be a Lyapunov function are then all fulfilled. The condition $|\sigma_{ij}| \leqslant \sigma$ can be replaced when every $|a_{ij}| \leqslant a$ by the condition that the $\lambda(\mathbf{A})$ are not merely separate but satisfy $|\lambda_i - \lambda_j| \geqslant \theta > 0$ for all i, j, $i \neq j$.

It will be noticed that when $\mathbf{A}' = 0$ (and therefore $\mathbf{S}' = 0$), $\mathbf{L}$ is always negative definite for every fixed $t \geqslant t_0$ and some $\eta > 0$. For

$$\mathbf{S}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathbf{S} = \mathbf{H}(\mathbf{D}^* + \mathbf{D})\mathbf{H}^*,$$

which is negative definite for any given t when every $\operatorname{Re}\lambda(\mathbf{A}) \leqslant -\epsilon < 0$ and the $\lambda(\mathbf{A})$ are distinct. A more obvious result, obtained by putting $\mathbf{P} = \mathbf{S}$, requires $\mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{S}' + \eta\mathbf{I}$ to be negative definite for some $\eta > 0$, but has the defect that $\mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{S}\mathbf{A}$ is not generally negative definite subject to the assumptions.

The matrix $\mathbf{S}$ is given by

$$(S_{ij}) = \left(\sum_{k=1}^{n} h_{ik}\bar{h}_{jk}\right) \qquad (34)$$

$$= \left(\sum_{k=1}^{n} \lambda_k^{\,i-1}\bar{\lambda}_k^{\,j-1}\right), \qquad (35)$$

where the bar denotes the complex conjugate. Because complex values of λ occur in pairs, $\mathbf{S}$ is a real symmetric matrix. By eqns. (24) and (9) it follows that $\mathbf{C}$ is real. Also $\mathbf{AS}$ is symmetric if the eigenvalues of $\mathbf{A}$ are real, for then:

$$\begin{aligned}\mathbf{AS}&=\mathbf{HDH}^{-1}\mathbf{HH}^*=\mathbf{HDH}^*\\&=\mathbf{HD}^*\mathbf{H}^*=\mathbf{HH}^*(\mathbf{H}^*)^{-1}\mathbf{D}^*\mathbf{H}^*=\mathbf{SA}^{\mathrm{T}}. \qquad (36)\end{aligned}$$

The characteristic equation of $\mathbf{A}$ is

$$\lambda^n+a_n(t)\lambda^{n-1}+\ldots+a_2(t)\lambda+a_1(t)=0, \qquad (37)$$

so that

$$\left.\begin{aligned}a_n\ &=-\sum_i\lambda_i,\\a_{n-1}&=\sum_{i>j}\sum_j\lambda_i\lambda_j,\\&\ldots\ldots\ldots\ldots\\a_1&=(-)^n\prod_j\lambda_j.\end{aligned}\right\} \qquad (38)$$

Thus by eqns. (35) and (38), $\mathbf{L}$ can be expressed in terms of the λ_i and λ_i'. Alternatively, $\mathbf{L}$ can be expressed in terms of $a_1(t)$, $a_2(t)$, ..., $a_n(t)$, though the formulae for $\mathbf{S}$ differ according to how many of λ_i are real. For example, when every λ_i is real:

$$\left.\begin{aligned}S_{11}&=\sum_i\lambda_i^0=n,\\S_{12}&=S_{21}=\sum_i\lambda_i=-a_n,\\S_{22}&=\sum_i\lambda_i^2=a_n^2-2a_{n-1},\quad \text{etc.}\end{aligned}\right\} \qquad (39)$$

On the other hand, when $n=2$ and the eigenvalues are complex:

$$\left.\begin{aligned}S_{11}&=2,\\S_{12}&=S_{21}=-a_2,\\S_{22}&=2a_1.\end{aligned}\right\} \qquad (40)$$

Then finally the following result can be stated:

Theorem 2

Let $\mathbf{x}'=\mathbf{A}(t)\mathbf{x}$, $\mathbf{A}(t)$ being given by eqn. (5), where for all $t\geqslant t_0$ every $a_i(t)$ satisfies $|a_i|\leqslant a$ and is differentiable, and the eigenvalues λ of $\mathbf{A}$ satisfy $\mathrm{Re}\,\lambda_i(\mathbf{A})\leqslant-\epsilon<0$, all i, and $|\lambda_i-\lambda_j|\geqslant\theta>0$, all i, j, $i\neq j$. Then the point $\mathbf{x}=0$ is asymptotically stable if the matrix $\mathbf{L}(t)$ is negative definite for every fixed $t\geqslant t_0$ and some $\eta>0$, where

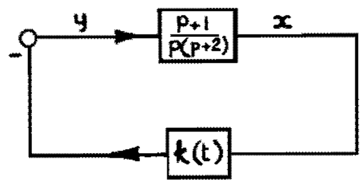
$$\mathbf{L}=\mathbf{SA}^{\mathrm{T}}+\mathbf{AS}-\mathbf{S}'+\eta\mathbf{I} \qquad (41)$$

and

$$(S_{ij})=\left(\sum_{k=1}^n\lambda_k^{i-1}\bar{\lambda}_k^{j-1}\right). \qquad (42)$$

§ 3. Example

Fig. 3



Consider the system shown in fig. 3, for which the equations are:

$$y' + y = x'' + 2x', \qquad (43)$$

$$y = -k(t)x, \quad k \geqslant \theta > 0, \qquad (44)$$

giving

$$x'' + (2+k)x' + (k+k')x = 0. \qquad (45)$$

The eigenvalues are

$$\lambda_1, \lambda_2 = \frac{-(2+k) \pm \sqrt{(4+k^2-4k')}}{2} \qquad (46)$$

and are real and distinct for $|k'| \leqslant 1$ and satisfy $|\lambda_1 - \lambda_2| \geqslant \theta > 0$. Then eqns. (41) and (39) give, after some manipulation:

$$-\mathbf{L} = 2\begin{pmatrix} 2+k & 2k-(2+k)^2 \\ 2k-(2+k)^2 & (2+k)^3 - 3k(2+k) \end{pmatrix} + k'\begin{pmatrix} 0 & 5 \\ 5 & 2-7(2+k) \end{pmatrix} + k''\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} - \eta\mathbf{I}. \qquad (47)$$

The eigenvalues of the first matrix on the right-hand side of eqn. (47) are given by

$$\lambda^2 - (2+k)(5+k+k^2)\lambda + k^3 + 4k = 0, \qquad (48)$$

whence the least eigenvalue satisfies

$$\min_i \lambda_i \geqslant \frac{k^3+4k}{(2+k)(5+k+k^2)}. \qquad (49)$$

Also for any symmetric matrix $\mathbf{B}$

$$\min_i \left[b_{ii} - \sum_{j \neq i} |b_{ij}|\right]\mathbf{x}^{\mathrm{T}}\mathbf{x} \leqslant \min_i \lambda_i(\mathbf{B})\mathbf{x}^{\mathrm{T}}\mathbf{x} \leqslant \mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x} \leqslant \max_i \lambda_i(\mathbf{B})\mathbf{x}^{\mathrm{T}}\mathbf{x} \leqslant \max_i \left[b_{ii} + \sum_{j \neq i} |b_{ij}|\right]\mathbf{x}^{\mathrm{T}}\mathbf{x}. \qquad (50)$$

Hence $-\mathbf{x}^{\mathrm{T}}\mathbf{L}\mathbf{x}$ may be estimated by

$$-\mathbf{x}^{\mathrm{T}}\mathbf{L}\mathbf{x} \geqslant \left\{\frac{2(k^3+4k)}{(2+k)(5+k+k^2)} - (17+7k)|k'| - 2|k''| - \eta\right\}\mathbf{x}^{\mathrm{T}}\mathbf{x}, \qquad (51)$$

and a sufficient condition for asymptotic stability is

$$\frac{2(k^3+4k)}{(2+k)(5+k+k^2)}-(17+7k)|k'|-2|k''|\geqslant\eta>0. \qquad (52)$$

This implies $|k'|\leqslant 1$, and hence makes the eigenvalues of $\mathbf{A}$ real and distinct as we have assumed.

An alternative procedure would be to apply the determinantal criterion to $\mathbf{L}$ to determine the conditions under which it is negative definite (Mirsky 1955). For the particular (second-order) example of eqn. (45) a better result is available (Grensted 1956).

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