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Relative Equilibria of a Rigid Satellite in a Circular Keplerian Orbit

Jeffrey A. Beck* and Christopher D. Hall†

Abstract

We examine relative equilibria of a rigid body free to rotate about its center of mass which is constrained to follow a Keplerian orbit in a central gravitational field. We derive a noncanonical Hamiltonian formulation of this system and show how it relates to the noncanonical system for an unconstrained rigid body (due to Wang et al) in a hierarchy of approximations of the two-body problem. For a particular approximation of the potential, the Keplerian system is equivalent to the classical approximation typically seen in the literature. We determine relative equilibria for this approximation and derive stability conditions for both arbitrary and axisymmetric bodies.

Introduction

The focus of the current work is the problem of two rigid bodies of finite extent acting under their mutual gravitational attraction. We are particularly interested in the application to an artificial satellite orbiting the Earth. Hence, we further restrict our investigation to the case where (i) one body (the *primary*) is assumed to have a total mass which is much greater than that of the other body (the *satellite*), and (ii) the mass distribution of the primary is spherically symmetric. The first condition implies the motion of the primary is essentially unaffected by the satellite and is perfectly valid for the artificial Earth satellite problem. The second condition implies the motion of the satellite is independent of the rotational dynamics of the primary and is somewhat more suspect. Clearly, the oblateness of the Earth is an issue which should be factored in at some point. However, the symmetry assumption allows considerable simplification and provides a valuable first approximation from which additional effects can be explored as perturbations at some later point.

Under these assumptions, we may treat the primary as a point mass fixed at the origin in some inertial frame and consider the motion of the satellite in the corresponding central

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gravitational field. Figure 1 shows a rigid body $\mathcal{B}$ in a central gravitational field with center of attraction O which we treat as the origin of an inertial frame of reference. We characterize the motion of $\mathcal{B}$ in the inertial frame in terms of a vector $\boldsymbol{\lambda}$ representing the position of the center of mass, C, and a direction cosine matrix representing the rotation of the body (and any associated body-fixed frame) relative to the inertial frame.

Although this system is not integrable for an arbitrary satellite, we can find particular solutions to the equations of motion which correspond to motions of practical interest. By recognizing the symmetry present in the problem and eliminating cyclic coordinates, we can reduce the order of the system. Equilibria of the reduced system are referred to as *relative equilibria*. For an arbitrary rigid body, the problem is invariant under rotations in three-dimensional space. The relative equilibria correspond to circular orbits about the primary with the body appearing stationary in an orbiting reference frame. Artificial satellites designed to remain in this relative equilibrium are referred to as *gravity-gradient satellites*. For axisymmetric bodies, the problem is also invariant under rotations about the axis of symmetry. In this case, the relative equilibria are steady spins about the symmetry axis which is fixed in the orbiting frame. Artificial satellites designed to take advantage of a stable relative equilibrium of this nature are referred to as *spin-stabilized satellites*.

For relative equilibria of an arbitrary rigid body, the general configuration is shown in Fig. 2 where O and C are as defined previously and O' is the orbit center. When O and O' coincide, the angular velocity vector, $\boldsymbol{\omega}$, and the position vector, $\boldsymbol{\lambda}$, are orthogonal. We refer to such a relative equilibrium as an *orthogonal relative equilibrium* and to the corresponding motion as an *orthogonal orbit*. When the two points are distinct, we refer to the relative equilibrium and its orbit as *oblique*.

One of the difficulties of the n-body problem for rigid bodies of finite extent is that the potential involves a volume integral which, in general, cannot be solved in closed form. However, the integral can be represented by a series expansion in powers of the ratio of body dimension to distance from the center of attraction. For problems such as that of an artificial Earth satellite, this ratio is very small and the series converges rapidly. We are therefore justified in truncating the series expansion to approximate the potential. This approximation is then used to derive the force and torque. Alternatively, we may expand and truncate the force and torque integrals directly.

The first analysis of relative equilibria for this problem was given by Lagrange [1]. Truncating the force and moment directly and using only the dominant term in each of the series expansions, he found that the translational and rotational equations decoupled. The resulting relative equilibria for this approximate system are orthogonal and the attitude is such that one principal axis of inertia is aligned with the radial direction and another is aligned

with the orbit normal. For a body with distinct moments of inertia there are twenty-four such relative equilibria and we refer to these as the *principal relative equilibria*. For this classical approximation, Likins and Roberson [2] proved it is not possible for oblique (or any other orthogonal) relative equilibria to exist.

For axisymmetric spin-stabilized satellites, the classical treatment admits several different classes of relative equilibria. As in the classical treatment of arbitrary satellites, all orbits are orthogonal. Differences appear in the orientation of the symmetry axis. Thomson [3] and Kane *et al* [4] first considered relative equilibria with the axis of symmetry normal to the orbital plane. Pringle [5] and Likins [6] identified two additional classes of relative equilibria, one with the symmetry axis in the plane formed by the radial and orbit normal directions, and the second with the symmetry axis in the plane formed by the tangential and orbit normal directions. These relative equilibria are denoted as *cylindrical*, *conical*, and *hyperbolic*, in reference to the figure of revolution traced out by the symmetry axis during a complete orbit (see Fig. 3).

Wang *et al* [7, 8] employed a generalized (noncanonical) Hamiltonian approach to re-examine relative equilibria of arbitrary rigid satellites. They showed that the gravitational center of attraction lies on the axis of rotation but does not always coincide with the center of the orbit. Their results are valid for the exact potential since they do not rely on a truncated series expansion. The discrepancy with the classical results arises from symmetries implicit in the truncation of the force and torque. Note that the displacement of the orbit center shown in Fig. 2 is greatly exaggerated for clarity. The displacement is much smaller than the greatest dimension of the satellite. However, Wang *et al* [8] have demonstrated that under certain circumstances the attitude can vary significantly from the classical solution. A similar noncanonical Hamiltonian approach has been applied to more complex systems: the gyrostat in a central gravitational field [9] and two rigid bodies under mutual gravitation [10].

Our research [11] explores the relationship between the classical approximation of a rigid satellite in a central gravitational field and the noncanonical system of Wang *et al*. In this paper, we present a hierarchy of systems approximating the motion of a rigid body in a central gravitational field and derive a new noncanonical formulation of the rigid body problem which is dynamically equivalent to the classical approximation. The new system is constrained to follow a Keplerian orbit. We apply generalized Hamiltonian methods to derive the classical results. This provides a point of departure for comparison of various approximations.

Hamiltonian Methods

Relative Equilibria

In this paper, we examine relative equilibria which arise as critical points of noncanonical Hamiltonian systems and explore their stability. The equations of motion for these systems take the special form

$$\dot{\mathbf{z}} = \mathbf{J}(\mathbf{z})\nabla H(\mathbf{z}) \tag{1}$$

where $\mathbf{z}$ is the phase variable vector, $\mathbf{J}(\mathbf{z})$ is the Poisson structure matrix, and $H(\mathbf{z})$ is the Hamiltonian. This is a generalization of the canonical Hamiltonian system for which the phase variable vector consists of n pairs of generalized coordinates and conjugate momenta (so that the phase space space is $\mathbb{R}^{2n}$) and the Poisson structure matrix is the $2n \times 2n$ standard symplectic matrix $\mathbf{J} = \left[\begin{smallmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{smallmatrix}\right]$. For the more general noncanonical systems, the variables are not necessarily paired and the phase space may be some manifold other than an even-dimensional real space. Furthermore, the skew-symmetric Poisson structure matrix is a function of the phase variable vector and may be singular. This property is very important in that it permits a special class of first integrals known as *Casimir* or *distinguished functions* for which the Poisson bracket with any other function is identically zero. That is, for a Casimir function $C(\mathbf{z})$ and any smooth function $F(\mathbf{z})$, we have

$$\{C, F\} = \nabla C(\mathbf{z}) \cdot \mathbf{J}(\mathbf{z})\nabla F(\mathbf{z}) = 0 \tag{2}$$

which implies

$$\mathbf{J}(\mathbf{z})\nabla C(\mathbf{z}) = \mathbf{0}. \tag{3}$$

Hence, the gradient of a Casimir function must lie in the null space of the structure matrix. Conversely, any vector in the null space must be the gradient of a Casimir function. Only a finite number of these functions are of interest. The difference between the dimension of the Hamiltonian system and the rank of the structure matrix is the number of Casimir functions which have linearly independent gradients. Such a set forms a basis for the null space of the structure matrix. In practice, this set can be found without much difficulty once the structure matrix has been determined.

We consider a relative equilibrium $\mathbf{z}_e$ which satisfies the condition

$$\mathbf{J}(\mathbf{z}_e)\nabla H(\mathbf{z}_e) = \mathbf{0}. \tag{4}$$

Hence, at a relative equilibrium $\mathbf{z}_e$ the gradient of the Hamiltonian lies in the null space of the structure matrix $\mathbf{J}(\mathbf{z}_e)$. If the structure matrix is singular, the gradient of the Hamiltonian

at relative equilibrium must equal some linear combination of the gradients of the Casimir functions:

$$\nabla H(\mathbf{z}_e) = \sum_{i=1}^{m} \mu_i \nabla C_i(\mathbf{z}_e). \tag{5}$$

Whereas condition (4) is a system of n algebraic equations in n unknowns (the phase variables at equilibrium), condition (5) is a system of n equations in $n + m$ unknowns (the phase variables and the multipliers, μ_i). We additionally require the m equations

$$C_i(\mathbf{z}) = k_i \qquad (i = 1, 2, \ldots, m) \tag{6}$$

where the k_i may be constants or parameters of the physical system. Conditions (5) and (6) are equivalent to condition (4).

Following Maddocks [12], we introduce the function

$$F(\mathbf{z}) = H(\mathbf{z}) - \sum_{i=1}^{m} \mu_i C_i(\mathbf{z}). \tag{7}$$

Critical points of this function are relative equilibria since the condition

$$\nabla F(\mathbf{z}_e) = \mathbf{0} \tag{8}$$

is equivalent to condition (5). It is immediately apparent that conditions (8) and (6) arise from the constrained variational principle

$$\text{make stationary } H(\mathbf{z}) \text{ subject to } C_i(\mathbf{z}) = k_i. \tag{9}$$

Here the μ_i serve as the Lagrange multipliers and the function $F(\mathbf{z})$ is the variational Lagrangian.[1] This variational characterization not only provides a direct method of determining the conditions for relative equilibrium, it will also prove useful in both the spectral and nonlinear stability analyses.

Along with the Hamiltonian system (1), we also consider its linearization

$$\dot{\delta\mathbf{z}} = \mathbf{A}(\mathbf{z}_e)\delta\mathbf{z} \tag{10}$$

where $\delta\mathbf{z} = \mathbf{z} - \mathbf{z}_e$ is the perturbation vector and $\mathbf{A}(\mathbf{z}) = d\mathbf{f}(\mathbf{z})/d\mathbf{z}$ is the Jacobian matrix (for

[1] We use the adjective *variational* to clearly distinguish this function from the Lagrangian function which arises in mechanics.

$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$). The Jacobian matrix (or linear system matrix) $\mathbf{A}(\mathbf{z}_e)$ for a Hamiltonian system may be expressed as the product of the structure matrix and the Hessian of the variational Lagrangian [12]

$$\mathbf{A}(\mathbf{z}_e) = \mathbf{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e) \tag{11}$$

so that it may be determined directly without performing linearization.

Stability of Relative Equilibria

Definitions

When we refer to the stability of a dynamical system, we are describing the tendency for the trajectory of the system to remain close to a given reference trajectory. Exactly in what sense the trajectories are close must be more clearly defined. Several definitions of stability which often appear in the literature are (See, *e.g.*, Holm *et al* [13]):

> **Definition 1 (Nonlinear Stability).** An equilibrium $\mathbf{z}_e$ is *nonlinearly stable* (or *Liapunov stable*) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $||\mathbf{z}(0) - \mathbf{z}_e|| < \delta$, then $||\mathbf{z}(t) - \mathbf{z}_e|| < \varepsilon$ for $t > 0$.

> **Definition 2 (Formal Stability).** An equilibrium $\mathbf{z}_e$ is *formally stable* if a conserved quantity is identified for which the first variation is zero and the second variation is positive or negative definite when evaluated at the equilibrium.

> **Definition 3 (Spectral Stability).** An equilibrium $\mathbf{z}_e$ is *spectrally stable* if the spectrum of the linearized operator $\mathbf{A}(\mathbf{z}_e)$ has no positive real part.

We also must consider the converse:

> **Definition 4 (Instability).** An equilibrium $\mathbf{z}_e$ is *unstable* if it is not nonlinearly stable.

For the finite-dimensional systems we consider here, we have [13]

$$\text{formal stability} \Rightarrow \text{nonlinear stability} \Rightarrow \text{spectral stability.}$$

The first relationship is a classical result due to Liapunov [14] and is a generalization of a theorem attributed to Lagrange and Dirichlet which states that an equilibrium which occurs at an isolated minimum of the potential function is nonlinearly stable (See, *e.g.*, Marsden and Ratiu [15]).

Our objective is to assess the nonlinear stability for all configurations which lie in some physical parameter space. Our approach consists of two steps. First, we analyze spectral stability for all configurations. Those regions which are not spectrally stable are unstable. Then we perform a nonlinear stability analysis which is, in general, applicable to some portion of the spectrally stable regions of the parameter space. The nonlinear stability of the other spectrally stable configurations remains undetermined. The result is that the configuration space is divided into three types of regions: nonlinearly stable, unstable, and indeterminate regions.

Spectral Stability

The eigenvalues of the linear system matrix are the roots of the characteristic equation

$$P(s) = \det\left[s\mathbf{1} - \mathbf{A}(\mathbf{z}_e)\right] = 0. \tag{12}$$

For low-order systems, we may solve for the eigenvalues explicitly. More typically, we apply an algorithm such as Routh's method to determine conditions on the coefficients of $P(s)$ which are necessary and sufficient to guarantee no eigenvalues lie in the right half-plane. For a general high-order polynomial, development of these conditions can be a daunting task. Fortunately, Hamiltonian systems have special properties with regard to both the form of the characteristic polynomial and the eigenstructure which greatly simplify this effort [11].

Property 1. *Eigenvalues of the linear system matrix are symmetric about both the real and imaginary axes.*

Property 2. *A zero eigenvalue exists for each linearly independent Casimir function.*

Property 3. *An additional pair of zero eigenvalues exists for each first integral which is associated with a symmetry of the Hamiltonian, and which is linear in the phase variables.*

Consideration of these properties leads us to conclude that the characteristic polynomial for a Hamiltonian system is of the form

$$P(s) = s^m(s^{n-m} + A_{n-m-2}s^{n-m-2} + A_{n-m-4}s^{n-m-4} + \cdots + A_2 s^2 + A_0) \tag{13}$$

where n is the dimension of the system and $n - m$ is the rank of the structure matrix. The even form of the polynomial inside the parentheses is a byproduct of the first property and allows the polynomial to be treated as a polynomial in $\sigma = -s^2$ of order $(n - m)/2$. For an additional integral of motion associated with a symmetry of the Hamiltonian as described in the third property, A_0 would go to zero so that an additional s^2 would factor out.

While these properties lead to considerable simplification of the characteristic polynomial, we can simplify still further in many cases. Often the linearized system decouples into (1) entirely independent subsystems so that the linear system matrix may be block-diagonalized, or (2) subsystems for which the coupling is only in one direction so that the linear system matrix may be block-triangularized. Recognition of this decoupling into block-diagonal or block-triangular form greatly simplifies the computation of the characteristic equation since, in either case, we may then treat each subsystem independently.

The above results guarantee that for an nth-order Hamiltonian system with m independent Casimir functions, we will only need to analyze polynomials of order $k \leq n - m$ in s. By the symmetry property of the roots, the system can only be spectrally stable when all the eigenvalues lie on the imaginary axis. We may determine under what conditions this requirement is satisfied by applying Routh's method [16] or a similar method based on Sturm sequences [17, 18, 19, 20].

The Hamiltonian systems considered in this paper are all ninth-order and decouple into smaller subsystems. Hence, we will never be concerned with polynomials greater than fourth order in s. Table 1 presents the spectral stability conditions derived by the method of Sturm sequences for second- and fourth-order polynomials. Recall that each coefficient is a function of the phase variables and physical parameters describing the system. Typically, the stability conditions will be further reduced to expressions involving these variables.

Nonlinear Stability

The variational Lagrangian $F(\mathbf{z}) = H(\mathbf{z}) - \sum_{i=1}^{m} \mu_i C_i(\mathbf{z})$ is a candidate Liapunov function for a Hamiltonian system since we have already shown that setting the first variation equal to zero gives precisely the conditions for relative equilibrium. Furthermore, because it is a linear combination of first integrals, it is also conserved. Finally, we have already computed the second variation as part of computing the linear system matrix $\mathbf{A}(\mathbf{z}_e) = \mathbf{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)$. Therefore, direct examination to determine positive- or negative-definiteness provides an immediate test for nonlinear stability with little more computational effort. This test provides sufficient conditions for nonlinear stability; however, the conditions are not very sharp and this method proves to be of limited utility.

We may take a variational approach and view the requirement for the second variation to be positive- or negative-definite as equivalently requiring that the relative equilibrium be an unconstrained minimum or maximum of the variational Lagrangian. This is a direct extension of the Lagrange-Dirichlet Theorem for canonical systems which states that a nondegenerate minimum or maximum of the Hamiltonian implies a stable equilibrium. However, as we described earlier, the relative equilibrium may also be viewed as a con-

strained extremum of the Hamiltonian subject to fixed values of the Casimir functions. The requirement to be a constrained minimum or maximum is less restrictive and thus might be successfully applied to a larger class of relative equilibria. Essentially, rather than considering general perturbations in phase space, this restricts consideration to perturbations on $TM|_{\mathbf{z}_e}$, the tangent space to the invariant manifold $M = \{\mathbf{z} : C_i(\mathbf{z}) = C_i(\mathbf{z}_e), i = 1, \ldots, m\}$ at the relative equilibrium, which may be identified with $\mathcal{R}(\mathbf{A}(\mathbf{z}_e))$, the range of the linearized system matrix. Then, by a theory due to Hestenes, we are assured of the existence of a suitable Liapunov function and may conclude nonlinear stability when $\mathbf{z}_e$ is a constrained minimum or maximum of the Hamiltonian (See Lemma 1 of Maddocks and Sachs [21]).[2]

Without loss of generality, we seek a constrained *minimum*. The second-variation condition may be stated as: For all $\mathbf{h} \in TM|_{\mathbf{z}_e}$ such that $\mathbf{h} \neq 0$ (*i.e.*, for all $\mathbf{h} \neq 0$ such that $\nabla C_i(\mathbf{z}_e) \cdot \mathbf{h} = 0$ $(i = 1, \ldots, m)$),

$$\mathbf{h} \cdot \nabla^2 F(\mathbf{z}_e)\mathbf{h} > 0. \tag{14}$$

In the absence of symmetries, the range of $\mathbf{A}(\mathbf{z})$ is identically the range of $\mathbf{J}(\mathbf{z})$, and conversely, the nullspace of $\mathbf{A}^\mathsf{T}(\mathbf{z})$ is identically the nullspace of $\mathbf{J}(\mathbf{z})$. Introducing the orthogonal projection matrix, $\mathbf{P}(\mathbf{z})$, onto the range of $\mathbf{A}(\mathbf{z})$, we may write the second-order condition as (recall that $\nabla C_i(\mathbf{z}_e)$ span $\mathcal{N}(\mathbf{J}(\mathbf{z}_e))$, the nullspace of $\mathbf{J}(\mathbf{z}_e)$): For all $\mathbf{h}$ such that $\mathbf{P}(\mathbf{z}_e)\mathbf{h} \neq \mathbf{0}$,

$$\mathbf{h} \cdot \mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)\mathbf{h} > 0. \tag{15}$$

Let $\mathbf{Q}(\mathbf{z})$ be the projection onto $\mathcal{N}\left(\mathbf{A}^\mathsf{T}(\mathbf{z})\right) = \mathcal{N}(\mathbf{J}(\mathbf{z}))$ given by

$$\mathbf{Q}(\mathbf{z}) = \mathbf{K}(\mathbf{z})\left(\mathbf{K}^\mathsf{T}(\mathbf{z})\mathbf{K}(\mathbf{z})\right)^{-1}\mathbf{K}^\mathsf{T}(\mathbf{z}) \tag{16}$$

where $\mathbf{K}(\mathbf{z})\left[\nabla C_1(\mathbf{z}) \quad \ldots \quad \nabla C_m(\mathbf{z})\right]$ is the matrix whose columns span the nullspace. Then the projection onto the range is

$$\mathbf{P}(\mathbf{z}) = \mathbf{1} - \mathbf{Q}(\mathbf{z}). \tag{17}$$

The *projected Hessian matrix* is then given by $\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)$. This $n \times n$ matrix will have m zero eigenvalues associated with the nullspace. If the remaining eigenvalues are all

[2]We present the case where all the known integrals are Casimir functions. For systems with symmetry integrals, a similar result applies; however, the analysis must be revised to account for the additional constraints and the function for which we seek a constrained extremum is no longer the Hamiltonian. The details of this analysis are presented by example below in the analysis of an axisymmetric rigid body in a central gravitational field.

positive, then $\mathbf{z}_e$ is a constrained minimum and the relative equilibrium is nonlinearly stable.

Hierarchy of Rigid-Body Problems

We next examine approaches to approximation of the two-body problem, establishing a three-dimensional hierarchy of related Hamiltonian systems. All approximations considered here consist of a rigid body moving in a central gravitational field. They differ in the degree of simplifications introduced. As shown in Fig. 4, the first dimension of the hierarchy is associated with increasing constraints on the trajectory of the satellite; the second dimension is decreasing order of approximation of the potential; and the third dimension is increasing symmetry of the satellite. The presentation given here finds its origin in the work of Beletskii [22] whose research examined many of the systems in the hierarchy using classical methods. The relationships between the various systems in the hierarchy can be inferred from the discussion in Ref. [22], but here we formally identify those relationships.

Trajectory Constraints

In Ref. [11], the Hamiltonian system for the two-body problem was developed and a process of reduction was applied (along with an assumption of spherical symmetry of one body) to transform the equations of motion from a twenty-fourth-order system with ten known first integrals to a ninth-order system with two known first integrals. In the limit as the mass of the second body (or satellite) becomes very small compared to the mass of the spherical body (or primary), this system reduces to that of a rigid body moving in a central gravitational field. We refer to this approximate system as the *free rigid-body system.*

The classical approach to treating the relative equilibria of a satellite moving about a large primary results in a different approximation. In this approach, the series expansions of the force and torque are each truncated after the lead term. The force which results is the same as the force for a particle of equal mass located at the center of mass. The center of mass therefore follows an (orthogonal) Keplerian orbit. If we constrain the center of mass to follow this orbit and consider the resulting attitude motion of the rigid body due to the gravitational torques, we have what we refer to as the *Keplerian system.* Since we are interested in relative equilibria similar to those of the free rigid-body system, we only consider circular orbits.

Finally, if we constrain a point in the body to remain fixed in inertial space, we have a system in which the centripetal acceleration effects are removed and only the central gravitational field effects remain. We refer to this as the *fixed-point system.* This system is a generalization of the classical heavy-top problem which treats motion about a fixed point in a uniform gravitational field.

Beletskii [22] first identified this categorization for the problems of a rigid body in a central

gravitational field. Note that in the free rigid-body and Keplerian systems we assume the point C is the center of mass, while no such assumption is made for the fixed-point system. In fact we could expand this dimension of the hierarchy to two dimensions by treating a series of possible constraints on the position of the center of mass relative to the origin of the body frame. Maddocks [12] considered such a series of constraints in a noncanonical treatment of the heavy-top problem. However, by restricting consideration to only those problems where C is either fixed or is the center of mass, the relationship between angular momentum and torque about C takes the simpler form

$$\dot{\boldsymbol{\pi}} = \mathbf{g}_c \tag{18}$$

in the inertial frame. For the free rigid-body system, the choice of the point C as the center of mass is simply a matter of convenience. However, for the constrained systems, the choice of C changes the dynamics.

The derivation of the Hamiltonian systems for the three problems comprising the first dimension is presented in Ref. [11]. The free rigid-body formulation is due to Wang *et al* [7], the fixed-point formulation is new but virtually identical to the heavy top system treated by Maddocks [12], and the Keplerian formulation is an entirely new derivation. The Keplerian derivation is also presented below.

Potential Approximations

Series Expansion of the Potential

The potential for a rigid body in a central gravitational field is given by

$$V(\boldsymbol{\Lambda}) = -\int_{\mathcal{B}} \frac{G_*}{|\boldsymbol{\Lambda} + \mathbf{a}|} dm \tag{19}$$

where G_* is the gravitational constant for the primary body. Here $\boldsymbol{\Lambda}$ is the vector from the center of attraction to the origin C of the body frame $\mathfrak{F}_b$ expressed in the body frame. We may expand the integrand as

$$|\boldsymbol{\Lambda} + \mathbf{a}|^{-1} = |\boldsymbol{\Lambda}|^{-1} \sum_{k=0}^{\infty} (-\varepsilon)^k P_k(\cos\phi) \tag{20}$$

where the P_k are the Legendre polynomials. Integrating term by term, the potential may be written as

$$V(\boldsymbol{\Lambda}) = -\frac{G_*}{|\boldsymbol{\Lambda}|} \sum_{k=0}^{\infty} \left[\int_{\mathcal{B}} (-\varepsilon)^k P_k(\cos\phi) dm \right] \tag{21}$$

as the series expansion of the potential.

For $\varepsilon \ll 1$ the higher-order terms in equation (21) quickly approach zero and it seems reasonable to truncate the series. We refer to a particular truncation of the potential as the *nth-order potential approximation* when it includes terms up to $\mathcal{O}(\varepsilon^n)$. We denote the nth-order approximation of the potential as V_n. Thus,

$$V_n(\mathbf{\Lambda}) = -\frac{G_*}{|\mathbf{\Lambda}|}\sum_{k=0}^{n}\left[\int_{\mathcal{B}}(-\varepsilon)^k P_k(\cos\phi)dm\right]. \tag{22}$$

Likewise, the associated approximate model of rigid body motion will be called the *nth-order model*.

Force and Torque

The force and torque acting on the rigid body are derived from the potential. We begin by considering the force acting on a differential mass element dm. The potential for this element, from equation (19), is given by

$$dV = -\frac{G_*}{R}dm \tag{23}$$

where $\mathbf{R} = \mathbf{\Lambda} + \mathbf{a}$ and $R = |\mathbf{R}|$. Let $\mathbf{F} = \mathbf{B}^\mathsf{T}\mathbf{f}$ and $\mathbf{G}_c = \mathbf{B}^\mathsf{T}\mathbf{g}_c$ be the total force and the torque about the point C expressed in the body frame. It should be clear that the force on the differential mass element can be expressed as

$$d\mathbf{F}(\mathbf{R}) = -\nabla dV(\mathbf{R}) = -\frac{G_*}{R^3}\mathbf{R}dm. \tag{24}$$

With a simple change of variables, it follows that $d\mathbf{F}(\mathbf{\Lambda}) = -\nabla dV(\mathbf{\Lambda})$. Let

$$\mathbf{a}^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

be the matrix form associated with the cross product in $\mathbb{R}^3$. Then the contribution to the torque about C is $d\mathbf{G}_c(\mathbf{\Lambda}) = \mathbf{a}^\times d\mathbf{F}(\mathbf{\Lambda}) = -\mathbf{\Lambda}^\times d\mathbf{F}(\mathbf{\Lambda})$. The last equality follows from the alignment of $d\mathbf{F}$ with $\mathbf{R}$ as shown in Fig. 5. Integration of the above expressions over the rigid body gives the total force and moment about C. Exchanging the order of differentiation

and integration, we find

$$\mathbf{F}(\mathbf{\Lambda}) = -\nabla V(\mathbf{\Lambda}) \tag{25}$$

$$\mathbf{G}_c(\mathbf{\Lambda}) = \mathbf{\Lambda}^{\times}\nabla V(\mathbf{\Lambda}). \tag{26}$$

Note that the torque is always orthogonal to $\mathbf{\Lambda}$ so that there is no torque component in the radial direction.

For some problems we consider, the distance $|\mathbf{\Lambda}|$ is fixed. In those instances we are only concerned with the torque and it is more convenient to work with the unit vector in the radial direction, $\boldsymbol{\gamma}$, rather than $\mathbf{\Lambda}$. Again, a change of variables in equation (26) gives

$$\mathbf{G}_c(\boldsymbol{\gamma}) = \boldsymbol{\gamma}^{\times}\nabla V(\boldsymbol{\gamma}). \tag{27}$$

This expression for the gravity torque was given in scalar form by Beletskii [22]. The force and torque given in equations (25), (26), and (27) are relatively simple expressions in terms of the potential and allow a direct substitution of any order approximation.

Rigid-Body Symmetry

The third dimension of the hierarchy is related to the level of symmetry present in the rigid body. Continuous rotational symmetries (axial or spherical symmetry) introduce additional integrals and allow further simplifications to be made. In the presence of spherical symmetry, the attitude motion is a constant spin since the components of angular momentum in the body frame are all conserved. The translation of the satellite decouples from the attitude dynamics and we may treat it as a point mass. If the body is axisymmetric, only the component of angular momentum in the symmetry axis direction is conserved. To treat this problem, the equations of motion are transformed from the body frame to a nodal frame in which the body spins at a constant rate about the symmetry axis. The noncanonical system in the nodal frame turns out to have the same Poisson structure, but the Hamiltonian has an additional term containing the spin rate and conserved momentum.

The stringency of the symmetry requirement is a function of the order of the potential approximation. For the second-order approximation of interest here, we require *dynamic symmetry* or equality of the principal moments of inertia.

Second-Order Keplerian System

In the following, we examine a particular system which appears in the hierarchy described above — the second-order Keplerian system. This system is dynamically equivalent to the classical approximation of the motion of a rigid-body in a central gravitational field. We

apply Hamiltonian methods to determine the relative equilibria and analyze their stability for both the arbitrary and axisymmetric rigid body.

Noncanonical Hamiltonian Formulation

We consider the motion of a rigid body about its center of mass, which is constrained to follow a circular Keplerian orbit about the center of attraction as shown in Fig. 6. The frequency of a Keplerian orbit with radius $|\mathbf{\Lambda}|$ is

$$\omega_c = \left(\frac{G_*}{|\mathbf{\Lambda}|^3} \right)^{1/2}. \tag{28}$$

In the canonical treatment (see, /eg, Chapter 12 of Meirovitch [23]), the nature of the constraint leads to a derivation of the equations in an orbiting frame. The problem has three degrees of freedom associated with rotation about the center of mass and hence a sixth-order system results. The Hamiltonian for the canonical system is not the total energy, but it is an integral of motion. The canonical system has no other known integrals of motion for an arbitrary rigid body.

In deriving the noncanonical system, we are also interested in motion relative to the orbital frame. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ be the body-frame representations of the orbital frame unit vectors as shown in Fig. 6. If $\mathbf{B}$ is the direction cosine matrix for the transformation from body to orbital frame, then $\mathbf{B}^\mathsf{T} = \begin{bmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma} \end{bmatrix}$. We then start by transforming equation (18) to the body frame to get Euler's equation for the rotation of the body:

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi}^\times \mathbf{I}^{-1} \mathbf{\Pi} + \mathbf{G}_c = \mathbf{\Pi}^\times \mathbf{I}^{-1} \mathbf{\Pi} + \boldsymbol{\gamma}^\times \nabla V(\boldsymbol{\gamma}) \tag{29}$$

where we have used the expression for the torque given in equation (27). The *relative* angular velocity and angular momentum may be defined as

$$\mathbf{\Omega}_r = \mathbf{\Omega} - \omega_c \boldsymbol{\beta} \tag{30a}$$

$$\mathbf{\Pi}_r = \mathbf{\Pi} - \omega_c \mathbf{I} \boldsymbol{\beta}. \tag{30b}$$

Differentiation of equation (30b) leads to

$$\dot{\mathbf{\Pi}}_r = \mathbf{\Pi}_r^\times \mathbf{I}^{-1} \mathbf{\Pi}_r + \omega_c \mathbf{\Pi}_r^\times \boldsymbol{\beta} + \omega_c (\mathbf{I}\boldsymbol{\beta})^\times \mathbf{I}^{-1} \mathbf{\Pi}_r + \omega_c^2 (\mathbf{I}\boldsymbol{\beta})^\times \boldsymbol{\beta} - \omega_c \mathbf{I} \dot{\boldsymbol{\beta}} + \boldsymbol{\gamma}^\times \nabla V(\boldsymbol{\gamma}). \tag{31}$$

We also have the rotational kinematic relations

$$\dot{\boldsymbol{\beta}} = \boldsymbol{\beta}^{\times}\mathbf{I}^{-1}\boldsymbol{\Pi}_r \tag{32a}$$

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}^{\times}\mathbf{I}^{-1}\boldsymbol{\Pi}_r. \tag{32b}$$

In Ref. [11], a proof is given to show that any vector $\boldsymbol{\beta}$ and symmetric matrix $\mathbf{I}$ satisfy the identity

$$\boldsymbol{\beta}^{\times}\mathbf{I} + \mathbf{I}\boldsymbol{\beta}^{\times} \equiv \left\{\left[\mathrm{tr}\left(\mathbf{I}\right)\mathbf{1} - \mathbf{I}\right]\boldsymbol{\beta}\right\}^{\times}. \tag{33}$$

Using equations (32a) and (33), equation (31) reduces to

$$\dot{\boldsymbol{\Pi}}_r = \left\{\boldsymbol{\Pi}_r - \omega_c\left[\mathrm{tr}\left(\mathbf{I}\right)\mathbf{1} - 2\mathbf{I}\right]\boldsymbol{\beta}\right\}^{\times}\mathbf{I}^{-1}\boldsymbol{\Pi}_r - \omega_c^2\boldsymbol{\beta}^{\times}\mathbf{I}\boldsymbol{\beta} + \boldsymbol{\gamma}^{\times}\nabla V(\boldsymbol{\gamma}). \tag{34}$$

Equations (32) and (34) form the noncanonical system

$$\begin{pmatrix}\dot{\boldsymbol{\Pi}}_r\\ \dot{\boldsymbol{\beta}}\\ \dot{\boldsymbol{\gamma}}\end{pmatrix} = \begin{bmatrix}\left\{\boldsymbol{\Pi}_r - \omega_c\left[\mathrm{tr}\left(\mathbf{I}\right)\mathbf{1} - 2\mathbf{I}\right]\boldsymbol{\beta}\right\}^{\times} & \boldsymbol{\beta}^{\times} & \boldsymbol{\gamma}^{\times}\\ \boldsymbol{\beta}^{\times} & \mathbf{0} & \mathbf{0}\\ \boldsymbol{\gamma}^{\times} & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\mathbf{I}^{-1}\boldsymbol{\Pi}_r\\ -\omega_c^2\mathbf{I}\boldsymbol{\beta}\\ \nabla V(\boldsymbol{\gamma})\end{pmatrix} \tag{35}$$

with Hamiltonian

$$H(\mathbf{z}) = \frac{1}{2}\boldsymbol{\Pi}_r\cdot\mathbf{I}^{-1}\boldsymbol{\Pi}_r - \frac{1}{2}\omega_c^2\boldsymbol{\beta}\cdot\mathbf{I}\boldsymbol{\beta} + V(\boldsymbol{\gamma}). \tag{36}$$

This is the same Hamiltonian found in the canonical system and it is a first integral. The nullspace of the structure matrix is three-dimensional:

$$\mathcal{N}\left[\mathbf{J}(\mathbf{z})\right] = \mathrm{span}\left\{\begin{pmatrix}\mathbf{0}\\ \boldsymbol{\beta}\\ \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \boldsymbol{\gamma}\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \boldsymbol{\gamma}\\ \boldsymbol{\beta}\end{pmatrix}\right\}. \tag{37}$$

From the spanning vectors, we identify three independent Casimir functions:

$$C_1(\mathbf{z}) = \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\beta} \qquad C_2(\mathbf{z}) = \frac{1}{2}\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} \qquad C_3(\mathbf{z}) = \boldsymbol{\beta}\cdot\boldsymbol{\gamma}. \tag{38}$$

In this problem, all three Casimir functions are trivial since $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are rows of the direction cosine matrix $\mathbf{B}$ so that

$$\boldsymbol{\beta}\cdot\boldsymbol{\beta} \equiv 1 \qquad \boldsymbol{\gamma}\cdot\boldsymbol{\gamma} \equiv 1 \qquad \boldsymbol{\beta}\cdot\boldsymbol{\gamma} \equiv 0. \tag{39}$$

We thus have a ninth-order system with four known first integrals.

The second-order approximation of the potential may be expressed as $V_2(\boldsymbol{\gamma}) = \frac{3}{2}\omega_c^2 \boldsymbol{\gamma} \cdot \mathbf{I}\boldsymbol{\gamma}$. This potential approximation may be substituted into the Keplerian system given in equation (35) to obtain

$$\begin{pmatrix} \dot{\bar{\boldsymbol{\Pi}}}_r \\ \dot{\boldsymbol{\beta}} \\ \dot{\boldsymbol{\gamma}} \end{pmatrix} = \begin{bmatrix} \left\{\bar{\boldsymbol{\Pi}}_r - \overline{\omega}_c \left[\mathrm{tr}\left(\bar{\mathbf{I}}\right)\mathbf{1} - 2\bar{\mathbf{I}}\right]\boldsymbol{\beta}\right\}^\times & \boldsymbol{\beta}^\times & \boldsymbol{\gamma}^\times \\ \boldsymbol{\beta}^\times & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\gamma}^\times & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \bar{\mathbf{I}}^{-1}\bar{\boldsymbol{\Pi}}_r \\ -\overline{\omega}_c^2 \bar{\mathbf{I}}\boldsymbol{\beta} \\ 3\overline{\omega}_c^2 \bar{\mathbf{I}}\boldsymbol{\gamma} \end{pmatrix}. \tag{40}$$

The overbar has been added to denote dimensional variables. To nondimensionalize the system, we choose the mass, length, and time scales

$$m = \overline{m} = \int_{\mathcal{B}} d\overline{m} \qquad l = \left(\frac{\mathrm{tr}\left(\bar{\mathbf{I}}\right)}{\overline{m}}\right)^{\frac{1}{2}} \qquad t = \overline{\omega}_c^{-1}. \tag{41}$$

The equations of motion then become

$$\begin{pmatrix} \dot{\boldsymbol{\Pi}}_r \\ \dot{\boldsymbol{\beta}} \\ \dot{\boldsymbol{\gamma}} \end{pmatrix} = \begin{bmatrix} \left[\boldsymbol{\Pi}_r - (\mathbf{1} - 2\mathbf{I})\boldsymbol{\beta}\right]^\times & \boldsymbol{\beta}^\times & \boldsymbol{\gamma}^\times \\ \boldsymbol{\beta}^\times & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\gamma}^\times & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{I}^{-1}\boldsymbol{\Pi}_r \\ -\mathbf{I}\boldsymbol{\beta} \\ 3\mathbf{I}\boldsymbol{\gamma} \end{pmatrix} \tag{42}$$

where $\boldsymbol{\Pi}_r = (m^{-1}l^{-2}t)\,\bar{\boldsymbol{\Pi}}_r$ and $\mathbf{I} = (m^{-1}l^{-2})\,\bar{\mathbf{I}}$. This is the nondimensional second-order model of the Keplerian system. The Casimir functions for this system as given in equation (38) are already in dimensionless form.

Relative Equilibria for an Arbitrary Rigid Body

Relative Equilibrium Conditions

The critical points of the Hamiltonian system (42) may be characterized by the first-variation condition $\nabla F(\mathbf{z}_e) = \mathbf{0}$ where

$$F(\mathbf{z}) = H(\mathbf{z}) - \mu_1 C_1(\mathbf{z}) - \mu_2 C_2(\mathbf{z}) - \mu_3 C_3(\mathbf{z}). \tag{43}$$

The resulting relative equilibrium conditions are

$$\mathbf{I}^{-1}\boldsymbol{\Pi}_{r_e} = \mathbf{0} \tag{44a}$$

$$-\mathbf{I}\boldsymbol{\beta}_e - \mu_1\boldsymbol{\beta}_e - \mu_3\boldsymbol{\gamma}_e = \mathbf{0} \tag{44b}$$

$$3\mathbf{I}\boldsymbol{\gamma}_e - \mu_2\boldsymbol{\gamma}_e - \mu_3\boldsymbol{\beta}_e = \mathbf{0} \tag{44c}$$

along with the Casimir Conditions (38). Equation (44a) gives $\mathbf{\Pi}_{r_e} = \mathbf{0}$. Physically, this corresponds to the situation where the body is stationary with respect to the orbital frame. Taking the dot product of $\boldsymbol{\gamma}_e$ with equation (44b) gives $\mu_3 = -\boldsymbol{\gamma}_e \cdot \mathbf{I}\boldsymbol{\beta}_e$ while the dot product of $\boldsymbol{\beta}_e$ with equation (44c) gives $\mu_3 = 3\boldsymbol{\beta}_e \cdot \mathbf{I}\boldsymbol{\gamma}_e$. Symmetry of the inertia matrix implies these two requirements can only be satisfied if μ_3 is identically zero. Then the relative equilibrium conditions on $\boldsymbol{\beta}_e$ and $\boldsymbol{\gamma}_e$ become $\mathbf{I}\boldsymbol{\beta}_e = -\mu_1\boldsymbol{\beta}_e$ and $\mathbf{I}\boldsymbol{\gamma}_e = \frac{\mu_2}{3}\boldsymbol{\gamma}_e$ so that $\boldsymbol{\beta}_e$ and $\boldsymbol{\gamma}_e$ must be eigenvectors of the inertia matrix. Thus, the principal relative equilibria are solutions for this system, as first shown by Lagrange [1], and are in fact the only solutions, as first demonstrated by Likins and Roberson [2].[3]

Without loss of generality, we choose the principal frame so that the relative equilibrium of interest is given as

$$\mathbf{\Pi}_{r_e} = \mathbf{0} \qquad \boldsymbol{\beta}_e = \mathbf{1}_2 \qquad \boldsymbol{\gamma}_e = \mathbf{1}_3 \tag{45a}$$

$$\mu_1 = -I_2 \qquad \mu_2 = 3I_3 \qquad \mu_3 = 0. \tag{45b}$$

Thus, at relative equilibrium the body frame and orbital frame are aligned.

Spectral Stability

For this system, equation (11) gives the linear system matrix at relative equilibrium as

$$\mathbf{A}(\mathbf{z}_e) = \begin{bmatrix} 0 & 0 & -(1-2I_2)/I_3 & 0 & 0 & I_2 - I_3 & 0 & -3(I_2 - I_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3(I_1 - I_3) & 0 & 0 \\ (1-2I_2)/I_1 & 0 & 0 & I_1 - I_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/I_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/I_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{46}$$

We find that the system decouples and the characteristic polynomial takes the form

$$P(s) = s^3\left(s^2 + A_0\right)\left(s^4 + B_2 s^2 + B_0\right). \tag{47}$$

[3]Our proof is more direct. Likins and Roberson began with the condition $3\boldsymbol{\gamma}_e^{\times}\mathbf{I}\boldsymbol{\gamma}_e - \boldsymbol{\beta}_e^{\times}\mathbf{I}\boldsymbol{\beta}_e = \mathbf{0}$ which is derived from the $\dot{\mathbf{\Pi}}_r$ equation with $\mathbf{\Pi}_r$ set equal to zero. We refer to this as the *Likins-Roberson Condition* for relative equilibrium. A similar condition arises for the free rigid-body system.

The three zero roots are associated with the three Casimir functions. The coefficients of the polynomial are

$$A_0 = 3k_2 \qquad B_2 = 1 + 3k_1 + k_1k_3 \qquad B_0 = 4k_1k_3 \tag{48}$$

where we introduce the Smelt inertia parameters

$$k_1 = \frac{I_2 - I_3}{I_1} \qquad k_2 = \frac{I_1 - I_3}{I_2} \qquad k_3 = \frac{I_2 - I_1}{I_3}. \tag{49}$$

Only two of these parameters are independent so that we may express k_2 as a function of k_1 and k_3:

$$k_2 = \frac{k_1 - k_3}{1 - k_1k_3}. \tag{50}$$

In addition, these parameters satisfy the constraints $|k_i| < 1$. The six regions of the k_1–k_3 plane formed by the $k_1 = k_3$, $k_1 = 0$, and $k_3 = 0$ lines each correspond to different choices of the major and minor axis directions. The spectral stability requirements from Table 1, upon eliminating strictly positive factors, reduce to

$$k_1 - k_3 \geq 0 \tag{51a}$$

$$1 + 3k_1 + k_1k_3 \geq 0 \tag{51b}$$

$$k_1k_3 \geq 0 \tag{51c}$$

$$(1 + 3k_1 + k_1k_3)^2 - 16k_1k_3 \geq 0. \tag{51d}$$

The regions satisfying Conditions (51) are shown in the Smelt parameter plane in Fig. 7. The dark gray region in the first quadrant corresponds to the stable relative equilibria identified by Lagrange [1]. The light gray region in the third quadrant was identified by DeBra and Delp [24].

Nonlinear Stability

In the process of computing the linear system matrix, it was necessary to compute the Hessian of the variational Lagrangian:

$$\nabla^2 F(\mathbf{z}) = \begin{bmatrix} \mathbf{I}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} - \mu_1\mathbf{1} & -\mu_3\mathbf{1} \\ \mathbf{0} & -\mu_3\mathbf{1} & 3\mathbf{I} - \mu_2\mathbf{1} \end{bmatrix}. \tag{52}$$

At relative equilibrium, this becomes

$$\nabla^2 F(\mathbf{z}_e) = \mathrm{diag}\,(\ 1/I_1,\ 1/I_2,\ 1/I_3,\ I_2 - I_1,\ 0,\ I_2 - I_3,\ 3(I_1 - I_3),\ 3(I_2 - I_3),\ 0\ ) \tag{53}$$

which is clearly not positive- or negative-definite. Following the procedure outlined above for computing the projected Hessian, we construct a matrix of basis vectors for $\mathcal{N}\left(\mathbf{J}(\mathbf{z}_e)\right) = \mathcal{N}\left(\mathbf{A}^{\mathsf{T}}(\mathbf{z}_e)\right)$ as $\mathbf{K}(\mathbf{z}_e) = \begin{bmatrix}\nabla C_1(\mathbf{z}_e) & \nabla C_2(\mathbf{z}_e) & \nabla C_3(\mathbf{z}_e)\end{bmatrix}$ and compute the projection $\mathbf{P}(\mathbf{z}_e)$ onto the tangent space $TM|_{\mathbf{z}_e}$. From this we may compute the projected Hessian

$$\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e) = \begin{bmatrix} 1/I_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 - I_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_2 - I_3 & 0 & -(I_2 - I_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3(I_1 - I_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(I_2 - I_3) & 0 & I_2 - I_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{54}$$

for which we find that the eigenvalues are

$$\{0,\ 0,\ 0,\ 1/I_1,\ 1/I_2,\ 1/I_3,\ I_2 - I_1,\ 2(I_2 - I_3),\ 3(I_1 - I_3)\}\,.$$

The three zero eigenvalues are associated with the nullspace and the remaining five are associated with the tangent space of the constraint manifold. The conditions for positive-definiteness on the tangent space are precisely the conditions for the Lagrange region ($I_2 > I_1 > I_3$). That is, in this region the Hamiltonian is a constrained minimum and we may conclude nonlinear stability.

Transformation to Nodal Frame

We now consider the Keplerian problem with the additional restriction that the body has an axis of symmetry. Dynamic symmetry is sufficient for the second-order approximation we treat here. Associated with the symmetry is an additional ignorable coordinate (and a conserved quantity). To treat this problem, the equations of motion are transformed from the body frame to a frame in which the body spins at a constant rate about the symmetry axis. We take one direction along the symmetry axis to be the third basis vector of both the

body frame and the new frame. We refer to the new frame as the *nodal frame*. The choice of the symmetry axis as a basis vector implies both the body and nodal frames are principal inertia frames.

The transformation to the nodal frame is given by

$$\mathbf{z} = \mathbf{M}\mathbf{z}_b \tag{55}$$

where $\mathbf{z}$ is the phase variable vector in the nodal frame, $\mathbf{z}_b$ is the phase variable vector in the body frame, and

$$\mathbf{M} = \begin{bmatrix} \mathbf{C}_{nb} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{nb} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{nb} \end{bmatrix} \qquad \mathbf{C}_{nb} = \begin{bmatrix} \cos\mu_4 t & -\sin\mu_4 t & 0 \\ \sin\mu_4 t & \cos\mu_4 t & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{56}$$

Here μ_4 is the rotation rate of the body about the symmetry axis relative to the nodal frame. The new conserved quantity is the angular momentum component in the direction of the symmetry axis, which is given by

$$C_4(\mathbf{z}) = \mathbf{\Pi} \cdot \mathbf{1}_3 = (\mathbf{\Pi}_r + \mathbf{I}\boldsymbol{\beta}) \cdot \mathbf{1}_3. \tag{57}$$

We will now show that the Hamiltonian form of the system remains unchanged with one exception — the Hamiltonian takes the form

$$H(\mathbf{z}) = H_b(\mathbf{z}) - \mu_4 C_4(\mathbf{z}) \tag{58}$$

where $H_b(\mathbf{z})$ is the previous Hamiltonian expressed in terms of the new phase variables. Differentiating equation (55) with respect to time gives

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{M}\dot{\mathbf{z}}_b + \dot{\mathbf{M}}\mathbf{z}_b \\ &= \mathbf{M}\mathbf{J}(\mathbf{z}_b)\nabla H_b(\mathbf{z}_b) + \dot{\mathbf{M}}\mathbf{M}^\mathsf{T}\mathbf{z} \\ &= \mathbf{M}\mathbf{M}^\mathsf{T}\mathbf{J}(\mathbf{z})\mathbf{M}\mathbf{M}^\mathsf{T}\nabla H_b(\mathbf{z}) + \mathbf{W}\mathbf{z} \\ &= \mathbf{J}(\mathbf{z})\nabla H_b(\mathbf{z}) + \mathbf{W}\mathbf{z} \end{aligned} \tag{59}$$

where

$$\mathbf{W} = \begin{bmatrix} \dot{\mathbf{C}}_{nb}\mathbf{C}_{nb}^\mathsf{T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{C}}_{nb}\mathbf{C}_{nb}^\mathsf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dot{\mathbf{C}}_{nb}\mathbf{C}_{nb}^\mathsf{T} \end{bmatrix} = \begin{bmatrix} \mu_4\mathbf{1}_3^\times & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mu_4\mathbf{1}_3^\times & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mu_4\mathbf{1}_3^\times \end{bmatrix}. \tag{60}$$

We may show that $\mathbf{W}\mathbf{z} = -\mu_4 \mathbf{J}(\mathbf{z})\nabla C_4(\mathbf{z})$ by direct computation so that

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{J}(\mathbf{z})\left[\nabla H_b(\mathbf{z}) - \mu_4 \nabla C_4(\mathbf{z})\right] \\ &= \mathbf{J}(\mathbf{z})\nabla H(\mathbf{z}) \end{aligned} \tag{61}$$

as claimed.

Relative Equilibria for an Axisymmetric Rigid Body

Relative Equilibrium Conditions

Given the transformation above, the relative equilibria of the axisymmetric system are now characterized by critical points of

$$F(\mathbf{z}) = H(\mathbf{z}) - \sum_{i=1}^{3} \mu_i C_i(\mathbf{z}) = H_b(\mathbf{z}) - \sum_{i=1}^{4} \mu_i C_i(\mathbf{z}). \tag{62}$$

Taking the gradient, we find the relative equilibrium conditions

$$\mathbf{I}^{-1}\mathbf{\Pi}_{r_e} - \mu_4 \mathbf{1}_3 = \mathbf{0} \tag{63a}$$

$$-\mathbf{I}\boldsymbol{\beta}_e - \mu_1 \boldsymbol{\beta}_e - \mu_3 \boldsymbol{\gamma}_e - \mu_4 \mathbf{I}\mathbf{1}_3 = \mathbf{0} \tag{63b}$$

$$3\mathbf{I}\boldsymbol{\gamma}_e - \mu_2 \boldsymbol{\gamma}_e - \mu_3 \boldsymbol{\beta}_e = \mathbf{0}. \tag{63c}$$

Along with the Casimir constraints (38) and the symmetry integral (57), these define a family of relative equilibria parameterized by μ_4 and the inertias. Equation (63a) gives $\mathbf{\Omega}_{r_e} = \mu_4 \mathbf{1}_3$ which implies at relative equilibrium the body rotates with angular velocity μ_4 about the symmetry axis relative to the orbital frame (*i.e.*, the nodal frame is fixed with respect to the orbital frame). We eliminate the Lagrange multipliers μ_1, μ_2, and μ_3 from equations (63b) and (63c) by taking the crossproduct of $\boldsymbol{\beta}_e$ with the first and $\boldsymbol{\gamma}_e$ with the second and adding to get

$$3\boldsymbol{\gamma}_e^{\times}\mathbf{I}\boldsymbol{\gamma}_e - \boldsymbol{\beta}_e^{\times}\mathbf{I}\boldsymbol{\beta}_e - \mu_4 \boldsymbol{\beta}_e^{\times}\mathbf{I}\mathbf{1}_3 = \mathbf{0}. \tag{64}$$

We refer to this as the *Pringle-Likins Condition* for axisymmetric relative equilibria since it is a generalization of the relative equilibrium conditions given by Pringle [5] and Likins [6] for specific choices of Euler angles. This condition is applicable for any parameterization of the transformation from orbital to nodal frame. Note the similarity to the Likins-Roberson Condition found for tri-inertial bodies. Let $I_1 = I_2 = I_t$ and $I_3 = I_a$ be the transverse and

axial principal moments of inertia. Then equation (64) becomes

$$\begin{pmatrix} (I_a - I_t)\,(3\gamma_2\gamma_3 - \beta_2\beta_3) - \mu_4 I_a \beta_2 \\ (I_a - I_t)\,(3\gamma_1\gamma_3 - \beta_1\beta_3) - \mu_4 I_a \beta_1 \\ 0 \end{pmatrix} = \mathbf{0}. \tag{65}$$

Let the inertia parameter k_t and the spin parameter Q be given by

$$k_t = (I_a - I_t)\,/I_t \tag{66a}$$

$$Q = -(I_a/I_t)\mu_4 = -\,(1 + k_t)\,\mu_4. \tag{66b}$$

Then, multiplying equation (65) by $1/I_t$, we may write the scalar form of the Pringle-Likins Condition as

$$3k_t\gamma_i\gamma_3 - k_t\beta_i\beta_3 + Q\beta_i = 0 \quad (i = 1, 2). \tag{67}$$

There are three classes of solutions to these conditions:

i. Cylindrical Relative Equilibria: $\beta_3^2 = 1$

ii. Hyperbolic Relative Equilibria: $\gamma_3 = 0 \quad \beta_3 = Q/k_t \quad (k_t \neq 0)$

iii. Conical Relative Equilibria: $\alpha_3 = 0 \quad \beta_3 = Q/(4k_t) \quad (k_t \neq 0)$.

These relative equilibria were shown in Fig. 3. To be complete, we note that the orbital frame unit vector $\boldsymbol{\alpha}$ is defined in terms of phase variables by $\boldsymbol{\alpha} = \boldsymbol{\beta}^{\times}\boldsymbol{\gamma}$. Only the components α_3, β_3, and γ_3 along the symmetry axis are determinate because there are infinitely many equally valid choices for the nodal frame. Given one choice, any rotation about the symmetry axis by a fixed amount gives another. We will introduce additional constraints to completely specify the nodal frame and simplify our computations; however, this is not a necessary step since we are only interested in the orientation of the symmetry axis in the orbital frame. Let $\boldsymbol{\eta}$ denote the unit vector along this axis so that $\boldsymbol{\eta} = \mathbf{B}\mathbf{1}_3 = [\boldsymbol{\alpha}\,\boldsymbol{\beta}\,\boldsymbol{\gamma}]^{\mathsf{T}}\,\mathbf{1}_3 = (\alpha_3, \beta_3, \gamma_3)$. Hence, this vector is completely specified for each class of solution and the physical orientations shown in Fig. 3 are readily apparent.

Solving for the Lagrange multipliers which we previously eliminated, we obtain

$$\mu_1 = -\left[I_t + (I_a - I_t)\beta_3^2 + \mu_4 I_a \beta_3\right] \qquad \mu_2 = 3\left[I_t + (I_a - I_t)\gamma_3^2\right] \qquad \mu_3 = 3(I_a - I_t)\beta_3\gamma_3. \tag{68}$$

In terms of the inertia and spin parameters defined above, these become

$$\mu_1 = -\frac{1 + k_t\beta_3^2 - Q\beta_3}{3 + k_t} \qquad \mu_2 = \frac{3\left(1 + k_t\gamma_3^2\right)}{3 + k_t} \qquad \mu_3 = \frac{3k_t\beta_3\gamma_3}{3 + k_t}. \tag{69}$$

These expressions are valid for all three relative equilibrium classes and will be used in the stability analysis to follow.

The additional integral for the axisymmetric system given in equation (57) is linear in the phase variables and has no influence on the Hessian of the variational Lagrangian $F(\mathbf{z})$ given in equation (52). The general form of the linear system matrix also remains valid for the axisymmetric system (recognizing that the variables are now expressed in the nodal frame rather than the body frame). However, inserting the relative equilibrium conditions results in different matrices for each class of relative equilibria. We treat the stability of the cylindrical, hyperbolic, and conical relative equilibria in turn.

Cylindrical Relative Equilibria

Spectral Stability For the cylindrical relative equilibria, the symmetry axis is aligned with the orbit normal so that $\beta_3^2 = 1$. Without loss of generality, let $\beta_3 = 1$. By the orthogonality of the unit vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$, this further implies $\alpha_3 = \gamma_3 = \beta_1 = \beta_2 = 0$. The choice of γ_1 and γ_2 is arbitrary as long as $\gamma_1^2 + \gamma_2^2 = 1$. For definiteness, we choose $\gamma_1 = 1$ and $\gamma_2 = 0$. The complete set of relative equilibrium conditions are then given as

$$\boldsymbol{\Pi}_{r_e} = \mu_4 I_a \mathbf{1}_3 \qquad \boldsymbol{\beta}_e = \mathbf{1}_3 \qquad \boldsymbol{\gamma}_e = \mathbf{1}_1 \tag{70a}$$

$$\mu_1 = -I_a(1 + \mu_4) \qquad \mu_2 = 3I_t \qquad \mu_3 = 0. \tag{70b}$$

For these values, the linear system matrix becomes

$$\mathbf{A}(\mathbf{z}_e) = \begin{bmatrix} 0 & 1 - k_t + Q & 0 & 0 & -\frac{kt-Q}{3+k_t} & 0 & 0 & 0 & 0 \\ -1 + k_t - Q & 0 & 0 & \frac{kt-Q}{3+k_t} & 0 & 0 & 0 & 0 & \frac{-3k_t}{3+k_t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(3 + k_t) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 + k_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3+k_t}{1+k_t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 + k_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{71}$$

This system matrix decouples and the characteristic equation takes the form

$$P(s) = s^5(s^4 + A_2 s^2 + A_0). \tag{72}$$

In addition to the three zero roots associated with the Casimir functions, two more roots are forced to zero by the symmetry of the body, so we will find that the characteristic polynomial takes this form for the other two classes of relative equilibria as well. The coefficients of the characteristic equation may be expressed as

$$A_0 = (k_t - Q)(4k_t - Q) \qquad A_2 = 1 + 3k_t + (k_t - Q)^2. \tag{73}$$

The spectral stability requirements from Table 1 reduce to

$$(k_t - Q)(4k_t - Q) \geq 0 \tag{74a}$$

$$1 + 3k_t + (k_t - Q)^2 \geq 0 \tag{74b}$$

$$(k_t - Q)^2(4k_t - Q)^2 - 4(k_t - Q)^2 - 4(1 + 3k_t) \geq 0. \tag{74c}$$

The stability regions are shown in Fig. 8 which plots the inertia parameter, k_t, versus the spin rate, μ_4.

Nonlinear Stability The projection method as described previously assumed the system had no symmetries. The presence of a symmetry introduces a new first integral (additional constraint) and requires some adjustments to our approach. Recall that in the projection method we construct the projection onto the tangent space at relative equilibrium (*i.e.*, onto $\mathcal{R}(\mathbf{A}(\mathbf{z}_e))$, the range of $\mathbf{A}(\mathbf{z}_e)$). For the arbitrary rigid body, $\mathcal{R}(\mathbf{A}(\mathbf{z}_e))$ coincided with $\mathcal{R}(\mathbf{J}(\mathbf{z}_e))$ as long as no two principal inertias were equal. With the addition of axial symmetry, $\mathcal{R}(\mathbf{A}(\mathbf{z}_e))$ no longer coincides with $\mathcal{R}(\mathbf{J}(\mathbf{z}_e))$, or conversely, $\mathcal{N}\left(\mathbf{A}^\mathsf{T}(\mathbf{z}_e)\right)$ no longer coincides with $\mathcal{N}(\mathbf{J}(\mathbf{z}_e))$. Two additional zero eigenvalues of $\mathbf{A}(\mathbf{z}_e)$ appear; however, the nullspace of $\mathbf{A}^\mathsf{T}(\mathbf{z}_e)$ is expanded by only one dimension spanned by the gradient of the first integral associated with the symmetry. That is, the algebraic multiplicity of the zero eigenvalue is $m + 2$ but the geometric multiplicity is only $m + 1$ and

$$\nabla C_{m+1}(\mathbf{z}_e) \cdot \mathbf{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e) = -\mathbf{J}(\mathbf{z}_e)\nabla C_{m+1}(\mathbf{z}_e) \cdot \nabla^2 F(\mathbf{z}_e) = \mathbf{0} \tag{75}$$

However, for the axial symmetry treated here, we have

$$\mathbf{J}(\mathbf{z})\nabla C_{m+1}(\mathbf{z}) = -(1/\mu_{m+1})\mathbf{W}\mathbf{z} \tag{76}$$

where $\mathbf{W}$ is the matrix defined in equation (60). Let

$$\boldsymbol{\xi}(\mathbf{z}) = -(1/\mu_{m+1})\mathbf{W}\mathbf{z} = \left(\mathbf{1}_3^{\times}\boldsymbol{\Pi}_r, \mathbf{1}_3^{\times}\boldsymbol{\beta}, \mathbf{1}_3^{\times}\boldsymbol{\gamma}\right). \tag{77}$$

Then we find

$$\boldsymbol{\xi}(\mathbf{z}_e) \cdot \nabla^2 F(\mathbf{z}_e) = \mathbf{0}. \tag{78}$$

Perturbations in the direction of $\boldsymbol{\xi}(\mathbf{z}_e)$ represent rotations about the symmetry axis and clearly are not prevented by the constraints. Thus, we find that the variational Lagrangian for axisymmetric systems must have at least a one-dimensional nullspace in the tangent space. This is related to the arbitrariness of the nodal frame and the concept of ignorable or cyclic coordinates in canonical systems. To treat this, we revise our definition of stability to omit perturbations in this direction.[4] Thus, in constructing the matrix of basis vectors for the constraints, we include $\boldsymbol{\xi}(\mathbf{z}_e)$.

The form of the Hessian of the variational Lagrangian given in equation (52) is still applicable for the axisymmetric system because the additional integral is linear in the phase variables. Substituting the current relative equilibrium conditions, we have

$$\nabla^2 F(\mathbf{z}_e) = \operatorname{diag}\left(3+k_t,\ 3+k_t,\ \frac{3+k_t}{1+k_t},\ \frac{k_t-Q}{3+k_t},\ \frac{k_t-Q}{3+k_t},\ \frac{-Q}{3+k_t},\ 0,\ 0,\ \frac{3k_t}{3+k_t} \right). \tag{79}$$

To construct the projection onto the portion of the tangent space orthogonal to the cyclic perturbation direction, we form the matrix of basis vectors for the $\mathcal{N}(\mathbf{A}(\mathbf{z}_e))$ augmented with the cyclic perturbation vector:

$$\mathbf{K}(\mathbf{z}_e) = \begin{bmatrix} \nabla C_1(\mathbf{z}_e) & \nabla C_2(\mathbf{z}_e) & \nabla C_3(\mathbf{z}_e) & \nabla C_4(\mathbf{z}_e) & \boldsymbol{\xi}(\mathbf{z}_e) \end{bmatrix}. \tag{80}$$

Then the projection matrix is given by equation (17) and we may compute the projected Hessian $\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)$ for which we find that the eigenvalues are

$$\left\{ 0,\ 0,\ 0,\ 0,\ 0,\ 3+k_t,\ 3+k_t,\ \frac{k_t-Q}{3+k_t},\ \frac{4k_t-Q}{2(3+k_t)} \right\}.$$

Four zero eigenvalues are associated with the constraints and one with the cyclic perturbation direction. The remaining four eigenvalues are associated with the restricted tangent space. Thus, the conditions for positive-definiteness on the restricted tangent space are $k_t - Q > 0$ and $4k_t - Q > 0$ which corresponds to the region shown in medium gray in Fig. 8. Recall that

[4] This is the *directional stability* of Hughes [25].

the variational Lagrangian was positive semi-definite in the first quadrant only. This problem clearly demonstrates that the variational Lagrangian need not even be positive semi-definite in order for a relative equilibrium to be a constrained minimum.

Hyperbolic Relative Equilibria

Spectral Stability For the hyperbolic relative equilibria, the symmetry axis lies within the plane formed by the tangential vector, $\boldsymbol{\alpha}$, and the orbit normal vector, $\boldsymbol{\beta}$, so that $\gamma_3 = 0$. Furthermore, we have $\beta_3 = Q/k_t$. This condition restricts the permissible range of Q/k_t to $|Q/k_t| \leq 1$ and results in portions of the k_t–μ_4 parameter space being physically unrealizable. For definiteness, we choose the nodal frame such that $\boldsymbol{\gamma}_e$ is in the direction of the first basis vector. The complete set of relative equilibrium conditions for the hyperbolic case are

$$\Pi_{r_1} = 0 \qquad \Pi_{r_2} = 0 \qquad \Pi_{r_3} = \mu_4 I_a \tag{81a}$$

$$\beta_1 = 0 \qquad \beta_2 \text{ unspecified} \qquad \beta_3 = Q/k_t \tag{81b}$$

$$\gamma_1 = 1 \qquad \gamma_2 = 0 \qquad \gamma_3 = 0 \tag{81c}$$

$$\mu_1 = -I_t \qquad \mu_2 = 3I_t \qquad \mu_3 = 0. \tag{81d}$$

By the second Casimir constraint, we must have $\beta_2^2 = 1 - \beta_3^2$. The choice of β_2 as positive or negative corresponds to the positive symmetry axis leaning forward or aft. This makes no difference in the analysis to follow and we leave it unspecified and treat both cases simultaneously. For these values, the linear system matrix becomes

$$\mathbf{A}(\mathbf{z}_e) = \begin{bmatrix} 0 & \frac{Q}{k_t} & -\beta_2 & 0 & 0 & \frac{-k_t\beta_2}{3+k_t} & 0 & 0 & 0 \\ \frac{-Q}{k_t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3k_t}{3+k_t} \\ (1+k_t)\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{(3+k_t)Q}{k_t} & \frac{(3+k_t)\beta_2}{1+k_t} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{(3+k_t)Q}{k_t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(3+k_t)\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3+k_t}{1+k_t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3+k_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{82}$$

This matrix does not decouple, but it does block-triangularize so that β_1, β_2, γ_1, and γ_2 are associated with trivial blocks on the diagonal. The characteristic equation again takes the form

$$P(s) = s^5(s^4 + A_2 s^2 + A_0). \tag{83}$$

In terms of the inertia parameter, k_t, and the spin parameter, Q, the coefficients of the characteristic equation are

$$A_0 = \frac{3}{k_t}(k_t - Q)(k_t + Q) \qquad A_2 = 1 + 3k_t \tag{84}$$

and the spectral stability requirements from Table 1 reduce to

$$k_t(k_t^2 - Q^2) \geq 0 \tag{85a}$$

$$1 + 3k_t \geq 0 \tag{85b}$$

$$k_t(k_t - 6k_t^2 + 9k_t^3 + 12Q^2) \geq 0. \tag{85c}$$

However, for physically realizable solutions, we require $|Q/k_t| \leq 1$. Thus Condition (85a) reduces to $k_t \geq 0$. Conditions (85b) and (85c) are then automatically satisfied. Therefore, we need only look at the bounds given by the Condition (85a) and the physical constraints. The stability regions are shown in Fig. 9.

Nonlinear Stability We now apply the projection method to show that the spectrally stable relative equilibria are also nonlinearly stable. Substituting the relative equilibrium conditions into (62), we have

$$\nabla^2 F(\mathbf{z}_e) = \operatorname{diag}\left(3 + k_t,\ 3 + k_t,\ \frac{3 + k_t}{1 + k_t},\ 0,\ 0,\ \frac{-k_t}{3 + k_t},\ 0,\ 0,\ \frac{3k_t}{3 + k_t} \right) \tag{86}$$

which is clearly indefinite. We again construct the projection onto the portion of the tangent space orthogonal to the cyclic perturbation direction by forming the matrix $\mathbf{K}(\mathbf{z}_e)$ of basis vectors for $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}(\mathbf{z}_e)\right)$ augmented with the cyclic perturbation vector. We then compute the projected variational Lagrangian $\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)$ for which the eigenvalues are

$$\left\{0,\ 0,\ 0,\ 0,\ 0,\ 3 + k_t,\ 3 + k_t,\ \frac{3(2k_t^2 - Q^2)}{2k_t(3 + k_t)},\ \frac{(3 + k_t)(k_t^2 - Q^2)}{k_t^2(2k_t^2 - Q^2) + 2k_t(4k_t^2 - Q^2) + (10k_t^2 - Q^2)}\right\}.$$

As before, four zero eigenvalues are associated with the constraints and one with the cyclic perturbation direction. The remaining four eigenvalues are associated with the restricted tangent space. The conditions for positive-definiteness on the restricted tangent space are $k_t > 0$ along with the physical constraint $k_t^2 - Q^2 > 0$. Thus, the relative equilibria for oblate configurations are nonlinearly stable.

Conical Relative Equilibria

Spectral Stability For the conical relative equilibria, the symmetry axis lies within the plane formed by the orbit normal vector, $\boldsymbol{\beta}$, and the radial vector, $\boldsymbol{\gamma}$, so that $\alpha_3 = 0$. Furthermore, we have $\beta_3 = Q/(4k_t)$. This condition restricts the permissible range of Q/k_t to $|Q/k_t| \leq 4$ and, just as for the hyperbolic case, results in portions of the k_t–μ_4 parameter space being physically unrealizable. We disregard the case where $\gamma_3 = 0$, since this is treated under the cylindrical relative equilibria. For definiteness, we choose the nodal frame such that the second basis vector is in the $\boldsymbol{\alpha}$ direction and the first basis vector is in the plane formed by $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. Then $\gamma_1 = \beta_3$ and $\gamma_3 = -\beta_1$. The complete set of relative equilibrium conditions for the conical case may then be given as

$$\Pi_{r_1} = 0 \qquad \Pi_{r_2} = 0 \qquad \Pi_{r_3} = \mu_4 I_a \tag{87a}$$

$$\beta_1 \text{ unspecified} \qquad \beta_2 = 0 \qquad \beta_3 = Q/(4k_t) \tag{87b}$$

$$\gamma_1 = Q/(4k_t) \qquad \gamma_2 = 0 \qquad \gamma_3 = -\beta_1 \tag{87c}$$

$$\mu_1 = \frac{-16k_t + 3Q^2}{16k_t(3+k_t)} \qquad \mu_2 = \frac{3(16k_t + 16k_t^2 - Q^2)}{16k_t(3+k_t)} \qquad \mu_3 = \frac{3Q\beta_1}{4(3+k_t)} \tag{87d}$$

where we have used the Casimir constraint $\beta_1^2 = 1 - \beta_3^3$ in specifying the μ_i. For these values, the linear system matrix becomes

$$\mathbf{A}(\mathbf{z}_e) = \begin{bmatrix}
0 & \frac{(1+3k_t)Q}{4k_t} & 0 & 0 & \frac{3Q}{4(3+k_t)} & 0 & 0 & -\frac{3k_t\beta_1}{3+k_t} & 0 \\
-\frac{(1+3k_t)Q}{4k_t} & 0 & \beta_1 & -\frac{3Q}{4(3+k_t)} & 0 & \frac{k_t\beta_1}{3+k_t} & \frac{3k_t\beta_1}{3+k_t} & 0 & -\frac{3Q}{4(3+k_t)} \\
0 & -(1+k_t)\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{(3+k_t)Q}{4k_t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{(3+k_t)Q}{4k_t} & 0 & -\frac{(3+k_t)\beta_1}{1+k_t} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (3+k_t)\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (3+k_t)\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-(3+k_t)\beta_1 & 0 & -\frac{(3+k_t)Q}{4k_t(1+k_t)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(3+k_t)Q}{4k_t} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{88}$$

This matrix does not decouple or block-triangularize. However, because we have three independent Casimir functions and a symmetry integral, we are assured the characteristic equation again takes the form

$$P(s) = s^5(s^4 + A_2 s^2 + A_0). \tag{89}$$

In terms of the inertia parameter, k_t, and the spin parameter, Q, the coefficients of the characteristic equation are expressed as

$$A_0 = \frac{-3(1-3k_t)(4k_t-Q)(4k_t+Q)}{16k_t} \qquad A_2 = \frac{16k_t - 96k_t^2 + 9Q^2 + 9k_tQ^2}{16k_t}. \tag{90}$$

The spectral stability requirements reduce to

$$-k_t(1-3k_t)(4k_t^2-Q^2) \geq 0 \tag{91a}$$

$$-k_t(-16k_t + 96k_t^2 - 9Q^2 - 9k_tQ^2) \geq 0 \tag{91b}$$

$$256k_t^2 + 96k_tQ^2 - 864k_t^2Q^2 - 1728k_t^3Q^2 + 81Q^4 + 162k_tQ^4 + 81k_t^2Q^4 \geq 0. \tag{91c}$$

However, for physically realizable solutions, we require $|Q/k_t| \leq 4$. Thus, Condition (91a) reduces to $-k_t(1-3k_t) \geq 0$. Unlike the hyperbolic case, this does not guarantee either Condition (91b) or Condition (91c) is satisfied. So we need to look at the bounds given by all three conditions. The stability regions are shown in Fig. 10. We find that a relatively large region of prolate configurations are spectrally stable while only a very select set of oblate configurations satisfy the conditions. At large rotation rates, these relative equilibria are not physically possible for most configurations.

Nonlinear Stability We again apply the projection method to determine nonlinear stability. We show that the spectrally stable relative equilibria corresponding to prolate configurations are also nonlinearly stable. Substituting the relative equilibrium conditions into (62), we have

$$\nabla^2 F(\mathbf{z}_e) = \begin{bmatrix}
3+k_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3+k_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3+k_t}{1+k_t} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-3Q^2}{16k_t(3+k_t)} & 0 & 0 & \frac{3Q\beta_1}{4(3+k_t)} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-3Q^2}{16k_t(3+k_t)} & 0 & 0 & \frac{3Q\beta_1}{4(3+k_t)} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-16k_t^2+3Q^2)}{16k_t(3+k_t)} & 0 & 0 & \frac{3Q\beta_1}{4(3+k_t)} \\
0 & 0 & 0 & \frac{3Q\beta_1}{4(3+k_t)} & 0 & 0 & \frac{3(16k_t^2-Q^2)}{16k_t(3+k_t)} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3Q\beta_1}{4(3+k_t)} & 0 & 0 & \frac{3(16k_t^2-Q^2)}{16k_t(3+k_t)} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3Q\beta_1}{4(3+k_t)} & 0 & 0 & \frac{3Q^2}{16k_t(3+k_t)}
\end{bmatrix} \tag{92}$$

which is not sign definite. Following the same procedure as in the cylindrical and hyperbolic cases, we construct the projection onto the portion of the tangent space orthogonal to the cyclic perturbation direction by forming the matrix $\mathbf{K}(\mathbf{z}_e)$ of basis vectors for $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}(\mathbf{z}_e)\right)$ augmented with the cyclic perturbation vector. We then compute the projected variational Lagrangian $\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)$ for which the eigenvalues are

$$\left\{0,\ 0,\ 0,\ 0,\ 0,\ 3+k_t,\ 3+k_t,\ \frac{-3k_t}{3+k_t},\ \frac{(1-3k_t)(3+k_t)(16k_t^2-Q^2)}{48k_t^4+224k_t^3+304k_t^2-Q^2-2k_tQ^2-k_t^2Q^2}\right\}.$$

As before, four zero eigenvalues are associated with the constraints and one with the cyclic perturbation direction. The remaining four eigenvalues are associated with the restricted tangent space. The conditions for positive-definiteness on the restricted tangent space are $k_t < 0$ along with the physical constraint $16k_t^2 - Q^2 > 0$. Thus, the relative equilibria for prolate configurations are nonlinearly stable. This completes the analysis for the Keplerian system.

Conclusions

The hierarchy of Hamiltonian systems we have developed provides a framework for comparison of various approximations of rigid-body motion in a central gravitational field. By demonstrating the equivalence of the second-order Keplerian approximation to the classical approximation commonly used in the literature, we have established a baseline to allow consideration of relative improvements provided by other approximations. A secondary contribution is the refinement of stability analysis techniques for relative equilibria of Hamiltonian systems. Sufficiency of constrained minima on the tangent space has been extended to minima on a restricted subspace orthogonal to a cyclic perturbation vector for bodies with symmetry. When applied to the second-order Keplerian system, this method led to the same results as previously provided by a classical Liapunov stability analysis. The advantage is that the nonlinear stability result was found directly with no need to specify a Liapunov function (for which the proper choice may not be obvious).

Acknowledgments

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List of Table Captions

$k = 2 : \ P(s) = s^2 + A_0$
$A_0 \geq 0$
$k = 4 : \ P(s) = s^4 + A_2 s^2 + A_0$
$A_0 \geq 0 \ \ A_2 \geq 0 \ \ A_2^2 - 4A_0 \geq 0$

List of Figure Captions

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Figure 5: Contribution to Torque About the Center of Mass

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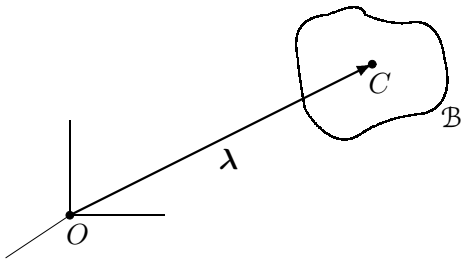
Figure 8: Stability Diagram for Cylindrical Relative Equilibria of Axisymmetric Keplerian System (medium gray = nonlinearly and spectrally stable, light gray = spectrally stable, white = unstable)

Figure 9: Stability Diagram for Hyperbolic Relative Equilibria of Axisymmetric Keplerian System (dark gray = physically impossible, medium gray = nonlinearly and spectrally stable, white = unstable)

Figure 10: Stability Diagram for Conical Relative Equilibria of Axisymmetric Keplerian System (dark gray = physically impossible, medium gray = nonlinearly and spectrally stable, light gray = spectrally stable, white = unstable)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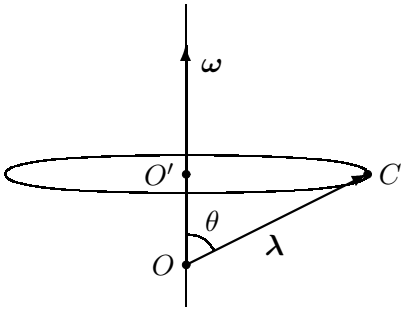
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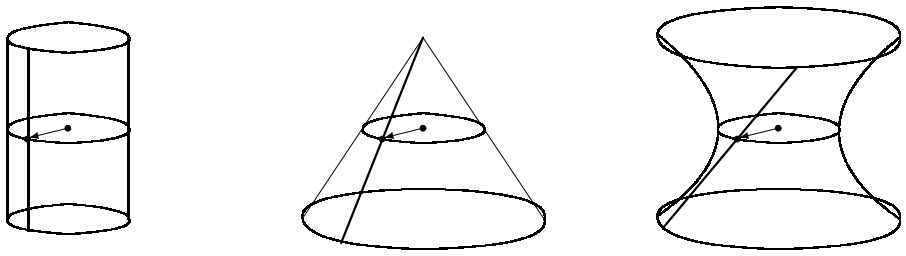
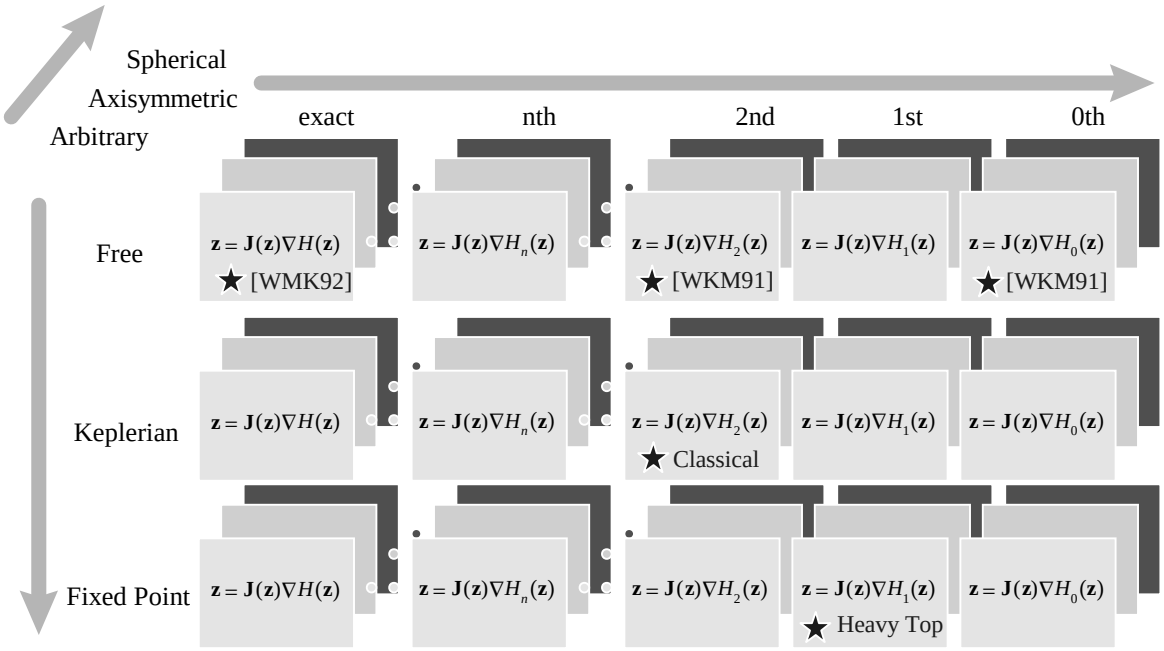


(a) Cylindrical (b) Conical (c) Hyperbolic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$\mathbf{z} = \mathbf{J}(\mathbf{z})\nabla H_0(\mathbf{z})$
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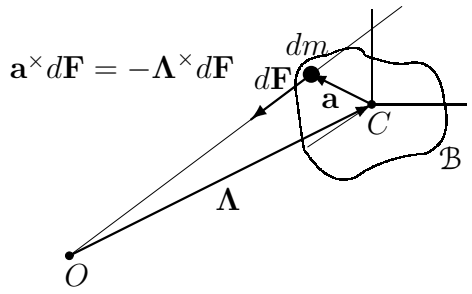
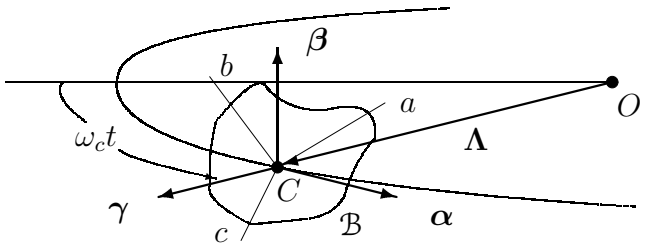
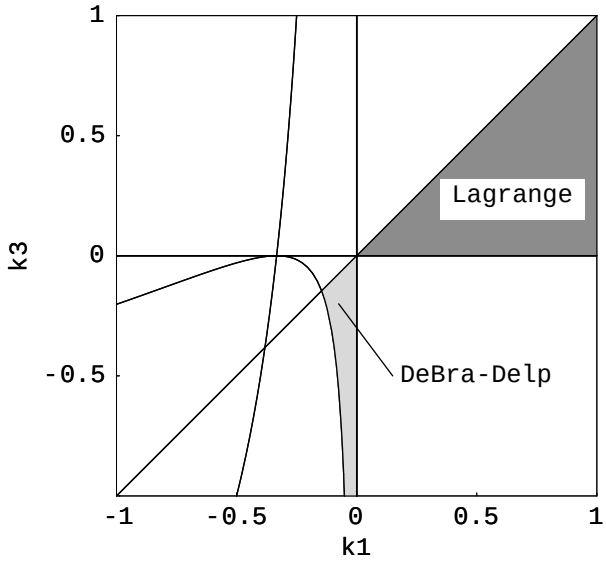
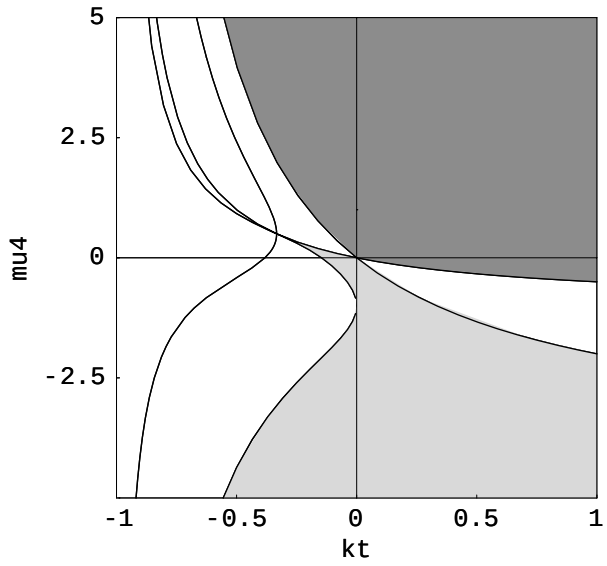
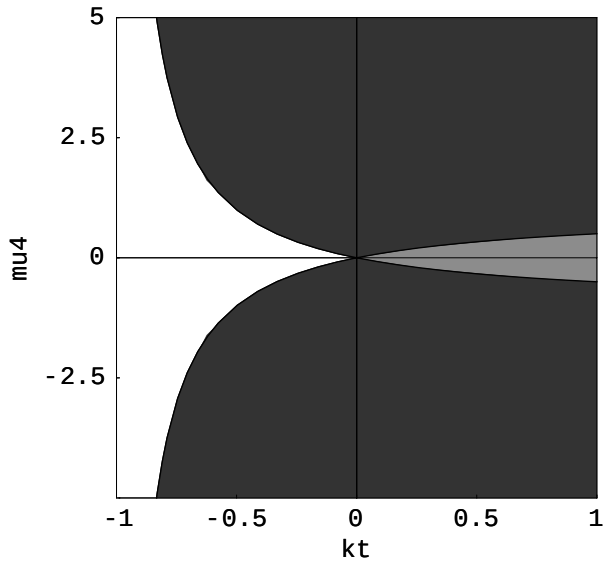
Classical
Heavy Top


$\mathbf{a}^{\times} d\mathbf{F} = -\mathbf{\Lambda}^{\times} d\mathbf{F}$
dm
$d\mathbf{F}$
$\mathbf{a}$
C
$\mathcal{B}$
$\mathbf{\Lambda}$
O


$\boldsymbol{\beta}$
b
a
O
$\omega_c t$
$\boldsymbol{\Lambda}$
C
$\boldsymbol{\gamma}$
$\mathcal{B}$
$\boldsymbol{\alpha}$
c


1
0.5
0
-0.5
k3
-1
-0.5
0
0.5
1
k1
Lagrange
DeBra-Delp


5
2.5
0
-2.5
mu4
-1
-0.5
0
0.5
1
kt


5
2.5
0
-2.5
mu4
-1
-0.5
0
0.5
1
kt