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Predefined-time control of nonlinear systems: A sigmoid function based sliding manifold design approach

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Sigmoid sliding surface;
Terminal sliding mode control

Abstract This work addresses the predefined-time convergence problem for a class of second order nonlinear systems subjected to matched disturbances. A novel family of sigmoid function based terminal sliding surfaces is proposed for which robust fixed time convergence is guaranteed. The stability properties are established through the Lyapunov framework. In the proposed technique, bound on the settling time is user-defined for arbitrary initial conditions. Furthermore, switching control law is designed to avoid singularity, thus sufficient condition for singularity free controller is determined. The utility of proposed control scheme is validated by simulating two numerical examples from recent relevant literature.

1. Introduction

In control systems design, settling time is one of the vital parameters for performance evaluation of controllers as it determines the time during which the system states settle within a small bound around the equilibrium point. Settling time has its significance for a wide range of applications like power systems, missile guidance system, submarines, spacecrafts, and formation containment of multiagent systems [1–8]. In recent developments, bound on the settling time is predefined in order to benchmark sophisticated control system performance for particular classes of systems. Bhat and Bernstein [9] evaluated bound on settling time function, dependent upon the initial conditions of the system using Lyapunov like formulation. In this case, *finite time* stability of system is guaranteed. The notion of finite time stability pertains to determine a bound on the time during which the system settles to an equilibrium point. The bound on the settling time is related to initial conditions by the so-called *settling time function*. Later on,

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necessary and sufficient conditions for existence of settling time function and settling time bound are determined for autonomous and non-autonomous systems by Moulay and Perruquetti [10,11]. Recently, the finite settling time controllers using barrier functions are introduced in [12,13].

In the context of analyzing the time required by a dynamic system to converge to an equilibrium point, the concept of *fixed time stability* is introduced in [14], and formally presented by Polyakov in [15]. In this case there exists a uniform bound on the settling time function independent of initial conditions which distinguishes fixed time stability from the finite time stability, discussed earlier. The work on fixed time stability is extended in [16] with derivation of less conservative bound on settling time function. Later, the implicit Lyapunov function is presented to guarantee the fixed time stability and robust stabilization of integrator dynamics in [17]. Zuo [18] proposed the methodology for fixed time control of second order systems in strict feedback form while ensuring nonsingular control law. In work of [19], the settling time bound for different finite and fixed time controllers is provided and existing methodologies are classified according to actions and structural properties. Another solution for evaluation of settling time bound ensuring fixed time stability for a similar class of systems under matched perturbations is presented in [20]. The sufficient condition for fixed time stability and settling time bound is established for affine nonlinear systems in [21].

In [22,23], the *predefined-time stability* is introduced for class of nonlinear systems to fulfill the hard time constraints of systems. The approach guarantees convergence of system states to the equilibrium point within the user-defined settling time bound. It is proved by Lyapunov methods that depending on system tunable controller parameters and initial conditions of system, the desired fixed time convergence is guaranteed for nonlinear systems. The predefined-time convergence controllers draw their significance from the fact that the tuning of control gains becomes straightforward, in order to achieve the desired settling time, unlike the case of traditional fixed time stability in which the obtained settling time has complex relationship with controller and system parameters [24,25]. An optimal method is further developed so that the control effort is distributed in reaching and sliding phase in extended work of [26]. Pal et al. proposed arbitrary and free-will arbitrary convergence time control design methodology in [27,28]. A Lyapunov based solution for convergence of a class of nonlinear autonomous systems is established. In [29], the upper bound on settling time is predefined for perturbed chain of integrators by addition of time varying controller gains. In [30], Seeber et al. developed a tuning method to assign user-defined bound on settling time for a class of second order differentiators. A fixed time control is designed using Lyapunov method and variable transformation for memristor based neural network in [31]. A robust fixed time converging neural network controller is presented in [32].

It is pertinent to mention that robustness to disturbances is generally not discussed for preset settling time techniques for autonomous systems, barring a few exceptions like [27–29,33]. The methodology of [27] caters for disturbance provided the initial states of system are known. In [28,33], free-will arbitrary time convergence is achieved by tuning the sliding phase only and the time of reaching phase of sliding mode control is assumed to be known based on simulations. Furthermore, [29] employed time varying piecewise controller to reject matched disturbances and predefined time is achieved with variation in system parameters. Extending the work of [22], this issue is recently addressed in [34] by constructing a novel terminal sliding surface based controller with predefined settling time for a class of nonlinear second order systems with matched disturbances. The performance of proposed control methodology is tested for single inverted pendulum (SIP) using identical model as [18] and [20]. Though predefined bound on settling time is fulfilled, the bounds were conservative resulting in large control effort. The work is extended to prove the ultimate boundedness with predefined time stability in [35]. Moreover, the predefined time in [34–36] is assumed to be function of initial states and system parameters. In [37], the backstepping approach is employed to achieve predefined time stability for higher order systems in strict feedback form. In [38] the predefined time stability is achieved for distributed order dynamical systems in presence of uncertainties. Moreover, a novel sliding mode control is designed in a recent work of [39] that results in predefined time convergence.

In the existing literature on predefined-time stability, disturbances are either not incorporated, or the predefined settling time is assumed to be a function of initial states. Besides this, the assumptions on convergence time are more conservative which result in relatively large control effort. The predefined-time convergence for perturbed nonlinear systems, with unknown initial conditions, is still an open problem to the best of our knowledge.

In view of the above, controller with predefined-time convergence is developed in this article for a class of second order nonlinear systems subjected to matched disturbances. A new family of terminal sliding surfaces is introduced based on sigmoid function. Fixed time convergence of sliding mode control is established using Lyapunov like method. Moreover, depending upon single adjustable parameter, the proposed controller guarantees the convergence of system within pre-assigned settling time with less conservative estimate.

A high-frequency switching of sliding mode excites the unmodeled dynamics in the closed-loop inducing chattering in the controller implementation that have a detrimental effect on the life of the control actuator. To deal with chattering, the chattering reduction methods have been discussed in [40] in detail. Though, chattering reduction does not require a detailed model of all components of the system [41]. Instead, the controller can be first designed assuming the ideal conditions of no unmodeled dynamics, and then the chattering reduction methods can be utilized. The focus of this paper is towards the design of predefined time controller using novel sliding surface therefore, the chattering reduction is not scope of this research. The novel contributions presented in this article are:

- Controller for predefined settling time is designed for a broad class of second order systems under matched disturbances not requiring knowledge of initial state of system.
- The settling time bound is predefined with a family of sliding surfaces based on antisymmetric sigmoid functions, different from [18] and [20].
- The switching control law is designed for the purpose of singularity avoidance different from the approach presented in [34]. Moreover, attributed to employing less conservative settling time bound, smaller control effort is guaranteed for the same preassigned time.

The overall presentation of this article is as follows: The nomenclature and related definitions are given in Section 2. The detailed contributions of this paper are given in Section 3 and Section 4. The validation of theoretical aspects is provided by two numerical examples in Section 5 which is followed by the Conclusions Section.

2. Notation and background

We start by defining notation used in the article along with basic technical definitions and brief review of existing results that have been presented in literature.

2.1. Nomenclature

We use the following notations throughout this article; real matrices of dimension $n \times m$ are denoted by $\mathbb{R}^{n\times m}$; $\mathbb{R}^{0+}$ defines set of non-negative real numbers; $\mathbb{R}^{+}$ represents the set of positive real numbers while $\mathbb{R}^{-}$ depicts the set of negative real numbers. The $\backslash$-th order derivative of function $f(*)$ is denoted by $\mathcal{D}^{\backslash}_{*}[f(*)]$. Furthermore, for any constant $\alpha \in \mathbb{R}^{0+}$, the scalar function $\lfloor x \rceil^{\star} : \mathbb{R} \to \mathbb{R}$ is defined by $\lfloor x \rceil^{\star} = |x|^{\star}\mathrm{sgn}(x)$ where $\mathrm{sgn}(x)$ is defined by

$$\mathrm{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ +1 & x > 0 \end{cases}.$$

The state space representation of a system is given as follows

$$\dot{x} = f(t,x); x(0) = x_0, \tag{1}$$

where $f : \mathbb{R}^{0+} \times \mathcal{Q} \to \mathbb{R}^{n}$ is defined continuously on open neighborhood $\mathcal{Q}$ of the origin $f(0) = 0$. Solution of (1) for initial condition x_0 is defined by $x(t, x_0)$.

The following definitions are adopted from literature and are presented here for convenience of reader.

2.2. Definition-1 (Finite time Stability) [1,5]

The origin of system (1) is said to be finite time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ for arbitrary initial condition x_0, reaches the origin, after time $T(x_0)$ such that $x(t, x_0) = 0, \forall t \geq T(x_0)$, where $T(\cdot)$ is settling time function.

2.3. Definition-2 (Fixed time Stability) [5]

The origin of system (1) is said to be globally fixed time stable if for any initial condition, it is globally uniformly finite time stable, and the settling time function is *globally bounded.* i.e., there exists $T_{max} \in \mathbb{R}$ such that $T \leq T_{max}$ and $x(t, x_0) = 0$ for all $t \geq T$ and all $x_0 \in \mathbb{R}^{n}$.

2.4. Definition-3 (Predefined-Time Stability) [27]

The origin of system (1) is said to be predefined-time stable if it is fixed time stable and the settling time bound T_{max} is user-assignable, depending upon the controller parameters.

2.5. Lemma-1 [5,16]

Consider the system (1) is fixed time stable. Let a positive definite, radially unbounded function $V(x) : \mathcal{Q} \to \mathbb{R}^{n}$, defined on an open neighborhood $\mathcal{Q}$ of the origin $f(0) = 0$, such that $V(x) = 0 \Rightarrow x = 0$ then

$$\dot{V}(x) \leq -k_1 V(x)^{\surd} - k_2 V(x)^{\text{II}},$$

where $k_1, k_2, \surd, \text{II} \in \mathbb{R}^{+}$; $\surd < 1; \text{II} > 1$. The settling time function of (1) is bounded by

$$T \leq \frac{1}{k_1\left(1 - \surd\right)} + \frac{1}{k_2(\text{II} - 1)},$$

Furthermore, if $\surd = 1 - \frac{1}{\mu}$ and $\text{II} = 1 + \frac{1}{\mu}$ are selected such that $\mu > 1$, then settling time can be estimated by less conservative bound:

$$T_{max} := \frac{\pi\mu}{2\sqrt{k_1 k_2}} .$$

2.6. Sigmoid function

A sigmoid function $\varsigma(x), \forall x \in \mathbb{R}$ is a sufficiently smooth, differentiable, bounded, real valued "S-shaped" function which exhibits nonnegative rate of change everywhere in the state space i.e.$\mathcal{D}^{1}_{x}[\varsigma(x)] \geq 0$. The sigmoid functions are convex for all negative real numbers and concave otherwise [42]. The plot of some of the common sigmoid functions is shown in Fig. 1. It can be observed that all these functions are sufficiently smooth, antisymmetric, monotonic, and uniformly bounded. Moreover, for $\kappa \in \mathbb{R}^{+}$, the following property holds for this class of functions:

$$\begin{aligned} \max_{x\in\mathbb{R}} \varsigma(x) &= \lim_{x\to\infty} \varsigma(x) = 1/\kappa \\ \min_{x\in\mathbb{R}} \varsigma(x) &= \lim_{x\to-\infty} \varsigma(x) = -1/\kappa \end{aligned} \tag{2}$$

It is further emphasized that the sigmoid functions used in this article belong to class K^{1} as defined in Definition-4 of Ref. [35], if scaled by κ i.e. $\kappa\varsigma(x) \in \mathrm{K}^{1}$.

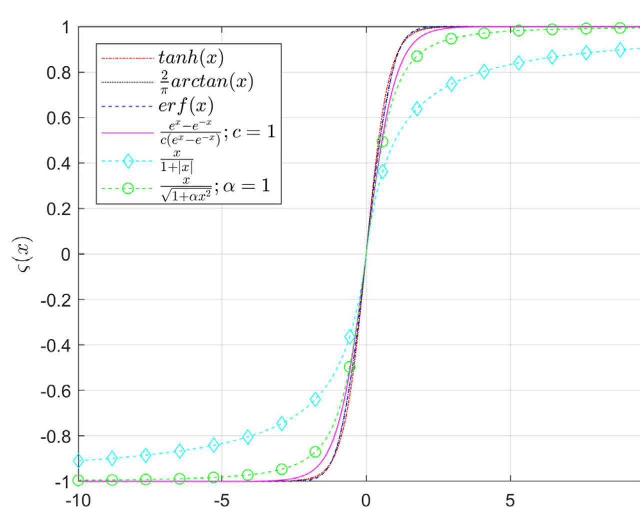

Fig. 1 Selected Sigmoid Functions.

3. Main results

Main results of the article are presented in this section.

3.1. Problem statement

We consider the following class of nonlinear second order systems

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x,t) + g(x,t)[\lceil(t,x) + u(t)], \end{aligned} \tag{3}$$

where$x := [x_1\ x_2]^T \in \mathcal{Q} \subseteq \mathbb{R}^{2\times 1}$;$\lceil\ : \mathbb{R}^{0+} \times \mathcal{Q} \to \mathbb{R}$; $t \in \mathbb{R}^{0+}$. $f : \mathcal{Q} \times \mathbb{R}^{0+} \to \mathbb{R}$ and $g : \mathcal{Q} \times \mathbb{R}^{0+} \to \mathbb{R}$ are smooth vector fields and $g(x,t) \neq 0$.

Assumption 1. The disturbance $\lceil(t,x)$ is bounded as follows:

$$|\lceil(t,x)| \le \rho(x) \quad \forall t \in \mathbb{R}^{0+}, \tag{4}$$

where $\rho(x)$ is known.

Assumption 2. $\dot{\varsigma}(x)$ is bounded by a sufficiently small constant.

Remark 1. To explain Assumption 2, consider $\varsigma(x) = \varsigma(cy)$ where c is a constant. $\varsigma(x)$ tends to $\mathrm{sgn}(x)$ as the constant c increases. As a result, $\dot{\varsigma}(x)$ tends to be indefinite and uniformity of proposed control is compromised. Therefore, the sigmoid functions with slow transitions are considered in this study. The purpose of this assumption is that the chosen sigmoid functions, in this paper, do not approach to signum function such that uniformity of the control law is preserved.

Objective-1

The goal is to drive a control law for the system (3) under Assumption-1 and Assumption-2 such that:

- The origin of the system is fixed time stable.
- The settling time bound is defined apriori, depending on controller parameters without knowledge of initial states.

Remark 2. According to Definition-3, the predefined-time convergence is established if fixed time stability is guaranteed with preassigned settling time depending on controller parameters, the Objective-1 then consequently, results in predefined-time convergence.

3.2. Theorem-1

The system (3) under Assumption-1 and Assumption-2 is predefined-time stable with robustness to matched perturbations and preassigned settling time bound T_s, with control law,

$$u(t) = -\frac{1}{g(t,x)} \begin{bmatrix} f(t,x) - \beta \frac{\mathcal{D}^2_{x_1}\left[\lfloor\varsigma(x_1)\rceil^{1/2}\mathrm{sgn}(x_1)\right]}{\left(\mathcal{D}^1_{x_1}\left[\lfloor\varsigma(x_1)\rceil^{1/2}\mathrm{sgn}(x_1)\right]\right)^2} x_2(t) \\ +\mu s(x)^2 + (\mu + \rho(x)|g(x,t)|)\ \mathrm{sgn}(s(x)) \end{bmatrix}, \tag{5}$$

and sliding surface,

$$s(x) = x_2 + 2\beta \frac{\lfloor\varsigma(x_1)\rceil^{1/2}}{\mathcal{D}^1_{x_1}[\varsigma(x_1)]}\ , \tag{6}$$

where $\varsigma(x_1)$ defines the antisymmetric sigmoid function with $\mu \le \frac{\pi}{(1-\eta)T_s}$; $\beta \le \frac{1}{\eta\sqrt{\kappa}T_s}$; $0 < \eta < 1$ and $\kappa = 1/\max(\varsigma(x_1))$.

Proof. Consider the following Lyapunov function:

$$V(x) = \frac{1}{2}s(x)^2,$$

then

$$\mathcal{D}^1_x[V(x)] = s(x)\mathcal{D}^1_x[s(x)]. \tag{7}$$

Solving $\mathcal{D}^1_x[s(x)]$

$$\mathcal{D}^1_x[s(x)] = \frac{\partial}{\partial x}\left[x_2 + 2\beta\frac{\lfloor\varsigma(x_1)\rceil^{1/2}}{\mathcal{D}^1_{x_1}[\varsigma(x_1)]}\right],$$

it is straightforward to show that above equation can be written in following form,

$$\mathcal{D}^1_x[s(x)] = \frac{\partial}{\partial x}\left[x_2 + \beta\left(\mathcal{D}^1_{x_1}\left[\lfloor\varsigma(x_1)\rceil^{1/2}\mathrm{sgn}(x_1)\right]\right)^{-1}\right].$$

Simplifying the right-hand side gives,

$$\mathcal{D}^1_x[s(x)] = \dot{x}_2 - \beta\frac{\mathcal{D}^2_{x_1}\left[\lfloor\varsigma(x_1)\rceil^{1/2}\mathrm{sgn}(x_1)\right]}{\left(\mathcal{D}^1_{x_1}\left[\lfloor\varsigma(x_1)\rceil^{1/2}\mathrm{sgn}(x_1)\right]\right)^2}\dot{x}_1.$$

Using the system dynamic equation (3) results in

$$\begin{aligned} \mathcal{D}^1_x[s(x)] &= f(x,t) + g(x,t)d(t,x) + g(x,t)u(t) \\ &\quad - \beta\frac{\mathcal{D}^2_{x_1}\left[\lfloor\varsigma(x_1)\rceil^{1/2}\mathrm{sgn}(x_1)\right]}{\left(\mathcal{D}^1_{x_1}\left[\lfloor\varsigma(x_1)\rceil^{1/2}\mathrm{sgn}(x_1)\right]\right)^2}x_2. \end{aligned}$$

Substituting control law (5) leads to

$$\begin{aligned} \mathcal{D}^1_x[s(x)] &= g(x,t)d(t,x) - \mu\lfloor s(x)\rceil^2 \\ &\quad - (\mu + \rho(x)|g(x,t)|)\ \mathrm{sgn}(s(x)). \end{aligned}$$

The rearrangement of terms on right side results in

$$\begin{aligned} \mathcal{D}^1_x[s(x)] &= [g(x,t)d(t,x) - \rho(x)|g(x,t)|\ \mathrm{sgn}(s(x))] \\ &\quad - \mu\left(\lfloor s(x)\rceil^2 + \mathrm{sgn}(s(x))\right). \end{aligned}$$

The following condition holds under Assumption-1:

$$g(x,t)d(t,x) - \rho(x)|g(x,t)|\mathrm{sgn}(s(x)) \le 0.$$

This condition helps the rejection of disturbances of the plant. Employing this condition, $\mathcal{D}^1_x[s(x)]$ becomes

$$\mathcal{D}^1_x[s(x)] \leqslant -\mu\left(\lfloor s(x)\rceil^2 + \mathrm{sgn}(s(x))\right)$$

Plugging in (7)

$$\mathcal{D}^1_x[V(x)] \leqslant -\mu s(x)\left[\lfloor s(x)\rceil^2 + \mathrm{sgn}(s(x))\right],$$

simplifying the right-side expression,

$$\mathcal{D}^1_x[V(x)] \leqslant -\mu\left[\lfloor s(x)\rceil^3 + \lfloor s(x)\rceil^1\right]$$

This inequality can be written in following form

$$\mathcal{D}^1_x[V(x)] \le -\mu V^{3/2} - \mu V^{1/2}.$$

Using Lemma-1, the maximum time required for states of system to attain sliding motion is

$$T \leq \frac{\pi}{\mu}.$$

Once the sliding motion is established on the surface $s(x) = 0$, the time for states to settle to origin can be calculated using (6) as follows;

$$\dot{x}_1 = -2\beta \frac{\lfloor \varsigma(x_1) \rceil^{1/2}}{\mathcal{D}^1_{x_1}[\varsigma(x_1)]}.$$

For the case $x_1(t) \geq 0$,

$$\frac{\mathcal{D}^1_{x_1}[\varsigma(x_1)]}{2[\varsigma(x_1)]^{1/2}} dx_1 = -\beta dt.$$

Integrating both sides

$$\sqrt{\varsigma(x_1(t))} - \sqrt{\varsigma(x_1(0))} = -\beta t.$$

At final time when states settle to zero, $t = T_f$, $x_1(t) = 0 \Rightarrow \sqrt{\varsigma(x_1)} = 0$. Hence,

$$\beta T_f = \sqrt{\varsigma(x_1(t=0))}$$

Since, $\varsigma(\cdot)$ is globally bounded, therefore

$$T_f \leq \frac{1}{\beta\sqrt{\kappa}}$$

Here κ is the inverse least upper bound of $\varsigma(x_1)$ as defined in (2) and eliminates the dependency of final time over the initial conditions. The same results are true for the case $x_1(t) \leq 0$ at the time $s = 0$ is attained. Hence, the fixed time convergence of proposed controller is guaranteed.

Now, the fixed time stability is ensured with uniformly bounded settling time:

$$T_f + T \leq \frac{1}{\beta\sqrt{\kappa}} + \frac{\pi}{\mu}$$

By substituting the constant values from Theorem-1,

$$T_f + T \leq \frac{\sqrt{\kappa} T_s \eta}{\sqrt{\kappa}} + \pi.\frac{(1-\eta)T_s}{\pi} = T_s$$

which marks the completion of proof of Theorem-1.

Remark 3. In the literature of sliding mode control (SMC), it is common to use sigmoid functions instead of signum function during sliding phase for chattering reduction as in [43,44]. As a result, the performance of SMC is compromised. In this study, we have employed the sigmoid functions in the terminal sliding manifold design. The advantage of using this type of surface is the achievement of predefined-time stability. It is noticeable that the signum function is still used during the sliding phase. Therefore, chattering is still observed in the controller.

Remark 4. In control law (5), the term $\mathcal{D}^2_{x_1}\left[\lfloor \varsigma(x_1) \rceil^{1/2} \mathrm{sgn}(x_1)\right] / \left(\mathcal{D}^1_{x_1}\left[\lfloor \varsigma(x_1) \rceil^{1/2} \mathrm{sgn}(x_1)\right]\right)^2$ exists for all values of x_1. The numerator and denominator are not defined individually at $x_1 = 0$, however, the overall term is defined at $x_1 = 0$ because the non-existent terms get canceled during the evaluation.

Remark 5. For piecewise continuous systems, whose solution is defined only in Filippov's sense, the $\mathrm{sgn}(\cdot)$ function is not suitable for analysis. In that case, the modified signum function $\mathrm{sgn}_{\{}(\cdot)$ can be used as follows:

$$\mathrm{sgn}_{\{}(x) = \begin{cases} -1 & x < 0 \\ +1 & x \geq 0 \end{cases}$$

4. Singularity avoidance

The terminal sliding control has associated singularity problem which occurs in condition where $x_1(t) = 0$ while $x_2(t) \neq 0$. A carefully designed switching scheme can resolve the singularity in terminal sliding mode control. The controller is switched to linear sliding mode instead of terminal sliding mode only when system trajectories enter in region resulting in singularity. One of the major drawbacks of switching control is that the sliding regime does not exist everywhere however, it has been shown in work of [45] that existence condition of sliding mode does not need to be satisfied everywhere in state space. Moreover, there is an increased convergence time as a cost of switching control [46]. The following lemma for switching control is useful:

4.1. Lemma-2 [20]

Consider the system (3) and the terminal sliding surface $\sigma(\S)$ of the form

$$\sigma(\S) = \S_2 + K\langle(\S_1); \quad K \in \mathbb{R}^+$$

The critical region $\Gamma_{\mathcal{S}}$, where singularity occurs in the terminal sliding mode control law, the following control input resolves the singularity problem:

$$u_{\mathcal{S}} = -\tfrac{1}{g(t,x)} f(t,x) + \rho(x)\mathrm{sgn}(x_2 g(x,t)) \tag{8}$$

Furthermore, the cost of singularity avoidance is the insertion of additional settling time $T_{\mathcal{S}}$ which is defined by:

$$T_{\mathcal{S}} \leq \max_{\S\in[0,\epsilon]} \left\{ \tfrac{K}{\langle(\S)} \S \right\} \tag{9}$$

where $\epsilon \in \mathbb{R}^+$ has arbitrarily small value.

Remark 6. For sliding surface of the form (6), it is obvious from the family of sigmoid curves of Figure-1 that

$$\max_{\S\in[0,\epsilon]} \left\{ \tfrac{\S}{\langle(\S)} \right\} = \max_{x\in[0,\epsilon]} \left\{ \tfrac{\mathcal{D}^1_x[\varsigma(x)]}{2\lfloor\varsigma(x)\rceil^{1/2}} \right\} \leq \zeta \tag{10}$$

where $\zeta \in \mathbb{R}^+$ defines the maxima.

4.2. Corollary-1

For the sigmoid functions of Table 1, a straightforward evaluation illustrates that for arbitrarily small value of ϵ, the bound on additional time (9) simplifies to $T_{\mathcal{S}} \leq \zeta\beta$. By construction, the results of Theorem-1 are valid for overall settling time $T^\star_s = T_s + T_{\mathcal{S}}$ with singularity free control input $u^\star$ defined by

Table 1 Selected Sigmoid Functions, associated sliding surface and bounds $\boldsymbol{\kappa}, \boldsymbol{\zeta}$.

Sigmoid Function $\varsigma(x_1)$	$\kappa = \frac{1}{\max(\varsigma(x_1))}$	Sliding surface	$\zeta \geq \max_{\S \in [0,\infty]} \left\{ \frac{\mathcal{D}_x^1[\varsigma(x)]}{2\lfloor \varsigma(x) \rceil^{1/2}} \right\}$
$\tanh(x_1)$	1	$s(x) = x_2 + 2\beta \cosh^2(x_1) \lfloor \tanh(x_1) \rceil^{1/2}$	0.2916
$Tan^{-1}(x_1)$	$2/\pi$	$s(x) = x_2 + 2\beta(1 + x_1^2) \lfloor Tan^{-1}(x_1) \rceil^{1/2}$	0.3010
$\mathrm{erf}(x_1) = \frac{2}{\sqrt{\pi}} \int_0^{x_1} e^{-t^2} dt$	1	$s(x) = x_2 + \beta \sqrt{\pi} e^{x_1^2} \lfloor \mathrm{erf}(x_1) \rceil^{1/2}$	0.3056
$\frac{e^{kx_1} - e^{-kx_1}}{c(e^{kx_1} + e^{-kx_1})} 0 < k \leq c/10 < 1; c \in \mathbb{R}^+$	c	$s(x) = x_2 + \beta \frac{c(e^{2kx_1} + 1)^2}{2ke^{2kx_1}} \lfloor \frac{e^{kx_1} - e^{-kx_1}}{c(e^{kx_1} + e^{-kx_1})} \rceil^{1/2}$	$\frac{0.2916}{\sqrt{c}}$
$\frac{x_1}{1+\lvert x_1 \rvert}$	1	$s(x) = x_2 + 2\beta(1 + \lvert x_1 \rvert)^2 \lfloor \frac{x_1}{1+\lvert x_1 \rvert} \rceil^{1/2}$	0.1925
$\frac{x_1}{\sqrt{1+\alpha x_1^2}} \alpha \geq 1$	$\sqrt{\alpha}$	$s(x) = x_2 + 2\beta(1 + \alpha x_1^2)^{\frac{3}{2}} \lfloor \frac{x_1}{\sqrt{1+\alpha x_1^2}} \rceil^{1/2}$	0.2675

$$u^\star = \begin{cases} u_S & \forall x \in \Gamma_S \\ u & otherwise \end{cases} \tag{11}$$

Furthermore, the new control parameters are defined by:

$$\eta = \frac{1}{T_s^\star} \sqrt{\frac{2\zeta}{\sqrt{\kappa}}}; \quad \beta = \frac{\eta T_s^\star}{2\zeta}; \quad \mu = \frac{\pi}{(1-\eta) T_s^\star} \tag{12}$$

Remark 7. To avoid the singularity in control law (6) the pre-assigned settling time must satisfy the following condition:

$$T_s^\star > \sqrt{\frac{2\zeta}{\sqrt{\kappa}}} \tag{13}$$

Some important sigmoid functions with corresponding sliding surfaces are given in Table 1. The associated constants κ (defined in (2)) and ζ (defined in (10)) are also provided alongside.

Remark 8. It is noticeable that the predefined-time convergence controller is proposed in this article in presence of external disturbances and without knowledge of initial states. The controller parameters η, β and μ can be selected using Table 1 and relations of (12). Moreover, the condition (13) is helpful in choosing the type of sliding manifold from Table 1 and does not change the global authenticity of the established theory. The controller parameters and their description is provided in Table 3, in Appendix A.

Remark 9. In presented methodology the settling time of controller directly relates to two parameters of sigmoid functions ζ and κ. From Table 1, these parameters depend on the upper bound of corresponding sigmoid function but not on the transition rate of sigmoid function. Therefore, the Assumption 2 will not affect the settling time of controller. The detailed block diagram of the proposed predefined-time controller is given in Fig. 2.

Remark 10. The results of Theorem 1 and hence Corollary-1 are valid for $\backslash^{th}$ order nonlinear MIMO square systems having $\backslash/2$ block diagonals, each with relative degree 2.

5. Numerical examples

In this section, two numerical examples are considered from existing literature. In Example-1a and Example-1b, we present the double integrator dynamics (without perturbation for analysis purpose). The performance of proposed controller for multiple initial conditions is analyzed in Example-1a. Whereas, the predefined-time convergence of controller is shown in Example-1b. In Example-2, we simulate the swing angle tracking control in SIP system under matched disturbances used by [18,20,34].

5.1. Example-1a (Double integrator dynamics)

First, we consider the example of double integrator dynamics with initial conditions $x_0 = \{(-10, 5), (10, 5), (-10, -5), (10, -5), (-1, -3), (4, 2)\}$. Let us pre-assign the settling time bound to be $8s$. By choosing the following sigmoid sliding surface $s(x)$

$$s(x) = x_2 + \beta \frac{c(e^{2kx_1} + 1)^2}{2ke^{2kx_1}} \lfloor \frac{e^{kx_1} - e^{-kx_1}}{c(e^{kx_1} + e^{-kx_1})} \rceil^{1/2}$$

For $k = 0.075; c = 0.75$; the control law (11) is applied with constant $\eta = 0.1102$, calculated using the guideline provided in *Remark-8*.

The results are shown in Fig. 3 and Fig. 4, which clearly depict that the goal of 8 s settling time have been achieved with very small over estimation. In Fig. 5, the control effort is shown for different initial conditions. It is evident from the results that a small control effort is required, when initial condition is close to the sliding surface. The proposed control has only drawback of chattering which cannot be fully eliminated while dealing with sliding mode control.

The sliding manifold $s = 0$ is shown in Fig. 6 along with the sate trajectories. It can be observed that starting from any initial condition, the system states move towards the sliding surface during the reaching phase. Once the trajectory reaches the sliding manifold, the controller ensures that it keeps moving on the nonlinear surface until the system is stabilized. All the sliding surfaces of Table 1 offer similar route to system trajectories.

The control effort for multiple initial conditions is given in Table 2. It is evident from Fig. 6 and Table 2 that the amount of control effort depends upon the initial distance of the states from the sliding manifold. For example, in Table 2 for initial conditions $(-10, 5)$ and $(-10, -5)$ are equidistant from the origin of system but the point $(-10, 5)$ is closer to the sliding surface and hence requires less control effort.

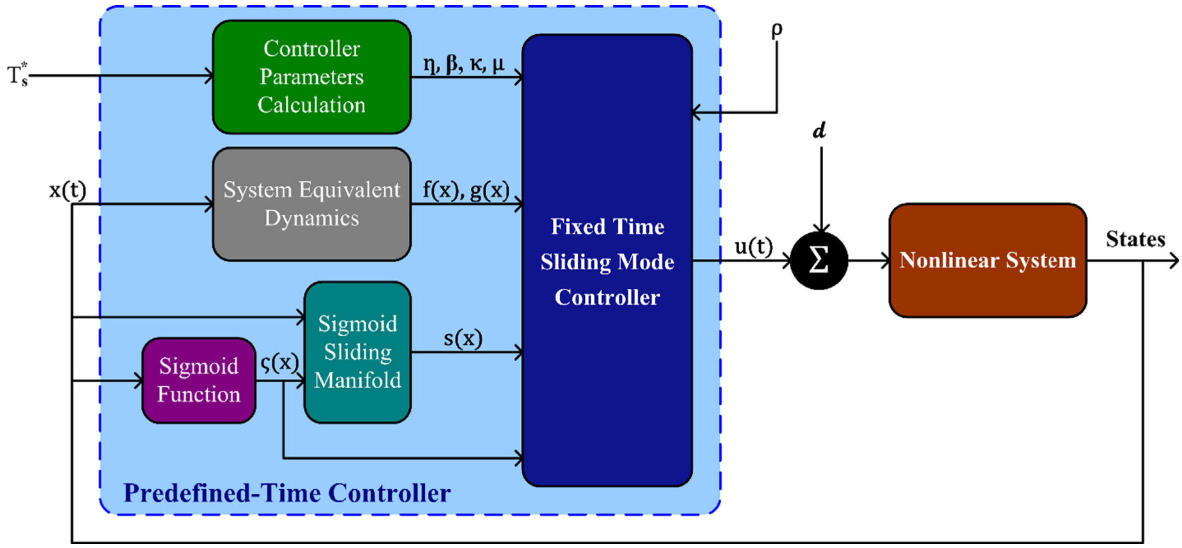

Fig. 2 Block Diagram of Predefined-time Controller.

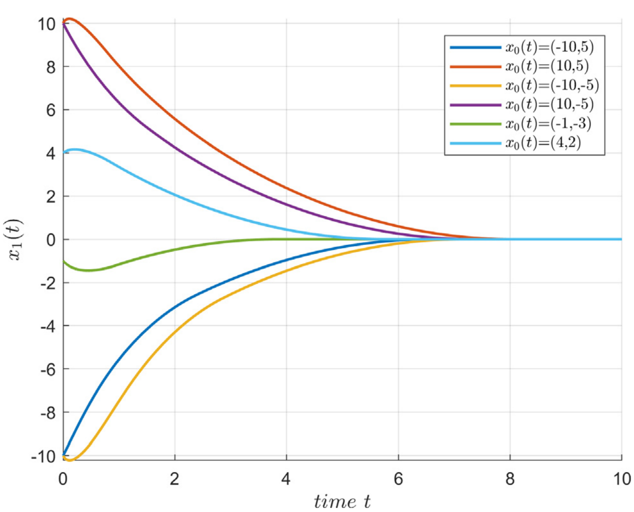

Fig. 3 Stabilization of x_1 state in integrator system.

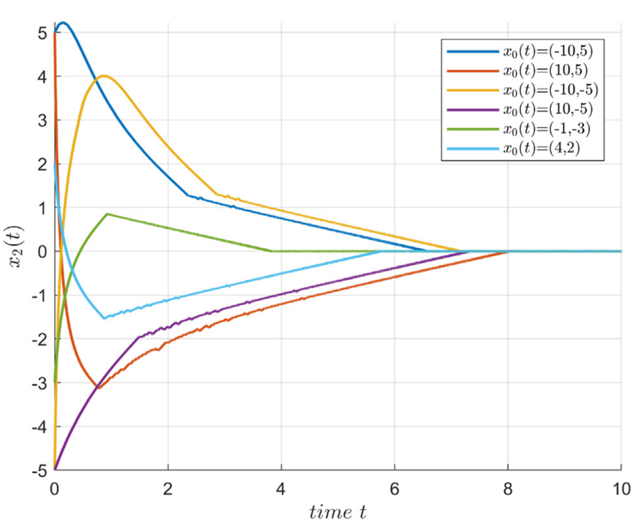

Fig. 4 Stabilization of x_2 state in integrator system.

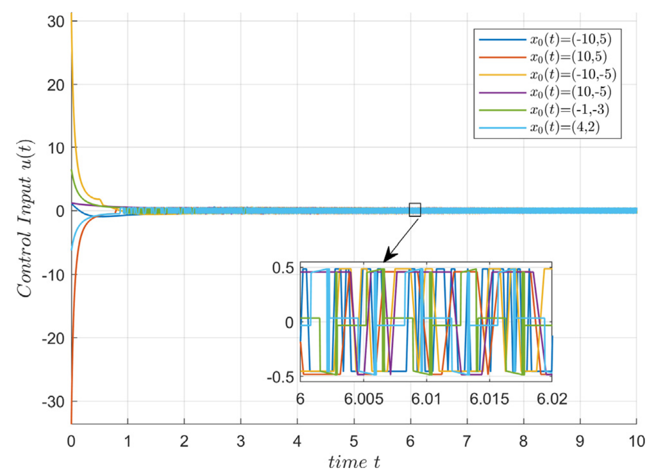

Fig. 5 Control effort for different initial conditions.

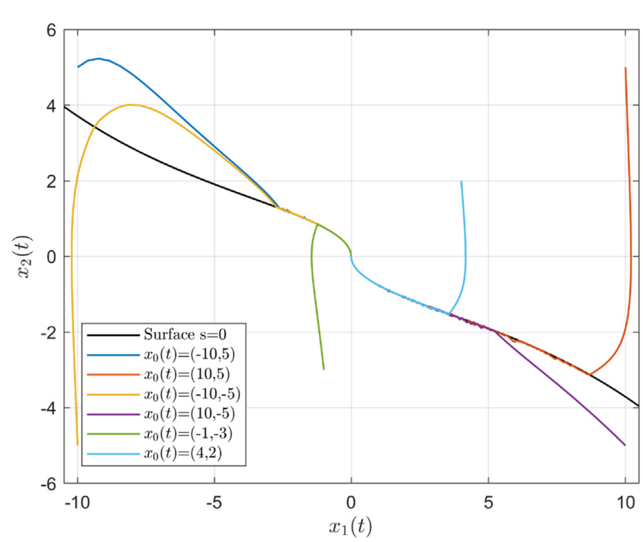

Fig. 6 Sliding manifold ($s = 0$) and state trajectories for different initial conditions.

Table 2 Control Effort for different initial conditions.

Initial Condition	$(-10, 5)$	$(10, 5)$	$(-10, -5)$	$(10, -5)$	$(-1, -3)$	$(4, 2)$
Maximum Control Effort	-1.132	-33.5763	31.2693	1.1938	6.5518	-6.2414

5.2. Example-1b

Now, we consider the example of double integrator dynamics for different predefined settling time. We define the initial condition of the system as $x_0 = (4, 2)$. By choosing the sigmoid sliding surface of Example-1a, the stabilization of the system is achieved by setting $T_s^\star = [4, 8, 12, 16, 20]$. The controller parameters are tuned using the guideline provided in *Remark-8*. In Fig. 7 and Fig. 8 the stabilization of $x_1(t)$ and $x_2(t)$ is presented, respectively. From the results, it can be deduced that states $x_1(t)$ and $x_2(t)$ stabilize to the origin within the prescribed upper bound of the settling time $T_s^\star$.

The control effort corresponding the $T_s^\star = [4, 8, 12, 16, 20]$ is provided in Fig. 9. By inspecting the simulation results, it is evident that control effort is dependent upon settling time i.e., with increase in the settling time, the less control effort is required and vice versa. Therefore, it is reaffirmed that calculation of less conservative bound on settling time bound has been advantageous to avoid extra control effort.

Example-2 (Swing angle Tracking of SIP [18])

Now we perform simulations for sinusoidal reference tracking of swing angle for SIP system as given in section 4 of [18]. The dynamics are transformed into error form that become similar to the system (3). The initial conditions of the system are set to $x_1(0) = 1$ and $x_2(0) = 0.5$. The disturbance $\lceil(t, x) = sin(10x_1(t)) + cos(x_2(t)) < 2 = \rho$. We choose *erf* type sliding manifold from Table 1.

First, we simulate for settling time $1.95s$ and compare results with those of [34]. Choosing $\eta = 0.4$ according to Theorem-1 and Corollary-1.

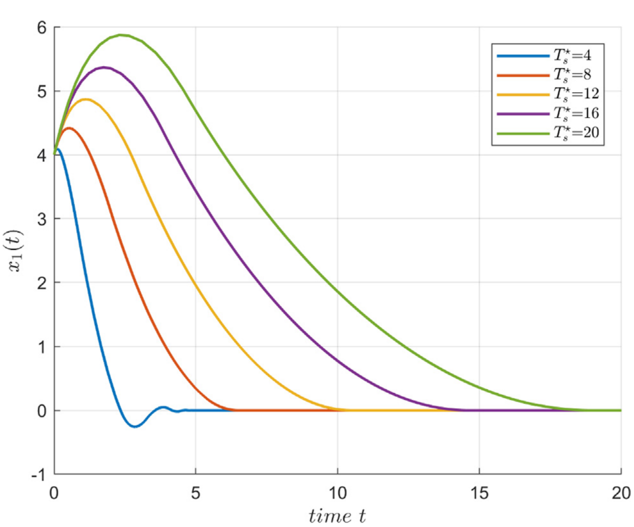

Fig. 7 Stabilization of x_1 state in integrator system for different predefined-time.

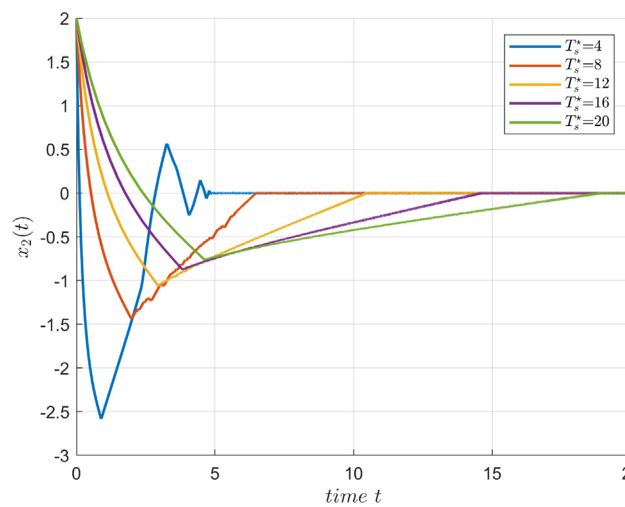

Fig. 8 Stabilization of x_2 state in integrator system for different predefined-time.

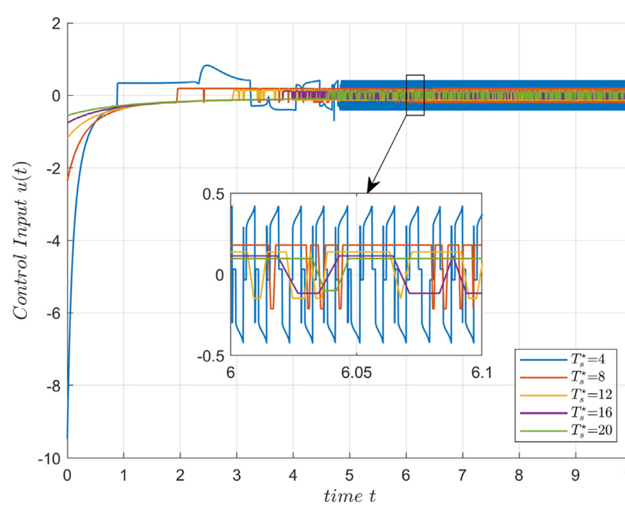

Fig. 9 Control effort for different predefined-time.

$$s(x) = x_2 + \beta\sqrt{\pi}e^{x_1^2}\lfloor \mathrm{erf}(x_1)\rceil^{1/2}$$

The parameter β is set to $\beta = 0.4280$ according to *Remark-8*. The resulting response of system is given in Fig. 10. It can be observed that exact settling time is almost $0.95s$ (1 s in case of [34]) and hence the desired response has been achieved. It is worth noting that the response for same desired settling time in [34] resulted in reference tracking in 0.4 s because of more conservative bound and it costs in requirement of more control effort i.e. $u_{max} = \max(u(t)) \approx 115$ in case of [34] compared to $u_{max} = 56$ in our case.

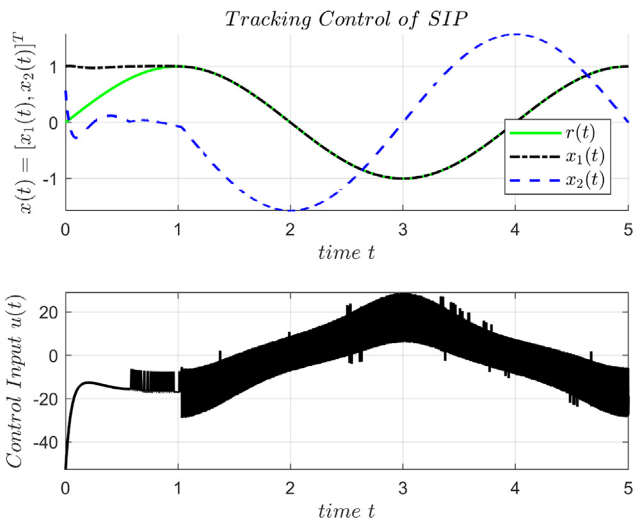

Fig. 10 Swing angle stabilization for SIP system ($T_s = 1$).

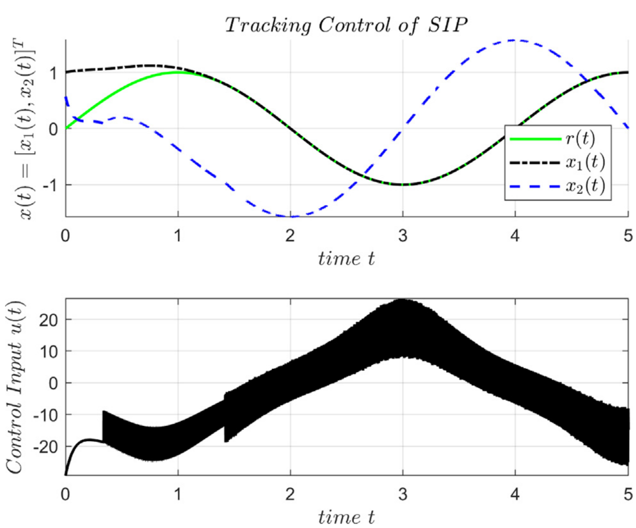

Fig. 11 Swing angle stabilization for SIP system ($T_s = 1.5$).

Table 3 Controller Parameters.

Control Parameter	Description	Defined in:
$T_s^\star$	Settling time of the predefined-time non-singular controller	–
T_s	Settling time of the fixed time controller without singularity avoidance	–
$T_\mathcal{S}$	Additional settling time due to singularity avoidance	–
κ	Inverse of maxima of sigmoid function	Table 1
ζ	Sliding manifold parameter that varies the $T_\mathcal{S}$	Table 1
η	Parameter that splits T_s into sliding time and reaching time	(Eq. (12))
β	Parameter that varies sliding time of the controller	(Eq. (12))
μ	Parameter that varies reaching time of the controller	(Eq. (12))
ρ	The upper bound on the disturbance	(Eq. (4))

Next, we set the upper bound to time as $1.5s$ to compare results with [18]. Using (11) the $\eta = 0.5212$ and $\beta = 0.6962$ according to *Remark-8*. The controller performance is comparable with results of [18]. It is important to note that, being far more conservative in approach, the calculated upper bound on settling time was $8s$ in [18] but system converged in $1.5s$. In our approach, we thus specified the predefined-time to be $1.5s$ with the goal to compare the control effort, in case of same convergence time. It is obvious from results of Fig. 11 that the control effort is comparable to that of as required in [18]. Hence, the effectiveness of proposed predefined-time controller is validated by results.

6. Conclusion

Predefined-time convergence for a class of nonlinear systems under unknown disturbances is studied in this paper. A new family of terminal sliding surfaces is proposed based on sigmoid functions, and convergence analysis is presented. The Lyapunov analysis revealed that designed controller not only ensures the fixed time stability of closed loop system but also the fixed time is less than the desired value. The singularity in control is countered using switching control and the sufficient condition for realization of nonsingular control is established. Exact sliding surfaces for some renowned sigmoid functions are provided and associated control parameters are calculated. The predefined-time convergence is established without varying the system parameters and without knowledge of initial states. The performance of designed controllers is evaluated for double integral dynamics and single inverted pendulum. The comparison of results with existing methods reveals the efficacy of proposed method. In future, the work will be extended to solve the higher order nonlinear systems.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Table 3.

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