A SET OF MODIFIED EQUINOCTIAL ORBIT ELEMENTS

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Abstract. Modified equinoctial elements are introduced which are suitable for perturbation analysis of all kinds of orbit. Equations of motion in Lagrangian and Gaussian forms are derived. Identities connecting the partial derivatives of the disturbing function with respect to equinoctial elements are established. Numerical comparisons of the evolution of a perturbed, highly eccentric, elliptic orbit analysed in equinoctial elements and by Cowell's method show satisfactory agreement.

1. Introduction

Perturbed satellite orbits are often studied theoretically or numerically by using Lagrange's planetary equations (Cornelisse *et al.*, 1979):

$$
\begin{aligned}
\frac{da}{dt} &= \frac{-2a^2}{\mu}\frac{\partial\tilde{R}}{\partial\tau} \\
\frac{de}{dt} &= \frac{-a(1-e)^2}{\mu e}\frac{\partial\tilde{R}}{\partial\tau} - \frac{1}{e}\sqrt{\frac{1-e^2}{\mu a}}\frac{\partial\tilde{R}}{\partial\omega} \\
\frac{di}{dt} &= \frac{1}{\sqrt{\mu a(1-e^2)}\cdot\sin i}\left(\cos i\,\frac{\partial\tilde{R}}{\partial\omega} - \frac{\partial\tilde{R}}{\partial\Omega}\right) \\
\frac{d\omega}{dt} &= \sqrt{\frac{1-e^2}{\mu a}}\left(\frac{1}{e}\frac{\partial\tilde{R}}{\partial e} - \frac{\cot i}{1-e^2}\frac{\partial\tilde{R}}{\partial i}\right) \\
\frac{d\Omega}{dt} &= \frac{1}{\sqrt{\mu a(1-e^2)}\sin i}\frac{\partial\tilde{R}}{\partial i}
\end{aligned}
\qquad (1)
$$

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These equations are important because they isolate the fast motion in the phase angle defining position in the orbit. This serves as the basis for many analytical developments. Disadvantages of employing classical Lagrangian elements, a, e, i, ω, Ω, τ are that the right ascension of the ascending node becomes indeterminate as the inclination tends to zero, and the argument of perigee becomes indeterminate as the eccentricity tends to zero.

Many schemes of orbital elements which avoid these difficulties have been developed, ranging from standard transformations applied to general canonical elements (Kaula, 1966), to specific sets of elements (e.g. Broucke and Cefola 1972; Cohen and Hubbard 1962; Giacaglia 1977). It is generally advisable to employ elements which are not too far removed from the classical ones; then transforming and interpreting them in terms of physically significant parameters is relatively easy. To this end, Cefola and his co-workers have made a good case for the use of equinoctial orbit elements.

For some purposes it is desirable to employ a 'fast variable' (phase angle) as the sixth 'element'. In particular, the Stroboscopic Method (Roth 1979) assumes such a formulation, so that a regular perturbation technique can be used with the fast variable as independent variable. In such cases it is natural to modify the equinoctial elements by choosing true longitude L, in place of mean longitude λ_0, as the element fixing position in the orbit. Further, by replacing semi-major axis a by semi-latus rectum p, we obtain a set of orbit elements of the prescribed form which are applicable to all orbits, and have non-singular equations of motion (excluding the case $i = \pi$; but this can be handled by an appropriate re-definition, as in Cefola, 1972). With these considerations, we define modified equinoctial elements thus*:

$$\begin{aligned} p &= a(1-e^2) \\ f &= e\cos(\omega + \Omega) \\ g &= e\sin(\omega + \Omega) \\ h &= \tan i/2 \cos\Omega \\ k &= \tan i/2 \sin\Omega \\ L &= \Omega + \omega + \nu \end{aligned} \tag{2}$$

where ν is true anomaly.

2. Review of Theory of Orbit Elements

Suppose α_i, $i = 1$ to 6, represent a set of orbit elements which, together with t, uniquely fix position and velocity of a satellite relative to a fixed Cartesian frame having the centre of attraction as origin. Further, suppose a perturbative acceleration (additional to the acceleration of the ideal Keplerian orbit) can be expressed as the gradient of a disturbing function $R = R(x)$. Then (using repeated suffix convention)

* This follows the European Space Agency's notation.

$$[\alpha_j, \alpha_i] \frac{d\alpha_i}{dt} = \frac{\partial R}{\partial x_i} \frac{\partial x_i}{\partial \alpha_j} = \frac{\partial \tilde{R}}{\partial \alpha_j} \tag{3}$$

where $[\alpha_j, \alpha_i]$ is a Lagrangian bracket, and $\tilde{R}$ is the disturbing function expressed in terms of α and t. Inverting equations (3) gives the equations of motion

$$\frac{d\alpha_i}{dt} = (\alpha_i, \alpha_j) \frac{\partial \tilde{R}}{\partial \alpha_j} \tag{4}$$

where (α_j, α_i) is a Poisson bracket. So derived, these equatinos of motion are unique. Nevertheless, the right hand sides can be expressed in many equivalent forms, and this can be important when checking the equations by alternative derivations. To see how such forms can arise, observe that R is independent of velocity $\dot{x}$, and hence $\tilde{R}$ always satisfies three identities

$$\frac{\partial \tilde{R}}{\partial \alpha_j} \frac{\partial \alpha_j}{\partial x_i} = 0 \tag{5}$$

By adding multiples of the left-hand sides of (5) into the right-hand sides of (4), many different forms of the equations may be generated. (This freedom does not exist in the Gaussian form of the equations, where the perturbative accelerations are uniquely specified). Three identities equivalent to (5) are given in the Appendix for classical elements: and similar identities for the equinoctial elements are also found.

3. Equinoctial Equations of Motion

By differentiating equations (2) with respect to time, the derivatives of the (modified) equinoctial elements can be found in terms of classical orbit elements and their derivatives. For the sixth equation it is convenient to use

$$\frac{d\nu}{dt} = \frac{\sqrt{\mu p}}{r^2} - \left(\frac{d\omega}{dt} + \frac{d\Omega}{dt} \cos i\right) . \tag{6}$$

Then using equations (1), and expressing classical elements in terms of equinoctial elements gives

$$\frac{dp}{dt} = 2\sqrt{\frac{p}{\mu}}\;\frac{\partial\tilde{R}}{\partial\omega}$$

$$\frac{df}{dt} = \frac{-p}{\mu(f^2+g^2)}\;\frac{\partial\tilde{R}}{\partial\tau} - \frac{(1-f^2-g^2)f}{(f^2+g^2)\sqrt{\mu p}}\;\frac{\partial\tilde{R}}{\partial\omega} - \frac{(1-f^2-g^2)g}{\sqrt{\mu p(f^2+g^2)}}\;\frac{\partial\tilde{R}}{\partial e} - \frac{\tan i/2}{\sqrt{\mu p}}\;\frac{\partial\tilde{R}}{\partial i}$$

$$\frac{dg}{dt} = \frac{-p}{\mu(f^2+g^2)}\;\frac{\partial\tilde{R}}{\partial\tau} - \frac{(1-f^2-g^2)g}{(f^2+g^2)\sqrt{\mu p}}\;\frac{\partial\tilde{R}}{\partial\omega} - \frac{(1-f^2-g^2)f}{(f^2+g^2)\sqrt{\mu p}}\;\frac{\partial\tilde{R}}{\partial e} - \frac{\tan i/2}{\sqrt{\mu p}}\;\frac{\partial\tilde{R}}{\partial i}$$

$$\frac{d}{dt} = \frac{h}{\sqrt{\mu p}}\;\frac{\cos i}{\sin^2 i}\;\frac{\partial\tilde{R}}{\partial\omega} - \frac{h}{\sqrt{\mu p}}\;\frac{1}{\sin^2 i}\;\frac{\partial\tilde{R}}{\partial\Omega} - \frac{k}{\sqrt{\mu p}}\;\frac{1}{\sin i}\;\frac{\partial\tilde{R}}{\partial i} \tag{7}$$

$$\frac{d}{dt} = \frac{k}{\sqrt{\mu p}}\;\frac{\cos i}{\sin^2 i}\;\frac{\partial\tilde{R}}{\partial\omega} - \frac{k}{\sqrt{\mu p}}\;\frac{1}{\sin^2 i}\;\frac{\partial\tilde{R}}{\partial\Omega} + \frac{h}{\sqrt{\mu p}}\;\frac{1}{\sin i}\;\frac{\partial\tilde{R}}{\partial i}$$

$$\frac{d}{dt} = \frac{\sqrt{\mu p}}{r^2} + \frac{\tan i/2}{\sqrt{\mu p}}\;\frac{\partial\tilde{R}}{\partial i}\,.$$

Finally, by regarding $\tilde{R}$ (a, e, i, ω, Ω, t-τ) as transformed into $\tilde{R}$(p, f, g, h, k, L), we obtain the required equations of motion: Define auxiliary (positive) variables

$$s^2 = 1 + h^2 + k^2 \quad \text{and} \quad w = p/r = 1 + f\cos L + g\sin L$$

then

$$\frac{dp}{dt} = 2\sqrt{\frac{p}{\mu}}\left(-g\,\frac{\partial\tilde{R}}{\partial f} + f\,\frac{\partial\tilde{R}}{\partial g} + \frac{\partial\tilde{R}}{\partial L}\right)$$

$$\frac{df}{dt} = \frac{1}{\sqrt{\mu p}}\left\{2\,pg\,\frac{\partial\tilde{R}}{\partial p} - (1-f^2-g^2)\,\frac{\partial\tilde{R}}{\partial g} - \frac{gs^2}{2}\left(h\,\frac{\partial\tilde{R}}{\partial h} + k\,\frac{\partial\tilde{R}}{\partial k}\right) + \right.$$

$$\left. + \left[f + (1+w)\cos L\right]\frac{\partial\tilde{R}}{\partial L}\right\}$$

$$\frac{dg}{dt} = \frac{1}{\sqrt{\mu p}}\left\{-\,2\,pf\,\frac{\partial\tilde{R}}{\partial p} + (1-f^2-g^2)\,\frac{\partial\tilde{R}}{\partial f} + \frac{fs^2}{2}\left(h\,\frac{\partial\tilde{R}}{\partial h} + k\,\frac{\partial\tilde{R}}{\partial k}\right)\right.$$

$$\left. + \left[g + (1+w)\sin L\right]\frac{\partial\tilde{R}}{\partial L}\right\} \tag{8}$$

$$\frac{dh}{dt} = \frac{s^2}{2\sqrt{\mu p}}\left\{h\left(g\,\frac{\partial\tilde{R}}{\partial f} - f\,\frac{\partial\tilde{R}}{\partial g} - \frac{\partial\tilde{R}}{\partial L}\right) - \frac{s^2}{2}\;\frac{\partial\tilde{R}}{\partial k}\right\}$$

$$\frac{dk}{dt} = \frac{s^2}{2\sqrt{\mu p}}\left\{k\left(g\,\frac{\partial\tilde{R}}{\partial f} - f\,\frac{\partial\tilde{R}}{\partial g} - \frac{\partial\tilde{R}}{\partial L}\right) + \frac{s^2}{2}\;\frac{\partial\tilde{R}}{\partial h}\right\}$$

$$\frac{dL}{dt} = \sqrt{\mu p}\left(\frac{w}{p}\right)^2 + \frac{s^2}{2\sqrt{\mu p}}\left\{h\,\frac{\partial\tilde{R}}{\partial h} + k\,\frac{\partial\tilde{R}}{\partial k}\right\}$$

Although we have described (8) as 'the' equations of motion, again there will be many equivalent forms of the right-hand sides of these equations.

This analysis holds in the first instance for elliptic motion only. However, the equinoctial elements used here are well-defined for parabolic and hyperbolic motion also; it follows by the principle of analytic continuation that equations (8) apply for all motions ($i = \pi$ excepted).

The Gaussian equations of motion corresponding to (8) can be obtained in the same way. They reduce to

$$\frac{dp}{dt} = \frac{2pC}{w}\sqrt{\frac{p}{\mu}}$$

$$\frac{df}{dt} = \sqrt{\frac{p}{\mu}}\left\{S \sin L + \frac{[(w+1)\cos L + f]C}{w} - \frac{g(h \sin L - k \cos L)N}{w}\right\}$$

$$\frac{dg}{dt} = \sqrt{\frac{p}{\mu}}\left\{-S \cos L + \frac{[(w+1)\cos L + g]C}{w} - \frac{f(h \sin L - k \cos L)N}{w}\right\}$$

$$\frac{dh}{dt} = \sqrt{\frac{p}{\mu}}\,\frac{s^2 N}{2w}\cos L \tag{9}$$

$$\frac{dk}{dt} = \sqrt{\frac{p}{\mu}}\,\frac{s^2 N}{2w}\sin L$$

$$\frac{dL}{dt} = \sqrt{\mu p}\left(\frac{w}{p}\right)^2 + \sqrt{\frac{p}{\mu}}\,\frac{(h \sin L - k \cos L)N}{w}$$

where C, S, N are components of perturbing acceleration in the directions perpendicular to the radius vector in the direction of motion, along the radius vector outwards, and normal to the orbital plane in the direction of the angular momentum vector.

4. Disturbing Function for Axial-symmetric Primary Body

For a primary body with axial symmetry (Cornelisse et al., 1979)

$$\tilde{R} = -\frac{\mu}{r}\sum_{n=2}^{\infty} J_n \left(\frac{R_e}{r}\right)^n P_n(\sin \phi) \tag{10}$$

where ϕ is the geocentric latitude, P_n is a Legendre polynomial, J_n is a coefficient for the zonal harmonic, and R_e is the primary equatorial radius. In equinoctial elements,

$$r = p/w \quad \text{and} \quad \sin\phi = \frac{2(h \sin L - k \cos L)}{s^2}.$$

Hence (prime denotes derivative)

$$\frac{\partial \tilde{R}}{\partial p} = \frac{\mu}{wr^2} \sum_{n=2}^{\infty} (n+1) J_n \left(\frac{R_e}{r}\right)^n P_n (\sin\phi)$$

$$\frac{\partial \tilde{R}}{\partial g} = \frac{-\mu \sin L}{wr} \sum_{n=2}^{\infty} (n+1) J_n \left(\frac{R_e}{r}\right)^n P_n (\sin\phi) \tag{11}$$

$$\frac{\partial \tilde{R}}{\partial h} = \frac{-2\mu}{rs^4} \left\{(1-h^2+k^2)\sin L + 2hk \cos L\right\} \sum_{n=2}^{\infty} J_n \left(\frac{R_e}{r}\right)^n P'_n (\sin\phi)$$

$$\frac{\partial \tilde{R}}{\partial k} = \frac{2\mu}{rs^4} \left\{(1+h^2-k^2)\cos L + 2hk \sin L\right\} \sum_{n=2}^{\infty} J_n \left(\frac{R_e}{r}\right)^n P'_n (\sin\phi)$$

$$\frac{\partial \tilde{R}}{\partial L} = \frac{-2\mu}{rs^2} (h \cos L + k \sin L) \sum_{n=2}^{\infty} J_n \left(\frac{R_e}{r}\right)^n P'_n (\sin\phi)$$

$$-\frac{\mu}{rw} (g \cos L - f \sin L) \sum_{n=2}^{\infty} (n+1) J_n \left(\frac{R_e}{r}\right)^n P_n (\sin\phi)$$

Substituting these equations in (8) theoretically determines the perturbations to an orbit described by a satellite of such a primary body.

5. Numerical Study

In order to check the validity of equations (8), we have made a numerical study of the perturbations on a highly eccentric elliptic orbit caused by an axial-symmetric primary body, using independent formulations in equinoctial elements and Cowell's method. The initial orbit chosen was (all units in kg, km, sec. system):

$a = 244$ $\quad a = 24419.205$

$e = 0.726$ $\quad e = 0.726683$

$i = 27.0$ deg

$\omega = 0$

$\Omega = 0$

$\nu = 0$ at initial time.

The primary body was taken to be earth:

$\mu = 398603.2$

$R_e = 6378.165$

$J_2 = 0.00108263$

$J_3 = -2.51.10^{-6}$

$J_4 = -1.60.10^{-6}$

$J_5 = -1.3.10^{-7}$

$J_6 = 5.0.10^{-7}$-

Digital computer programs were written in FORTRAN, with double precision variables (nominally 16 significant decimal digits). Integrations were performed using a fourth-order Runge-Kutta-Merson routine, incorporating automatic step-length adjustment. Truncation tolerances for each step of integration were set at

10^{-6} (p and Cartesian co-ordinates and velocities)

10^{-9} (non-dimensional elements f, g, h, k, L).

Table I compares the orbit state after a time interval of two days (four and a half complete orbits), as found by both methods.

TABLE I

Formulation	Equinoctial	Cowell's method
a	24331.443	24331.443
e	0.72557888	0.72557888
i, deg.	26.988272	26.988272
ω, deg.	1.199160	1.199160
Ω, deg.	359.280136	359.280136
ν, deg.	186.307367	186.307368

To the accuracy quoted, no discrepancy greater than three units in the last decimal place was found over the whole time interval: agreement is very satisfactory. In addition, for the equinoctial formulation, the total specific energy $-\mu/2a - \tilde{R}$ was found to be constant to 10 significant decimal digits, and the polar component of angular momentum $\sqrt{\mu p}\cos i$ was found to be constant to 14 significant decimal digits. Again, this checks very satisfactorily. Figure 1 illustrates the short-period variation in elements a, e, and i, over the first complete orbit. Figure 2 illustrates short-period and secular variations in elements ω en Ω over half the time interval considered.

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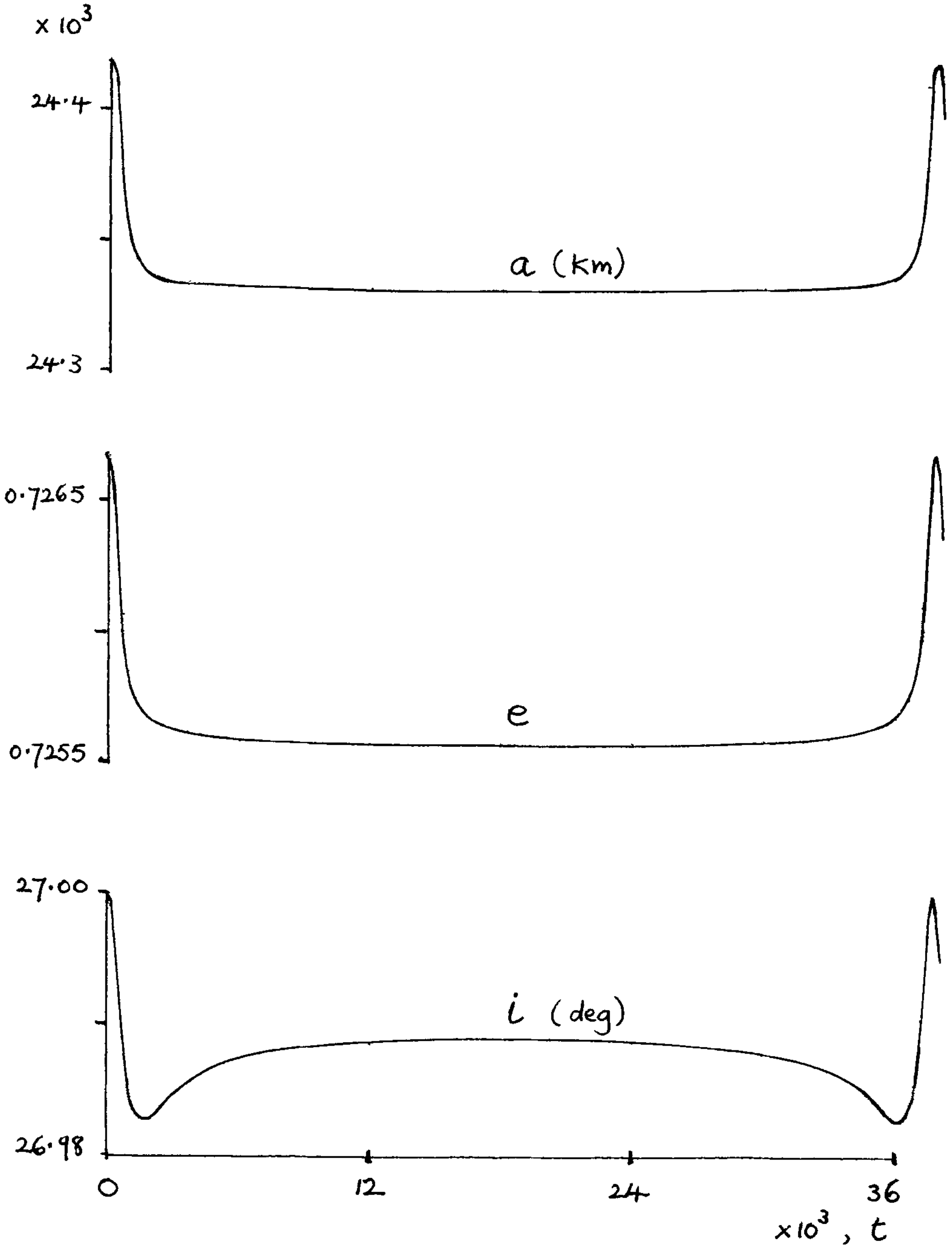

Fig. 1

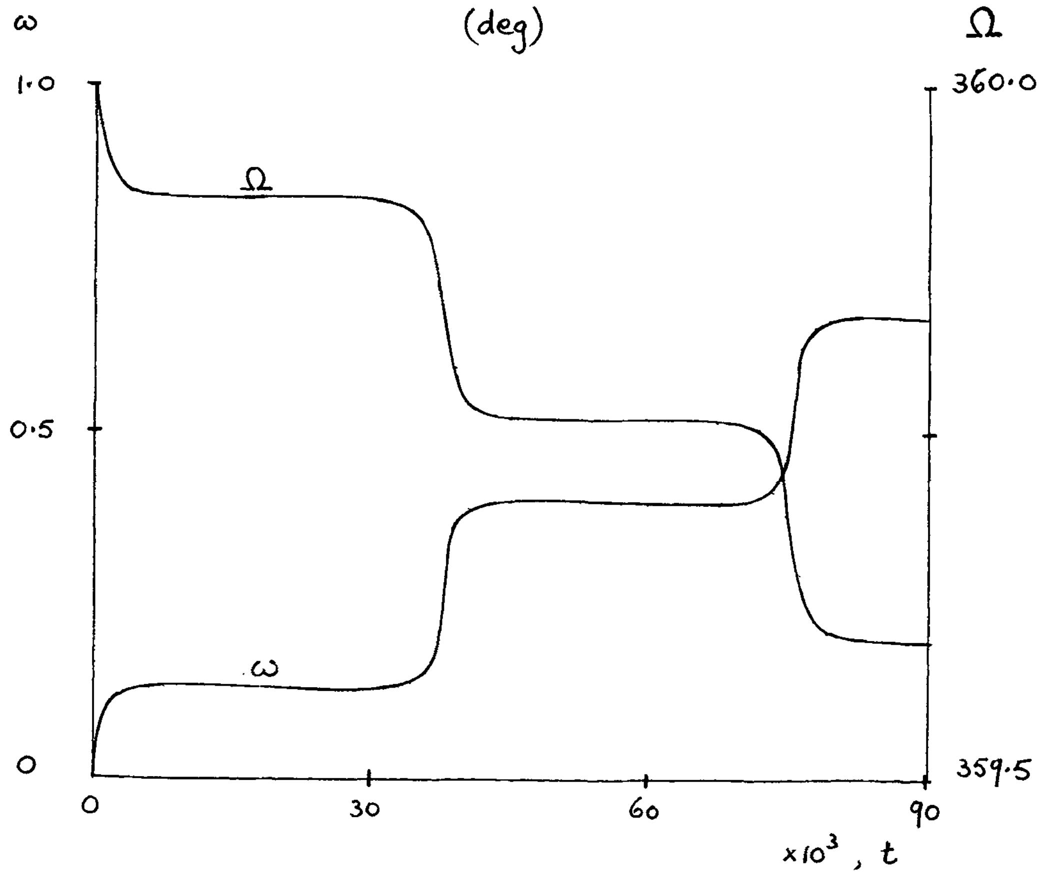

Fig. 1 (continued).

Appendix

Identities between Partial Derivatives of the Disturbing Function

Set $F = [S\ C\ N]^T$, the column vector of perturbative acceleration components relative to a moving radial frame of reference. Then the gradient of $R(x)$, expressed as a column vector, gives

$$\left[\frac{\partial \tilde{R}}{\partial x_i}\right] = M F ,$$

where M is the matrix representing the transformation from the radial reference frame to the inertial reference frame. Introduce the state vector

$$X = [x \quad x]^T .$$

It is convenient to regard R as a joint dissipation-potential function $R(\chi)$, in which the (augmented) gradient components in the last three places correspond to zero perturbative acceleration components in the radial reference frame:

$$\left[\frac{\partial R}{\partial X_i}\right] = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} F \\ 0 \end{bmatrix}$$

In matrix notation, the transformation between grad $\tilde{R}(\alpha)$ and grad $R(\chi)$ is

$$\left[\frac{\partial \tilde{R}}{\partial \alpha_i}\right] = \left[\frac{\partial X_j}{\partial \alpha_i}\right] \left[\frac{\partial R}{\partial X_j}\right]$$

so

$$\left[\frac{\partial \tilde{R}}{\partial \alpha_i}\right] = Q \begin{bmatrix} F \\ 0 \end{bmatrix} \text{, where } Q = \left[\frac{\partial X_j}{\partial \alpha_i}\right] \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$$

Then

$$Q^{-1} \left[\frac{\partial \tilde{R}}{\partial \alpha_i}\right] + \begin{bmatrix} F \\ 0 \end{bmatrix}$$

The last three rows of Q^{-1} generate three identities connecting the partial derivatives $\partial\tilde{R}/\partial\alpha_i$. In the interests of simplification it is permissible to scale these last three rows, or add multiples of them into the first three rows, without destroying the validity of these equations.

We have found Q, and hence Q^{-1} (appropriately simplified ad described), for both classical and equinoctial elements. For classical elements a, e, i, ω, Ω, τ

$$-\frac{1}{a\cos\nu}\frac{\partial\tilde{R}}{\partial e} + \frac{\tan\nu}{p}(2 + e\cos\nu)\frac{\partial\tilde{R}}{\partial\omega} = S$$

$$\frac{1}{r}\frac{\partial\tilde{R}}{\partial\omega} = C.$$

$$\frac{1}{r\sin(\omega+\nu)}\frac{\partial\tilde{R}}{\partial i} = N$$

$$\sin\nu\frac{\partial\tilde{R}}{\partial e} - \left(\frac{2e + e^2\cos\nu + \cos\nu}{e(1-e^2)}\right)\frac{\partial\tilde{R}}{\partial\omega} - \frac{a}{e}\sqrt{\frac{p}{\mu}}\cos\nu\frac{\partial\tilde{R}}{\partial\tau} = 0$$

$$\frac{a^2\cos\nu}{r}\frac{\partial\tilde{R}}{\partial a} + \frac{\partial\tilde{R}}{\partial e} - \frac{\sin\nu(2 + e\cos\nu)}{1-e^2}\frac{\partial\tilde{R}}{\partial\omega} - \frac{3a\cos\nu}{2r}\cdot(t-\tau)\frac{\partial\tilde{R}}{\partial\tau} = 0$$

$$\frac{\partial \tilde{R}}{\partial i} - \frac{\tan(\omega+\nu)}{\tan i}\frac{\partial \tilde{R}}{\partial \omega} + \frac{\tan(\omega+\nu)}{\sin i}\frac{\partial \tilde{R}}{\partial \Omega} = 0.$$

For equinoctial elements p, f, g, h, k, L

$$-\frac{w \cos L}{r}\frac{\partial \tilde{R}}{\partial f} - \frac{w \sin L}{r}\frac{\partial \tilde{R}}{\partial g} = S$$

$$-\frac{g}{r}\frac{\partial \tilde{R}}{\partial f} + \frac{f}{r}\frac{\partial \tilde{R}}{\partial g} + \frac{1}{r}\frac{\partial \tilde{R}}{\partial L} = C$$

$$\frac{s^2 h}{2r(h \sin L - k \cos L)}\frac{\partial \tilde{R}}{\partial h} + \frac{s^2 k}{2r(h \sin L - k \cos L)}\frac{\partial \tilde{R}}{\partial k} = N$$

$$\sin L \frac{\partial \tilde{R}}{\partial f} - \cos L \frac{\partial \tilde{R}}{\partial g} = 0$$

$$\frac{\partial \tilde{R}}{\partial p} + \frac{f + (w+1)\cos L}{2p}\frac{\partial \tilde{R}}{\partial f} + \frac{g + (w+1)\sin L}{2p}\frac{\partial \tilde{R}}{\partial g} = 0$$

$$-g\frac{\partial \tilde{R}}{\partial f} + f\frac{\partial \tilde{R}}{\partial g} + \frac{s^2 \cos L}{2(h \sin L - k \cos l)}\frac{\partial \tilde{R}}{\partial h} + \frac{s^2 \sin L}{2(h \sin L - k \cos L)}\frac{\partial \tilde{R}}{\partial k} + \frac{\partial \tilde{R}}{\partial L} = 0.$$

Here

$$s^2 = 1 + h^2 + k^2 \quad \text{and} \quad w = 1 + f \cos L + g \sin L.$$

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