LQR Optimal Control for High Precision Rendezvous and Docking

Gaurav Kumar

I. System Description

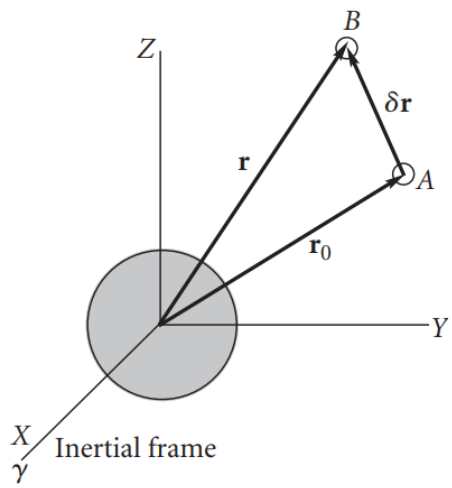


Fig. 1 Orbital framework of Docking: A is target and B is chaser.

From the figure,

$$\vec{r} = \vec{r}_0 + \vec{\delta} r$$

with

$$\frac{\delta r}{r_0} << 1$$

We define a moving frame(CWH frame) xyz with origin on A with,

$$\hat{i} = \frac{\vec{r}_0}{r_0}$$

y-axis is the local horizon and $\hat{k} = \hat{i} \times \hat{j}$

Let angular velocity of A(and so of the CWH frame) be Ω

II. Mathematical Model

The equations of motion for the chaser spacecraft are nonlinear and can be expressed in vector form as:

$$\ddot{\vec{r}} = -\mu\frac{\vec{r}}{r^3} + \frac{\vec{F}}{m} \tag{1}$$

$\vec{F}$ is force applied on the spacecraft.

Putting $\vec{r} = \vec{r}_0 + \vec{\delta}r$ in (1),

$$\ddot{\vec{\delta r}} = -\ddot{\vec{r}}_0 - \mu\frac{\vec{r}_0 + \delta\vec{r}}{r^3} + \frac{\vec{F}}{m} \tag{2}$$

Now,

$$r^{-3} = r_0^{-3}\left(1 + \frac{2\vec{r}_0 \cdot \vec{\delta}r}{r_0^2}\right)^{-3/2}$$

by neglecting $\left(\frac{\delta}{r_0}\right)^2$,

Using Binomial theorem this simplifies as:

$$\left(1 + \frac{2\vec{r}_0 \cdot \vec{\delta}r}{r_0^2}\right)^{-3/2} = 1 + \left(\frac{-3}{2}\right)\left(1 + \frac{2\vec{r}_0 \cdot \vec{\delta}r}{r_0^2}\right)$$

$$\therefore\ \frac{1}{r^3} = \frac{1}{r_0^3} - \frac{3}{r_0^5}\vec{r}_0 \cdot \vec{\delta}r$$

Putting this in (2):

$$\ddot{\vec{\delta r}} = \ddot{\vec{r}}_0 - \mu\left(\frac{1}{r_0^3} - \frac{3}{r_0^5}\vec{r}_0 \cdot \vec{\delta}r\right)(\vec{r}_0 + \delta\vec{r}) + \frac{\vec{F}}{m}$$

$$= -\ddot{\vec{r}}_0 - \mu\left[\frac{\vec{r}_0}{r_0^3} + \frac{\vec{\delta}r}{r_0^3} - \frac{3(\vec{r}_0 \cdot \vec{\delta}r)\vec{r}_0}{r_0^5} + \text{Higher order terms in } \delta r \text{ (which we neglect)}\right] + \frac{\vec{F}}{m}$$

$$\therefore\ \ddot{\vec{\delta r}} = -\ddot{\vec{r}}_0 - \mu\left[\frac{\vec{r}_0}{r_0^3} + \frac{\vec{\delta}r}{r_0^3} - \frac{3(\vec{r}_0 \cdot \vec{\delta}r)\vec{r}_0}{r_0^5}\right] + \frac{\vec{F}}{m}$$

Now dynamical equation of the target vehicle is

$$\ddot{\vec{r}}_0 = -\mu\frac{\vec{r}_0}{r_0^3}$$

Substituting above,

$$\ddot{\vec{\delta r}} = -\frac{\mu}{r_0^3}\left[\delta\vec{r} - \frac{3(\vec{r}_0 \cdot \vec{\delta}r)\vec{r}_0)}{r_0^2}\right] + \frac{\vec{F}}{m} \tag{3}$$

Now,

$$\vec{\delta}r = \delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k}$$

Using vector kinematics, we know that,

$$\ddot{\vec{r}} = \ddot{\vec{r}}_0 + \vec{\Omega} \times \delta r + \dot{\vec{\Omega}} \times (\vec{\Omega} \times \vec{\delta} r) + 2\vec{\Omega} \times \delta \vec{v}_{rel} + \vec{\delta} a_{rel}$$

Putting $\vec{r} = \vec{r}_0 + \delta r$ above,

$$\vec{\delta}\ddot{r} = \dot{\vec{\Omega}} \times \vec{\delta} r + \vec{\Omega} \times (\vec{\Omega} \times \vec{\delta} r) + 2\vec{\Omega} \times \vec{\delta} v_{rel} + \vec{\delta} a_{rel}$$

Assuming target vehicle to be in **circular orbit**, $\dot{\vec{\Omega}} = 0$,

$$\vec{\delta}\ddot{r} = \vec{\Omega} \times (\vec{\Omega} \times \vec{\delta} r) + 2\vec{\Omega} \times \vec{\delta} v_{rel} + \vec{\delta} a_{rel}$$

Expanding the RHS,

$$\vec{\delta}\ddot{r} = \vec{\Omega}(\vec{\Omega} \cdot \vec{\delta} r) - \Omega^2 \vec{\delta} r + 2\vec{\Omega} \times \vec{\delta} v_{rel} + \vec{\delta} a_{rel}$$

Putting $\vec{\Omega} = n\hat{k}$, $\vec{\delta} v_{rel} = \delta\dot{x}\hat{i} + \delta\dot{y}\hat{j} + \delta\dot{z}\hat{k}$ and $\vec{\delta} a_{rel} = \delta\ddot{x}\hat{i} + \delta\ddot{y}\hat{j} + \delta\ddot{z}\hat{k}$ and recollecting terms this simplifies as,

$$\vec{\delta}\ddot{r} = (-n^2\delta x - 2n\delta\dot{y} + \delta\ddot{x})\hat{i} + (-n^2\delta y - 2n\delta\dot{x} + \delta\ddot{y})\hat{j} + \delta\ddot{z}\hat{k} \tag{4}$$

Now from (3),by substituting values of $\vec{\delta} r$, $\vec{r}_0 \cdot \delta r = r_0 \delta x$ and using $n^2 = \frac{\mu}{r_0^3}$ we get,

$$\vec{\delta}\ddot{r} = -n^2 \left[\delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k} - \frac{3}{r_0^2}(r_0\delta x) r_0 \hat{i} \right] + \frac{F_x}{m}\hat{i} + \frac{F_y}{m}\hat{j} + \frac{F_z}{m}\hat{k} = (2n^2\delta x + \frac{F_x}{m})\hat{i} + (-n^2\delta y + \frac{F_y}{m})\hat{j} + (-n^2\delta z + \frac{F_z}{m})\hat{k} \tag{5}$$

Equating the value of $\vec{\delta}\ddot{r}$ from equation (4) and (5) and collecting terms,

$$(\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y})\hat{i} + (\delta\ddot{y} + 2n\delta\dot{x})\hat{j} + (\delta\ddot{z} + n^2\delta z)\hat{k} = \frac{F_x}{m}\hat{i} + \frac{F_y}{m}\hat{j} + \frac{F_z}{m}\hat{k}$$

This can be simplified to give linear CWH equation:

$$\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y} = \frac{F_x}{m} = u_x \tag{6}$$

$$\delta\ddot{y} + 2n\delta\dot{x} = \frac{F_y}{m} = u_y \tag{7}$$

$$\delta\ddot{z} + n^2\delta z = \frac{F_z}{m} = u_z \tag{8}$$

This model can be formulated as:

$$\dot{X} = AX + BU$$

where X $\in \mathcal{R}^6$ is the state vector and U $\in \mathcal{R}^3$ is the control vector.

$$X = \begin{pmatrix} \delta x \\ \delta y \\ \delta z \\ \delta \dot{x} \\ \delta \dot{y} \\ \delta \dot{z} \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

A. Controllability:

The derived system satisfies the necessary and sufficient conditions for controllability, i.e,

$$rank \begin{bmatrix} A & A^2B\dots & A^{(n-1)}B \end{bmatrix} = 6$$

III. Optimal Control Design Using Krotov Conditions

The cost function is:

$$min\ J = \int_0^\infty X^T QX + U^T RU\, dt \tag{9}$$

For Optimization of the cost functional with respect to the system dynamics with possible constraints, generally Calculus of Variation is employed, but they give only necessary conditions for local optimum. We now develop optimal control by using Krotov functions which will give sufficient conditions and the control law that will be derived provide global optimum.

From the equivalency principle, this optimization problem can be restated as:

$$min\ J_{eq} = L(x(0),t)) + \int_0^\infty S(X,U,t))\, dt \tag{10}$$

where q is a Krotov function which when chosen appropriately reduces the non-convex function and

$$s = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial X}[AX + BU] + X^T QX + U^T RU$$

So, the problem reduces to

$$\min_{(X,U)\in\mathcal{R}^6\times\mathcal{R}^3} S(X,U,t), \forall t \in [t,\infty] \tag{11}$$

This gives us a sufficient condition as minimum S will minimise the cost function. As S is non-convex, we set $L = X^T PX$,

$$\therefore S = X^T \dot{P}X + X^T (P + P^T)[AX + BU] + X^T QX + U^T RU$$

$$= X^T \dot{P}X + X^T PAX + X^T P^T AX + X^T PBX + X^T P^T BX + X^T QX + U^T RU$$

Adding and subtracting $X^T(\frac{1}{2}PBR^{-1}B^TP + \frac{1}{4}PBR^{-1}B^TP^T + \frac{1}{4}P^TBR^{-1}B^TP)X$ and collecting terms:

$$S = X^T(\dot{P} + A^TP + Q - \frac{1}{2}PBR^{-1}B^TP - \frac{1}{4}PBR^{-1}B^TP^T - \frac{1}{4}P^TBR^{-1}B^TP)X + \text{scalar terms}$$

Now S will be convex if,

$$\dot{P} + A^T P + Q - \frac{1}{2}PBR^{-1}B^T P - \frac{1}{4}PBR^{-1}B^T P^T - \frac{1}{4}P^T BR^{-1}B^T P \geq 0$$

For infinite horizon case, $\dot{P} = 0$, For optimality P must satisfy,

$$A^T P + Q - \frac{1}{2}PBR^{-1}B^T P - \frac{1}{4}PBR^{-1}B^T P^T - \frac{1}{4}P^T BR^{-1}B^T P = 0 \tag{12}$$

Therefore The optimal Control solution is:

$$U^* = -\frac{1}{2}R^{-1}B^T (P^T + P)X^* \tag{13}$$

A. Stability Analysis

For proving stability of the calculated control solution, we choose Lyapunov function $V = X^T (P^T + P)X$, then:

$$\dot{V} = X^T (P^T + P)\dot{X} + \dot{X}^T (P^T + P)X$$

Now, using $\dot{X} = AX + BU$ where U is substituted from (13)

$$\dot{V} = 2X^T (-Q - \frac{1}{2}PBR^{-1}B^T P - \frac{1}{4}PBR^{-1}B^T P^T - \frac{1}{4}P^T BR^{-1}B^T P)X$$

It is observed that for $P + P^T > 0$ the closed loop system is Lyapunov stable for $Q \geq 0$ and asymptotically stable for Q>0

IV. Results

Considering our system dynamics,(12) and (13) is solved by selecting appropriate gains Q and R.

A. Gain Selection

Gain matrix Q and R are selected as diagonal matrix with Q>0 for asymptotic stability.

$$Q = \begin{pmatrix} \frac{\alpha_1^2}{X_{max}^2} & 0 & & 0 & 0 & 0 \\ 0 & \frac{\alpha_2^2}{X_{max}^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_3^2}{X_{max}^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_4^2}{X_{max}^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha_5^2}{X_{max}^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha_2^2}{X_{max}^2} \end{pmatrix}$$

V. Infinite horizon unconstrained LQR

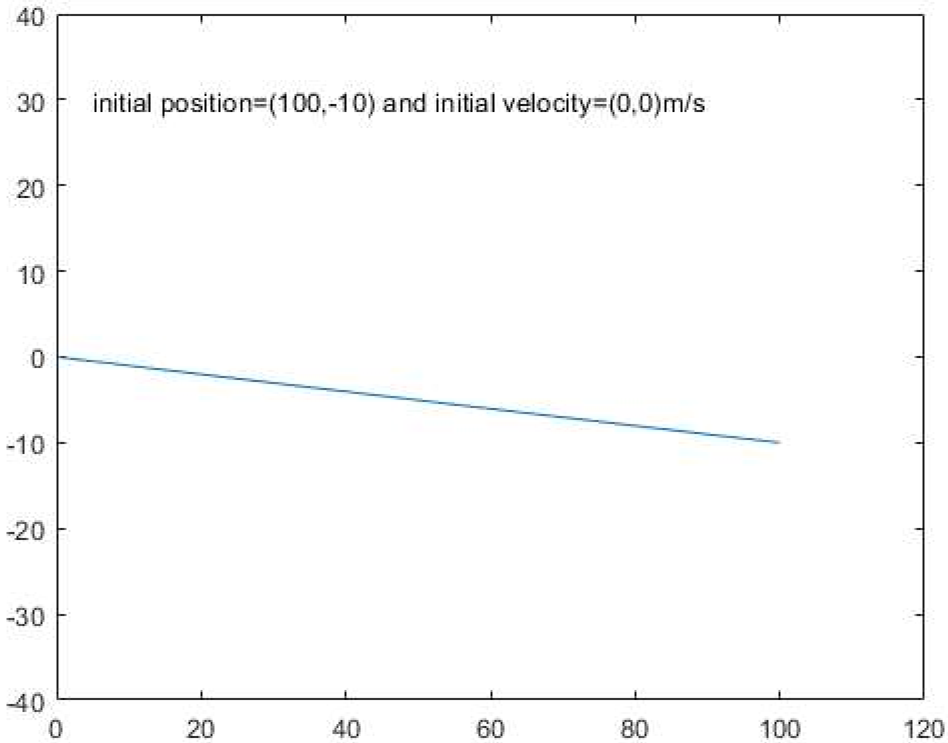
The cost function is:

$$min\ J = \sum_{k=0}^{\infty} X(k)^T QX(k) + U(k)^T RU(k) \tag{14}$$

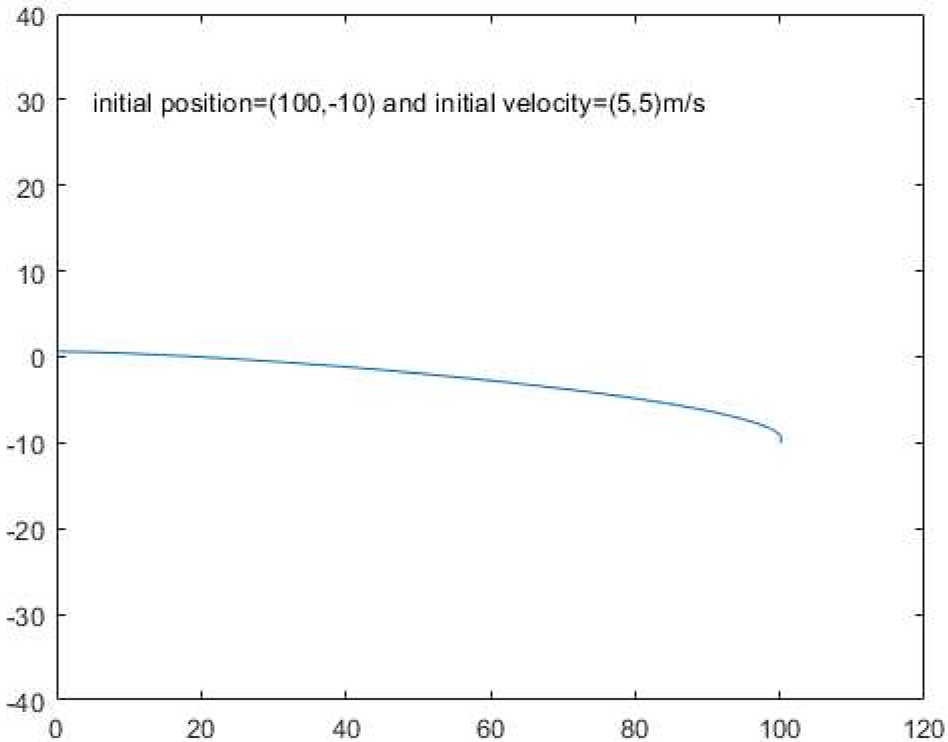
This problem is solved in MATLAB with appropriate gains Q and R with the following results:

LQR applied on linear CWH equation for rendezvous and Docking

```
n=0.001;  %orbital rate for 500 KM orbit
% for X'=AX + BU:
A=[0 0 1 0;0 0 0 1;3*n*n 0 0 2*n;0 0 -2*n 0];
B=[0 0;0 0;1 0; 0 1];
C1 = [1, 0, 0, 0];
C2=[0, 1, 0, 0];
h = 0.001; t = 0; time(1) = 0;
%For the cost function: J=Sigma(X'QX+U'RU)
Q= 30*[1 0 0 0;0 1 0 0;0 0 0.1 0;0 0 0 0.1]; %state error weight
R=100*eye(2); %control error weight
[P,K,L]=icare(A,B,Q,R,[],[]);% solving algebraic Ricaatti eqn
X=[100; -10 ;0 ;0]; %initial condition 1
y(1)=-10;
x(1)=100;

for i = 1:100000

    X=X+(A-B*K)*X*0.001;
    z(i+1)=sqrt((C2*X)^2+(C1*X)^2);
    x(i+1)=C1*X;
    y(i+1)=C2*X;
    t = t+h;
    time(i+1)=t;
end
plot(x,y);
str = {'initial position=(100,-10) and initial velocity=(0,0)m/s'};
text(5,30,str)
xlim([0 120])
ylim([-40 40])
```




```
 clf
 X=[100; -10 ;5 ;5];
 y(1)=-10;
x(1)=100;  %initial condition 2
 for i = 1:100000

     X=X+(A-B*K)*X*0.001;
     z(i+1)=sqrt((C2*X)^2+(C1*X)^2);
     x(i+1)=C1*X;
     y(i+1)=C2*X;
     t = t+h;
     time(i+1)=t;
end
plot(x,y);
str = {'initial position=(100,-10) and initial velocity=(5,5)m/s'};
text(5,30,str)
xlim([0 120])
ylim([-40 40])
```



Published with MATLAB® R2020a