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OPTIMAL LOW-THRUST RENDEZVOUS USING EQUINOCTIAL ORBIT ELEMENTS†

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Abstract—The problem of optimal low-thrust rendezvous using continuous constant acceleration is presented based on the use of the non-singular equinoctial orbit elements. State and adjoint equations are integrated numerically by applying the thrust vector along a direction that maximizes the variational Hamiltonian at each instant of time. The two-point-boundary-value problem is solved via a Newton–Raphson scheme after guessing the initial values of the Lagrange multipliers as well as the total flight time. The transversality condition for the Hamiltonian at the final time is also used to carry out the 7×7 search.

Equinoctial elements have previously been used to carry out minimum-time orbit transfers by considering only the five slowly varying elements. The inclusion of the sixth element representing the fast variable has allowed us to extend the applicability of the non-singular formulation to problems of orbital rendezvous. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

Analytic solutions for the low-thrust rendezvous problem in the vicinity of the initial circular or elliptic orbit have been obtained in the early sixties by Gobetz and Edelbaum [1,2]. In [3] Marec and Vinh have presented both analytic and numerical techniques to solve the more general low-thrust problem with rigorous treatment of the analysis of the conjugate point.

In this paper the problem of minimum-time low-thrust orbit rendezvous using continuous constant acceleration is presented. The variation of parameter equations are written in terms of a set of non-singular equinoctial orbit elements which are free of singularities for zero eccentricity and zero and 90° inclination. This orbit theory was developed early in the 1970s by Broucke and Cefola and later extended by Edelbaum and his colleagues to solve five-state orbit transfer problems [4–7]. The full set of six-state adjoint and state equations using the mean longitude as the fast variable is used in this paper to solve the optimal rendezvous problem. It is assumed that there exists enough control authority to prevent the use of coasting arcs and that the relative geometry between the two orbits is such that a direct rendezvous trajectory is possible. This can be achieved by waiting in the initial orbit until the proper orbital configuration is reached for direct trajectory rendezvous initiation. If coasting arcs are allowed, the optimal initial waiting period will be obtained from the optimization process. Nevertheless, in our simpler problem, the thrust is on continuously from time zero, and as an example we consider a rendezvous problem in near-circular condition to show the effectiveness of using the equinoctial elements as opposed to using the classical elements which would undoubtedly introduce undesirable singularities during the numerical integration of the various exact equations of motion.

2. THE DIFFERENTIAL EQUATIONS IN TERMS OF THE EQUINOCTIAL ELEMENTS

The basic set of the six-state differential equations of motion have been developed in [4,6,7] and are repeated in this section for completeness.

The equinoctial elements in terms of the classical elements are given by

$$a = a \tag{1}$$

$$h = e\sin(\omega + \Omega) \tag{2}$$

$$k = e\cos(\omega + \Omega) \tag{3}$$

$$p = \tan\left(\frac{i}{2}\right)\sin\Omega \tag{4}$$

$$q = \tan\left(\frac{i}{2}\right)\cos\Omega \tag{5}$$

$$\lambda = M + \omega + \Omega \tag{6}$$

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The last expression that defines the mean longitude can be replaced by

$$F = E + \omega + \Omega \tag{7}$$

or

$$L = \theta^* + \omega + \Omega \tag{8}$$

defining the eccentric longitude and true longitude, respectively. Here θ^* represents the true anomaly. The inverse transformation is given by

$$a = a \tag{9}$$

$$e = (h^2 + k^2)^{1/2} \tag{10}$$

$$i = 2\tan^{-1}(p^2 + q^2)^{1/2} \tag{11}$$

$$\Omega = \tan^{-1}\left(\frac{p}{q}\right) \tag{12}$$

$$\omega = \tan^{-1}\left(\frac{h}{k}\right) - \tan^{-1}\left(\frac{p}{q}\right) \tag{13}$$

$$M = \lambda - \tan^{-1}\left(\frac{h}{k}\right). \tag{14}$$

If F or L are used as the fast variable defining the sixth equinoctial element instead of λ, then the last expression in eqn (14) should be replaced by

$$E = F - \tan^{-1}\left(\frac{h}{k}\right) \tag{15}$$

or

$$\theta^* = L - \tan^{-1}\left(\frac{h}{k}\right). \tag{16}$$

The direct equinoctial frame is shown in Fig. 1 with unit vectors $\hat{\mathbf{f}}$, $\hat{\mathbf{g}}$, $\hat{\mathbf{w}}$, such that $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$ are contained in the instantaneous orbit plane with the direction of $\hat{\mathbf{f}}$ obtained through a clockwise rotation of an angle Ω from the direction of the ascending node. The position vector and velocity $\dot{\mathbf{r}}$ are such that

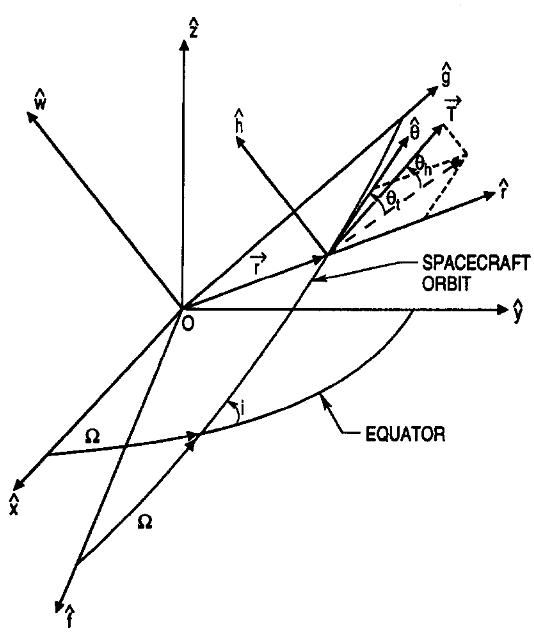

Fig. 1. Equinoctial frame and thrust geometry.

$$\mathbf{r} = X_1\hat{f} + Y_1\hat{g} \tag{17}$$

$$\dot{\mathbf{r}} = \dot{X}_1\hat{f} + \dot{Y}_1\hat{g} \tag{18}$$

where the components are written in terms of the equinoctial elements and the eccentric longitude F, with s_F and c_F standing for sin F and cos F, etc...

$$X_1 = a[(1 - h^2\beta)c_F + hk\beta s_F - k] \tag{19}$$

$$Y_1 = a[hk\beta c_F + (1 - k^2\beta)s_F - h] \tag{20}$$

$$\dot{X}_1 = a^2 n r^{-1}[hk\beta c_F - (1 - h^2\beta)s_F] \tag{21}$$

$$\dot{Y}_1 = a^2 n r^{-1}[(1 - k^2\beta)c_F - hk\beta s_F] \tag{22}$$

with β given by eqn (23) and where $n = \mu^{1/2}a^{-3/2}$ represents the orbit mean motion; μ is the earth gravity constant

$$\beta = \frac{1}{(1 + G)} \tag{23}$$

and where

$$G = (1 - h^2 - k^2)^{1/2}.$$

Here $r = a(1 - kc_F - hs_F)$. Letting $\mathbf{z} = (a, h, k, p, q, \lambda)^T$ represent the orbit state vector at time t, and letting $\mathbf{f}$ and m stand for the thrust vector and spacecraft mass, respectively, with $\mathbf{f} = f\hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ is a unit vector in the direction of the thrust, the variation of parameters equations for the thrust perturbation are given by eqn (24) with the addition of an n term to $\dot{\lambda}$

$$\dot{\mathbf{z}} = \left(\frac{\partial \mathbf{z}}{\partial \dot{\mathbf{r}}}\right)\hat{\mathbf{u}}f_t \tag{24}$$

where $f_t = f/m$ represents the instantaneous perturbation acceleration, with both $\partial\mathbf{z}/\partial\dot{\mathbf{r}}$ and $\hat{\mathbf{u}}$ expressed in the direct equinoctial frame and where

$$\frac{\partial a}{\partial \dot{\mathbf{r}}} = 2a^{-1}n^{-2}(\dot{X}_1\hat{\mathbf{f}} + \dot{Y}_1\hat{\mathbf{g}})$$

$$= M_{11}\hat{\mathbf{f}} + M_{12}\hat{\mathbf{g}} + M_{13}\hat{\mathbf{w}} \tag{25}$$

$$\frac{\partial h}{\partial \dot{\mathbf{r}}} = Gn^{-1}a^{-2}\left[\left(\frac{\partial X_1}{\partial k} - h\beta\frac{\dot{X}_1}{n}\right)\hat{\mathbf{f}} + \left(\frac{\partial Y_1}{\partial k} - h\beta\frac{\dot{Y}_1}{n}\right)\hat{\mathbf{g}}\right]$$

$$+ k(qY_1 - pX_1)n^{-1}a^{-2}G^{-1}\hat{\mathbf{w}}$$

$$= M_{21}\hat{\mathbf{f}} + M_{22}\hat{\mathbf{g}} + M_{23}\hat{\mathbf{w}} \tag{26}$$

$$\frac{\partial k}{\partial \dot{\mathbf{r}}} = -Gn^{-1}a^{-2}\left[\left(\frac{\partial X_1}{\partial h} + k\beta\frac{\dot{X}_1}{n}\right)\hat{\mathbf{f}} + \left(\frac{\partial Y_1}{\partial h} + k\beta\frac{\dot{Y}_1}{n}\right)\hat{\mathbf{g}}\right] - h(qY_1 - pX_1)n^{-1}a^{-2}G^{-1}\hat{\mathbf{w}} = M_{31}\hat{\mathbf{f}} + M_{32}\hat{\mathbf{g}} + M_{33}\hat{\mathbf{w}} \tag{27}$$

$$\frac{\partial p}{\partial \dot{\mathbf{r}}} = KY_1\frac{n^{-1}a^{-2}G^{-1}}{2}\hat{\mathbf{w}} = M_{41}\hat{\mathbf{f}} + M_{42}\hat{\mathbf{g}} + M_{43}\hat{\mathbf{w}} \tag{28}$$

$$\frac{\partial q}{\partial \dot{\mathbf{r}}} = KX_1\frac{n^{-1}a^{-2}G^{-1}}{2}\hat{\mathbf{w}} = M_{51}\hat{\mathbf{f}} + M_{52}\hat{\mathbf{g}} + M_{53}\hat{\mathbf{w}} \tag{29}$$

$$\frac{\partial \lambda}{\partial \dot{\mathbf{r}}} = n^{-1}a^{-2}\left[-2X_1 + G\left(h\beta\frac{\partial X_1}{\partial h} + k\beta\frac{\partial X_1}{\partial k}\right)\right]\hat{\mathbf{f}} + n^{-1}a^{-2}\left[-2Y_1 + G\left(h\beta\frac{\partial Y_1}{\partial h} + k\beta\frac{\partial Y_1}{\partial k}\right)\right]\hat{\mathbf{g}} + n^{-1}a^{-2}G^{-1}(qY_1 - pX_1)\hat{\mathbf{w}} = M_{61}\hat{\mathbf{f}} + M_{62}\hat{\mathbf{g}} + M_{63}\hat{\mathbf{w}} \tag{30}$$

with $K = 1 + p^2 + q^2$ and

$$\frac{\partial X_1}{\partial h} = a\left[-(hc_F - ks_F)\left(\beta + \frac{h^2\beta^3}{(1-\beta)}\right) - \frac{a}{r}c_F(h\beta - s_F)\right] \tag{31}$$

$$\frac{\partial X_1}{\partial k} = -a\left[(hc_F - ks_F)\frac{hk\beta^3}{(1-\beta)} + 1 + \frac{a}{r}s_F(s_F - h\beta)\right] \tag{32}$$

$$\frac{\partial Y_1}{\partial h} = a\left[(hc_F - ks_F)\frac{hk\beta^3}{(1-\beta)} - 1 + \frac{a}{r}c_F(k\beta - c_F)\right] \tag{33}$$

$$\frac{\partial Y_1}{\partial k} = a\left[(hc_F - ks_F)\left(\beta + \frac{k^2\beta^3}{(1-\beta)}\right) + \frac{a}{r}s_F(c_F - k\beta)\right]. \tag{34}$$

The system of differential equations for the state variables is given by

$$\dot{a} = \left(\frac{\partial a}{\partial \dot{\mathbf{r}}}\right)^{\mathrm{T}} \cdot \hat{\mathbf{u}}f_t \tag{35}$$

$$\dot{h} = \left(\frac{\partial h}{\partial \dot{\mathbf{r}}}\right)^{\mathrm{T}} \cdot \hat{\mathbf{u}}f_t \tag{36}$$

$$\dot{k} = \left(\frac{\partial k}{\partial \dot{\mathbf{r}}}\right)^{\mathrm{T}} \cdot \hat{\mathbf{u}}f_t \tag{37}$$

$$\dot{p} = \left(\frac{\partial p}{\partial \dot{\mathbf{r}}}\right)^{\mathrm{T}} \cdot \hat{\mathbf{u}}f_t \tag{38}$$

$$\dot{q} = \left(\frac{\partial q}{\partial \dot{\mathbf{r}}}\right)^{\mathrm{T}} \cdot \hat{\mathbf{u}}f_t \tag{39}$$

$$\dot{\lambda} = n + \left(\frac{\partial \lambda}{\partial \dot{\mathbf{r}}}\right)^{\mathrm{T}} \cdot \hat{\mathbf{u}}f_t \tag{40}$$

where n, the mean motion, is oscillating too since it is dependent on the semi-major axis through eqn (35). Since we are integrating the mean longitude λ as the sixth element and due to the fact that X_1, Y_1, $\dot{X}_1$, $\dot{Y}_1$, r as well as the partials $\partial X_1/\partial h$ through $\partial Y_1/\partial k$ are given in terms of the eccentric longitude F, it is necessary at each integration step to solve Kepler's equation

$$\lambda = F - ks_F + hc_F \tag{41}$$

for F, through a Newton–Raphson iteration scheme.

The intermediate results for the partial derivatives of the position vector with respect to the six elements used to generate the partials in eqn (24) are given by

$$\frac{\partial \mathbf{r}}{\partial a} = \frac{\partial X_1}{\partial a}\hat{\mathbf{f}} + \frac{\partial Y_1}{\partial a}\hat{\mathbf{g}} = \left(\frac{X_1}{a} - \frac{3}{2}\frac{t}{a}\dot{X}_1\right)\hat{\mathbf{f}} + \left(\frac{Y_1}{a} - \frac{3}{2}\frac{t}{a}\dot{Y}_1\right)\hat{\mathbf{g}}$$

$$\frac{\partial \mathbf{r}}{\partial h} = \frac{\partial X_1}{\partial h}\hat{\mathbf{f}} + \frac{\partial Y_1}{\partial h}\hat{\mathbf{g}}$$

$$\frac{\partial \mathbf{r}}{\partial k} = \frac{\partial X_1}{\partial k}\hat{\mathbf{f}} + \frac{\partial Y_1}{\partial k}\hat{\mathbf{g}}$$

$$\frac{\partial \mathbf{r}}{\partial \lambda} = \frac{\dot{X}_1}{n}\hat{\mathbf{f}} + \frac{\dot{Y}_1}{n}\hat{\mathbf{g}} = \frac{\dot{\mathbf{r}}}{n}$$

$$\frac{\partial \mathbf{r}}{\partial p} = X_1\frac{\partial \hat{\mathbf{f}}}{\partial p} + Y_1\frac{\partial \hat{\mathbf{g}}}{\partial p} = \frac{2}{K}[q(Y_1\hat{\mathbf{f}} - X_1\hat{\mathbf{g}}) - X_1\hat{\mathbf{w}}]$$

$$\frac{\partial \mathbf{r}}{\partial q} = X_1\frac{\partial \hat{\mathbf{f}}}{\partial q} + Y_1\frac{\partial \hat{\mathbf{g}}}{\partial q} = \frac{2}{K}[p(X_1\hat{\mathbf{g}} - Y_1\hat{\mathbf{f}}) + Y_1\hat{\mathbf{w}}]$$

3. THE EULER–LAGRANGE DIFFERENTIAL EQUATIONS

The Euler–Lagrange or adjoint differential equations for the set a, h, k, p, q and λ have been derived in Refs [8] and [9] and are shown in this section for the purpose of the discussion. The necessary condition for optimality, which consists of maximizing the Hamiltonian of the system, can be found in Refs [10–12].

The Hamiltonian of the system is given by

$$H = \boldsymbol{\lambda}_z^{\mathrm{T}}\dot{\mathbf{z}} = \boldsymbol{\lambda}_z^{\mathrm{T}}M(\mathbf{z}, F)f_t\hat{\mathbf{u}} + \lambda_\lambda n \tag{42}$$

where the seventh differential equation for the mass flow rate has been neglected. The Euler–Lagrange equations are then given by

$$\dot{\boldsymbol{\lambda}}_z = -\frac{\partial H}{\partial \mathbf{z}} = -\boldsymbol{\lambda}_z^{\mathrm{T}}\frac{\partial M}{\partial \mathbf{z}}f_t\hat{\mathbf{u}} - \lambda_\lambda\frac{\partial n}{\partial \mathbf{z}}. \tag{43}$$

The optimal thrust direction $\hat{\mathbf{u}}$ is chosen such that it is at all times parallel to $\boldsymbol{\lambda}_z^{\mathrm{T}}M(\mathbf{z}, F)f_t$ in order to maximize the Hamiltonian. The matrix $\mathbf{M}$ for the set (a, h, k, p, q, λ) is as given by eqns (25)–(30). The partials $\partial M/\partial \mathbf{z}$ appeared in Ref. [8] and are also shown here in the Appendix. If we let u_f, u_g, u_w represent the components of $\mathbf{u}$ in the equinoctial frame, then

$$\begin{aligned}H = [&\lambda_a(M_{11}u_f + M_{12}u_g + M_{13}u_w)\\ &+ \lambda_h(M_{21}u_f + M_{22}u_g + M_{23}u_w)\\ &+ \lambda_k(M_{31}u_f + M_{32}u_g + M_{33}u_w)\\ &+ \lambda_p(M_{41}u_f + M_{42}u_g + M_{43}u_w)\\ &+ \lambda_q(M_{51}u_f + M_{52}u_g + M_{53}u_w)\\ &+ \lambda_\lambda(M_{61}u_f + M_{62}u_g + M_{63}u_w)]f_t + \lambda_\lambda n. \end{aligned} \tag{44}$$

The Euler–Lagrange equations are now written as

$$\begin{aligned}\dot{\lambda}_a = -\frac{\partial H}{\partial a} = \bigg[&-\lambda_a\left(\frac{\partial M_{11}}{\partial a}u_f + \frac{\partial M_{12}}{\partial a}u_g + \frac{\partial M_{13}}{\partial a}u_w\right)\\ &-\lambda_h\left(\frac{\partial M_{21}}{\partial a}u_f + \frac{\partial M_{22}}{\partial a}u_g + \frac{\partial M_{23}}{\partial a}u_w\right)\\ &-\lambda_k\left(\frac{\partial M_{31}}{\partial a}u_f + \frac{\partial M_{32}}{\partial a}u_g + \frac{\partial M_{33}}{\partial a}u_w\right)\\ &-\lambda_p\left(\frac{\partial M_{41}}{\partial a}u_f + \frac{\partial M_{42}}{\partial a}u_g + \frac{\partial M_{43}}{\partial a}u_w\right)\\ &-\lambda_q\left(\frac{\partial M_{51}}{\partial a}u_f + \frac{\partial M_{52}}{\partial a}u_g + \frac{\partial M_{53}}{\partial a}u_w\right)\\ &-\lambda_\lambda\left(\frac{\partial M_{61}}{\partial a}u_f + \frac{\partial M_{62}}{\partial a}u_g + \frac{\partial M_{63}}{\partial a}u_w\right)\bigg]f_t\\ &-\lambda_\lambda\frac{\partial n}{\partial a}. \end{aligned} \tag{45}$$

where

$$\frac{\partial n}{\partial a} = -\frac{3n}{2a} = -\frac{3}{2}\mu^{1/2}a^{-5/2}$$

$$\begin{aligned}\dot{\lambda}_h = -\frac{\partial H}{\partial h} = \bigg[&-\lambda_a\left(\frac{\partial M_{11}}{\partial h}u_f + \frac{\partial M_{12}}{\partial h}u_g + \frac{\partial M_{13}}{\partial h}u_w\right)\\ &-\lambda_h\left(\frac{\partial M_{21}}{\partial h}u_f + \frac{\partial M_{22}}{\partial h}u_g + \frac{\partial M_{23}}{\partial h}u_w\right)\\ &-\lambda_k\left(\frac{\partial M_{31}}{\partial h}u_f + \frac{\partial M_{32}}{\partial h}u_g + \frac{\partial M_{33}}{\partial h}u_w\right)\\ &-\lambda_p\left(\frac{\partial M_{41}}{\partial h}u_f + \frac{\partial M_{42}}{\partial h}u_g + \frac{\partial M_{43}}{\partial h}u_w\right)\\ &-\lambda_q\left(\frac{\partial M_{51}}{\partial h}u_f + \frac{\partial M_{52}}{\partial h}u_g + \frac{\partial M_{53}}{\partial h}u_w\right)\\ &-\lambda_\lambda\left(\frac{\partial M_{61}}{\partial h}u_f + \frac{\partial M_{62}}{\partial h}u_g + \frac{\partial M_{63}}{\partial h}u_w\right)\bigg]f_t \end{aligned} \tag{46}$$

$$\begin{aligned}\dot{\lambda}_k = -\frac{\partial H}{\partial k} = \bigg[&-\lambda_a\left(\frac{\partial M_{11}}{\partial k}u_f + \frac{\partial M_{12}}{\partial k}u_g + \frac{\partial M_{13}}{\partial k}u_w\right)\\ &-\lambda_h\left(\frac{\partial M_{21}}{\partial k}u_f + \frac{\partial M_{22}}{\partial k}u_g + \frac{\partial M_{23}}{\partial k}u_w\right)\\ &-\lambda_k\left(\frac{\partial M_{31}}{\partial k}u_f + \frac{\partial M_{32}}{\partial k}u_g + \frac{\partial M_{33}}{\partial k}u_w\right)\\ &-\lambda_p\left(\frac{\partial M_{41}}{\partial k}u_f + \frac{\partial M_{42}}{\partial k}u_g + \frac{\partial M_{43}}{\partial k}u_w\right)\\ &-\lambda_q\left(\frac{\partial M_{51}}{\partial k}u_f + \frac{\partial M_{52}}{\partial k}u_g + \frac{\partial M_{53}}{\partial k}u_w\right)\\ &-\lambda_\lambda\left(\frac{\partial M_{61}}{\partial k}u_f + \frac{\partial M_{62}}{\partial k}u_g + \frac{\partial M_{63}}{\partial k}u_w\right)\bigg]f_t \end{aligned} \tag{47}$$

$$\begin{aligned}\dot{\lambda}_p = -\frac{\partial H}{\partial p} = \bigg[&-\lambda_a\left(\frac{\partial M_{11}}{\partial p}u_f + \frac{\partial M_{12}}{\partial p}u_g + \frac{\partial M_{13}}{\partial p}u_w\right)\\ &-\lambda_h\left(\frac{\partial M_{21}}{\partial p}u_f + \frac{\partial M_{22}}{\partial p}u_g + \frac{\partial M_{23}}{\partial p}u_w\right)\\ &-\lambda_k\left(\frac{\partial M_{31}}{\partial p}u_f + \frac{\partial M_{32}}{\partial p}u_g + \frac{\partial M_{33}}{\partial p}u_w\right)\\ &-\lambda_p\left(\frac{\partial M_{41}}{\partial p}u_f + \frac{\partial M_{42}}{\partial p}u_g + \frac{\partial M_{43}}{\partial p}u_w\right)\\ &-\lambda_q\left(\frac{\partial M_{51}}{\partial p}u_f + \frac{\partial M_{52}}{\partial p}u_g + \frac{\partial M_{53}}{\partial p}u_w\right)\\ &-\lambda_\lambda\left(\frac{\partial M_{61}}{\partial p}u_f + \frac{\partial M_{62}}{\partial p}u_g + \frac{\partial M_{63}}{\partial p}u_w\right)\bigg]f_t \end{aligned} \tag{48}$$

$$\begin{aligned}\dot{\lambda}_q = -\frac{\partial H}{\partial q} = \bigg[&-\lambda_a\left(\frac{\partial M_{11}}{\partial q}u_f + \frac{\partial M_{12}}{\partial q}u_g + \frac{\partial M_{13}}{\partial q}u_w\right)\\ &-\lambda_h\left(\frac{\partial M_{21}}{\partial q}u_f + \frac{\partial M_{22}}{\partial q}u_g + \frac{\partial M_{23}}{\partial q}u_w\right)\\ &-\lambda_k\left(\frac{\partial M_{31}}{\partial q}u_f + \frac{\partial M_{32}}{\partial q}u_g + \frac{\partial M_{33}}{\partial q}u_w\right)\\ &-\lambda_p\left(\frac{\partial M_{41}}{\partial q}u_f + \frac{\partial M_{42}}{\partial q}u_g + \frac{\partial M_{43}}{\partial q}u_w\right)\\ &-\lambda_q\left(\frac{\partial M_{51}}{\partial q}u_f + \frac{\partial M_{52}}{\partial q}u_g + \frac{\partial M_{53}}{\partial q}u_w\right)\\ &-\lambda_\lambda\left(\frac{\partial M_{61}}{\partial q}u_f + \frac{\partial M_{62}}{\partial q}u_g + \frac{\partial M_{63}}{\partial q}u_w\right)\bigg]f_t \end{aligned} \tag{49}$$

$$\begin{aligned}\dot{\lambda}_\lambda = -\frac{\partial H}{\partial \lambda} = \bigg[&-\lambda_a\left(\frac{\partial M_{11}}{\partial \lambda}u_f + \frac{\partial M_{12}}{\partial \lambda}u_g + \frac{\partial M_{13}}{\partial \lambda}u_w\right)\\ &-\lambda_h\left(\frac{\partial M_{21}}{\partial \lambda}u_f + \frac{\partial M_{22}}{\partial \lambda}u_g + \frac{\partial M_{23}}{\partial \lambda}u_w\right)\\ &-\lambda_k\left(\frac{\partial M_{31}}{\partial \lambda}u_f + \frac{\partial M_{32}}{\partial \lambda}u_g + \frac{\partial M_{33}}{\partial \lambda}u_w\right)\\ &-\lambda_p\left(\frac{\partial M_{41}}{\partial \lambda}u_f + \frac{\partial M_{42}}{\partial \lambda}u_g + \frac{\partial M_{43}}{\partial \lambda}u_w\right)\\ &-\lambda_q\left(\frac{\partial M_{51}}{\partial \lambda}u_f + \frac{\partial M_{52}}{\partial \lambda}u_g + \frac{\partial M_{53}}{\partial \lambda}u_w\right)\\ &-\lambda_\lambda\left(\frac{\partial M_{61}}{\partial \lambda}u_f + \frac{\partial M_{62}}{\partial \lambda}u_g + \frac{\partial M_{63}}{\partial \lambda}u_w\right)\bigg]f_t \end{aligned} \tag{50}$$

The following partials are used to generate the $\partial M/\partial \mathbf{z}$ partials of the Appendix. Here F must be considered to be independent of a but not of h, k and λ because $\partial\lambda/\partial a$ is now equal to zero.

$$\frac{\partial r}{\partial a}=\frac{r}{a}$$

$$\frac{\partial r}{\partial h}=\frac{a^2}{r}(h-s_F)$$

$$\frac{\partial r}{\partial k}=\frac{a^2}{r}(k-c_F)$$

$$\frac{\partial r}{\partial F}=a(ks_F-hc_F)$$

$$\frac{\partial F}{\partial a}=0$$

$$\frac{\partial F}{\partial h}=-\frac{a}{r}c_F$$

$$\frac{\partial F}{\partial k}=\frac{a}{r}s_F$$

$$\frac{\partial F}{\partial \lambda}=\frac{a}{r}.$$

It is also true that $\partial\lambda/\partial F = r/a$. The components of the unit vector $\hat{\mathbf{u}}$ are obtained from

$$\hat{\mathbf{u}}=\frac{(\boldsymbol{\lambda}_z^{\mathrm{T}}M)^{\mathrm{T}}}{|\boldsymbol{\lambda}_z^{\mathrm{T}}M|}. \tag{51}$$

From Fig. 1, the thrust vector $\mathbf{T}=\mathbf{f}=f\hat{\mathbf{u}}$ is defined by the thrust pitch and thrust yaw angles θ_t and θ_h, respectively, in the rotating $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{h}}$ frame; $\hat{\mathbf{r}}$ is a unit vector along the instantaneous position vector $\mathbf{r}$, with $\hat{\boldsymbol{\theta}}$ in the orbit plane and along the direction of motion, and $\hat{\mathbf{h}}$ along the angular momentum vector. Therefore

$$\theta_t=\tan^{-1}\left(\frac{u_r}{u_\theta}\right) \tag{52}$$

$$\theta_h=\tan^{-1}\left(\frac{u_w}{u_\theta}\right) \tag{53}$$

with u_r, u_θ and u_h representing the components of the unit vector $\hat{\mathbf{u}}$ along the $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{h}}$ directions. We have

$$\hat{\mathbf{r}}=\frac{X_1}{|\mathbf{r}|}\hat{\mathbf{f}}+\frac{Y_1}{|\mathbf{r}|}\hat{\mathbf{g}} \tag{54}$$

$$\hat{\boldsymbol{\theta}}=\hat{\mathbf{h}}\times\hat{\mathbf{r}}=-\frac{Y_1}{|\mathbf{r}|}\hat{\mathbf{f}}+\frac{X_1}{|\mathbf{r}|}\hat{\mathbf{g}} \tag{55}$$

such that

$$u_r=\frac{X_1}{r}u_f+\frac{Y_1}{r}u_g \tag{56}$$

$$u_\theta=-\frac{Y_1}{r}u_f+\frac{X_1}{r}u_g \tag{57}$$

$$u_h=u_w \tag{58}$$

where

$$r=|\mathbf{r}|=(X_1^2+Y_1^2)^{1/2}$$

4. NUMERICAL RESULTS

Since we are minimizing the total rendezvous time, i.e.

$$J=\int_{t_0}^{t_f}\mathrm{d}t=(t_f-t_0) \tag{59}$$

this is equivalent to maximizing

$$J=\int_{t_0}^{t_f}L\,\mathrm{d}t=-\int_{t_0}^{t_f}\mathrm{d}t=-(t_f-t_0). \tag{60}$$

Therefore $L=-1$ such that the transversality condition at the unknown final time t_f namely $H_f=0$ for the augmented Hamiltonian $H=-1+\boldsymbol{\lambda}_z^{\mathrm{T}}\dot{\mathbf{z}}$ is equivalent to $H_f=1$ for our Hamiltonian $H=\boldsymbol{\lambda}_z^{\mathrm{T}}\dot{\mathbf{z}}$. The two-point boundary value problem consists of guessing the initial values of the six Lagrange multipliers $(\lambda_a)_0, (\lambda_h)_0, (\lambda_k)_0, (\lambda_p)_0, (\lambda_q)_0, (\lambda_\lambda)_0$ as well as the rendezvous time t_f (we assume here $t_0=0$) and integrating numerically the 12 differential equations of motion namely eqns (35)–(40) and eqns (45)–(50) using the optimal control given in eqn (51) such that the six state parameters at time t_f namely $(a)_f$, $(h)_f, (k)_f, (p)_f, (q)_f, (\lambda)_f$, and $H_f=1$ are satisfied. This consists therefore of a 7×7 search with given initial parameters $(a)_0, (h)_0, (k)_0, (p)_0, (q)_0, (\lambda)_0$.

For this purpose, we make use of the minimization algorithm UNCMIN of Ref. [13] which is designed for the unconstrained minimization of a real-valued function $F(x)$ of n variables denoted by the vector $\mathbf{x}$. This subroutine is based on a general descent method and uses a quasi-Newton algorithm. In the Newton method, the step p is computed from the solution of a set of n linear equations known as the Newton equations

$$\nabla^2F(x)p=-\nabla F(x). \tag{61}$$

Therefore, the solution is updated by using

$$\begin{aligned}x_{k+1}&=x_k+p\\&=x_k-[\nabla^2F(x_k)]^{-1}\nabla F(x_k)\end{aligned} \tag{62}$$

Here $\nabla F(x)$ denotes the gradient of F at x while $\nabla^2F(x)$, the constant matrix of the second partial derivatives of F at x represents the Hessian matrix.

The Newton direction given by p is guaranteed to be a descent direction only if $[\nabla^2F]^{-1}$ is positive definite, i.e. $z^{\mathrm{T}}[\nabla^2F]^{-1}z>0$ for all $z\neq 0$, since then for small ϵ,

$$\begin{aligned}F(x+\epsilon p)&=F(x)+\epsilon\nabla F^{\mathrm{T}}p+o(\epsilon^2)\\&=F(x)-\epsilon\nabla F^{\mathrm{T}}[\nabla^2F]^{-1}\nabla F+o(\epsilon^2)\end{aligned} \tag{63}$$

such that $F(x+\epsilon p)<F(x)$. This is equivalent to requiring the linear term in Newton's quadratic approximation for the function F to be negative. This is, of course, the second term in the Taylor series expansion for F at $x+p$.

The algorithm UNCMIN builds a secant approximation B_k to the Hessian as the function is being minimized such that at x_k

$$B_k p = -\nabla F_k \tag{64}$$

with the matrix B_k positive definite since there is no guarantee that $[\nabla^2 F]^{-1}$ will always be positive definite for each x. Once the descent direction is established, a line search on α is established such that $F(x_k + \alpha p) < F_k$ and the solution updated via $x_{k+1} = x_k + \alpha p$. The approximate Hessian B_k is updated next using x_{k+1} and the gradient $\nabla F(x_{k+1})$. For example, if F is quadratic [13],

$$B_{k+1}(x_{k+1} - x_k) = \nabla F_{k+1} - \nabla F_k. \tag{65}$$

Only gradient values are needed for the update of the approximate Hessian, which is achieved by finite differencing such that the user must provide only the function F itself.

The function to be minimized is chosen as

$$F = \frac{|a - a_T|}{a_T} + |h - h_T| + |k - k_T| + |p - p_T| + |q - q_T| + \frac{|\lambda - \lambda_T|}{2\pi} + |H - 1| \tag{66}$$

where all the values are at the final unknown time t_f, and where the subscript T is used for the target parameters that must be matched. These are, of course, the same as $(a)_f$, $(h)_f$, $(k)_f$, $(p)_f$, $(q)_f$ and $(\lambda)_f$. Starting from $a_0 = 42000$ km, $e_0 = 0$, $i_0 = 28.5°$, $\Omega_0 = 30°$, $\omega_0 = 10°$, $M_0 = 0°$, and using a constant acceleration $f_t = 3.5 \times 10^{-7}$ km/s², we seek the minimum time solution that achieves the final target parameters $a_f = 42767.073$ km, $e_f = 1.64459 \times 10^{-4}$, $i_f = 28.343°$, $\Omega_f = 29.999°$, $\omega_f = 247.299°$ and $M_f = 120.905°$.

The value of F is minimized at $F = 0.38989 \times 10^{-5}$ with the solution given as $(\lambda_a)_0 = 7347.174908$ s/km, $(\lambda_h)_0 = -6.994059338 \times 10^5$ s, $(\lambda_k)_0 = 8.246922768 \times 10^5$ s, $(\lambda_p)_0 = -3.631161518 \times 10^8$ s, $(\lambda_q)_0 = -6.198272749 \times 10^8$ s, $(\lambda_\lambda)_0 = -1.107002405 \times 10^6$ s/rad, and $t_f = 86402.453$ s. Figure 2 depicts the evolution of the semi-major axis and eccentricity during the 24-h transfer, while Fig. 3 shows the evolution of the inclination and the right ascension of the ascending node. The relative minima in the Ω curve correspond to the stationary points of the i curve with the maxima corresponding to the largest negative gradients of the inclination. Figure 4 shows ω and M, which exhibit large gradients as soon as the eccentricity reaches the near-circular condition. Figure 5 shows the time history of the Hamiltonian which is a constant in this formulation due to the independence of the accessory variable namely the eccentric longitude on the semi-major axis. Figure 6 shows the control angles θ_t and θ_h that achieve the desired orbit transfer. Finally, Figs 7–9 show the evolution of the Lagrange multipliers during the transfer.

5. CONCLUDING REMARKS

The minimum-time low-thrust orbit rendezvous using a set of non-singular elements is presented. This theory can be extended in order to allow the appearance of initial and intermediate time coasting arcs to solve minimum-fuel time-fixed rendezvous problems

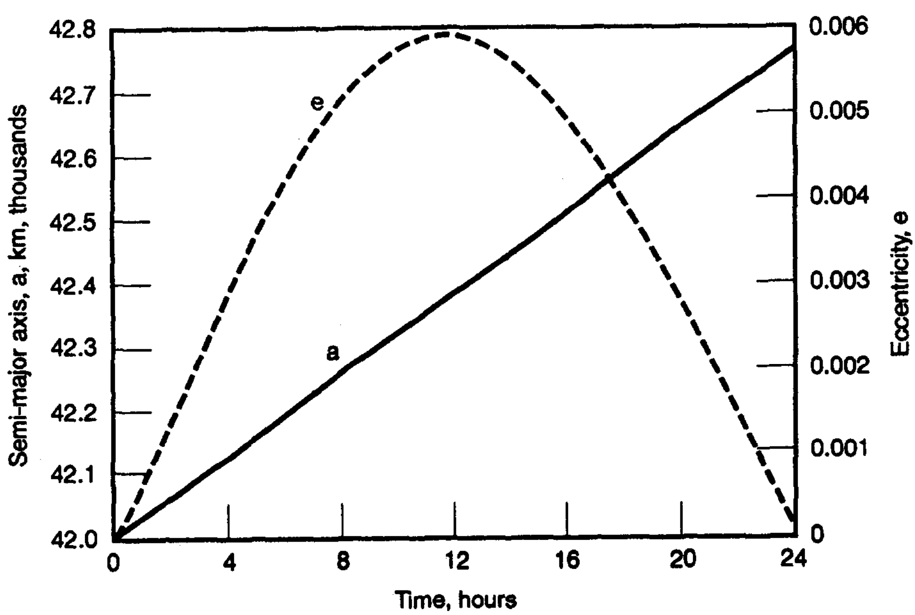

Fig. 2. Evolution of semi-major axis and eccentricity.

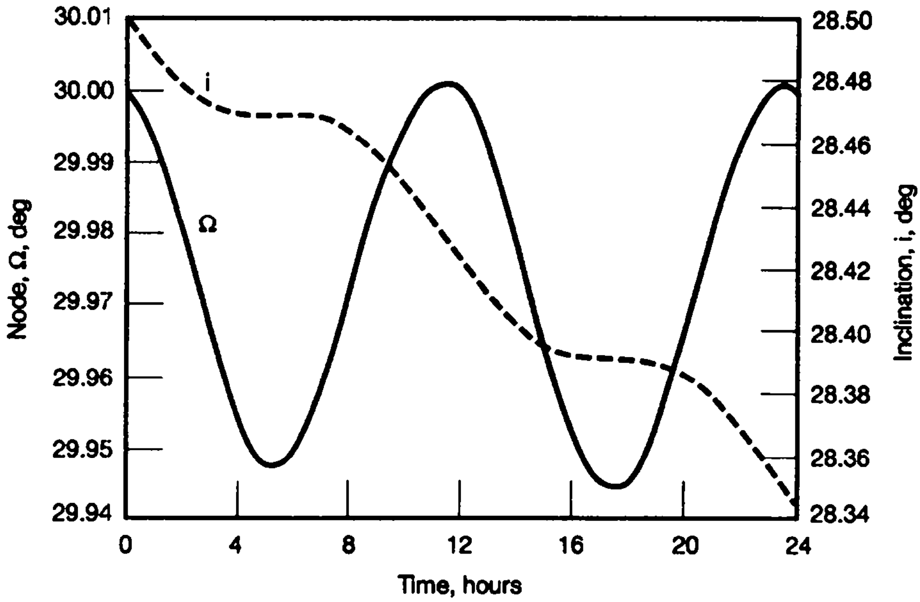

Fig. 3. Evolution of orbit inclination and node.

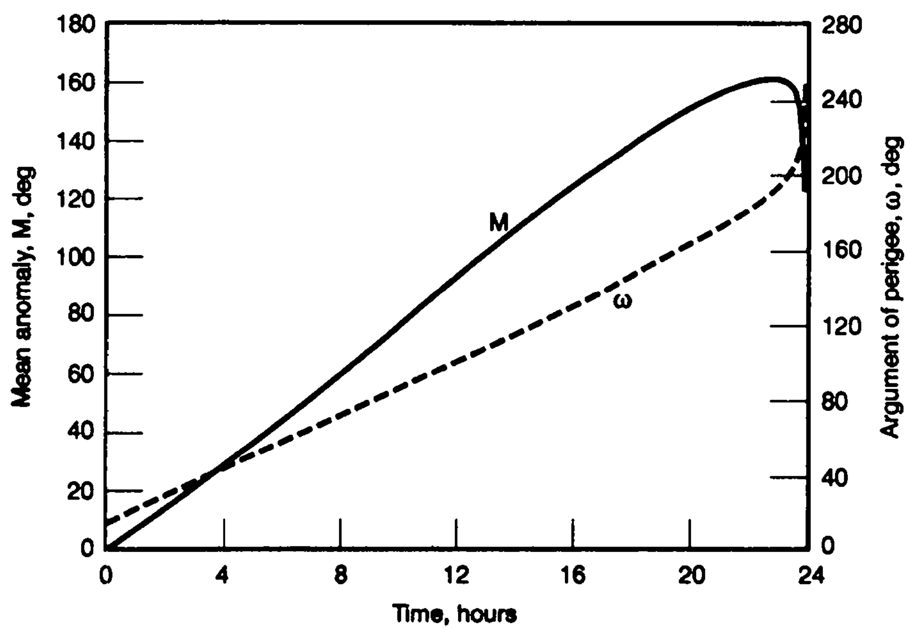

Fig. 4. Evolution of mean anomaly and argument of perigee.

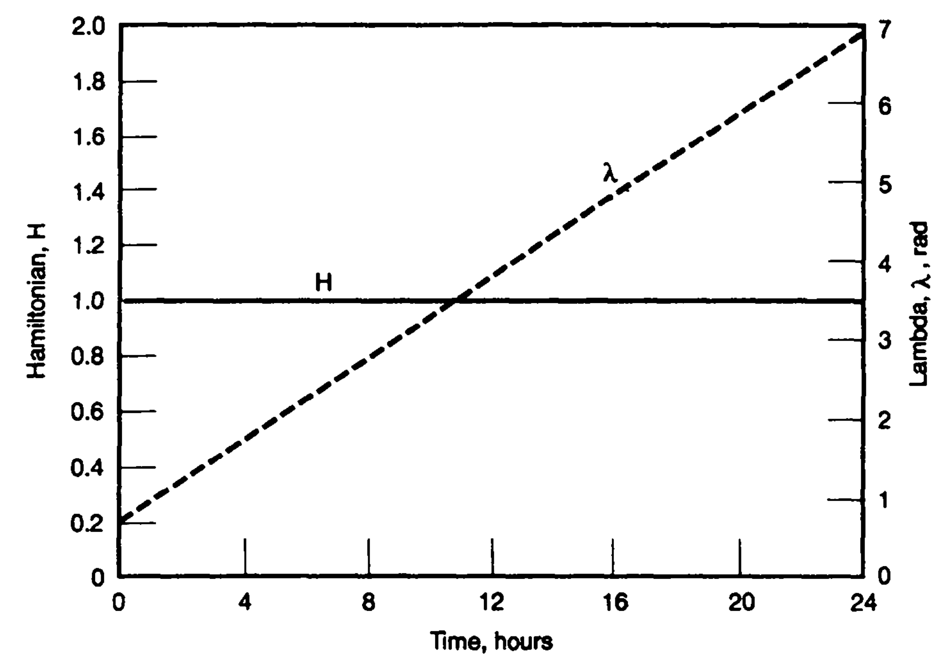

Fig. 5. Evolution of optimal Hamiltonian during transfer.

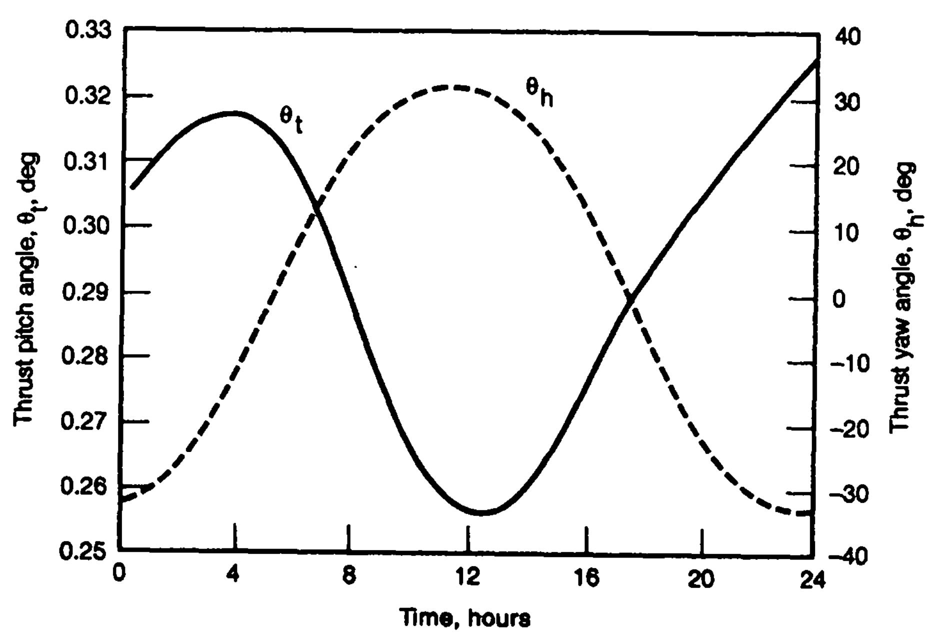

Fig. 6. Optimal thrust pitch and yaw control angles during transfer.

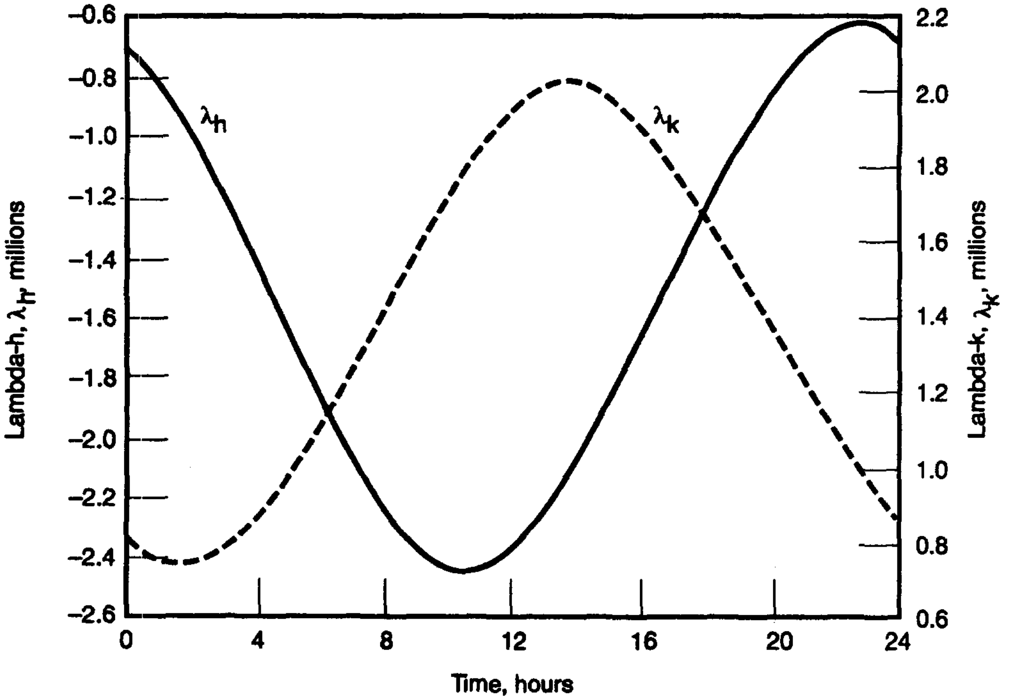

Fig. 7. Time history of λ_h and λ_k.

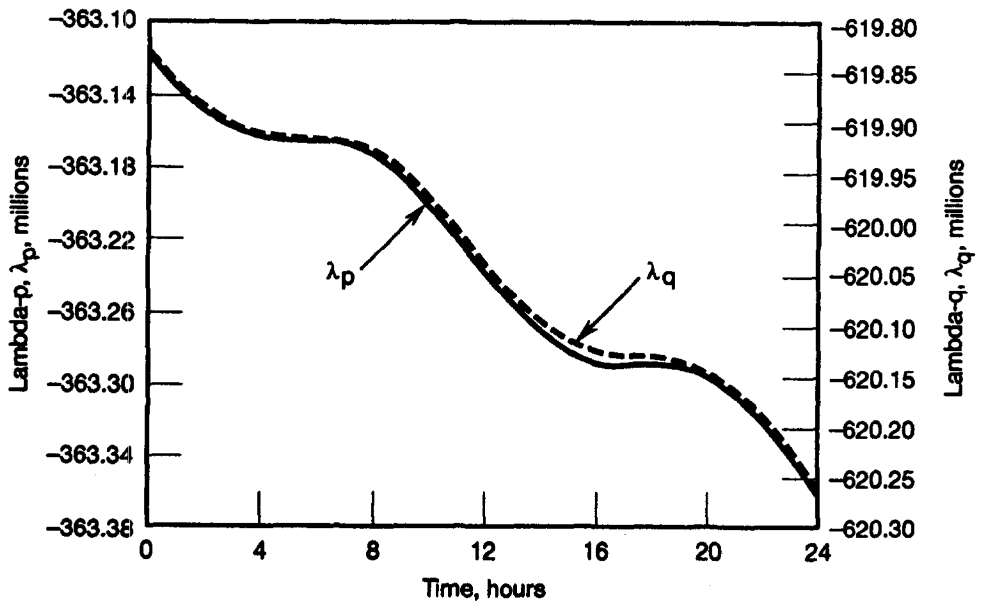

Fig. 8. Time history of λ_p and λ_q.

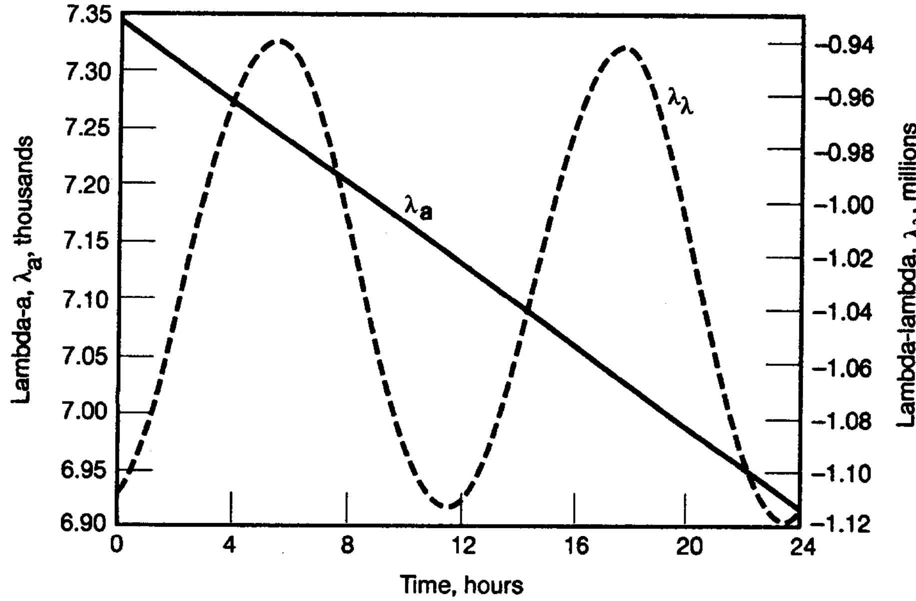

Fig. 9. Time history of λ_a and λ_λ.

as well. Further improvements would necessitate the consideration of the J_2 zonal harmonic and possibly the acceleration due to air drag for low-earth orbit applications. Although the present work presents exact solutions using precision integration, long duration rendezvous applications can effectively make use of the averaging technique for rapid calculation.

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APPENDIX

The Partials of the M Matrix

The partial derivatives of M with respect to h

$$\frac{\partial M_{11}}{\partial h}=\frac{2}{n^2a}\frac{\partial \dot{X}_1}{\partial h} \tag{A1}$$

$$\frac{\partial M_{12}}{\partial h}=\frac{2}{n^2a}\frac{\partial \dot{Y}_1}{\partial h} \tag{A2}$$

$$\frac{\partial M_{13}}{\partial h}=0 \tag{A3}$$

$$\frac{\partial M_{21}}{\partial h}=\frac{-h}{na^2(1-h^2-k^2)^{1/2}}\left(\frac{\partial X_1}{\partial k}-\frac{h\beta}{n}\dot{X}_1\right)+\frac{(1-h^2-k^2)^{1/2}}{na^2}\left[\frac{\partial^2 X_1}{\partial h\,\partial k}-\frac{\dot{X}_1}{n}\left(\beta+\frac{h^2\beta^3}{1-\beta}\right)-\frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial h}\right]. \tag{A4}$$

In a similar way

$$\frac{\partial M_{22}}{\partial h}=\frac{-h}{na^2(1-h^2-k^2)^{1/2}}\left(\frac{\partial Y_1}{\partial k}-\frac{h\beta}{n}\dot{Y}_1\right)+\frac{(1-h^2-k^2)^{1/2}}{na^2}\left[\frac{\partial^2 Y_1}{\partial h\,\partial k}-\frac{\dot{Y}_1}{n}\left(\beta+\frac{h^2\beta^3}{1-\beta}\right)-\frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial h}\right] \tag{A5}$$

$$\frac{\partial M_{23}}{\partial h}=\frac{hk(1-h^2-k^2)^{-3/2}}{na^2}(qY_1-pX_1)+\frac{k\left(q\frac{\partial Y_1}{\partial h}-p\frac{\partial X_1}{\partial h}\right)}{na^2(1-h^2-k^2)^{1/2}} \tag{A6}$$

$$\frac{\partial M_{31}}{\partial h}=\frac{h}{na^2(1-h^2-k^2)^{1/2}}\left(\frac{\partial X_1}{\partial h}+k\beta\frac{\dot{X}_1}{n}\right)-\frac{(1-h^2-k^2)^{1/2}}{na^2}\left(\frac{\partial^2 X_1}{\partial h^2}+\frac{hk\beta^3}{1-\beta}\frac{\dot{X}_1}{n}+\frac{k\beta}{n}\frac{\partial\dot{X}_1}{\partial h}\right) \quad \text{(A7)}$$

$$\frac{\partial M_{32}}{\partial h}=\frac{h}{na^2(1-h^2-k^2)^{1/2}}\left(\frac{\partial Y_1}{\partial h}+k\beta\frac{\dot{Y}_1}{n}\right)-\frac{(1-h^2-k^2)^{1/2}}{na^2}\left(\frac{\partial^2 Y_1}{\partial h^2}+\frac{hk\beta^3}{1-\beta}\frac{\dot{Y}_1}{n}+\frac{k\beta}{n}\frac{\partial\dot{Y}_1}{\partial h}\right) \quad \text{(A8)}$$

$$\frac{\partial M_{33}}{\partial h}=\frac{-1}{na^2(1-h^2-k^2)^{1/2}}\left[(qY_1-pX_1)+h\left(q\frac{\partial Y_1}{\partial h}-p\frac{\partial X_1}{\partial h}\right)\right]-\frac{h^2(qY_1-pX_1)}{na^2(1-h^2-k^2)^{3/2}} \quad \text{(A9)}$$

$$\frac{\partial M_{43}}{\partial h}=\frac{(1+p^2+q^2)}{2na^2(1-h^2-k^2)^{1/2}}\left[\frac{\partial Y_1}{\partial h}+\frac{hY_1}{(1-h^2-k^2)}\right] \quad \text{(A10)}$$

$$\frac{\partial M_{53}}{\partial h}=\frac{(1+p^2+q^2)}{2na^2(1-h^2-k^2)^{1/2}}\left[\frac{\partial X_1}{\partial h}+\frac{hX_1}{(1-h^2-k^2)}\right]. \quad \text{(A11)}$$

The partials

$$\frac{\partial M_{41}}{\partial h}=\frac{\partial M_{42}}{\partial h}=\frac{\partial M_{51}}{\partial h}=\frac{\partial M_{52}}{\partial h}=0$$

are all identically zero.

$$\frac{\partial M_{61}}{\partial h}=\frac{1}{na^2}\left\{-2\frac{\partial X_1}{\partial h}-h\beta(1-h^2-k^2)^{-1/2}\cdot\left(h\frac{\partial X_1}{\partial h}+k\frac{\partial X_1}{\partial k}\right)+(1-h^2-k^2)^{1/2}\cdot\left[\left(\beta+\frac{h^2\beta^3}{1-\beta}\right)\frac{\partial X_1}{\partial h}+\frac{hk\beta^3}{1-\beta}\frac{\partial X_1}{\partial k}+\beta\left(h\frac{\partial^2 X_1}{\partial h^2}+k\frac{\partial^2 X_1}{\partial h\,\partial k}\right)\right]\right\} \quad \text{(A12)}$$

$$\frac{\partial M_{62}}{\partial h}=\frac{1}{na^2}\left\{-2\frac{\partial Y_1}{\partial h}-h\beta(1-h^2-k^2)^{-1/2}\cdot\left(h\frac{\partial Y_1}{\partial h}+k\frac{\partial Y_1}{\partial k}\right)+(1-h^2-k^2)^{1/2}\cdot\left[\left(\beta+\frac{h^2\beta^3}{1-\beta}\right)\frac{\partial Y_1}{\partial h}+\frac{hk\beta^3}{1-\beta}\frac{\partial Y_1}{\partial k}+\beta\left(h\frac{\partial^2 Y_1}{\partial h^2}+k\frac{\partial^2 Y_1}{\partial h\,\partial k}\right)\right]\right\} \quad \text{(A13)}$$

$$\frac{\partial M_{63}}{\partial h}=\frac{(1-h^2-k^2)^{-1/2}}{na^2}\cdot\left[\left(q\frac{\partial Y_1}{\partial h}-p\frac{\partial X_1}{\partial h}\right)+h(1-h^2-k^2)^{-1}(qY_1-pX_1)\right]. \quad \text{(A14)}$$

The partial derivatives of M with respect to k

$$\frac{\partial M_{11}}{\partial k}=\frac{2}{n^2a}\frac{\partial\dot{X}_1}{\partial k} \quad \text{(A15)}$$

$$\frac{\partial M_{12}}{\partial k}=\frac{2}{n^2a}\frac{\partial\dot{Y}_1}{\partial k} \quad \text{(A16)}$$

$$\frac{\partial M_{13}}{\partial k}=0 \quad \text{(A17)}$$

$$\frac{\partial M_{21}}{\partial k}=\frac{-k}{na^2(1-h^2-k^2)^{1/2}}\left(\frac{\partial X_1}{\partial k}-\frac{h\beta}{n}\dot{X}_1\right)+\frac{(1-h^2-k^2)^{1/2}}{na^2}\left[\frac{\partial^2 X_1}{\partial k^2}-\frac{hk\beta^3}{n(1-\beta)}\dot{X}_1-\frac{h\beta}{n}\frac{\partial\dot{X}_1}{\partial k}\right] \quad \text{(A18)}$$

$$\frac{\partial M_{22}}{\partial k}=\frac{-k}{na^2(1-h^2-k^2)^{1/2}}\left(\frac{\partial Y_1}{\partial k}-\frac{h\beta}{n}\dot{Y}_1\right)+\frac{(1-h^2-k^2)^{1/2}}{na^2}\left[\frac{\partial^2 Y_1}{\partial k^2}-\frac{hk\beta^3}{n(1-\beta)}\dot{Y}_1-\frac{h\beta}{n}\frac{\partial\dot{Y}_1}{\partial k}\right] \quad \text{(A19)}$$

$$\frac{\partial M_{23}}{\partial k}=\frac{(qY_1-pX_1)}{na^2(1-h^2-k^2)^{1/2}}+\frac{1}{na^2(1-h^2-k^2)^{1/2}}\cdot\left[k\left(q\frac{\partial Y_1}{\partial k}-p\frac{\partial X_1}{\partial k}\right)+\frac{k^2(qY_1-pX_1)}{(1-h^2-k^2)}\right] \quad \text{(A20)}$$

$$\frac{\partial M_{31}}{\partial k}=\frac{k}{na^2(1-h^2-k^2)^{1/2}}\left(\frac{\partial X_1}{\partial h}+k\beta\frac{\dot{X}_1}{n}\right)-\frac{(1-h^2-k^2)^{1/2}}{na^2}\cdot\left[\frac{\partial^2 X_1}{\partial k\,\partial h}+\left(\beta+\frac{k^2\beta^3}{1-\beta}\right)\frac{\dot{X}_1}{n}+\frac{k\beta}{n}\frac{\partial\dot{X}_1}{\partial k}\right] \quad \text{(A21)}$$

$$\frac{\partial M_{32}}{\partial k}=\frac{k}{na^2(1-h^2-k^2)^{1/2}}\left(\frac{\partial Y_1}{\partial h}+k\beta\frac{\dot{Y}_1}{n}\right)-\frac{(1-h^2-k^2)^{1/2}}{na^2}\cdot\left[\frac{\partial^2 Y_1}{\partial k\,\partial h}+\left(\beta+\frac{k^2\beta^3}{1-\beta}\right)\frac{\dot{Y}_1}{n}+\frac{k\beta}{n}\frac{\partial\dot{Y}_1}{\partial k}\right] \quad \text{(A22)}$$

$$\frac{\partial M_{33}}{\partial k}=\frac{-h}{na^2(1-h^2-k^2)^{1/2}}\left(q\frac{\partial Y_1}{\partial k}-p\frac{\partial X_1}{\partial k}\right)-\frac{hk}{na^2(1-h^2-k^2)^{3/2}}(qY_1-pX_1) \quad \text{(A23)}$$

$$\frac{\partial M_{43}}{\partial k}=\frac{(1+p^2+q^2)}{2na^2(1-h^2-k^2)^{1/2}}\frac{\partial Y_1}{\partial k}+\frac{k(1+p^2+q^2)}{2na^2(1-h^2-k^2)^{3/2}}Y_1 \quad \text{(A24)}$$

$$\frac{\partial M_{53}}{\partial k}=\frac{(1+p^2+q^2)}{2na^2(1-h^2-k^2)^{1/2}}\frac{\partial X_1}{\partial k}+\frac{k(1+p^2+q^2)}{2na^2(1-h^2-k^2)^{3/2}}X_1 \quad \text{(A25)}$$

$$\frac{\partial M_{41}}{\partial k}=\frac{\partial M_{42}}{\partial k}=\frac{\partial M_{51}}{\partial k}=\frac{\partial M_{52}}{\partial k}=0$$

$$\frac{\partial M_{61}}{\partial k}=\frac{1}{na^2}\left\{-2\frac{\partial X_1}{\partial k}-k\beta(1-h^2-k^2)^{-1/2}\cdot\left(h\frac{\partial X_1}{\partial h}+k\frac{\partial X_1}{\partial k}\right)+(1-h^2-k^2)^{1/2}\cdot\left[\left(\beta+\frac{k^2\beta^3}{1-\beta}\right)\frac{\partial X_1}{\partial k}+\frac{hk\beta^3}{1-\beta}\frac{\partial X_1}{\partial h}+\beta\left(h\frac{\partial^2 X_1}{\partial k\,\partial h}+k\frac{\partial^2 X_1}{\partial k^2}\right)\right]\right\} \quad \text{(A26)}$$

$$\frac{\partial M_{62}}{\partial k}=\frac{1}{na^2}\left\{-2\frac{\partial Y_1}{\partial k}-k\beta(1-h^2-k^2)^{-1/2}\cdot\left(h\frac{\partial Y_1}{\partial h}+k\frac{\partial Y_1}{\partial k}\right)+(1-h^2-k^2)^{1/2}\cdot\left[\left(\beta+\frac{k^2\beta^3}{1-\beta}\right)\frac{\partial Y_1}{\partial k}+\frac{hk\beta^3}{1-\beta}\frac{\partial Y_1}{\partial h}+\beta\left(h\frac{\partial^2 Y_1}{k\partial h}+k\frac{\partial^2 Y_1}{\partial k^2}\right)\right]\right\} \tag{A27}$$

$$\frac{\partial M_{63}}{\partial k}=\frac{(1-h^2-k^2)^{-1/2}}{na^2}\left[\left(q\frac{\partial Y_1}{\partial k}-p\frac{\partial X_1}{\partial k}\right)+k(1-h^2-k^2)^{-1}(qY_1-pX_1)\right] \tag{A28}$$

The partial derivatives of M with respect to p

The only non-zero partials are

$$\frac{\partial M_{23}}{\partial p}=\frac{-kX_1}{na^2(1-h^2-k^2)^{1/2}} \tag{A29}$$

$$\frac{\partial M_{33}}{\partial p}=\frac{hX_1}{na^2(1-h^2-k^2)^{1/2}} \tag{A30}$$

$$\frac{\partial M_{43}}{\partial p}=\frac{pY_1}{na^2(1-h^2-k^2)^{1/2}} \tag{A31}$$

$$\frac{\partial M_{53}}{\partial p}=\frac{pX_1}{na^2(1-h^2-k^2)^{1/2}} \tag{A32}$$

$$\frac{\partial M_{63}}{\partial p}=\frac{-X_1}{na^2(1-h^2-k^2)^{1/2}}. \tag{A33}$$

The partial derivatives of M with respect to q

The non-zero partials are

$$\frac{\partial M_{23}}{\partial q}=\frac{kY_1}{na^2(1-h^2-k^2)^{1/2}} \tag{A34}$$

$$\frac{\partial M_{33}}{\partial q}=\frac{-hY_1}{na^2(1-h^2-k^2)^{1/2}} \tag{A35}$$

$$\frac{\partial M_{43}}{\partial q}=\frac{qY_1}{na^2(1-h^2-k^2)^{1/2}} \tag{A36}$$

$$\frac{\partial M_{53}}{\partial q}=\frac{qX_1}{na^2(1-h^2-k^2)^{1/2}} \tag{A37}$$

$$\frac{\partial M_{63}}{\partial q}=\frac{Y_1}{na^2(1-h^2-k^2)^{1/2}}. \tag{A38}$$

The partial derivatives of $\dot{X}_1$ with respect to h and k are

$$\frac{\partial \dot{X}_1}{\partial h}=\frac{a}{r}\dot{X}_1\left[s_F+\frac{a}{r}c_F(ks_F-hc_F)\right]+\frac{na^2}{r}\cdot\left\{h\beta s_F+(kc_F+hs_F)\left(\beta+\frac{h^2\beta^3}{1-\beta}\right)+\frac{a}{r}c_F[hk\beta s_F+(1-h^2\beta)c_F]\right\} \tag{A39}$$

$$\frac{\partial \dot{X}_1}{\partial k}=-\frac{\dot{X}_1}{r}a\left[-c_F+\frac{a}{r}s_F(ks_F-hc_F)\right]+\frac{na^2}{r}\cdot\left\{\frac{hk\beta^3}{1-\beta}(kc_F+hs_F)+h\beta c_F-\frac{a}{r}s_F[hk\beta s_F+(1-h^2\beta)c_F]\right\} \tag{A40}$$

The partials of $\dot{Y}_1$ with respect to h and k are

$$\frac{\partial \dot{Y}_1}{\partial h}=-\frac{\dot{Y}_1}{r}a\left[-s_F-\frac{a}{r}c_F(ks_F-hc_F)\right]+\frac{na^2}{r}\cdot\left\{\frac{-hk\beta^3}{1-\beta}(kc_F+hs_F)-k\beta s_F+[hk\beta c_F+(1-k^2\beta)s_F]\frac{a}{r}c_F\right\} \tag{A41}$$

$$\frac{\partial \dot{Y}_1}{\partial k}=\frac{-\dot{Y}_1}{r}a\left[-c_F+\frac{a}{r}s_F(ks_F-hc_F)\right]+\frac{na^2}{r}\cdot\left\{-\left(\beta+\frac{k^2\beta^3}{1-\beta}\right)(kc_F+hs_F)-k\beta c_F-\frac{a}{r}s_F[hk\beta c_F+(1-k^2\beta)s_F]\right\} \tag{A42}$$

The second partials of X_1 and Y_1 with respect to h and k are

$$\frac{\partial^2 X_1}{\partial h^2}=a\left\{-\frac{2a}{r}c_F\left(\beta+\frac{h^2\beta^3}{1-\beta}\right)-\frac{h\beta^3}{1-\beta}(hc_F-ks_F)\left[3+\frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right]+\frac{a^2}{r^2}c_F(h\beta-s_F)\left[-s_F+\frac{a}{r}(h-s_F)\right]-\frac{a^2}{r^2}c_F^3\right\}. \tag{A43}$$

$$\frac{\partial^2 X_1}{\partial k^2}=-a\left\{-\frac{2a}{r}s_F\frac{hk\beta^3}{(1-\beta)}+(hc_F-ks_F)\cdot\left[1+\frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right]\frac{h\beta^3}{(1-\beta)}+\frac{a^2}{r^2}s_F(h\beta-s_F)\left[-c_F+\frac{a}{r}(k-c_F)\right]+\frac{a^2}{r^2}c_Fs_F^2\right\} \tag{A44}$$

$$\frac{\partial^2 X_1}{\partial h\,\partial k}=-a\left\{\frac{a}{r}c_F\frac{hk\beta^3}{(1-\beta)}+(hc_F-ks_F)\cdot\left[1+\frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right]\frac{k\beta^3}{(1-\beta)}+(s_F-h\beta)\left[\frac{a}{r}(s_F^2-hs_F)-c_F^2\right]\frac{a^2}{r^2}-\frac{a^2}{r^2}s_Fc_F^2-\frac{a}{r}s_F\left(\beta+\frac{h^2\beta^3}{1-\beta}\right)\right\} \tag{A45}$$

$$\frac{\partial^2 X_1}{\partial k\,\partial h}=a\left\{\frac{a}{r}s_F\left(\beta+\frac{h^2\beta^3}{(1-\beta)}\right)-(hc_F-ks_F)\cdot\left[1+\frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right]\frac{k\beta^3}{(1-\beta)}+\frac{a^2}{r^2}\left[\frac{a}{r}(kc_F-c_F^2)+s_F^2\right](h\beta-s_F)-\frac{a}{r}c_F\frac{hk\beta^3}{(1-\beta)}+\frac{a^2}{r^2}c_F^2s_F\right\} \tag{A46}$$

$$\frac{\partial^2 Y_1}{\partial h^2}=a\left\{\frac{2a}{r}c_F\frac{hk\beta^3}{(1-\beta)}+(hc_F-ks_F)\frac{k\beta^3}{(1-\beta)}\cdot\left[1+\frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right]+\frac{a^2}{r^2}c_F\cdot\left[-\frac{a}{r}(h-s_F)+s_F\right](k\beta-c_F)-\frac{a^2}{r^2}s_Fc_F^2\right\} \tag{A47}$$

$$\frac{\partial^2 Y_1}{\partial k^2} = a\left\{\frac{-2a}{r} s_F\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + (hc_F - ks_F) \cdot \left[3 + \frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right]\frac{k\beta^3}{(1-\beta)} + \frac{a^2}{r^2}s_F \cdot \left[-\frac{a}{r}(k - c_F) + c_F\right](c_F - k\beta) - \frac{a^2}{r^2}s_F^3\right\} \tag{A48}$$

$$\frac{\partial^2 Y_1}{\partial h\,\partial k} = a\left\{\frac{a}{r} c_F\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + (hc_F - ks_F) \cdot \frac{h\beta^3}{(1-\beta)}\left[1 + \frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] - \frac{a^2}{r^2}\left[\frac{a}{r}s_F(h - s_F) + c_F^2\right] \cdot (c_F - k\beta) + \frac{a^2}{r^2}c_F s_F^2 - \frac{a}{r}s_F\frac{hk\beta^3}{(1-\beta)}\right\} \tag{A49}$$

$$\frac{\partial^2 Y_1}{\partial k\,\partial h} = a\left\{-\frac{a}{r}s_F\frac{hk\beta^3}{(1-\beta)} + (hc_F - ks_F)\frac{h\beta^3}{(1-\beta)} \cdot \left[1 + \frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] - \frac{a^2}{r^2}\left[\frac{a}{r}c_F(k - c_F) + s_F^2\right] \cdot (k\beta - c_F) + \frac{a}{r}c_F\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + \frac{a^2}{r^2}c_F s_F^2\right\}. \tag{A50}$$

Next, the accessory partials $\partial^2 X_1/\partial a\,\partial k$, $\partial^2 X_1/\partial a\,\partial h$, $\partial^2 Y_1/\partial a\,\partial k$ and $\partial^2 Y_1/\partial a\,\partial h$, are generated with $\partial F/\partial a = 0$. In all the following partials, $\partial X_1/\partial a = X_1/a$ and $\partial Y_1/\partial a = Y_1/a$

$$\frac{\partial^2 X_1}{\partial a\,\partial k} = \frac{1}{a}\frac{\partial X_1}{\partial k} \tag{A51}$$

$$\frac{\partial^2 X_1}{\partial a\,\partial h} = \frac{1}{a}\frac{\partial X_1}{\partial h} \tag{A52}$$

$$\frac{\partial^2 Y_1}{\partial a\,\partial k} = \frac{1}{a}\frac{\partial Y_1}{\partial k} \tag{A53}$$

$$\frac{\partial^2 Y_1}{\partial a\,\partial h} = \frac{1}{a}\frac{\partial Y_1}{\partial h}. \tag{A54}$$

The partial derivatives of M with respect to a

$$\frac{\partial M_{11}}{\partial a} = \frac{4}{n^2a^2}\dot{X}_1 + \frac{2}{n^2a}\frac{\partial \dot{X}_1}{\partial a} \tag{A55}$$

$$\frac{\partial M_{12}}{\partial a} = \frac{4}{n^2a^2}\dot{Y}_1 + \frac{2}{n^2a}\frac{\partial \dot{Y}_1}{\partial a} \tag{A56}$$

$$\frac{\partial M_{13}}{\partial a} = 0 \tag{A57}$$

$$\frac{\partial M_{21}}{\partial a} = \frac{(1-h^2-k^2)^{1/2}}{na^2} \cdot \left[-\frac{1}{2a}\frac{\partial X_1}{\partial k} + \frac{\partial^2 X_1}{\partial a\,\partial k} - \frac{h\beta}{na}\dot{X}_1 - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial a}\right] \tag{A58}$$

$$\frac{\partial M_{22}}{\partial a} = \frac{(1-h^2-k^2)^{1/2}}{na^2} \cdot \left[-\frac{1}{2a}\frac{\partial Y_1}{\partial k} + \frac{\partial^2 Y_1}{\partial a\,\partial k} - \frac{h\beta}{na}\dot{Y}_1 - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial a}\right] \tag{A59}$$

$$\frac{\partial M_{23}}{\partial a} = \frac{k}{na^2(1-h^2-k^2)^{1/2}} \cdot \left[-\frac{1}{2a}(qY_1 - pX_1) + q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right] \tag{A60}$$

$$\frac{\partial M_{31}}{\partial a} = -\frac{(1-h^2-k^2)^{1/2}}{na^2} \cdot \left[-\frac{1}{2a}\frac{\partial X_1}{\partial h} + \frac{\partial^2 X_1}{\partial a\,\partial h} + \frac{k\beta}{na}\dot{X}_1 + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial a}\right] \tag{A61}$$

$$\frac{\partial M_{32}}{\partial a} = -\frac{(1-h^2-k^2)^{1/2}}{na^2} \cdot \left[-\frac{1}{2a}\frac{\partial Y_1}{\partial h} + \frac{\partial^2 Y_1}{\partial a\partial h} + \frac{k\beta}{na}\dot{Y}_1 + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial a}\right] \tag{A62}$$

$$\frac{\partial M_{33}}{\partial a} = \frac{-h}{na^2(1-h^2-k^2)^{1/2}} \cdot \left[-\frac{1}{2a}(qY_1 - pX_1) + q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right] \tag{A63}$$

$$\frac{\partial M_{41}}{\partial a} = 0 \tag{A64}$$

$$\frac{\partial M_{42}}{\partial a} = 0 \tag{A65}$$

$$\frac{\partial M_{43}}{\partial a} = \frac{(1+p^2+q^2)}{2na^2(1-h^2-k^2)^{1/2}}\left(-\frac{1}{2a}Y_1 + \frac{\partial Y_1}{\partial a}\right) \tag{A66}$$

$$\frac{\partial M_{51}}{\partial a} = 0 \tag{A67}$$

$$\frac{\partial M_{52}}{\partial a} = 0 \tag{A68}$$

$$\frac{\partial M_{53}}{\partial a} = \frac{(1+p^2+q^2)}{2na^2(1-h^2-k^2)^{1/2}}\left(-\frac{1}{2a}X_1 + \frac{\partial X_1}{\partial a}\right) \tag{A69}$$

$$\frac{\partial M_{61}}{\partial a} = \frac{-M_{61}}{2a} + \frac{1}{na^2} \cdot \left[-2\frac{\partial X_1}{\partial a} + (1-h^2-k^2)^{1/2} \cdot \left(h\beta\frac{\partial^2 X_1}{\partial a\,\partial h} + k\beta\frac{\partial^2 X_1}{\partial a\,\partial k}\right)\right] \tag{A70}$$

$$\frac{\partial M_{62}}{\partial a} = -\frac{M_{62}}{2a} + \frac{1}{na^2} \cdot \left[-2\frac{\partial Y_1}{\partial a} + (1-h^2-k^2)^{1/2} \cdot \left(h\beta\frac{\partial^2 Y_1}{\partial a\,\partial h} + k\beta\frac{\partial^2 Y_1}{\partial a\,\partial k}\right)\right] \tag{A71}$$

$$\frac{\partial M_{63}}{\partial a} = -\frac{M_{63}}{2a} + \frac{1}{na^2} \cdot \left[\left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right)(1-h^2-k^2)^{-1/2}\right] \tag{A72}$$

with

$$\frac{\partial \dot{X}_1}{\partial a} = -\frac{1}{2}\frac{na}{r} \cdot [hk\beta c_F - (1-h^2\beta)s_F] \tag{A73}$$

$$\frac{\partial \dot{Y}_1}{\partial a} = \frac{1}{2}\frac{na}{r} \cdot [hk\beta s_F - (1-k^2\beta)c_F] \tag{A74}$$

The partial derivatives of M with respect to λ

$$\frac{\partial M_{11}}{\partial \lambda} = \frac{2}{n^2r}\frac{\partial \dot{X}_1}{\partial F} \tag{A75}$$

$$\frac{\partial M_{12}}{\partial \lambda} = \frac{2}{n^2r}\frac{\partial \dot{Y}_1}{\partial F} \tag{A76}$$

$$\frac{\partial M_{13}}{\partial \lambda} = 0 \tag{A77}$$

$$\frac{\partial M_{21}}{\partial \lambda} = \frac{(1-h^2-k^2)^{1/2}}{nar}\left(\frac{\partial^2 X_1}{\partial F\partial k} - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial F}\right) \tag{A78}$$

$$\frac{\partial M_{22}}{\partial \lambda} = \frac{(1-h^2-k^2)^{1/2}}{nar}\left(\frac{\partial^2 Y_1}{\partial F\partial k} - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial F}\right) \tag{A79}$$

$$\frac{\partial M_{23}}{\partial \lambda} = \frac{k\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)}{nar(1-h^2-k^2)^{1/2}} \tag{A80}$$

$$\frac{\partial M_{31}}{\partial \lambda} = -\frac{(1-h^2-k^2)^{1/2}}{nar}\left(\frac{\partial^2 X_1}{\partial F \partial h} + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial F}\right) \tag{A81}$$

$$\frac{\partial M_{32}}{\partial \lambda} = -\frac{(1-h^2-k^2)^{1/2}}{nar}\left(\frac{\partial^2 Y_1}{\partial F \partial h} + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial F}\right) \tag{A82}$$

$$\frac{\partial M_{33}}{\partial \lambda} = \frac{-h\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)}{nar(1-h^2-k^2)^{1/2}} \tag{A83}$$

$$\frac{\partial M_{41}}{\partial \lambda} = \frac{\partial M_{42}}{\partial \lambda} = 0 \tag{A84}$$

$$\frac{\partial M_{43}}{\partial \lambda} = \frac{(1+p^2+q^2)}{2nar(1-h^2-k^2)^{1/2}}\frac{\partial Y_1}{\partial F} \tag{A85}$$

$$\frac{\partial M_{51}}{\partial \lambda} = \frac{\partial M_{52}}{\partial \lambda} = 0 \tag{A86}$$

$$\frac{\partial M_{53}}{\partial \lambda} = \frac{(1+p^2+q^2)}{2nar(1-h^2-k^2)^{1/2}}\frac{\partial X_1}{\partial F} \tag{A87}$$

$$\frac{\partial M_{61}}{\partial \lambda} = \frac{1}{nar}\left[-2\frac{\partial X_1}{\partial F} + (1-h^2-k^2)^{1/2} \cdot \left(h\beta\frac{\partial^2 X_1}{\partial F \partial h} + k\beta\frac{\partial^2 X_1}{\partial F \partial k}\right)\right] \tag{A88}$$

$$\frac{\partial M_{62}}{\partial \lambda} = \frac{1}{nar}\left[-2\frac{\partial Y_1}{\partial F} + (1-h^2-k^2)^{1/2} \cdot \left(h\beta\frac{\partial^2 Y_1}{\partial F \partial h} + k\beta\frac{\partial^2 Y_1}{\partial F \partial k}\right)\right] \tag{A89}$$

$$\frac{\partial M_{63}}{\partial \lambda} = \frac{\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)}{nar(1-h^2-k^2)^{1/2}} \tag{A90}$$

The auxiliary partials are

$$\frac{\partial X_1}{\partial F} = a[hk\beta c_F - (1-h^2\beta)s_F] \tag{A91}$$

$$\frac{\partial Y_1}{\partial F} = a[-hk\beta s_F + (1-k^2\beta)c_F] \tag{A92}$$

$$\frac{\partial \dot{X}_1}{\partial F} = -\frac{a}{r}(ks_F - hc_F)\dot{X}_1 + \frac{a^2 n}{r}[-hk\beta s_F - (1-h^2\beta)c_F] \tag{A93}$$

$$\frac{\partial \dot{Y}_1}{\partial F} = -\frac{a}{r}(ks_F - hc_F)\dot{Y}_1 + \frac{a^2 n}{r}[-hk\beta c_F - (1-k^2\beta)s_F] \tag{A94}$$

$$\frac{\partial^2 X_1}{\partial F \partial h} = a\left[(hs_F + kc_F)\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) + \frac{a^2}{r^2}(h\beta - s_F)(s_F - h) + \frac{a}{r}c_F^2\right] \tag{A95}$$

$$\frac{\partial^2 X_1}{\partial F \partial k} = -a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta} + \frac{a^2}{r^2}(s_F - h\beta)(c_F - h) + \frac{a}{r}s_F c_F\right] \tag{A96}$$

$$\frac{\partial^2 Y_1}{\partial F \partial h} = a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta} - \frac{a^2}{r^2}(k\beta - c_F)(s_F - h) + \frac{a}{r}s_F c_F\right] \tag{A97}$$

$$\frac{\partial^2 Y_1}{\partial F \partial k} = a\left[-(hs_F + kc_F)\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + \frac{a^2}{r^2}(c_F - k\beta)(c_F - k) - \frac{a}{r}s_F^2\right]. \tag{A98}$$